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An exact transfer matrix method for coupled bending and bending vibrations of a twisted Timoshenko beam

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Abstract. In this study, an exact transfer matrix expression for a twisted uniform beam considering the effect of shear deformation and rotary inertia is developed. The particular transfer matrix is derived by applying the distributed mass and transcendental function while using a local coordinate system. The results obtained from this method are independent for a number of subdivided elements, and this method can determine the required number of exact solutions for the free vibration characteristics of a twisted uniform Timoshenko beam using a single element. In addition, it can be used as a useful numerical method for the computation of high-order natural frequencies. To validate the accuracy of the proposed method, the computed results are compared with those reported in the existing literature, and the comparison results indicate notably good agreement. In addition, the method is used to investigate the effects of shear deformation and rotary inertia for a twisted beam

Keywords: Timoshenko beam, transfer matrix method, exact solution, twisted beam

1. Introduction

The analysis of dynamic characteristics for a twisted beam has been widely studied in engineering designs, including turbine blades, impellers, propellers, and helicopter rotor blades. To examine the static and dynamic characteristics for these designs, various methods have been proposed for a twisted beam (Carnegie and Thomas 1972, Zhu 2011, Mustapha and Zhong 2012, Yoon and Lee 2014, Huang *et al*. 2017, Huang *et al*. 2019). Using the governing differential equation of motion derived by Hamilton's principle, Murthy (1976) determined the solution of the natural vibration characteristics for non-uniform rotor blades using the transfer matrix method (TMM). Subrahmanyam *et al*. (1981) used the Reissner method to investigate the dynamic characteristics for the coupled bending-bending vibration of a twisted Timoshenko blade. Banerjee (2001,2004) used a dynamic stiffness matrix method to investigate free vibration characteristics for a twisted Bernoulli-Euler and Timoshenko beams. The dynamic stiffness method proposed by Banerjee (2001,2004) is deduced in a similar way with the proposed method in this study, but his method is formulated based on the finite element method and the present method is formulated based on the TMM. Thus, these two methods have a significant difference in their usage (Lee and Lee 2016, 2017a). Lin (1997) and Lin *et al*. (2001) used the TMM to verify the effect of twist for non-uniform Euler-Bernoulli and Timoshenko beams, respectively. Oh *et al*. (2003) considered the effect of pre-twist and presetting for rotating thin-walled composite beams. The background studies for a twisted beam can be found in a review paper by Rosen (1991).

Yoo *et al*. (2001) performed a vibration analysis for rotating pre-twisted blades with a concentrated mass using a modeling method based on hybrid deformation variables. A finite element model using Gauss–Legendre numerical integration was considered by Yardimoglu and Yildirim (2004) to determine the solution of the coupled bending vibration analysis for a twisted beam. The differential transform based on Hamilton's principle is used by Ho and Chen (2006) to solve problems for the free transverse vibration of an axially loaded non-uniform spinning twisted Timoshenko beam. Chen (2014) investigated the effect of damping via a parametric study on vibration characteristics for the bending vibration of axially loaded twisted Timshenko beams under Kelvin-Voigt damping. Sinha and Turner (2011) studied the effect of a pre-twist angle for bending and torsional vibrations of a twisted plate. Oh and Yoo (2016), Shenas *et al*. (2017a), and Shenas *et al*. (2017b) studied the vibration characteristics of pretwisted beams made of functionally graded materials. Mustapha (2017) analyzed Dynamic behaviors of spinning pretwisted Rayleigh micro-beams, and Chen and Li (2019) analyzed the vibration characteristics of a rotating pre-twisted composite laminated blade. Gu *et al*. (2019) investigated the vibration characteristics of a cantilever pre-twisted panel with initial exponential function type geometric imperfection based on the Rayleigh–Ritz method and continuous algebraic polynomial functions. In addition, many researchers investigated the influences of transverse shear deformation (Chaabane *et al*. 2019, Bellifa *et al*. 2017, Bourada *et al*. 2019) and thickness stretching (Bouhadra *et al*. 2018, Ait Atmane *et al*. 2017) using the

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efficient beam theory, where no shear correction factors are needed.

Among these methods, the TMM can be used as a tool of the simple and powerful numerical analysis to determine high-order natural frequencies for one-dimensional vibration structures. The non-physical numerical instabilities of the TMM can be simply overcome by employing the results discussed in previous works (Lee *et al*. 2017b, Lee and Lee 2019).

The results obtained from the TMM developed by Lee and Lee (2016) are independent of the number of subdivided elements for a twisted uniform beam, but this method can only be used for a twisted Bernoulli-Euler beam because the theory assumed that the beam element has small cross-sectional dimensions compared to its length. For these reasons, an exact TMM is proposed to determine the exact solution for a twisted uniform Timoshenko beam.

The objective of this study is to develop the TMM to determine the exact eigenpairs of the dynamic characteristics for the above-mentioned problems. Under the derivation of the transfer matrix, it becomes possible to obtain the exact solution for free vibration characteristics of a twisted uniform Timoshenko beam by considering the local coordinate system, the distributed mass, and the transcendental function. Except for the method discussed herein, the transfer matrix method (TMM) with the distributed mass for a twisted uniform Timoshenko beam has not been studied. To demonstrate the accuracy of the present theory, the computed results are compared with those reported in the literature (Subrahmanyam *et al.* 1981, Lin *et al*. 2001, Oh *et al*. (2003), Yardimoglu and Yildirim 2004, Ho *et al*. 2006). In addition, the effect of shear deformation and rotary inertia for the non-dimensional natural frequencies with regard to the variation of the inverse in the slenderness ratio is studied for a twisted beam.

2. Theory

The theory offered herein considers not only rotary inertia and shear deformation but also the pure bending displacements coupled in two principal planes. The notation and coordinate systems shown in Fig. 1 are used to determine the exact solutions of free vibration characteristics for a uniform Timoshenko beam with pre-twist, where XYZ is the global coordinate system, and xyz is the local coordinate system. The same X and x axes are used for the global and local coordinate systems, respectively; the twisted angle at point O and at distance L is 0° and β, respectively.

