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Dynamic characteristics of multiple inerter-based dampers for suppressing harmonically forced oscillations

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Abstract. Based on the ball-screw mechanism, a tuned viscous mass damper (TVMD) has been proposed, which has functions of amplifying physical mass of the system and frequency tuning. Considering the sensitivity of a single TVMD's effectiveness to frequency mistuning like that of the conventional tuned mass damper (TMD) and according to the concept of the conventional multiple tuned mass damper (MTMD), in the present paper, multiple tuned mass viscous dampers (MTVMD) consisting of many tuned mass dampers (TVMD) with a uniform distribution of natural frequencies are considered for attenuating undesirable vibration of a structure. The MTVMD is manufactured by keeping the stiffness and damping constant and varying the mass associated with the lead of the ball-screw type inerter element in the damper. The structure is represented by its mode-generalized system in a specific vibration mode controlled using the mode reduced-order method. Modal properties and fundamental characteristics of the MTVMD-structure system are investigated analytically with the parameters, i.e., the frequency band, the average damping ratio, the tuning frequency ratio, the total number of TVMD and the total mass ratio. It is found that there exists an optimum set of the parameters that makes the frequency response curve of the structure flattened with smaller amplitudes in a wider input frequency range. The effectiveness and robustness of the MTVMD are also discussed in comparison with those of the usual single TVMD (STVMD) and the results shows that the MTVMD is more effective and robust with the same level of total mass.

Keywords: MTVMD; TVMD; inerter; dynamic characteristic; passive vibration control

1. Introduction

Many passive energy-dissipation devices such as metal dampers, friction dampers and viscous dampers, have been developed and widely applied to building structures to suppress the excessive wind- or earthquake-induced vibration (Soong and Constantinou 1994, Liu *et al*. 2018). Tuned mass dampers (TMD) are also one of the most widely used control devices especially in high-rise buildings subjected to wind excitations. To address its high sensitivity to frequency mistuning or to variation of the primary structural parameters, multiple tuned mass dampers (MTMD) with distributed natural frequencies were proposed by Xu and Igusa (1992). It has been shown that the MTMD can be made more effective and robust than a conventional optimal single TMD (Abe and Fujino 1994, Kim and Lee 2018, Li 2002, Yamaguchi and Harnpornchai 1993, Zuo *et al*. 2017).

The conventional dampers are effective with respect to the relative displacement or relative velocity between two nodes. As an extension of this mechanical concept and by using the force-current analogy between mechanical and electrical networks, Smith (2002) firstly introduced the inerter element, which had the property that the force generated at its two terminals was proportional to the relative acceleration between them. The constant of proportionality is termed "inertance" and is measured in mass units. Various physical realizations of the ideal inerter have been proposed, such as the rack-pinion type (Smith 2002), the ball-screw type (Chen *et al*. 2009) and the hydraulic type (Wang *et al.* 2010). All these types of inerter transform the linear motion into the high-speed rotational motion and significantly amplifies the physical mass of the system. Initially, this device was employed in the suspension systems of Formula 1 racing car (Chen *et al*. 2009) and then successfully applied to many mechanical systems such as motorcycles (Evangelou *et al*. 2007), trains (Wang *et al.* 2009), optical tables (Wang and Wu 2014) and aircraft landing gears (Xin *et al.* 2015). Later, Wang *et al.* (2007) introduced this device to control the vibration of civil engineering structures. Then, Takewaki *et al.* (2012) further investigated the fundamental mechanism of earthquake response reduction in building structures with inerters and Chen *et al*. (2014) studied the influence of inerter on natural frequencies of vibration systems. Due to its performances of mass amplification and frequency tuning, the inerter-based damper attracts more researchers' attentions.

Considering different combinations among mass, spring,

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damper and inerter elements, various types of inerter-based damper are proposed. Hwang *et al*. (2007) developed a rotational inertia viscous damper (RIVD) for structural vibration control, in which a ball screw was utilized to transform the linear motion into high-speed rotational motion of an internal tube immersed in the viscous fluid that provided a significant damping force to the system. Based on the concept of RIVD, Ikago *et al*. (2012a) further proposed a tuned viscous mass damper (TVMD) to mitigate seismic induced vibrations of buildings, and they derived the closed-form solution of an optimum design for the TVMD incorporated into a SDOF structure using the fixed-point method (Den 1985). Some researchers introduced inerter into the conventional TMD system to increase its inertial property without evidently increasing its physical mass. Marian and Giaralis (2014) proposed a tuned mass-damper-inerter (TMDI) by utilizing the mass amplification effect of the inerter, and they analytically and numerically demonstrated that the TMDI was more effective than the TMD with the same amount of additional auxiliary mass in reducing the vibrations of stochastically support-excited structural systems. Giaralis and Taflanidis (2017) further showed the enhanced robustness of the TMDI to building uncertainties. A better performance was also found for suppressing the wind-induced vibrations of tall buildings by using TMDI (Giaralis and Petrini 2017). Lazar *et al*. (2014) developed a tuned inerter damper (TID), a special case of TMDI without the attached mass, to reduce the seismic-induced vibrations of a single degree-of-freedom (SDOF) system. Numerical results showed that the TID could be an attractive alternative to the TMD and other damper systems due to its small mass and overall size. Similarly, following the traditional layout of the TMD, Hu and Chen (2015) proposed different configurations of inerter-based dynamic vibration absorbers (IDVAs) by replacing the damper in the TMD with some inerter-based mechanical networks, which were one-terminal devices different from the TMDI and TID and have been successfully applied to the passive vibration control of vehicle suspension systems (Shen *et al.* 2016), beams (Jin *et al*. 2016) and cables (Lazar *et al.* 2016, Sun *et al.* 2017). It should be noted that all these aforementioned applications are focus on using one inerter-based control device in SDOF or multiple degree-of-freedom (MDOF) structures.

In practical application, more than one inerter-based damper may be used to achieve a better improvement in the response reduction of the primary structure. At present, the effectiveness of multiple inerter-based devices for suppressing vibration of a building has not been systematically studied. Ikago *et al.* (2012b) placed TVMDs between every storey in a multi-storey building, but all the TVMDs had the same parameters, which in essence still correspond to a single TVMD with the mass being the sum of the TVMDs. Zhang *et al*. (2016) introduced two parallel TIDs into the bottom storey of an idealized five-storey building and the optimized TIDs showed the remarkable effectiveness when the total inertance was smaller than 1500 kg. Wen *et al*. (2017) presented a comparable performance of distributed TVMDs and TIDs to control multiple modal vibrations relative to a singe TVMD and TID. However, there is less detailed discussion on the effect of parameters of TVMDs or TIDs, e.g., the frequency band and the total number of TVMD or TID included, on the vibration control of the primary structure to investigate the fundamental characteristics of multiple inerter-based dampers.

According to the concept of MTMD, multiple tuned viscous mass damper (MTVMD) consisting of many TVMDs with distributed natural frequencies is proposed in the present paper. The control performance of the proposed system is investigated through analytical case studies in the frequency domain. The arrangement of this paper is as follows: Section 2 gives the governing equation of the MTVMD-structure system as well as the frequency response functions of the structure and individual TVMD. To easily obtain a physical understanding of the dynamic characteristics of the system, some assumptions, i.e., the stiffness and damping coefficient of each TVMD are same and the natural frequencies of the MTVMD are uniformly distributed, are introduced in Section 3. Based on the simplified MTVMD-structure system, the effect of the parameters of the MTVMD on the modal properties is investigated in Section 4. Resulting from this effect on the modal properties, some fundamental characteristics of the MTVMD are presented in Section 5. Sections 6 and 7 show the effectiveness and robustness of the MTVMD. At last, Section 8 summarizes the major findings of the present work.

2. Frequency responses of the MTVMD-structure system

In the present study, MTVMD consisting of n TVMDs is considered for the control of the specific vibration mode of a structure. The analytical model is shown in Fig. 1 and the notations used in this study are shown in Table 1. In addition, assumption that natural frequencies of the structure are well-separated is adopted such that the structure can be adequately represented by an equivalent SDOF system. The main system is characterized by the mode generalized stiffness k_s, damping coefficient c_s and mass m_s, respectively. Each TVMD is also modelled as a SDOF system which is set to have different dynamic characteristics. Especially, the natural frequencies of the MTVMD are to be equally distributed around the natural frequency corresponding to the structural vibration mode to be mitigated. As a result, the total number of degree-of-freedom of this MTVMD-structure system is n+1. The analysis that follows is based on this simplified system.

The equations of motion for the analytical model subjected to a force $f(t)$ applied directly on the structural mass can be expressed as

$$\begin{cases} m_s\ddot{x}_s + c_s\dot{x}_s + k_s x_s + \sum_{j=1}^{n} k_j(x_s - x_j) = f(t) \\ m_j\ddot{x}_j + c_j\dot{x}_j + k_j(x_j - x_s) = 0 \quad j = 1, 2, ..., n \end{cases} \tag{1}$$