The four second-order differential equations for the coupled bending-bending vibration of a twisted Timoshenko beam are obtained by (Banerjee 2004)

$$-\rho A\ddot{v} + \kappa_y AGv'' - k^2\kappa_z AGv + kAG(\kappa_y+\kappa_z)w' + \kappa_y AG\psi' - k\kappa_z AG\phi = 0 \quad (1)$$

$$-\rho A\ddot{w} + \kappa_z AGw'' - k^2\kappa_y AGw - kAG(\kappa_y+\kappa_z)v' - k\kappa_y AG\psi - \kappa_z AG\phi' = 0 \quad (2)$$

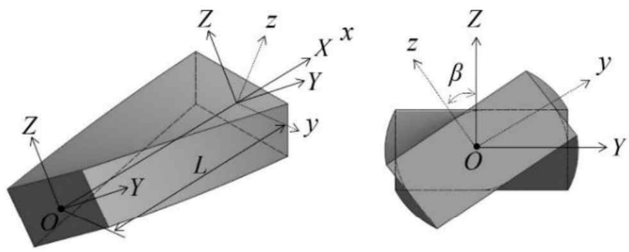

Fig. 1 Notation and coordinate systems used for a twisted Timoshenko beam

$$-\rho I_{yy}\ddot{\phi} + EI_{yy}\phi'' - k^2 EI_z\phi - \kappa_z AG\phi - k\kappa_z AGv + \kappa_z AGw' + k(EI_{yy}+EI_{zz})\psi' = 0 \quad (3)$$

$$-\rho I_{zz}\ddot{\psi} + EI_{zz}\psi'' - k^2 EI_y\psi - \kappa_y AG\psi - k\kappa_y AGw - \kappa_y AGv' - k(EI_{yy}+EI_{zz})\phi' = 0 \quad (4)$$

where $k(=\beta/L)$ is the rate of twist, β is the twist angle at distance L, κ_y and κ_z are the shear shape coefficients, EI_{yy} and EI_{zz} are the out-of-plane stiffness and in-plane bending stiffness, respectively, $\kappa_y AG$ and $\kappa_z AG$ are the out-of-plane stiffness and in-plane shear stiffness, respectively, and I_{yy} and I_{zz} are the geometric moments of inertia for the beam cross-section on the xy- and yz-planes, respectively. E, G and ρ are the Young's modulus, the shear modulus, and the density of the beam material, respectively. m is the mass per unit length, and $w(=w(x,t))$ and $v(=v(x,t))$ are the bending displacements in the z- and y-directions, respectively. $\phi(=\phi(x,t))$ and $\psi(=\psi(x,t))$ are the slope of the deflection curves on the y- and z-axes, respectively; the primes denote differentiation with respect to distance x, and the double dots indicate differentiation with respect to time t.

The variables related to distance and time in Eqs. (1)–(4) can be integrated by a separation of variables method. If $w(x,t)$, $v(x,t)$, $\phi(x,t)$, and $\psi(x,t)$ are assumed to be harmonic vibrations with angular frequency (ω):

$$v(x,t) = V(x)\mathrm{e}^{i\omega t}, w(x,t) = W(x)\mathrm{e}^{i\omega t}, \quad \phi(x,t) = \Phi(x)\mathrm{e}^{i\omega t}, \psi(x,t) = \Psi(x)\mathrm{e}^{i\omega t} \quad (5)$$

By substituting Eq. (5) into Eqs. (1)–(4), these equations can be rewritten as described below

$$V'' - k^2(\mu - p^2 s_z^2)V + k(1+\mu)W' + \Psi' - k\mu\Phi = 0 \quad (6)$$

$$\mu W'' - k^2(1 - p^2 s_z^2)W - k(1+\mu)V' - k\Psi - \mu\Phi' = 0 \quad (7)$$

$$\mu s_y^2\Phi'' - k^2(s_z^2 + \mu - p^2 r_y^2 s_z^2)\Phi - \mu k^3 V + \mu k^2 W' + k(\mu s_y^2 + s_z^2)\Psi' = 0 \quad (8)$$

$$s_z^2\Psi'' - k^2(\mu s_y^2 + 1 - p^2 r_z^2 s_z^2)\Psi - k^3 W - k^2 V' - k(\mu s_y^2 + s_z^2)\Phi' = 0 \quad (9)$$

where

$$\mu = \frac{\kappa_z}{\kappa_y}, p^2 = \frac{\rho A\omega^2}{EI_{zz}k^4}, q^2 = \frac{\rho A\omega^2}{EI_{yy}k^4},$$
$$s_z^2 = \frac{EI_{zz}k^2}{\kappa_y AG}, s_y^2 = \frac{EI_{yy}k^2}{\kappa_z AG}, r_y^2 = \frac{I_{yy}k^2}{A}, r_z^2 = \frac{I_{zz}k^2}{A} \tag{10}$$

By eliminating all variables except for one of variables $W, V, \Phi,$ and Ψ in Eqs. (6)–(9), four second-order differential equations can be integrated into an eighth-order differential equations, as described below:

$$DU^8 + C_1 DU^6 + C_2 DU^4 + C_3 DU^2 + C_4 U = 0 \tag{11}$$

where

$$U = V(x), W(x), \Phi(x), or\ \Psi(x) \tag{12}$$

$$D = \frac{\mathrm{d}}{\mathrm{d}x} \tag{13}$$

and

$$C_1 = k^2\{4 + p^2(r_z^2 + s_z^2) + q^2(s_y^2 + r_y^2)\}$$
$$\begin{aligned} C_2 = k^4\{&(6 - p^2 - q^2) + p^2q^2(s_y^2 + r_y^2)(s_z^2 + r_z^2) \\ &+ 2(p^2r_z^2 + q^2r_y^2) \\ &+ p^2(s_z^2 - r_y^2 + p^2r_z^2s_z^2) \\ &+ q^2(s_y^2 - r_z^2 + q^2r_y^2s_y^2)\} \end{aligned} \tag{14}$$