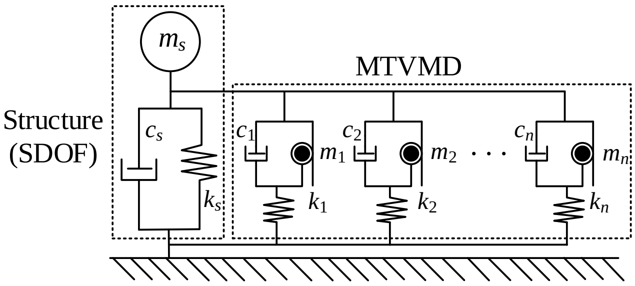


Fig. 1 MTVMD-structure system

Table 1 Notation

0	:	Mass of the primary system
c_s	:	Damping coefficient of the primary system
k_s	:	Stiffness of the primary system
x_s	:	Displacement of the primary system
$f(t)$	:	Applied force on the mass of primary system
$\ddot{x}_g(t)$	:	Base acceleration excitation
n	:	Total number of TVMD included
m_j	:	Equivalent mass of each TVMD
c_j	:	Damping coefficient of each TVMD
k_j	:	Stiffness of each TVMD
x_j	:	Relative displacement of the two inerter terminals of each TVMD
ω	:	Exciting frequency
ω_s	:	Natural frequency of the primary system
ξ_s	:	Damping ratio of the primary system
ω_j	:	N atural frequency of each TVMD
ξ_j	:	Damping ratio of each TVMD
$\gamma_j = \omega_j / \omega_s$	:	Tuning frequency ratio
$\lambda = \omega / \omega_s$	:	Input frequency ratio
$\mu_j = m_j / m_s$	:	Mass ratio
ω_T	:	Average frequency of all the TVMDs
ξ_T	:	Average damping ratio of the TVMDs
β	:	Non-dimensional frequency spacing of the TVMDs
$f = \omega_T / \omega_s$	:	Average tuning frequency ratio
m_T	:	Total mass of TVMDs
$\mu = m_T / m_s$	:	Total mass ratio
c_T	:	Constant damping for each TVMD
k_T	:	Constant stiffness for each TVMD
$H_s(\lambda)$	:	Displacement frequency response function of the primary system
$H_j(\lambda)$	:	Displacement frequency response function of each TVMD
I	:	Rotational moment of inertia of the physical mass in TVMD
ρ_j	:	Lead of the ball screw in TVMD
$\mathbf{M}$	:	Mass matrix of the MTVMD-structure system
$\mathbf{C}$	:	Damping matrix of the MTVMD-structure system
$\mathbf{K}$	:	Stiffness matrix of the MTVMD-structure system
$\mathbf{r}$	:	Influence coefficient vector due to the ground motion
Ω	:	Eigenvalues of the MTVMD-structure system
$\widehat{\omega}_i$	:	Natural frequency of the MTVMD-structure system
$\hat{\xi}_i$	:	Modal damping ratio of the MTVMD-structure system
$\boldsymbol{\phi}_i$	:	Mode vector of the MTVMD-structure system
$\phi_{j,i}$	:	The j^{th} component of $\boldsymbol{\phi}_i$
$\eta_{s,i}$	:	Modal participation factor in the frequency-domain

which can be further written in a matrix form as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}(t) \tag{2}$$

where the vector $\mathbf{x} = [x_s\ x_1\ x_2\ \ldots\ x_n]^{\mathrm{T}}$ is a n+1 dimensional displacement vector and x_j denotes the relative displacement of the two inerter terminals; $\mathbf{f}(t) = [f(t)\ 0\ 0\ \ldots\ 0]^{\mathrm{T}}$ is the applied loading vector. Since the TVMD is not activated by the ground motion but the displacement in the primary structure, the present study is also suitable for the case of base acceleration excitations. In such case, only the primary structure is subjected to the base excitation and $\mathbf{f}(t) = [-m_s\ddot{x}_g(t)\ 0\ 0\ \ldots\ 0]^{\mathrm{T}} = -\mathbf{M}\mathbf{r}\ddot{x}_g(t)$, where $\mathbf{r} = [1\ 0\ 0\ \ldots\ 0]^{\mathrm{T}}$ denotes the influence coefficient vector due to the ground motion and $\ddot{x}_g(t)$ is an ground acceleration. **M**, **C** and **K** are the mass, damping and stiffness matrices, respectively. They are given by

$$\mathbf{M} = \mathrm{diag}(m_s \quad m_1 \quad m_2 \quad \cdots \quad m_n)$$

$$\mathbf{C} = \mathrm{diag}(c_s \quad c_1 \quad c_2 \quad \cdots \quad c_n)$$

$$\mathbf{K} = \begin{bmatrix} k_s + \sum_{j=1}^{n} k_j & -k_1 & -k_2 & \cdots & -k_n \\ -k_1 & k_1 & & & \\ -k_2 & & k_2 & & \\ \vdots & & & \ddots & \\ -k_n & & & & k_n \end{bmatrix}$$

Herein, diag(·) denotes the diagonal matrix with diagonal elements given in braces.

The steady-state harmonic response of the system can be obtained by substituting $f(t) = e^{i\omega t}$ into Eq. (2) and using

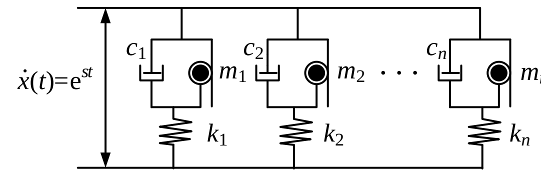


Fig. 2 Configuration for impedance definition

Cramer's rule for the solution, as done by Xu and Igusa (1992) and Yamguchi and Harnpornchai (1993). Alternatively, it is more convenient to use Laplace transformation. According to Eq. (1), the equation of motion for the primary structure in the Laplace domain can be expressed as

$$\left[m_s s^2 + c_s s + k_s + Z(s)\right] X_s(s) = f(s) \tag{3}$$

where $Z(s)$ is the impedance of the MTVMD. In this paper, impedance is defined to be the ratio of force to velocity as shown in Fig. 2. From this figure, $Z(s)$ can be derived as

$$Z(s) = \sum_{j=1}^{n} \frac{k_j (m_j s^2 + c_j s)}{m_j s^2 + c_j s + k_j} \tag{4}$$

Then, one can obtain the normalized displacement transfer function of the primary structure as

$$H_s(s) = \frac{X_s(s)}{X_0(s)} = \frac{k_s}{m_s s^2 + c_s s + k_s + \sum_{j=1}^{n} \frac{k_j (m_j s^2 + c_j s)}{m_j s^2 + c_j s + k_j}} \tag{5}$$

where $X_0(s) = f(s)/k_s$ represents the static displacement of the primary system.

Similarly, based on the Laplace transformation of Eq. (1), the frequency-domain displacement corresponding to x_j in the TVMD can be obtained as

$$X_j(s) = \frac{k_j X_s(s)}{m_j s^2 + c_j s + k_j} \tag{6}$$

For generality and convenience, the following parameters are introduced:

$$\omega_s^2 = \frac{k_s}{m_s} \quad \xi_s = \frac{c_s}{2 m_s \omega_s} \quad \omega_j^2 = \frac{k_j}{m_j} \quad \xi_j = \frac{c_j}{2 m_j \omega_j}$$

By substituting s=iω in Eq. (5), the frequency response function of the structure with MTVMD can be represented as

$$H_s(\omega) = \frac{\omega_s^2}{-\omega^2 + \mathrm{i}2\xi_s \omega_s \omega + \omega_s^2 + Z(\omega)/m_s} \tag{7}$$

where

$$Z(\omega) = \sum_{j=1}^{n} \frac{k_j(-\omega^2 + \mathrm{i}2\xi_j \omega_j \omega)}{-\omega^2 + \mathrm{i}2\xi_j \omega_j \omega + \omega_j^2}$$

Introducing the tuning frequency ratio of the jth TVMD, i.e., $\gamma_j = \omega_j / \omega_s$, the mass ratio of the jth TVMD, i.e., $\mu_j = m_j / m_s$, and the input frequency ratio of the excitation, i.e., $\lambda = \omega / \omega_s$, and carrying out mathematical manipulations, the frequency response functions of the structure and individual TVMD can then be written in the explicit and general form as

$$H_s(\lambda) = \frac{1}{-\lambda^2 + \mathrm{i}2\xi_s\lambda + 1 + \sum_{j=1}^{n} \frac{\mu_j \gamma_j^2 (-\lambda^2 + \mathrm{i}2\xi_j \gamma_j \lambda)}{-\lambda^2 + \mathrm{i}2\xi_j \gamma_j \lambda + \gamma_j^2}} \tag{8}$$

and

$$H_j(\lambda) = \frac{\gamma_j^2}{-\lambda^2 + \mathrm{i}2\xi_j \gamma_j \lambda + \gamma_j^2} H_s(\lambda) \tag{9}$$

respectively. Note that the response ratio of each TVMD in the above equation is exactly the same as that of a single TVMD (STVMD) because there is no direct interaction among the TVMDs in a MTVMD.

The displacement amplitudes in the structure and each TVMD can be represented by $|H_s(\lambda)|$ and $|H_j(\lambda)|$, respectively. Simultaneously, they are dynamic magnification factors of structural and TVMD responses. In this paper, the reduction in the amplitudes of the structural frequency response or the dynamic magnification factor of structural response is discussed numerically for different characteristics of MTVMD in order to investigate its performance.

3. Simplification for the MTVMD-structure system

In this paper, interest is on fundamental characteristics on the mechanism of control of this kind of MTVMD for the suppression of harmonically forced structural vibration. The harmonic responses of the MTVMD-structure system can always be evaluated using the exact expressions in Eqs. (8) and (9). However, when the TVMDs in the MTVMD have arbitrary properties, the system responses can be erratic and complicated. It is then difficult to obtain a physical understanding of the dynamic characteristics of the system. Therefore, in this section a special class of TVMD is examined. It is assumed that the stiffness and damping coefficient of each TVMD in the MTVMD are same and the natural frequencies of the MTVMD are uniformly distributed. As a result, the MTVMD is made by keeping the stiffness, damping constant and mass variation, i.e. $k_1 = k_2 = \ldots = k_n$; $c_1 = c_2 = \ldots = c_n$; $m_1 \neq m_2 \neq \ldots \neq m_n$. Note that for a ball-screw type TVMD sketched in Fig. 3, its equivalent mass or inertance is

$$m_j = I\left(\frac{2\pi}{\rho_j}\right)^2 \tag{10}$$

where I is the rotational moment of inertia of the physical mass in the damper and ρ_j is the lead of the ball screw. In the MTVMD, I can be fixed and ρ_j varies such that various equivalent masses of the TVMDs can be obtained.

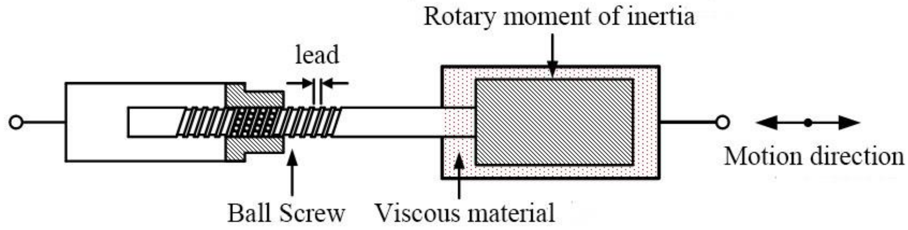


Fig. 3 Configuration of TVMD

Let ω_T be the average frequency of all the TVMDs, i.e., $\omega_T = \sum_{j=1}^{n} \omega_j / n$. The natural frequency of the jth TVMD can be presented as follows

$$\omega_j = \omega_T \left[1 + \left(j - \frac{n+1}{2} \right) \frac{\beta}{n-1} \right] \tag{11}$$

where the parameter β is the non-dimensional frequency spacing of the MTVMD, defined as

$$\beta = \frac{\omega_n - \omega_1}{\omega_T} \tag{12}$$

in which, ω_1 and ω_n represent natural frequencies of the first and the last TVMD, respectively.

If k_T and c_T are the constant stiffness and damping of each TVMD, respectively, then the mass and damping ratio of the jth TVMD are expressed as

$$m_j = \frac{k_T}{\omega_j^2} \quad \xi_j = \frac{c_T}{2 m_j \omega_j} = \frac{c_T}{2 k_T} \omega_j \tag{13}$$

Note that the mass of the TVMD in the above equation follows an inverse quadratic relation to the natural frequency of the TVMD and the damping ratio of each TVMD is proportional to its natural frequency such that the distribution of damping ratios of the MTVMD is consistent with that of its natural frequencies. The damping ratio spacing can be given by $\Delta\xi = \beta\xi_T$, where ξ_T is the average damping ratio of the TVMDs in the MTVMD, defined as

$$\xi_T = \frac{1}{n} \sum_{j=1}^{n} \xi_j = \frac{c_T \omega_T}{2 k_T} \tag{14}$$

The ratio of the total mass of the MTVMD to the mass of the main structure (i.e., the generalized mass of the structure in the mode that needs to be controlled) is referred to as the total mass ratio, i.e.,

$$\mu = \frac{m_T}{m_s} \tag{15}$$

where $m_T = \sum_{j=1}^{n} m_j$ is the total mass of the MTVMD.

The constant stiffness and damping of each TVMD may be evaluated using

$$k_T = \mu m_s \Big/ \sum_{j=1}^{n} \frac{1}{\omega_j^2} \qquad c_T = 2 \xi_T \mu m_s \Big/ \left(\omega_T \sum_{j=1}^{n} \frac{1}{\omega_j^2} \right) \tag{16}$$

The ratio of the average frequency of the MTVMD to the natural frequency of the main structure is defined as the tuning frequency ratio, i.e.,

$$f = \frac{\omega_T}{\omega_s} \tag{17}$$

Then, utilizing Eqs. (11)-(17), the parameters required in the frequency response functions in Eqs. (8) and (9) can be expressed as

$$\mu_j = \mu \Big/ \left(\kappa_j^2 \sum_{j=1}^{n} \frac{1}{\kappa_j^2} \right) \quad \gamma_j = f \kappa_j \quad \xi_j = \kappa_j \xi_T \tag{18}$$

where

$$\kappa_j = 1 + \left(j - \frac{n+1}{2} \right) \frac{\beta}{n-1}$$

After introducing these assumptions on MTVMD, it is apparent that the frequency response functions of the MTVMD-structure system only depend on the frequency band β, the average damping ratio ξ_T, the tuning frequency ratio f, the total number of TVMD n and the total mass ratio μ. The problem is thus simplified significantly.

If the natural frequencies of the TVMDs are equal to each other, this configuration degenerates to a STVMD with mass $m_0 = m_T$, damping $c_0 = c_T$ or damping ratio $\xi_0 = c_T / (2 m_T \omega_T)$ and natural frequency $\omega_0 = \omega_T$, as shown in Fig. 4. The frequency response function of the structure becomes

$$H_s(\lambda) = \frac{1}{1 - \lambda^2 + \mathrm{i}2\xi_s\lambda + \dfrac{\mu\gamma_0^2(-\lambda^2 + \mathrm{i}2\xi_0\gamma_0\lambda)}{-\lambda^2 + \mathrm{i}2\xi_0\gamma_0\lambda + \gamma_0^2}} \tag{19}$$

which is consistent with the result provided by Ikago *et al.* (2012a) and this type damper has been effectively used in controlling motion of SDOF and MDOF structures for the optimally tuned situation (Ikago *et al.* 2012a,b).

Like studies on MTMD (Abe and Fujino 1994, Kim and Lee 2018, Li 2002, Yamaguchi and Harnpornchai 1993, Zuo *et al.* 2017), these assumptions on MTVMD are introduced just to simplify the problem and do not change the fundamental characteristics of MTVMD. In this paper, following studies on MTVMD are based on the simplified analytical model.

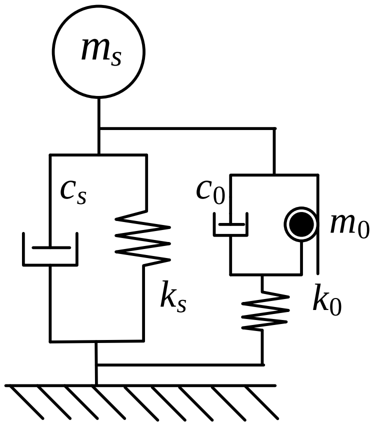


Fig. 4 STVMD-structure system

4. Modal properties of the MTVMD-structure system

The poles of $H_s(\lambda)$ correspond to the eigenvalues Ω (normalized by ω_s) of the MTVMD-structure system, according to Eq. (8), which can be obtained by solving the following 2(n+1) order polynomial with respect to Ω

$$\left(-\Omega^2+\mathrm{i}2\xi_s\Omega+1+\sum_{j=1}^{n}\mu_j\gamma_j^2\right)\prod_{j=1}^{n}\left(-\Omega^2+\mathrm{i}2\xi_j\gamma_j\Omega+\gamma_j^2\right)-\sum_{j=1}^{n}\mu_j\gamma_j^4\prod_{\substack{i=1\\ i\neq j}}^{n}\left(-\Omega^2+\mathrm{i}2\xi_i\gamma_i\Omega+\gamma_i^2\right)=0 \tag{20}$$

We know that there are 2(n+1) solutions to the above equation in the complex-valued field, which are real- or complex-valued depending on the amount of damping in the MTVMD-structure system. For simplification, in the present study attention is restricted to under-critical damping such that the eigenvalues of the system are complex-valued and appear in conjugate pairs. In such case, the i^{th} eigenvalue can be expressed as (Chen *et al.* 2017)

$$\Omega_i,\bar{\Omega}_i=-\frac{\hat{\omega}_i}{\omega_s}\hat{\xi}_i\pm\mathrm{i}\frac{\hat{\omega}_i}{\omega_s}\sqrt{1-\hat{\xi}_i^2} \tag{21}$$

where the superposed bar denotes complex conjugate; $\hat{\omega}_i$ and $\hat{\xi}_i$ are the i^{th} natural frequency and modal damping ratio, respectively. They can be obtained by

$$\frac{\hat{\omega}_i}{\omega_s}=\left|\Omega_i\right| \qquad \hat{\xi}_i=-\frac{\mathrm{Re}(\Omega_i)}{\left|\Omega_i\right|} \tag{22}$$

herein, |·| and Re(·) represent the modulus and real part of a complex value, respectively.

Substituting the calculated eigenvalues individually into the eigenvector equation, i.e.,

$$(-\Omega_i^2\omega_s^2\mathbf{M}+\mathrm{i}\Omega_i\omega_s\mathbf{C}+\mathbf{K})\boldsymbol{\phi}_i=\mathbf{0} \tag{23}$$

the corresponding mode vector, $\boldsymbol{\phi}_i$, can then be obtained. Referring to vibration shapes, these modes can be divided into two categories. One is associated with the main structure, named as structure-dominant mode, representing the whole motion of the MTVMD-structure system and the other is called TVMD-dominant mode, which reflects the local motion of individual TVMD. In this study, each mode is normalized such that the largest modulus of entities in the mode vector is unit. Since no interaction exists among these TVMDs, each TVMD-dominant mode is close to an n+1 dimensional vector, $\mathbf{e}_i$, with the i^{th} entity being unit and others zero, where i is the DOF identifier of each TVMD.

To investigate the effect of each mode on the structural response, it is convenient to express the frequency response function shown in Eq. (8) in terms of mode superposition as

$$H_s(\lambda)=-\sum_{i=1}^{n}\left(\frac{1}{a_i}\frac{\phi_{s,i}^2}{\mathrm{i}\lambda-\Omega_i}+\frac{1}{\bar{a}_i}\frac{\bar{\phi}_{s,i}^2}{\mathrm{i}\lambda-\bar{\Omega}_i}\right) \tag{24}$$

where

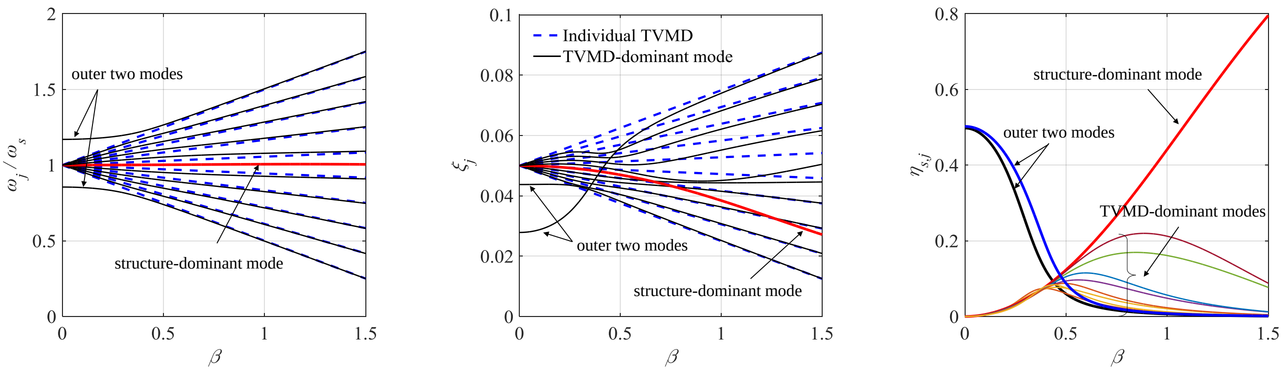
$$a_i=2\sum\nolimits_{j=1}^{n}\mu_j\phi_{j,i}^2(\Omega_i+\xi_j\omega_j/\omega_s)$$