$$\begin{aligned} C_3 = k^6\{&(4 + 6p^2 + 6q^2) - p^2q^2(r_y^2 + r_z^2)(s_y^2 + s_z^2) \\ &+ p^2q^2(q^2r_y^2s_y^2 - 1)(s_z^2 + r_z^2) \\ &+ 2p^2q^2(s_y^2s_z^2 + r_y^2r_z^2) \\ &- 2(p^2r_y^2 + q^2r_z^2) \\ &+ p^2r_z^2(1 - p^2s_z^2) \\ &- p^2s_z^2(1 + p^2r_y^2) \\ &+ q^2r_y^2(1 - q^2s_y^2) \\ &- q^2s_y^2(1 + q^2r_z^2) \\ &+ p^2q^2(p^2r_z^2s_z^2 - 1)(s_y^2 + r_y^2)\} \end{aligned} \tag{15}$$

$$\begin{aligned} C_4 = k^8[&(1 - p^2)(1 - q^2) \\ &+ (s_y^2 \\ &+ r_z^2)\{p^2q^2(s_z^2 - p^2r_y^2s_z^2 + r_y^2) \\ &- q^2\} + p^2s_z^2(p^2 - 1) \\ &+ q^4s_y^2(1 + r_z^2) \\ &+ (p^2s_z^2 - 1)(p^2r_y^2 + p^2q^4r_y^2r_z^2s_y^2) \\ &+ p^2q^2(r_y^2 + r_z^2 - q^2s_y^2(r_y^2 + r_z^2s_z^2) \\ &- p^2r_z^2s_z^2)] \end{aligned} \tag{16}$$

If the general solution of Eq. (16) is the following equation,

$$U = \mathrm{e}^{\lambda x} \tag{17}$$

then by substituting Eq. (17) into Eq. (11), an eighth-order polynomial can be found as the following equation.

$$\lambda^8 + C_1\lambda^6 + C_2\lambda^4 + C_3\lambda^2 + C_4 = 0 \tag{18}$$

To determine the solution of Eq. (18), an eighth-order polynomial in λ can be changed to a fourth-order polynomial, as follows:

$$\xi^4 + C_1\xi^3 + C_2\xi^2 + C_3\xi + C_4 = 0 \tag{19}$$

where

$$\lambda = \pm\sqrt{\xi} \tag{20}$$

Hence, the solutions of Eq. (11) for $V(x)$, $W(x)$, $\Phi(x)$, and $\Psi(x)$ are given by

$$V(x) = \sum_{j=1}^{8} A_j \mathrm{e}^{\lambda_j x} \tag{21}$$

$$W(x) = \sum_{j=1}^{8} B_j \mathrm{e}^{\lambda_j x} \tag{22}$$

$$\Phi(x) = \sum_{j=1}^{8} F_j \mathrm{e}^{\lambda_j x} \tag{23}$$

$$\Psi(x) = \sum_{j=1}^{8} H_j \mathrm{e}^{\lambda_j x} \tag{24}$$

where $\lambda_j (for\ j = 1{\sim}8)$ is obtained from Eq. (20), and D_j, E_j, F_j, and H_j are constant values. The relationship between the four constants can be obtained by substituting Eqs. (21)–(24) into Eqs. (6)–(9). A_j, B_j, F_j, and H_j are related by the following equations:

$$B_j = \alpha_j A_j, \qquad F_j = \beta_j A_j, \qquad H_j = \gamma_j A_j \tag{25}$$

where

$$\alpha_j = -\frac{(\overline{a}_j\overline{b}_j + k^2\lambda_j^2\overline{c}^2)\overline{d}_j + \overline{h}_j(k^3\lambda_j^2\overline{c} - \mu k^3\overline{a}_j)}{(\overline{a}_j\overline{b}_j + k^2\lambda_j^2\overline{c}^2)\overline{e}_j + \overline{h}_j(\mu k^2\lambda_j\overline{a}_j + k^4\lambda_j\overline{c})},$$
$$\beta_j = \frac{\overline{d}_j + \overline{e}_j\alpha_j}{\overline{h}_j}, \qquad \gamma_j = \frac{k^3\alpha_j + k^2\lambda_j + k\lambda_j\overline{c}\beta_j}{\overline{a}_j} \tag{26}$$

and

$$\overline{a}_j = s_z^2\lambda_j^2 - k^2(\mu s_y^2 + 1 - p^2r_z^2s_z^2),$$
$$\overline{b}_j = \mu s_y^2\lambda_j^2 - k^2(s_z^2 + \mu - p^2r_y^2s_z^2),$$
$$\overline{c} = \mu s_y^2 + s_z^2, \qquad \overline{d}_j = k^3p^2s_z^2 - \mu k^3 - \mu k\lambda_j^2, \tag{27}$$
$$\overline{e}_j = k^2\mu\lambda_j + \mu\lambda_j^3 + \lambda_j k^2p^2s_z^2, \qquad \overline{h}_j = \mu(\lambda_j^2 + k^2)$$

Substituting Eqs. (25) and (26) into Eqs. (22)–(24), these equations can be rewritten as follows:

Table 1 Comparison results of the first five natural frequencies for a uniform cantilever beam with pre-twist

ω	Natural frequencies (Hz)							
	β=45°							
	Present			Lin *et al.* (2001)	Subrahmanyam *et al.* (1981)	Oh *et al.* (2003)	Yardimoglu *et al.* (2004)	Ho *et al.* (2006)
	ω^T	ω^B	ω^T/ω^B					
1	62.0	62.1	0.9984	62.0	61.9	62.0	61.8	62.0
2	305.0	305.5	0.9984	304.6	304.7	305.1	304.8	305.1
3	943.1	949.9	0.9928	938.0	937.0	949.0	944.5	943.2
4	1199.9	1221.9	0.9820	1186.2	1205.1	1206.1	1193.0	1199.7

Table 2 Comparison of the natural frequencies for a twisted beam given by using a different number of elements