$\phi_{j,i}$ is the j^{th} component of the i^{th} mode and $\phi_{s,i}$ denotes the component associated with the structure. Considering the modal participation factor in the modal superposition approach in the time-domain, define the modal participation factor in the frequency-domain as follows

$$\eta_{s,i}=\frac{\left|\phi_{s,i}^2/a_i\right|}{\sum_{j=1}^{n}\left|\phi_{s,j}^2/a_j\right|} \tag{25}$$

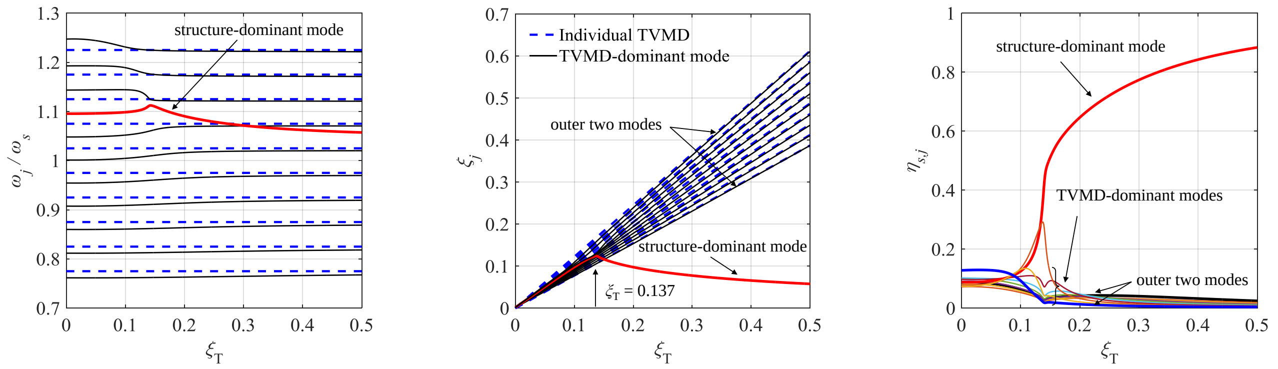
which reflects the relative contribution of the i^{th} mode to the structural response. Since the natural frequencies and modal damping ratios are also included in the above expression, $\eta_{s,i}$ should, to some degree, be viewed as a comprehensive indicator. This is very different from the conventional modal participation factor, which only represents the contribution of each mode to the spatial distribution of the applied loading.

Recalling the assumptions introduced into the MTVMD-structure system, it is apparent that the modal properties (the natural frequency, modal damping ratio and modal participation factor) are functions of the frequency band, the average damping ratio, the tuning frequency ratio, the total number of TVMD and the total mass ratio. In the following, the effect of these parameters on the modal properties is examined with only one parameter varying and others being constant. The analytical results are illustrated by several numerical examples. In all the examples, the natural frequencies are sorted in descending order and the modes are divided into two categories, i.e., structure-dominant mode and TVMD-dominant mode. The former is always at the center of all the modes and the latter further consists of the outer two modes and the intermediate modes. For the sake of comparison, the natural frequency and damping ratio of individual TVMD calculated by Eqs. (11) and (13) are also presented with dashed curves or discrete dots in the subsequent figures.



(a) frequency ratio (b) damping ratio (c) modal participation factor

Fig. 5 Effect of the frequency band on the modal properties (ξ_T=0.05, f=1.0, μ=0.1, n=10, ξ_s=0.02)

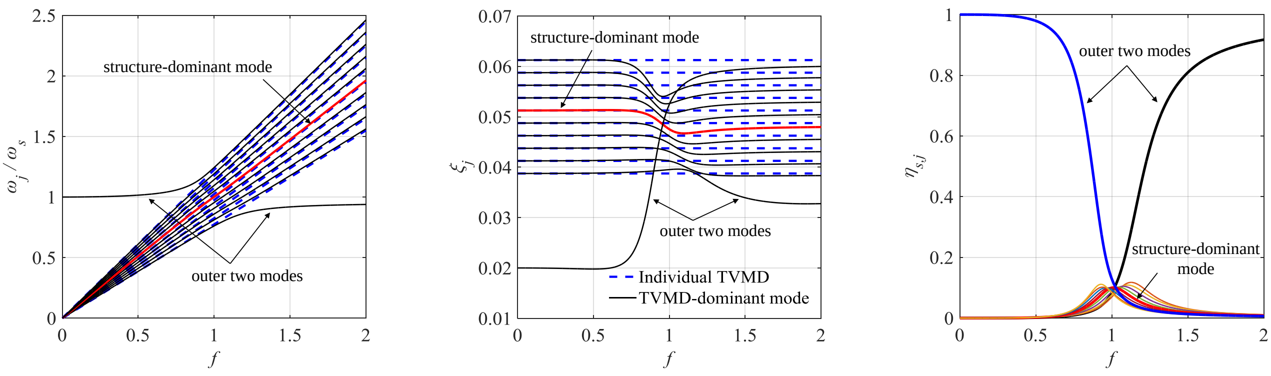


(a) frequency ratio (b) damping ratio (c) modal participation factor

Fig. 6 Effect of the average damping ratio on the modal properties (β=0.46, f=1.0, μ=0.1, n=10, ξ_s=0.02)

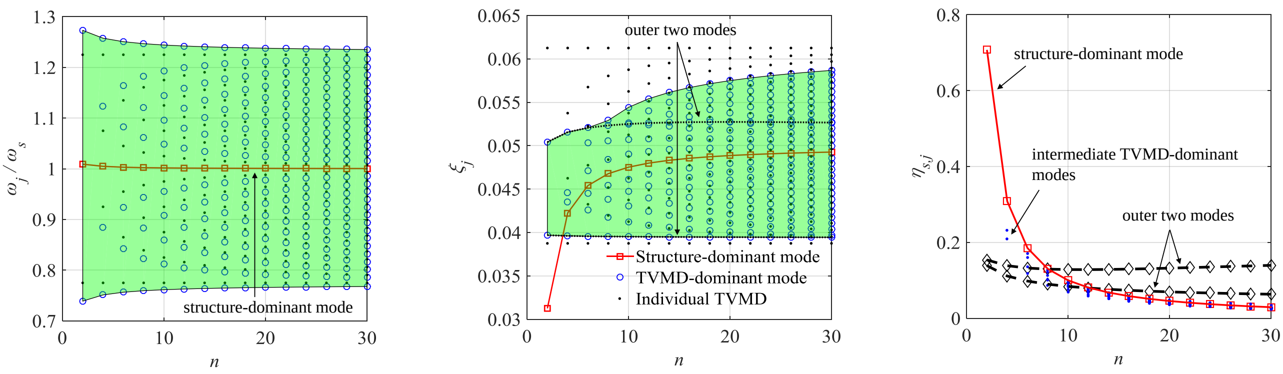
Fig. 5 shows how changes in the frequency band of the MTVMD affect the modal properties. Ten TVMDs are used with the average damping ratio ξ_T = 0.05, the tuning frequency ratio f = 1.0, the total mass ratio μ = 0.1 and the structural damping ratio ξ_s = 0.02. It can be observed that for the TVMD-dominant modes, the distribution or spacing of the natural frequencies increases continually as β increases, which means that the MTVMD-structure system becomes a system with well-separated natural frequencies from one with closely spaced frequencies and the interaction between the structure and the TVMD then becomes weaker for the input frequency around the structural natural frequency. Note that $\hat{\omega}_i$ and $\hat{\xi}_i$ in the first half modes decrease linearly with β increasing, which indicates some secondary peaks corresponding to these modes in the structural frequency response curve may appear in the case of larger frequency bands. It can also be found that the outer two (TVMD-dominant) modes significantly differ from other intermediate ones, particularly when β<0.46, their modal damping ratios are far lower than the average value, and their contributions to the structural response are very important as shown in Fig. 5(c). For the structure-dominant mode, the modal damping ratio decreases constantly with β increasing. In addition, the contribution of this mode becomes remarkable when β>0.46 (see Fig. 5(c)). As a result, it is easy to predict that two primary peaks locating at the natural frequencies of the two outer (TVMD-dominant) modes will appear in the frequency response curve of structure when β is small, otherwise a single primary peak locating at the structural natural frequency will appear in conjunction with some secondary peaks resulting from the nonignorable contribution of intermediate TVMD-dominant modes shown in Fig. 5(c). In other words, the structural response is firstly controlled by the outer two (TVMD-dominant) modes and then by the structure-dominant mode with β increasing from zero.

Fig. 6 depicts the analyzed modal properties for different average damping ratios. In this study, parameters required about MTVMD except the frequency band β = 0.46 are the same as those used in Fig. 5. It can be seen in Fig. 6(a) that the natural frequencies corresponding to the TVMD-dominant modes are hardly affected by the change of MVTMD damping and very close to the natural frequency of individual TVMD, especially in the case of high MTVMD damping, while the natural frequency of the structure-dominant mode significantly decreases after ξ_T = 0.137. Fig. 6(b) shows that all the damping ratios of TVMD-dominant modes increase linearly as ξ_T increases and are also very close to the damping ratio of individual TVMD. However, after the average damping ratio reaches a certain value (ξ_T is approximately 0.137 in this case), the damping ratio of the structure-dominant mode, starts decreasing gradually. This indicates that there exits a proper average damping ratio making the damping ratio of the structure-dominant mode maximum. After that, the contribution of the structure-dominant mode to the structural response increases sharply as shown in Fig. 6(c).



(a) frequency ratio (b) damping ratio (c) modal participation factor

Fig. 7 Effect of the tuning frequency ratio on the modal properties (β=0.46, ξ_T=0.05, μ=0.1, n=10, ξ_s=0.02)



(a) frequency ratio (b) damping ratio (c) modal participation factor

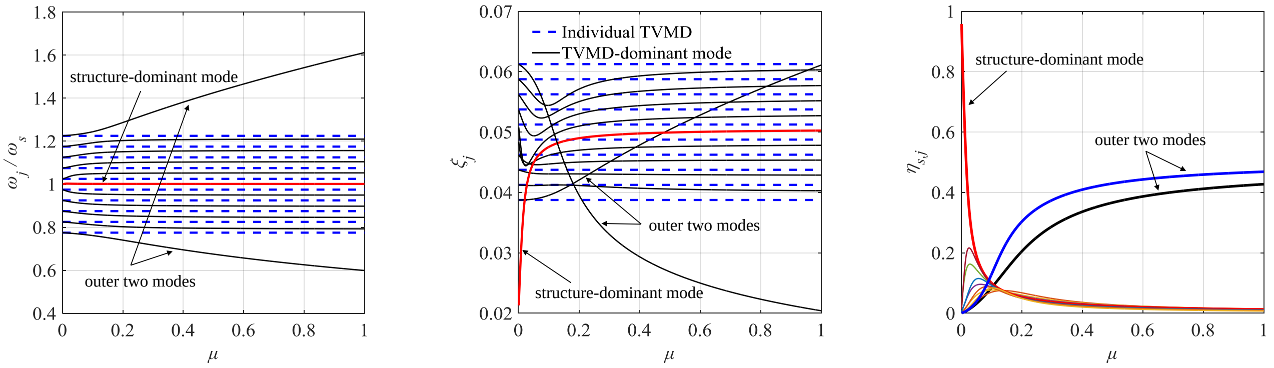
Fig. 8 Effect of the total number of TVMD on the dynamic characteristics (β=0.46, ξ_T=0.05, f=1.0, μ=0.1, ξ_s=0.02)

From this figure, we can also get that the structure-dominant mode almost dominates the whole structural response when the average damping ratio is larger.

As shown in Fig. 7, it is obvious that the effect of tuning frequency ratio on the outer two (TVMD-dominant) modes is very different from other modes. When f decreases gradually from 1.0, the natural frequency and damping ratio of the second of the outer two modes tend towards to those of the structure-dominant mode, while its contribution to the structural response becomes larger and larger. When f increases gradually from 1.0, the modal properties of the first of the outer two modes has the same trend as those of the second of the outer two modes. Therefore, when f varies around 1.0, the maximum value of the two primary peaks in the frequency response of the structure will swing from one side to another.

Fig. 8 presents the variation of the modal properties with respect to the total number of TVMD included in the MTVMD-structure system with β = 0.46, ξ_T = 0.05, f = 1.0, μ = 0.1 and ξ_s = 0.02. From this figure, it can be seen that the natural frequencies, damping ratios and modal participation factors of the outer two (TVMD-dominant) modes almost contain constant with the change of the number of TVMD when n is beyond 10. Likewise, the effect of the number of TVMD on the natural frequency of the structure-dominant mode is also ignorable. However, its modal damping ratio heavily depends on the number of TVMD, particularly in the case of smaller number of TVMD. Fig. 8(c) shows that the structure-dominant mode plays an important role in the structural response when a smaller number of TVMD is used and the contribution decreases with the number of TVMD increasing as same as that of the intermediate TVMD modes such that secondary peaks in the structural frequency response curve can be suppressed.

Displayed in Fig. 9 is the variation of the modal properties with the total mass ratio. Other parameters required in this numerical example about the MTVMD-structure system are the same as those used in the above studies. From Fig. 9(a), the total mass ratio mainly affects the natural frequencies of the outer two (TVMD-dominant) modes and one increases as μ increases, while the other is reverse. It can be found in Fig. 9(b) that when the total mass ratio is small, the damping ratio of the structure-dominant mode is very sensitive to μ and increases sharply as μ increases. What's more, in such case, the structure-dominant mode almost dominates the structural response as shown in Fig. 9(c). However, the contribution of the outer two (TVMD-dominant) modes to the structural response becomes more and more important with μ increasing. Therefore, when μ increases from a very low level, one primary peak firstly appearing in the structural frequency response curve, which corresponds to the structure-dominant mode, will decrease constantly resulting from the increase of the modal damping ratio or the decrease of the modal contribution, then two primary peaks will appear and increase gradually, which can be verified by Fig. 15 depicting the effect of total mass ratio on the structural response.



(a) frequency ratio (b) damping ratio (c) modal participation factor

Fig. 9 Effect of the total mass ratio on the dynamic characteristics (β=0.46, ξ_T=0.05, f=1.0, n=10, ξ_s=0.02)

From the previous discussions, the frequency band is a key design parameter because it controls the natural frequency spacing of the MTVMD-structure system and affects the effectiveness of MTVMD to suppress the vibration of structure. As shown in the above figures, the effect of the parameters of MTVMD on the structure-dominant mode and the outer two TVMD-dominant modes is very different from other modes and more contribution to the structural response comes from the structure-dominant mode or the outer two TVMD-dominant modes. The increases of both the frequency band and the average damping ratio make the damping ratios of the outer two TVMD-dominant modes increase (for the frequency band, one damping ratio of the outer two TVMD-dominant modes decreases, but the contribution of this mode also decreases). The total number of TVMD and the total mass ratio play an important role in the increase of the damping ratio of the structure-dominant mode, while larger frequency band and average damping ratio significantly reduce the damping ratio of the structure-dominant mode. All the design parameters can adjust the relative contributions of the structure-dominant mode and the outer two TVMD-dominant modes. These modal properties determine the performances of MTVMD in the following section.

5. Performance of MTVMD

As shown in Eqs. (8) and (18), the structural frequency response is associated with the frequency band β, the average damping ratio ξ_T, the tuning frequency ratio f, the total number of TVMD n and the total mass ratio μ. In this section, the fundamental characteristics of MTVMD are investigated numerically with these parameters by the reduction in the structural frequency response. For convenience of study, only one parameter varies and other four parameter are fixed in each numerical example. Moreover, the structural damping ratio keeps constant and ξ_s = 0.02 is also selected as the same as that in the aforementioned studies of modal properties.

5.1 Effects of frequency band

Fig. 10 depicts the analyzed frequency response curves of the structure for five different values of frequency band.

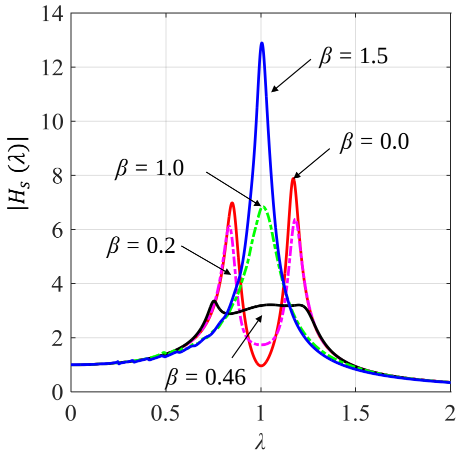


Fig. 10 Frequency response curves of structure for different frequency bands of MTVMD (ξ_T=0.05, f=1.0, μ=0.1, n=20, ξ_s=0.02)

The total mass ratio μ is 0.1 and the tuning frequency f=1.0. To adequately suppress secondary peaks, the total number n of TVMD should be large enough such that the frequency response curves can be smoothed adequately (see Fig. 13 for more details). In this example, n is fixed to be 20. Similarly, the average damping ratio ξ_T is set to be 0.05 according to Fig. 11, which is relatively very small compared with the optimum damping ratio (about 0.20) for a single TVMD.

As can be seen from Fig. 10, the controlled structural response is transformed from a two-peak characteristic to a one-peak characteristic with the increase in frequency band of MTVMD due to the change in contributions of the outer two TVMD-dominant modes and the structure-dominant mode to the structural response shown in Fig. 5(c). From the perspective of physical significance, if the frequency band is large enough, the natural frequencies of the MTVMD-structure system are well-separated such that the interaction between the structure and TVMD is very weak and the system behaves like a SDOF system with natural frequency around that of the structure. If the frequency band is zero, all the natural frequencies of TVMD are practically equal to each other and the behavior of the MTVMD-structure system becomes that of the STVMD-structure

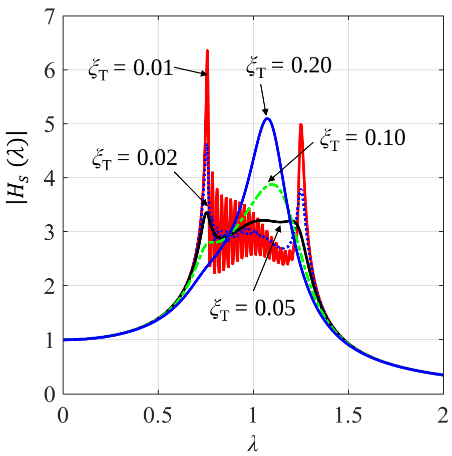


Fig. 11 Frequency response curves of structure for different average damping ratios (ξ_T=0.05, f=1.0, μ=0.1, n=20, ξ_s=0.02)

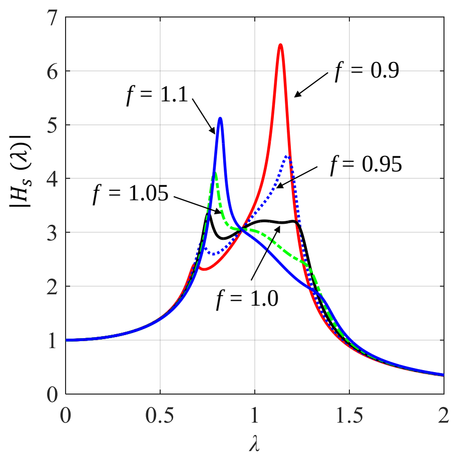


Fig. 12 Frequency response curves of structure for different tuning frequency ratios (β=0.46, ξ_T=0.05, μ=0.1, n=20, ξ_s=0.02)

system, i.e., the maximum structural response is controlled by two primary peaks. Apparently, there is a proper or optimum value of the frequency band of MTVMD (0.46 for the case of Fig. 10) which makes the frequency response curve flat for a wide range of frequency (the maximum structural response is about 3.35 and the frequency range of the flat region is about 0.45). This is also the mechanism of vibration control by MTVMD. Because of this control mechanism, the effectiveness of MTVMD is very much dependent on the frequency band.