ω	Natural frequency (Hz)			
	β=45°			
	1 element	100 elements	2 elements	
			3:7	6:4
1	62.0	62.0	62.0	62.0
2	305.0	305.0	305.0	305.0
3	943.1	943.1	943.1	943.1
4	1199.9	1199.9	1199.9	1199.9

$$W(x) = \sum_{j=1}^{8} \alpha_j A_j e^{\lambda_j x} \tag{28}$$

$$\Phi(x) = \sum_{j=1}^{8} \beta_j A_j e^{\lambda_j x} \tag{29}$$

$$\Psi(x) = \sum_{j=1}^{8} \gamma_j A_j e^{\lambda_j x} \tag{30}$$

The shear force and moments are given by (Banerjee 2004)

$$M_y(x,t) = -EI_{yy}\phi'(x,t) - kEI_{yy}\psi(x,t) \tag{31}$$

$$M_z(x,t) = -EI_{zz}\psi'(x,t) + kEI_{zz}\phi(x,t) \tag{32}$$

$$\begin{aligned} V_y(x,t) = &-EI_{zz}\psi''(x,t) + k^2 EI_{yy}\psi(x,t) \\ &+ k\left(EI_{yy} + EI_{zz}\right)\phi'(x,t) \\ &+ \rho I_{zz}\ddot{\psi}(x,t) \end{aligned} \tag{33}$$

$$\begin{aligned} V_z(x,t) = &EI_{yy}\phi''(x,t) - k^2 EI_{zz}\phi(x,t) \\ &+ k\left(EI_{yy} + EI_{zz}\right)\psi'(x,t) \\ &- \rho I_{yy}\ddot{\phi}(x,t) \end{aligned} \tag{34}$$

where $V_y(x,t)$ and $V_z(x,t)$ are the shear force in the y- and z-directions, respectively, and $M_y(x,t)$ and $M_z(x,t)$ are the bending moments on the y- and z-axes, respectively.

Substituting Eqs. (5) (29), and (30) into Eqs. (31)–(34), the following equations are obtained:

$$M_y(x) = -EI_{yy}\{\beta_j\lambda_j + k\gamma_j\}A_j e^{\lambda_j x} \tag{31}$$

$$M_z(x) = -EI_{zz}\{\gamma_j\lambda_j - k\beta_j\}A_j e^{\lambda_j x} \tag{32}$$

$$\begin{aligned} V_y(x) = -EI_{zz}\{&\left(\lambda_j^2 - k^2 r + k^2 p^2 r_z^2\right)\gamma_j \\ &- k(r+1)\beta_j\lambda_j\}A_j e^{\lambda_j x} \end{aligned} \tag{33}$$

$$\begin{aligned} V_z(x) = EI_{zz}\{&\left(r\lambda_j^2 - k^2 + k^2 p^2 r_y^2\right)\beta_j \\ &+ k(r+1)\lambda_j\gamma_j\}A_j e^{\lambda_j x} \end{aligned} \tag{34}$$

where

$$r = \frac{EI_{yy}}{EI_{zz}} \tag{35}$$

The distance x of the element at the left-hand and right-hand end, as shown in Fig. 1, are zero and L, respectively. Therefore, when $x = 0$ is substituted into Eqs. (21), (28)–(30), and (31)–(34), these equations can be written in matrix form as follows:

$$\left\{\begin{matrix} V(x) \\ W(x) \\ \Phi(x) \\ \Psi(x) \\ V_y(x) \\ V_z(x) \\ M_y(x) \\ M_z(x) \end{matrix}\right\}_{x=0} = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} & P_{15} & P_{16} & P_{17} & P_{18} \\ P_{21} & P_{22} & P_{23} & P_{24} & P_{25} & P_{26} & P_{27} & P_{28} \\ P_{31} & P_{32} & P_{33} & P_{34} & P_{35} & P_{36} & P_{37} & P_{38} \\ P_{41} & P_{42} & P_{43} & P_{44} & P_{45} & P_{46} & P_{47} & P_{48} \\ P_{51} & P_{52} & P_{53} & P_{54} & P_{55} & P_{56} & P_{57} & P_{58} \\ P_{61} & P_{62} & P_{63} & P_{64} & P_{65} & P_{66} & P_{67} & P_{68} \\ P_{71} & P_{72} & P_{73} & P_{74} & P_{75} & P_{76} & P_{77} & P_{78} \\ P_{81} & P_{82} & P_{83} & P_{84} & P_{85} & P_{86} & P_{87} & P_{88} \end{bmatrix} \left\{\begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \end{matrix}\right\} \tag{36}$$

where

$$
\begin{aligned}
&P_{1j} = 1,\ P_{2j} = \alpha_j,\ P_{3j} = \beta_j,\ P_{4j} = \gamma_j \\
&P_{5j} = -EI_{zz}\{(\lambda_j^2 - k^2 r + k^2 p^2 r_z^2)\gamma_j - k(r+1)\beta_j\lambda_j\} \\
&P_{6j} = EI_{zz}\{(r\lambda_j^2 - k^2 + k^2 p^2 r_y^2)\beta_j + k(r+1)\lambda_j\gamma_j\} \\
&P_{7j} = -EI_{yy}\{\beta_j\lambda_j + k\gamma_j\},\ P_{8j} = -EI_{zz}\{\gamma_j\lambda_j - k\beta_j\}
\end{aligned}
\tag{37}
$$

Eq. (36) can be simplified as follows:

$$\mathbf{Z}_{x=0} = \mathbf{PA} \tag{38}$$

From Eq. (38), it is obvious that the arbitrary constants $\mathbf{A}$ is given by

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{Z}_{x=0} \tag{39}$$

In a similar way, when $x = L$ is substituted into Eqs. (21), (28)–(30), and (31)–(34), these equations can be written in matrix form as follows:

$$
\begin{Bmatrix} V(x) \\ W(x) \\ \Phi(x) \\ \Psi(x) \\ V_y(x) \\ V_z(x) \\ M_y(x) \\ M_z(x) \end{Bmatrix}_{x=L} =
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} & Q_{14} & Q_{15} & Q_{16} & Q_{17} & Q_{18} \\
Q_{21} & Q_{22} & Q_{23} & Q_{24} & Q_{25} & Q_{26} & Q_{27} & Q_{28} \\
Q_{31} & Q_{32} & Q_{33} & Q_{34} & Q_{35} & Q_{36} & Q_{37} & Q_{38} \\
Q_{41} & Q_{42} & Q_{43} & Q_{44} & Q_{45} & Q_{46} & Q_{47} & Q_{48} \\
Q_{51} & Q_{52} & Q_{53} & Q_{54} & Q_{55} & Q_{56} & Q_{57} & Q_{58} \\
Q_{61} & Q_{62} & Q_{63} & Q_{64} & Q_{65} & Q_{66} & Q_{67} & Q_{68} \\
Q_{71} & Q_{72} & Q_{73} & Q_{74} & Q_{75} & Q_{76} & Q_{77} & Q_{78} \\
Q_{81} & Q_{82} & Q_{83} & Q_{84} & Q_{85} & Q_{86} & Q_{87} & Q_{88}
\end{bmatrix}
\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \end{Bmatrix}
\tag{40}
$$

where

$$
\begin{aligned}
&Q_{1j} = \mathrm{e}^{\lambda_j L}, \quad Q_{2j} = \alpha_j \mathrm{e}^{\lambda_j L}, \quad Q_{3j} = \beta_j \mathrm{e}^{\lambda_j L}, \\
&Q_{4j} = \gamma_j \mathrm{e}^{\lambda_j L} \\
&Q_{5j} = -EI_{zz}\{(\lambda_j^2 - k^2 r + k^2 p^2 r_z^2)\gamma_j - k(r+1)\beta_j\lambda_j\}\mathrm{e}^{\lambda_j L} \\
&Q_{6j} = EI_{zz}\{(r\lambda_j^2 - k^2 + k^2 p^2 r_y^2)\beta_j + k(r+1)\lambda_j\gamma_j\}\mathrm{e}^{\lambda_j L} \\
&Q_{7j} = -EI_{yy}\{\beta_j\lambda_j + k\gamma_j\}\mathrm{e}^{\lambda_j L},\ Q_{8j} = -EI_{zz}\{\gamma_j\lambda_j - k\beta_j\}\mathrm{e}^{\lambda_j L}
\end{aligned}
\tag{41}
$$

Eq. (40) can be simplified as follows:

$$\mathbf{Z}_{x=L} = \mathbf{QA} \tag{42}$$

By substituting Eq. (39) into Eq. (42), the relationship between the state vectors $\mathbf{Z}_{x=L}$ and $\mathbf{Z}_{x=0}$ is as follows:

$$\mathbf{Z}_{x=L} = \mathbf{TZ}_{x=0} \tag{43}$$

where $\mathbf{T} = \mathbf{QP}^{-1}$, and $\mathbf{T}$ is the transfer matrix.

The transfer matrix of Eq. (43) can compute the exact eigenpairs of free vibration characteristics for a twisted Timoshenko beam using various boundary conditions. The frequency determinants for the end condition can be found in the previous works (Lee and Lee 2016, 2017a), but as an example, the fixed-free end condition is considered in this study.

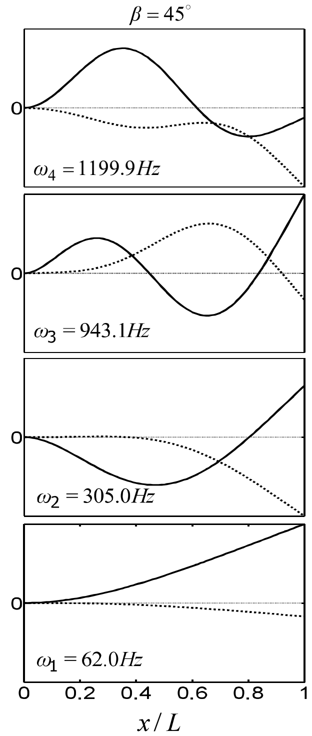

Fig. 2 The first four natural frequencies and mode shapes for a twisted blade when β=45°. (—) mode shapes of in-plane (XZ plane) bending, and (----) mode shapes of out-of-plane (XY plane) bending

3. Scope and limitation of present theory

The present theory is considered for a twisted uniform Timoshenko beam where the bending displacements are coupled in two principal planes by the effect of pre-twisting. Therefore, the torsional and axial deformations are not investigated because the theory is assumed for only pure bending displacements. It is difficult to estimate the eigenpairs for practical structures such as turbine blades and helicopter rotor blades, and this difficulty has significant effects for coupled problems (bending, torsional and axial deformations) and rotation speeds. Therefore, the applicability of the proposed TMM is limited. However, the natural frequencies and mode shapes for the bending vibration of a twisted non-uniform Timoshenko beam can be computed by idealizing the segments as many stepped uniform twisted beams.

4. Results and discussion

The natural frequencies and mode shapes of a twisted Timoshenko beam with fixed-free boundary conditions can be computed from the matrix of Eq. (43). The results

Table 3 The non-dimensional natural frequencies of a twisted Timoshenko beam with the fixed-free condition, 1/r =1.001

β	Non-dimensional natural frequency							
	1st	2nd	3rd	4th	5th	6th	7th	8th
0°	3.499973	3.500035	21.35731	21.36234	57.47073	57.49440	106.9269	106.9680
10°	3.499974	3.500034	21.35733	21.36232	57.47082	57.49434	106.9270	106.9679
20°	3.499974	3.500034	21.35739	21.36228	57.47108	57.49417	106.9272	106.9676
30°	3.499975	3.500033	21.35748	21.36219	57.47150	57.49389	106.9277	106.9670
40°	3.499976	3.500031	21.35761	21.36208	57.47206	57.49352	106.9283	106.9663
50°	3.499978	3.500029	21.35776	21.36195	57.47274	57.49307	106.9290	106.9654
60°	3.499980	3.500026	21.35794	21.36180	57.47352	57.49256	106.9298	106.9644
70°	3.499982	3.500023	21.35813	21.36163	57.47436	57.49201	106.9311	106.9633
80°	3.499984	3.500020	21.35834	21.36145	57.47523	57.49143	106.9327	106.9622
90°	3.499987	3.500017	21.35855	21.36126	57.47609	57.49086	106.9343	106.9610