It should also be noted that the effect of MTVMD is reduced if the frequency band increases too much as shown in Fig. 10. In addition, there are several slight secondary peaks in the response curve in the case of β = 1.0 and 1.5 as a result of the decrease of damping ratios of some lower TVMD-dominant modes (see Fig. 5(b)) or the increase of their contribution to the structural response (see Fig. 5(c)).

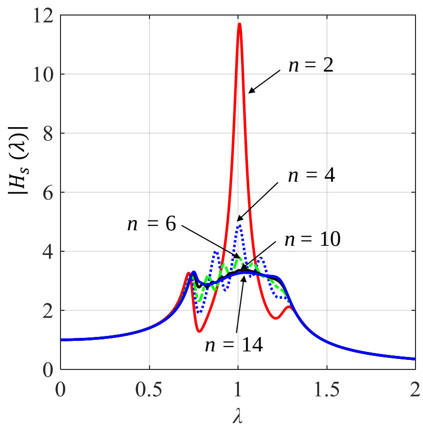


Fig. 13 Frequency response curves of structure for different numbers of TVMD (β=0.46, ξ_T=0.05, f=1.0, μ=0.1, ξ_s=0.02)

5.2 Effects of average damping ratio

Fig. 11 shows the changes in the frequency response curve of the structure due to the different values of average damping ratio of the MTVMD. The frequency band, the tuning frequency ratio, the total mass ratio and the total number of TVMD selected are 0.46, 1.0, 0.1 and 20, respectively. From this figure, it can be seen in the case of very small damping ratio that there exist a large number of significant secondary peaks, caused by the resonances of the intermediate TVMD, between the outer two peaks, which correspond to the outer two TVMD-dominant modes, and that one of the outer two peaks gives the maximum response of the structure. On the contrary, these secondary peaks vanish if the average damping ratio increases, since the damping ratios of the intermediate TVMD-dominant modes significantly increase as well (see Fig. 6(b)). As the average damping ratio continues to grow, the maximum response of the structure occurs at the resonance frequency of the structure due to the important contribution of the structure-dominant mode as shown in Fig. 6(c) and the frequency response curve with two primary peaks turns into one with a single primary peak. The amplitude of the single primary peak, however, becomes larger if the damping ratio is too large. This is because the damping ratio of the structure-dominant mode becomes very small, and its contribution to the structural response is very remarkable in such case. In fact, in the case of a very large damping ratio, each TVMD is almost fixed to the structure such that the system becomes a SDOF one and the vibration energy in the structure cannot be dissipated. It can be concluded that the damping in MTVMD plays an important role in reducing the secondary peaks and the two primary peaks if the damping ratio is smaller, while the excessive damping increases the structural resonant peak. Therefore, an optimum average damping ratio exists in the MTVMD.

5.3 Effects of tuning frequency ratio

Fig. 12 presents the variation of frequency response curves of the structure with regard to the tuning frequencyratio. The frequency band, the average damping ratio, the total mass ratio and the total number of TVMD are set to 0.46, 0.05, 0.1 and 20, respectively. It is observed that

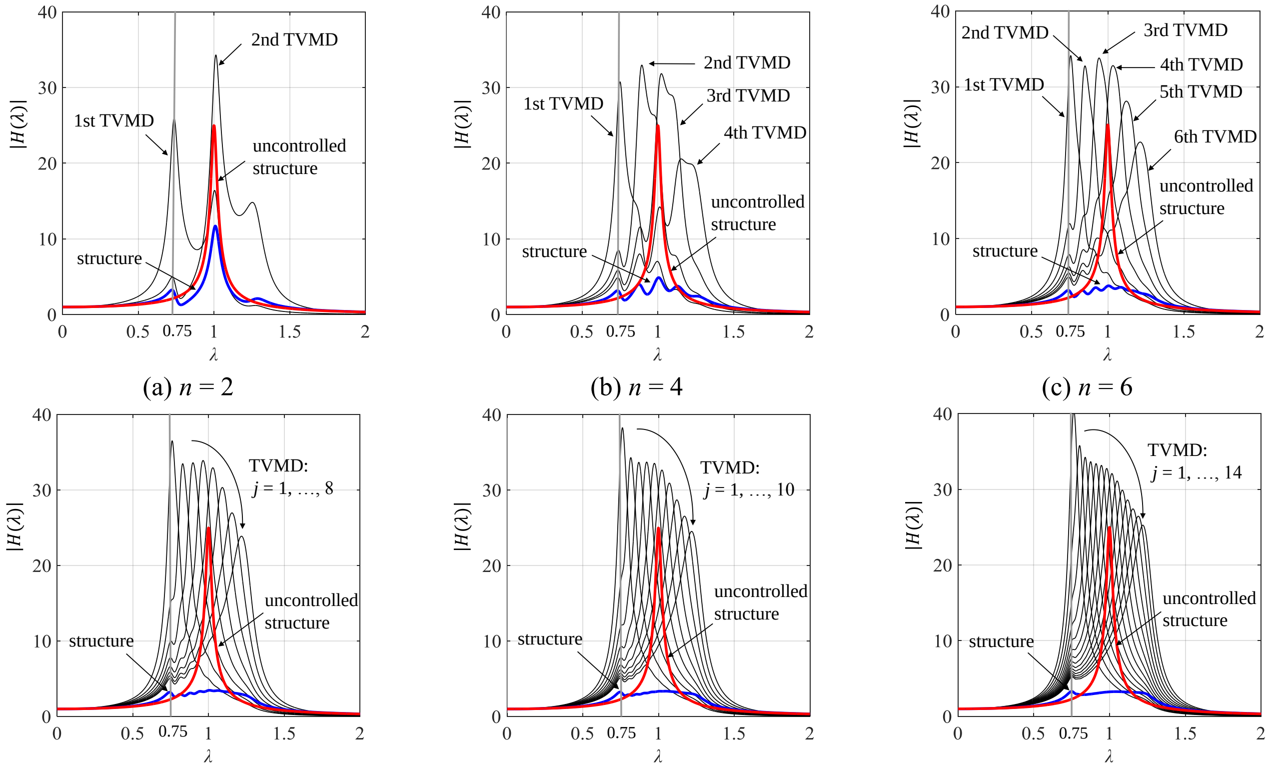


Fig. 14 Frequency response curves of structure and TVMD for different numbers of TVMD

changing f can shift the amplitudes of the two primary peaks, one peak going up and the other going down due to the change in the contributions of the two outer TVMD-dominant modes as shown in Fig. 7. The most favorable case is where the tuning frequency has equal amplitudes at the two primary peaks such that the response curve has the smallest vibration peak. Thus, the tuning frequency ratio plays a role in adjusting the heights of two primary peaks, which is identical to that in the conventional TDM, MTMD and STVMD systems.

5.4 Effects of total number of TVMD

The effectiveness of MTVMD is also very dependent on the total number of TVMD used in the MTVMD-structure system, as can be seen from Fig. 13, which depicts different response curves of structure for different total numbers of TVMD under the condition of β = 0.46, ξ_T = 0.05, f = 1.0 and μ = 0.1. It is clear that increasing the total number of TVMD has the effect of reducing the magnitude of the secondary peaks due to the decrease in the contribution of the intermediate TVMD-dominant modes to the structural response (see Fig. 8(c)), which is very similar to the effect of the average damping ratio of MTVMD as shown in Fig. 11. Also, as n increases continually, the primary peak is reduced remarkably. This is because the primary peak is dominated by the structure-dominant mode, whose damping ratio becomes higher with the increase of n as shown in Fig. 8(b). However, when n reaches a certain value, increasing n continually no longer reduces the primary peak significantly since the damping ratio of the structure-dominant mode

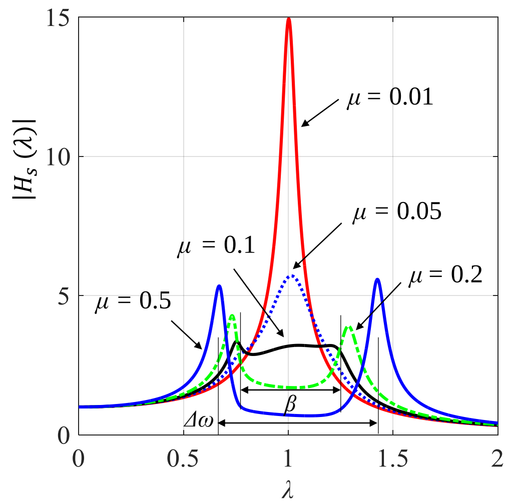


Fig. 15 Frequency response curves of structure for different total mass ratios (β=0.46, ξ_T=0.05, f=1.0, n=20, ξ_s=0.02)

almost keeps constant (see Fig. 8(b)). Therefore, there is a minimum number of TVMD for the effectiveness of MTVMD.

Like the traditional TMD, the working principle of MTVMD is also using resonance to introduce the vibration of structure into the TVMDs and as a result, the vibration energy is dissipated through the damping in the TVMDs. For instance, at λ = 0.75 and 1.0 in Fig. 14, there are always peaks with different amplitudes in all the frequency response curves of TVMD, which indicates more structural vibration is absorbed into the MTVMD at the two input frequencies. Hence, the peaks in the frequency response curve of the structure can be reduced continually, particularly the peak at λ = 1.0, as the total number of TVMD increases. This process is presented in Fig. 14. It should be noted that the effective frequency range of each TVMD (in the frequency range, more vibration occurs in the TVMD) is limited such that a reasonable number of TVMD is required to flatten the structural response curve in a wider input frequency band. In addition, we know that TVMD is a two-terminal device and activated by the displacement in the structure. However, from Fig. 14, the displacement in the damper is far larger than that in the structure. This is resulting from the negative stiffness characteristic of the inerter element such that the relative velocity and displacement in the viscous damper element connected to the inerter element can be amplified largely and more energy thus are dissipated.