Table 4 The non-dimensional natural frequencies of a twisted Timoshenko beam with the fixed-free condition, 1/r =10

β	Non-dimensional natural frequency					
	1st	2nd	3rd	4th	5th	6th
0°	3.49980	10.63974	21.35465	54.39871	57.47046	106.9264
10°	3.50073	10.60021	21.43488	53.47082	58.46238	106.5927
20°	3.50352	10.48566	21.67078	51.84401	60.27993	105.6665
30°	3.50817	10.30691	22.04914	50.14413	62.29297	104.3132
40°	3.51471	10.07854	22.55148	48.47692	64.38838	102.6956
50°	3.52313	9.81574	23.15711	46.87097	66.52665	100.9359
60°	3.53345	9.53217	23.84524	45.33820	68.68056	99.1202
70°	3.54569	9.23902	24.59576	43.88566	70.82158	97.3125
80°	3.55983	8.94482	25.38895	42.51935	72.91291	95.9697
90°	3.57588	8.65580	26.20449	41.24622	74.90118	93.9540

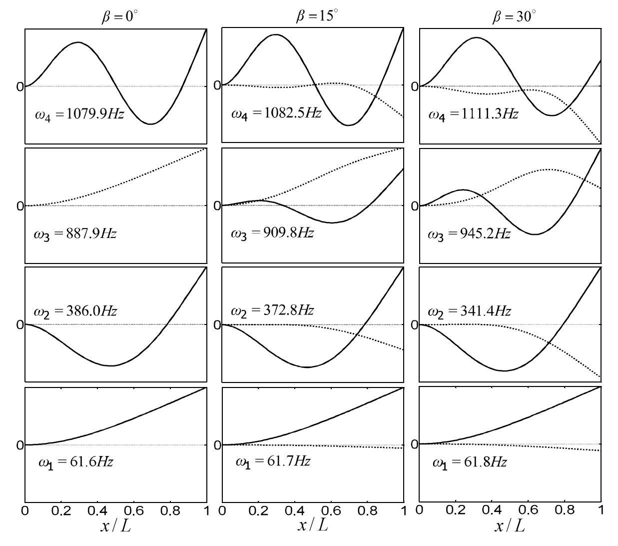

Fig. 3 The first four natural frequencies and mode shapes for the example blade when β=0, 15, and 30°. (—) mode shapes of in-plane (XZ plane) bending and (----) mode shapes of out-of-plane (XY plane) bending

obtained from the present method are compared with those reported in selected literature articles to demonstrate the accuracy of the proposed transfer matrix theory by using the following examples.

Example 1. The angles of twist applied are 0° at the fixed end and 45° at the free end, and the other properties used are as follows (Subrahmanyam *et al.* 1981):

$$E = 206.85GPa, G = 82.74GPa, L = 0.1524m, \rho = 7857.6\ kg/m^3, m = 0.3447\ kg/m,$$

$$I_{yy} = 1.0906 \times 10^{-11}m^4, I_{zz} = 2.3586 \times 10^{-9}m^4, A = 4.3871 \times 10^{-5}m^2, k_y = k_z = 0.847458.$$

As shown in Table 1, the first four natural frequencies are compared with the results reported in the literature, and the comparison results exhibit very good agreement. To confirm the effect of shear deformation and rotary inertia for a twisted beam, the results (ω^T) given from the proposed Timoshenko theory are compared with the results (ω^B) computed by using the Bernoulli-Euler theory . The comparison results have a slight effect on the natural frequencies of the beam element because the values of the main parameters are small: $s_y = 0.0044, s_z = 0.0649, r_y = 0.0026, and\ r_z = 0.03$. In addition, the effects were calculated as the frequency ratio (ω^T/ω^B), with the value provided in Table 1.

Because the present solution method should produce the exact same results without regard to the number of

Table **5** Effect of the shear deformation and rotary inertia on the non-dimensional natural frequencies for a twisted uniform beam as a function of the radius of gyration: $1/r = 10$ and $\beta = 40°$

	r_y	r_z	Non-dimensional natural frequency					
			1st	2nd	3rd	4th	5th	6th
Ref. [13]		-	3.5341	10.4599	23.5604	54.6941	78.6049	117.3949
Present	0.00002	0.000063	3.5341	10.4599	23.5604	54.6941	78.6049	117.3949
	0.0002	0.000632	3.5341	10.4598	23.5603	54.6935	78.6026	117.3931
	0.002	0.006325	3.5339	10.4558	23.5495	54.6281	78.3826	117.2173
	0.01	0.031623	3.5292	10.3601	23.2921	53.0639	73.7224	113.1828
	0.02	0.063246	3.5147	10.0785	22.5515	48.4769	64.3884	102.6956
	0.1	0.316228	3.1404	6.2098	13.3005	16.4150	26.5732	29.5680
	0.2	0.632456	2.4994	3.6559	6.7091	8.8933	12.1271	15.2944
	0.5	1.581139	1.3810	1.5515	2.3046	3.9891	4.7322	5.1536

subdivided elements, a detailed study was performed to confirm the agreement between the natural frequencies obtained using the subdivided elements with equal (one and one hundred elements) and unequal lengths (two elements). For two elements, the ratios of the length applied are 6:4 and 3:7, and this structural element has an equal ratio of twist along its length; the angle of twist applied is 45°. The first four natural frequencies are observed, and the accuracy of the new transfer matrix method (TMM) is verified because the comparison results that are obtained are exactly the same in each system, as shown in Table 2. The present theory has not affected by the number of elements needed to obtain the desired results. In this context, the mode shapes for the twisted cantilever blade (for 45°) shown in Fig. 2 were calculated using the proposed TMM, and the comparison results with those given by Subrahmanyam *et al.* (1981) using the Reissner method reveal very good agreement (see Figs. 6-9 in Subrahmanyam *et al.* (1981)).

Based on the results mentioned above for mode shapes, a detailed study for the natural frequencies and mode shapes of this blade was performed, and the twist of angle was reduced from 45° to 0° in 15° intervals. As shown in Fig. 3, all others, except for the first natural frequency and mode shape, were, as expected, considerably affected due to the increase in the angle of twist.

Example 2. To investigate the non-dimensional natural frequencies for a twisted beam by considering shear deformation and rotary inertia, the following values are assumed:

$$EI_{zz}/EI_{yy} = 1.001 \; and \; 10, E = 1, m = 1, L = 1,$$

$$k_z = k_y = 1, A = 2500, G = 0.25.$$

The non-dimensional natural frequencies for a twisted Timoshenko beam are shown in Tables 3 and 4, when the inverse of the ratios for bending stiffness $\left(1/r = EI_{zz}/EI_{yy}\right)$ are 1.001 and 10, respectively. When the results are compared with those computed by using the Bernoulli-Euler theory, it was confirmed that the non-dimensional natural frequencies are significantly affected by the shear deformation and rotary inertia (see Tables 3-4 in Lee and Lee (2016)).

In this context, the effect of twisting for the coupled bending-bending vibration of a twisted Timoshenko beam was investigated. As shown in Fig. 4, these results were

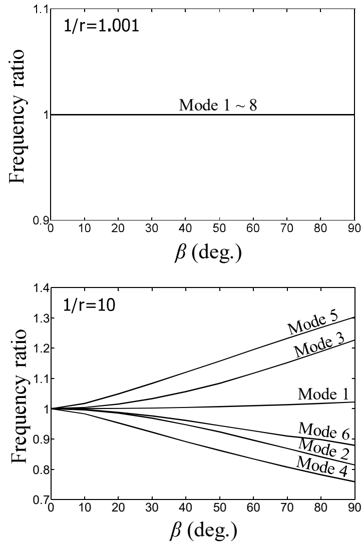

Fig. 4 The effect of pre-twist for the non-dimensional frequencies of a twisted beam with shear deformation and rotary inertia for 1/r=1.001 and 10

presented as a frequency ratio with regard to the variation of the angle of twist. The angle of twist is increased from 0° to 90° in 10° intervals, and the frequency ratio used herein was calculated as $\omega^T_{twist}/\omega^T_{straight}$. ω^T_{twist} is the non-dimensional natural frequency given from the proposed Timoshenko theory for a twisted beam ($\beta = 0$~90°), and $\omega^T_{straight}$ is the non-dimensional natural frequency obtained by the present theory for an untwisted beam ($\beta =$ 0°). Except for $1/r = 1.001$, the more the ratio of bending stiffness increases, the larger the effect of pre-twist on the non-dimensional natural frequencies. Of course, there is a slight effect on the first frequency, regardless of the stiffness ratio.

A study to confirm the effect of the shear deformation and rotary inertia for a twisted beam is performed, and the non-dimensional natural frequencies $(\omega^T_j, for\, j = 1\sim4)$ obtained from the proposed method are computed as the frequency ratios (ω^T_j/ω^B_j) with regards to the variation of the inverse of the slenderness ratio (r_G/L). ω^B_j are the

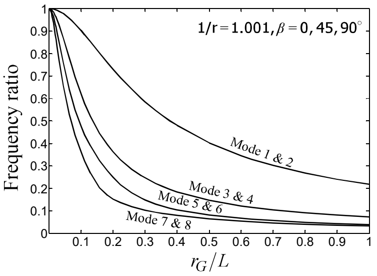

Fig. 5 Effect of shear deformation and rotary inertia as indicated by the frequency ratio 1/r=1.001

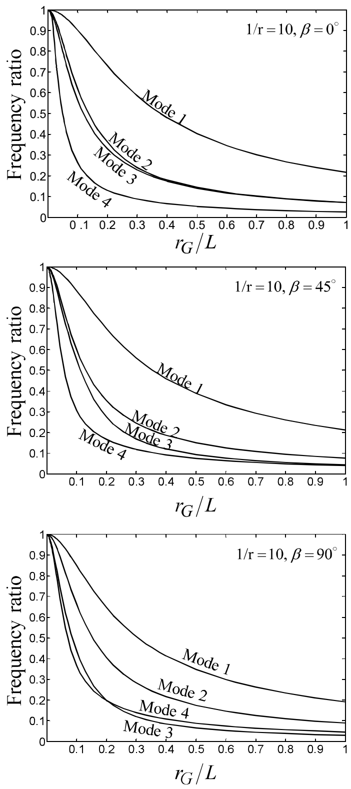

Fig. 6 The effect of shear deformation and rotary inertia on the frequency ratio for a twisted beam 1/r=10

non-dimensional natural frequencies obtained from the Bernoulli-Euler theory. ω_j^B and ω_j^T have the same results when r_y in the proposed Timoshenko theory is smaller than 0.00002. $r_G\left(=\sqrt[4]{I_y \cdot I_z/A^2}\right)$ is the radius of gyration for the two coupled planes. As shown in Figs. 5 and 6, when the beam has a very large length compared to the cross-sectional dimensions, as expected, this beam is hardly affected by them. However, in the opposite case, the non-dimensional natural frequencies of the beam are considerably affected. The effect of the shear deformation and rotary inertia for twisted $(\beta = 45\, and\, 90°)$ and untwisted $(\beta = 0°)$ beams when $1/r = 1.001$ and 10 is shown in Figs. 5 and 6, respectively. As the results show in Fig. 5, the effect of shear deformation and rotary inertia exhibit the same results, regardless of the angle of twist because the beam with the square cross-section is nearly affected by twist.