5.5 Effects of total mass ratio

Fig. 15 shows the effect of different total mass ratios on the frequency response of the structure. Herein, β = 0.46, ξ_T = 0.05, f = 1.0 and n = 20. From this figure, it is obvious that increasing μ can constantly reduce the structural response within the input frequency range of β, while two primary peaks associated with the outer two TVMDs grow significantly and their spacing $\Delta\omega=(\omega_n-\omega_1)/\omega_s$, where ω_1 and ω_n are natural frequencies of the outer two (TVMD-dominant) modes, also increases. This is identical to the description of Fig. 9. In general, the total mass ratio is known or given in advance. However, it should be noticed that excessive mass ratios may result in larger peaks if the damping ratio cannot be changed simultaneously.

According to the previous discussions based on the analytically calculated results of structural response curve, it can be said that the maximum response of the structure is controlled by the structure-dominant mode or one of the outer two TVMD-dominant mode, heavily depending on the selection of these parameters, unlike the MTMD in which the secondary peaks may dominate the maximum response of the structure (Yamaguchi and Harnpornchai 1993).The frequency band and the average damping ratio can be designed to reduce the amplitudes of the two primary peaks associated with the outer two TVMD-dominant modes, while larger values in the two parameters lead to the transformation of the two primary peaks to a single primary peak corresponding to the structure-dominant mode and the peak increases with the increase of the two parameters. The relative heights of the two primary peaks can be adjusted by the tuning frequency ratio. The number of TVMD and the total mass ratio can be used to suppress the primary peak at the structure-dominant mode, but larger values in the two parameters cannot further improve the effectiveness of MTVMD. Moreover, the effective input frequency range where the MTVMD works well is mainly determined by the frequency band. Therefore, a appreciate set of the parameters is one that makes the frequency response curve of the structure flattened and amplitudes smaller in a wider input frequency range with a lesser number of TVMD and total mass.

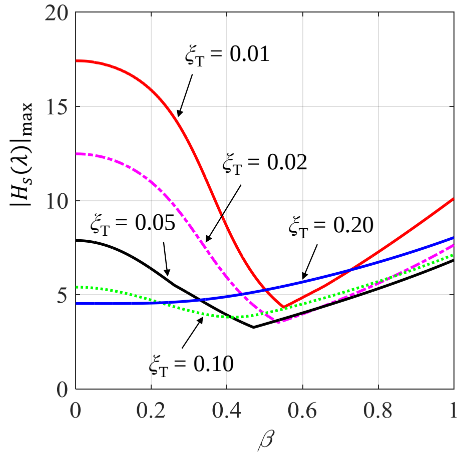


Fig. 16 Effectiveness of MTVMD versus frequency band for different damping ratios

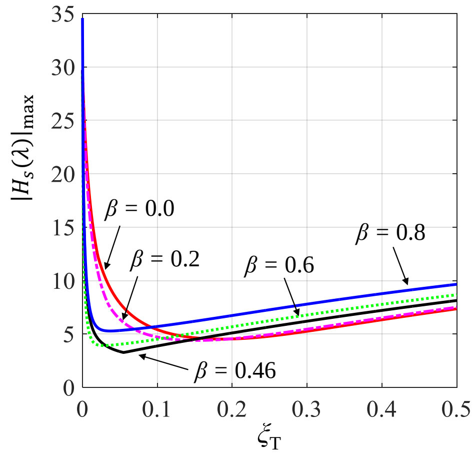


Fig. 17 Effectiveness of MTVMD versus damping ratio for different frequency bands

6. Effectiveness and robustness of MTVMD

6.1 Effectiveness of MTVMD

The maximum peak value in the frequency response curve is used as one of the performance indices which represent the effectiveness of a MTVMD. It is noted that the MTVMD which gives the smaller value of $|H_s(\lambda)|_{max}$ is more effective. In this study, the total mass ratios is μ = 0.1. Since much larger number of TVMD is unnecessary for the vibration control as shown in Fig. 13, the effect of the total number of TVMD on $|H_s(\lambda)|_{max}$ is not considered here. To suppress secondary peaks, the total number of TVMD n = 20 is selected. The structural damping is still 0.02.

Fig. 16 shows the relation between the effectiveness of a MTVMD, or the maximum structural response, and the frequency band of the MTVMD for different values of damping ratio with the tuning frequency ratio f = 1.0. It can be clearly seen that there is a well distinguished optimum frequency band which gives the minimum value of the maximum structural response or the maximum effectiveness, for each damping ratio ξ_T. This figure also shows that the optimum frequency band becomes smaller with the increase

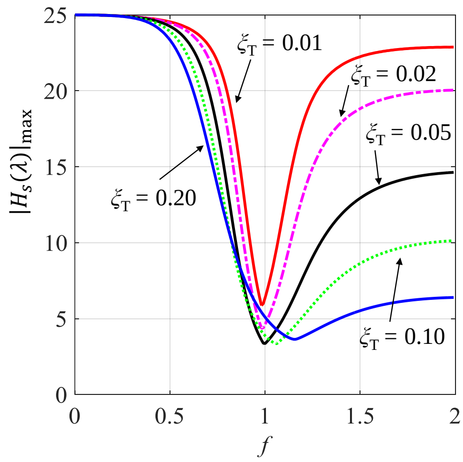


Fig. 18 Effectiveness of MTVMD versus tuning frequency ratio for different damping ratios

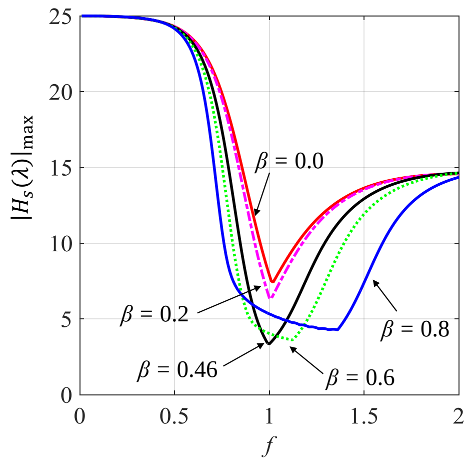
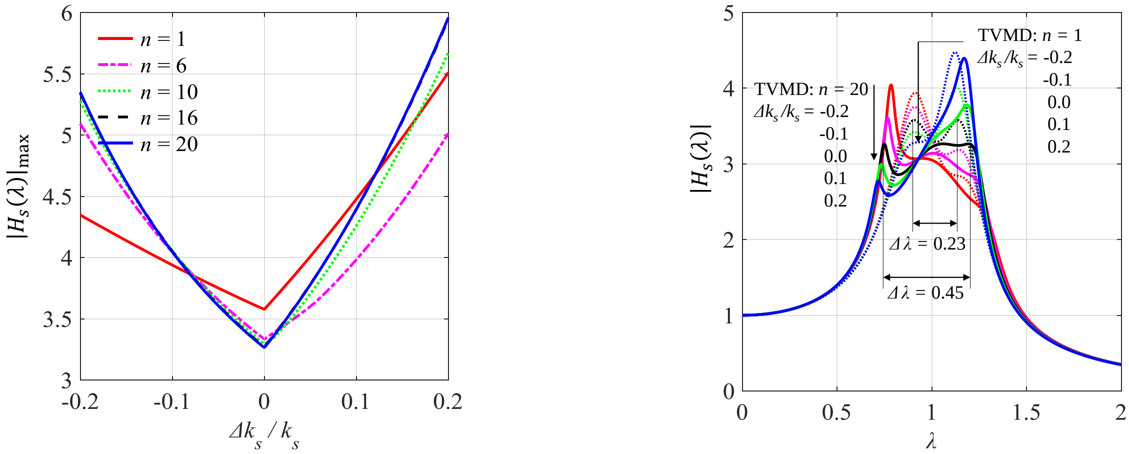


Fig. 19 Effectiveness of MTVMD versus tuning frequency ratio for different frequency bands



(a) The maximum structural response (b) The structural frequency response

Fig. 20 Robustness of MTVMD systems to variations in the structural stiffness k_s

of ξ_T and the effectiveness of the MTVMD tends to that of a single TVMD.

The effect of damping ratio on the effectiveness of a MTVMD is depicted more clearly in Fig. 17 for different values of the frequency band with the tuning frequency ratio f = 1.0. There exists an optimum value of damping ratio which minimizes the maximum structural response for the given frequency band of a MTVMD. As shown in this figure, the optimum damping ratio increases continually and the minimum value of $|H_s(\lambda)|_{max}$ becomes smaller when the frequency range of a MTMD decreases and approaches its optimum value. If the frequency band is too small, the optimum damping ratio becomes much larger and closer to the optimum damping ratio of a single TVMD of 0.20.

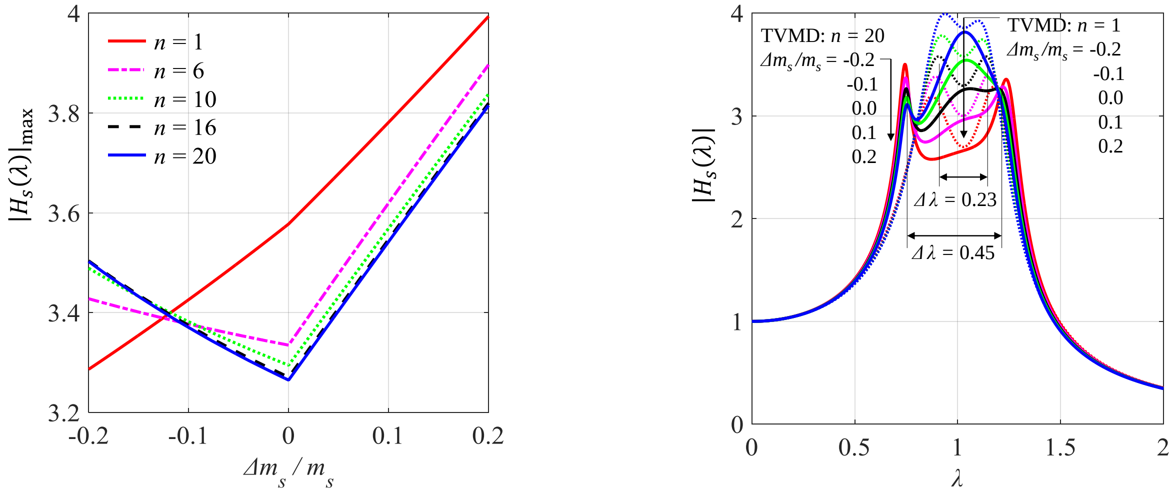
Figs. 18 and 19 show the effect of tuning frequency ratio on effectiveness of MTVMD with different frequency bands and damping ratios (in Fig. 18, β = 0.46 and in Fig. 19, ξ_T = 0.05). From these figures, it can be observed clearly that if the damping ratio and frequency band are not too large, the optimum tuning frequency ratio of the MTVMD is not much affected by the damping ratio and frequency band and is around 1.0.