Example 3. Based on the above-mentioned results, an additional study was performed to confirm the effect of shear deformation and rotary inertia on the natural frequencies of the non-uniform twisted beam shown in Fig. 7. A non-uniform twisted beam is modeled by idealizing the segments as several uniform twisted beams. The properties considered is as follows:

$$E = 200GPa, G = 76.923GPa, L = 0.4\,m,$$

$$\rho = 7850\,kg/m^3, \upsilon = 0.3, k_y = k_z = 0.8496732$$

As shown in Fig. 7(a) and 7(b), the dimensions of the beam cross-section are $b_0 = 0.03m, h_0 = 0.02m$ at the distance $x = 0$, and are $b_L = 0.015m, h_L = 0.01m$ at the distance $x = L$, repectively.

In this regard, we investigated the number of subdivisions required to converge the first six natural frequencies. The convergence is tabulated in Table 6, and the natural frequencies on the non-uniform twisted beam can be found to be rapidly converged as shown in Fig. 8. The results computed from 40 sub-divided elements is within 0.06% compared with those given using 250 subdivisions, and when using 100 segments, the percentage difference of these results is within 0.0073%. Therefore, using 100 subdivisions to obtain more accurate results, we examined the effect of twisting on the first six natural frequencies for the coupled bending and bending vibration

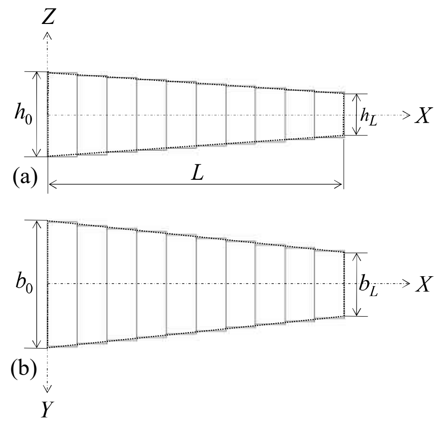

Fig. 7 Geometry of non-uniform twisted beam modeled by idealizing the stepped and uniform twisted beam: (a) side view, (b) top view

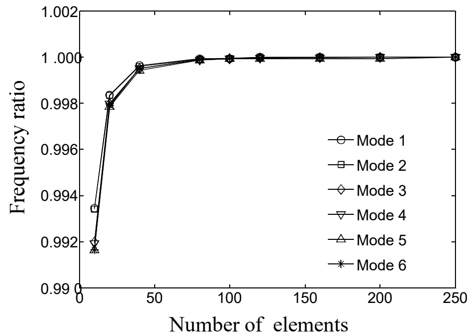

Fig. 8 Convergence of the first six natural frequencies for non-uniform twisted Timoshenko beams.

Table 6 Convergence of the first six natural frequencies on the non-uniform twisted beam

$\beta = 30^{\circ}$		No. of elements	Natural frequency (Hz)					
			1st	2nd	3rd	4th	5th	6th
Uniform	Present	1	101.90	151.59	634.19	922.02	1748.7	2479.0
	ANSYS	15,456	102.23	151.97	636.24	924.43	1754.9	2486.0
	Diff. (%)		0.3212	0.2530	0.3215	0.2609	0.3509	0.2813
Nonuniform	Present	5	130.88	194.16	549.77	800.95	1354.1	1939.3
		10	133.26	197.81	562.19	819.28	1385.4	1984.0
		20	133.92	198.79	565.59	824.28	1394.1	1996.5
		40	134.09	199.04	566.45	825.54	1396.3	1999.7
		100	134.13	199.11	566.69	825.89	1397.0	2000.6
		200	134.14	199.12	566.73	825.94	1397.0	2000.7
		250	134.14	199.12	566.73	825.95	1397.1	2000.7
		400	134.14	199.12	566.73	825.95	1397.1	2000.7
	ANSYS	37,380	134.43	199.55	568.23	828.05	1401.2	2006.2
	Diff. (%)		0.2139	0.2144	0.2633	0.2532	0.2952	0.2730

Table 7 Effect of twisting on the first six natural frequencies of non-uniform twisted beams

β	Natural frequency (Hz)					
	1st	2nd	3rd	4th	5th	6th
0°	133.85	200.17	562.33	834.19	1382.8	2026.9
10°	133.85	200.14	562.54	833.66	1383.9	2024.8
20°	133.96	199.75	564.10	830.69	1388.8	2015.4
30°	134.13	199.11	566.69	825.89	1397.0	2000.6
40°	134.38	198.24	570.26	819.46	1408.0	1981.4
50°	134.69	197.15	574.77	811.66	1421.6	1959.1
60°	135.06	195.88	580.14	802.73	1437.4	1934.5
70°	135.50	194.45	586.30	792.97	1455.2	1908.4
80°	136.00	192.89	593.16	782.63	1474.5	1881.7
90°	136.55	191.22	600.60	771.95	1495.0	1854.9

of a non-uniform twisted Timoshenko beam. The twist angle for an example to investigate the effect of twisting is increased from 0° to 90° in intervals of 10°, and the results calculated from the present method are listed in Table 7.

Based on the examined results, the present method using approximately 40 elements can be computed the natural frequencies of such non-uniform twisted beams with the sufficient accuracy.

4. Conclusions

A transfer matrix method (TMM) developed in the study can be used to compute exact natural frequencies and mode shapes of free vibration characteristics for a twisted uniform Timoshenko beam. A specific transfer matrix using the distributed mass and transcendental function is developed, which contains the mass and stiffness information in a single matrix. The use of the developed TMM is somewhat limited because the coupled problems (bending, torsional and axial deformations) and the effects of the rotation speed are not included in the topic of this study. However, this method is an exact and simple way of solving the coupled bending vibrations of nonrotating twisted Timoshenko beams. When using the proposed TMM for a twisted uniform Timoshenko beam, the computed results are independent for the number of subdivisions. The application of the developed TMM was demonstrated from the compared results above. The effects of shear deformation and rotary inertia were investigated through the comparison results between the Timoshenko and Bernoulli-Euler theories.

Moreover, as confirmed from the example for a non-uniform beam with pre-twisting, the method can also calculate the natural frequencies and mode shapes for a twisted non-uniform Timoshenko beam by idealizing the subdivided components as many uniform stepped beams with twisting. Therefore, the present method can be used as a useful tool to demonstrate other numerical approaches and the computed results can be used as a benchmark solution.

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