According to the above discussions, it is concluded that the effectiveness of a MTVMD is much dependent on the frequency band, the damping ratio and the tuning frequency ratio and that there is an optimum MTVMD, whose effectiveness is a maximum with respect to the three above parameters for a given total mass ratio and total number of TVMD. This is equivalent to the existence of an optimum TVMD with respect to the tuning ratio and the damping ratio in the case of a single TVMD.

6.2 Robustness of MTVMD

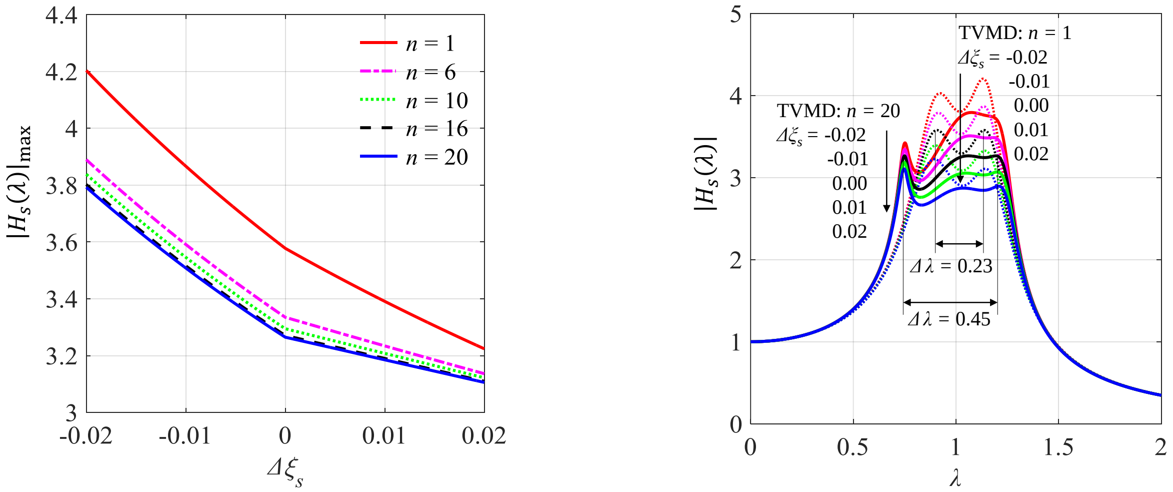
In real applications, structural properties are possibly in error due to uncertainty in modeling or measurement. Hence, one concern with the MTVMD in practice is the robustness to the parameter change of the primary system. Utilizing a numerical searching technique shown in Section 7, the optimum parameters of the MTVMD can be obtained based on a given total mass ratio (in this study μ=0.1) and a total number of TVMD. Then, the variations of the structural response as the mass m_s, stiffness k_s and damping ratio ξ_s of the primary system deviate from their nominal values are discussed and shown in Figs. 20-22.

Fig. 20 depicts the robustness of the MTVMD-structure system with different numbers of TVMD against the change or the estimation error in the structural stiffness. As shown



(a) The maximum structural response (b) The structural frequency response

Fig. 21 Robustness of MTVMD systems to variations in the structural mass m_s



(a) The maximum structural response (b) The structural frequency response

Fig. 22 Robustness of MTVMD systems to variations in the structural damping ratio ξ_s

in Fig. 20(a), both the optimum single TVMD and MTVMD are not robust in the sense that the error in the stiffness of the structure increases the maximum structural response and the increase in the total number of TVMD cannot improve the robustness. However, it should be noted that the effectiveness of MTVMD is better than that of the optimal single TVMD when the shift of the structural stiffness is not large, approximately ±0.1. Furthermore, the effective frequency range of MTVMD is much wider than that of the single TVMD as shown in Fig. 20(b).

From Fig. 21(a), the robustness of the single TVMD against the change in the structural mass is very different from that of the MTVMD, that is, negative error in the structural mass makes the single TVMD more effective. However, the effectiveness of the single TVMD is much less than that of the MTVMD in a large error range of the structural mass and the effective frequency range of MTVMD is wider. In this sense, the MTVMD is more robust compared with the single TVMD. The same conclusion can also be drawn for the change in the structural damping ratio presented in Fig. 22.

7. Application to passive control

For a single TMD- or TVMD-structure system, the fixed-point method is commonly used to analytically obtain the optimal parameters. Since there do not exist fixed points with respect to the damping ratio for MTVMD, it is difficult to obtain simple and analytical representations for optimal parameters. However, the value of the optimal parameters of MTVMD can be determined by numerical searching techniques. In this paper, the optimum objective selected is to minimize the maximum magnitude of the structural frequency response, which is known as H_∞ optimization. For a given total mass ratio and total number of TVMD, the optimization problem can be expressed as

$$\begin{aligned}&\min_{\beta,\xi_T,f}\ \max_{\lambda_j}|H_s(\lambda_j)|\\&\text{s.t.}\quad 0<\beta,0<\xi_T<1,0<f\end{aligned} \tag{26}$$

where λ_j, j = 1, 2, ..., N, are the solutions of the following equation

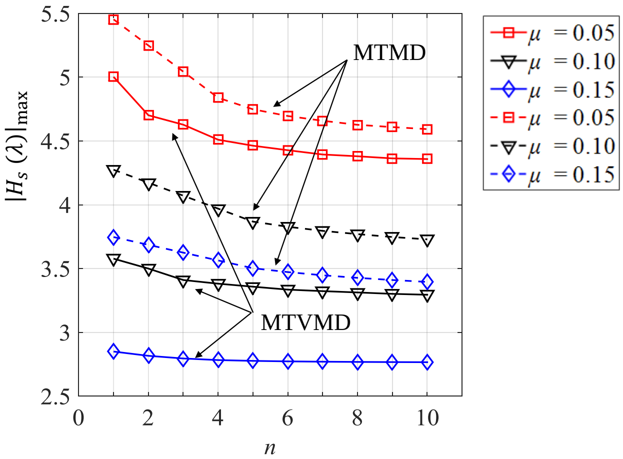


Fig. 23 Optimized structural responses of MTVMD- and MTMD-structure system

$$\frac{\partial |H_s(\lambda)|}{\partial \lambda} = 0 \tag{27}$$

These conditions satisfy (i) the natural frequencies of the TVMD are positive real and (ii) the TVMD are under-damped.

Note that the solution set of Eq. (27), i.e., λ_j, j = 1, 2, ..., N, contains the resonance frequencies, anti-resonance frequencies, and other frequencies that the curves horizontally pass through. Since the largest magnitude of the frequency response, representing the H_∞ norm, only occurs at the resonance frequencies, the eigenvalue set Ω, i.e., the solutions of Eq. (20), can be used to replace λ_j. The optimization condition about λ_j then becomes $\lambda_j \in \Omega$. The above optimization problem is actually a multi-objective optimization problem. In this paper, the Matlab solver *fminimax* is employed to solve this problem.

Taking advantage of the previous optimization procedure, the optimized structural response $|H_s(\lambda)|_{max}$ of the MTVMD-structure system can be obtained, which is shown in Fig. 23 together with that of the MTMD-structure system. In the present study, the structural damping ratio ξ_s =0:02 is taken into account and the total mass ratio μ=0.05, 0.10 and 0.15 are considered. From this figure, it can be seen that the effectiveness of MTVMD is much dependent on the total mass ratio and the effect of the number of TVMD on the structural response is also associated with the mass ratio. When the mass ratio is smaller, for instance, μ=0.05, the structural response can be suppressed continually as the number of TVMD increases. It is worth noting that when the number of TVMD is beyond a certain value, the improvement of the structural response becomes slight. Also, with the increase of the mass ratio, increasing the number of TVMD cannot reduce the structural response significantly. Therefore, much larger number of TVMD is unnecessary in practice. Compared with MTMD, the effectiveness of MTVMD is always better than that of MTDM no mater what the mass ratio and the number of damper are used.

8. Conclusions

The fundamental characteristics of MTVMD have been investigated analytically with emphasis on its mechanism of vibration control, effectiveness and robustness for harmonically forced structural oscillations. The major conclusions obtained through the present study are as follows:

1. The MTVMD design parameters consist of the frequency band, the average damping ratio and the tuning frequency ratio. The frequency band controls the natural frequency spacing of the MTVMD-structure system and directly dominates the effectiveness of frequency tuning. A proper average damping ratio can significantly reduce the secondary peaks and the outer two primary peaks of the structural response, and the relative amplitudes of the outer two primary peaks can be shifted by the tuning frequency ratio.
2. The maximum response of the structure is dominated by the structure-dominant mode or the outer two TVMD-dominant modes. Their relative contributions heavily depend the frequency band and the average damping ratio, which can result in the transformation between the two-peak characteristic and the one-peak characteristic of structural response.
3. There exists an optimum MTVMD, which makes the structural frequency response curve flattened with minimum amplitude in a wider input frequency range, for a given total number of TVMD and a total mass ratio.
4. The optimum MTVMD, in a strict sense, is as unrobust as the optimum single TVMD against the changes in the structural parameters. However, the optimum MTVMD is more effective than the optimum single TVMD in a wider range of both the estimation error in the structural property and the input frequency. In this sense, the MTVMD is more robust than the singe TVMD.
5. Compared with the traditional MTMD, MTVMD is more effective. The effectiveness is much dependent on the total mass ratio. When the mass ratio is small, increasing the number of TVMD can suppress the structural response significantly, while the effectiveness of increasing the number of TVMD cannot be improved further as the mass ratio increases.

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