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Dispersion of axisymmetric longitudinal waves in a "hollow cylinder + surrounding medium" system with inhomogeneous initial stresses

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Abstract. The paper studies the dispersion of the axisymmetric longitudinal wave propagating in the "hollow cylinder + surrounding medium" system with inhomogeneous initial stresses caused by the uniformly distributed radial compressional forces acting at infinity. Up to now in the world literature, there exist only a few investigations related to the wave dispersion in a hollow cylinder with inhomogeneous initial stresses. Therefore, this paper is one of the first attempts in this field in the sense of the development of investigations for the case where the cylinder is surrounded with an infinite medium. The three-dimensional linearized theory of elastic waves is used for describing the considered wave propagation problem and, for a solution to the corresponding mathematical problem, the discrete-analytical solution method is developed and employed. The corresponding dispersion equation is obtained and this equation is solved numerically and, as a result of this solution, the dispersion curves are constructed for the first and second modes. By analyzing these curves, the character of the influence of the inhomogeneous initial stresses on the dispersion curves is established. In particular, it is established that as a result of the inhomogeneity of the initial stresses both new dispersion curves and the "band gap" for the wave frequencies can appear.

Keywords: inhomogeneous initial stresses; "hollow cylinder + surrounding medium" system; discrete-analytical method; wave dispersion; dispersion curves; band gap

1. Introduction

The development of mining at greater depths in the Earth, and the creation of tunnels for high-speed underground trains and other types of moving wheels, as well as taking into account safety under earthquakes and powerful underground explosions require study of the dynamics of such underground structures taking into account initial-state particularities, one of which is the inhomogeneous initial stresses in these structures caused by the Earth's gravity. As usual, in related investigations, the underground structures are modelled as a hollow cylinder which is surrounded with an infinite elastic medium. In this model, the cylinder is the underground structure and the infinite elastic medium is the soil. Consequently, the corresponding investigations are made for the bi-material elastic system consisting of the hollow cylinder and the surrounding infinite elastic medium. At present, investigations accounting for the aforementioned inhomogeneous initial stresses are almost completely absent. Therefore, in this paper an attempt is made in this field relating to the study of the influence of the inhomogeneous initial stresses on the dispersion of the axisymmetric wave propagating in the "hollow cylinder + surrounding elastic medium". As detailed formulation of the problem will be considered in the next section, here we consider a brief review of related investigations.

First of all, we note that up to now the dynamics of the "cylinder + surrounding elastic medium" system has been investigated from various points of view, one of which is the study of the response of this system to the moving load acting in the interior of the cylinder. Moreover, we note that almost all of these investigations have been made within the scope of the classical linear theory of elastodynamics. These studies include many works (Parnes 1969, 1980, Pozhuev 1980, Hasheminejad and Komedili 2009, Hussein *et al*. 2014, Yuan *et al*. 2017) and others listed therein. A detailed review of these works is given in recent papers (Akbarov and Mehdiyev 2018a, 2018b, 2018c, Akbarov *et al*. 2018, Ozisik *et al*. 2018) in which the dynamics of the moving and oscillating moving load acting in the interior of the cylinder surrounded with elastic medium are also investigated. In these recent works, not only the axisymmetric but also the non-axisymmetric 3D problems are studied within the scope of the three-dimensional exact equations and relations of elastodynamics (except the work by Akbarov and Mehdiyev 2018a). However, in this work (Akbarov and Mehdiyev 2018a), the corresponding axisymmetric moving load problem for the initially stretched or compressed "hollow cylinder + surrounding elastic medium" system was studied within the scope of the so-called 3D linearized theory of elastic waves in bodies with initial stresses. Under this study, it is assumed that the

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initial stresses in the cylinder and in the surrounding medium, which appear as a result of the initial stretching or compressing of the system along the cylinder's axis, are homogeneous. Besides all of the above, we note that up to now the corresponding problems for the hydro-elastic and hydro-viscoelastic systems have also been investigated and a review of these are given in the paper (Akbarov 2018).

With this we restrict ourselves to reviewing investigations regarding the response of the "cylinder + surrounding elastic medium" system to the moving and oscillating moving load acting in the interior of the cylinder.

Now we consider a brief review especially of the investigations regarding the dynamics of the wave propagation and dispersion in the system "hollow cylinder + surrounding medium" as well as in the hollow cylinder with inhomogeneous initial stresses. We begin this review with the paper (Abdulkadirov 1981) which studied the axisymmetric wave dispersion in the hollow cylinder which is surrounded by an elastic medium within the framework of linear elastodynamics. Concrete numerical results are presented and discussed for the cases where the modulus of elasticity of the cylinder material is greater than that of the surrounding medium.

The paper (Parnes 1981) deals with the study of the dispersion of axisymmetric waves in the "rod + embedded medium" and under this study the equation of motion of the rod is written within the framework of the Bernoulli hypothesis, however, the motion of the surrounding medium is written within the scope of the exact equations of elastodynamics.

In the paper (Akbarov and Guliev 2010) the axisymmetric wave propagation in the initially stressed "solid cylinder + surrounding medium" is investigated within the scope of the 3D linearized theory of elastic waves in initially stressed bodies. Note that under this investigation it is assumed that the initial stresses in the system are homogenous and are caused by the stretching or compressing forces acting at infinity in the cylinder's central axis direction. The materials of the cylinder and surrounding medium are taken as compressible and highly elastic, the elasticity relations of which are described with the harmonic potential (John 1960). These and many other related results on wave propagation in bi-material elastic systems are detailed in the monograph (Akbarov 2015). Note that all the investigations which are considered in this monograph have been made by utilizing the 3D linearized theory of elastic waves in bodies with initial stresses, the basic equations and relations of which are described in the monographs (Eringen and Suhubi 1975, Guz 1986a, 1986b, 1999, 2004) and others listed therein. Moreover, note that the corresponding investigations related to tubes and plates are reviewed and analyzed in the papers (Li *et al*. 2017, Negin 2018) and in many other ones listed in therein.

We recall that in all the investigations indicated above it was assumed that the initial stresses in the investigated objects are homogeneous. There are a few works, namely the works (Engin and Suhibi, 1978, Shearer *et al*. 2013, Wu *et al*. 2018 and Akbarov and Bagirov, 2019). In the first three of these, axisymmetric wave propagation in the hollow cylinder made of incompressible highly elastic material with inhomogeneous initial stresses, which appear as a result of the internal and external hydrostatic pressures acting on the internal and external surfaces of the cylinder, is studied. Note that in the papers (Engin and Erdogan 1978, Shearer *et al*. 2013) dispersion of torsional waves is investigated and the elasticity relations for the cylinder's material is described through the Mooney–Rivlin strain energy function. Moreover, note that in the paper (Engin and Suhibi 1978) for the solution to the corresponding mathematical problem, the variational method is employed, however, in the paper (Shearer *et al*. 2013) the Liouville–Green method is employed.

The paper (Wu *et al*. 2018) indicated above, studies the longitudinal axisymmetric wave propagation in a hollow cylinder made of an incompressible highly elastic functionally graded material, the elasticity relations of which are also written through the Mooney–Rivlin strain energy function. In this work, the initial stresses in the cylinder are taken as determined in the papers (Batra and Bahrami 2009, Chen *et al*. 2017). For the solution to the linearized wave propagation equations, the state-space formalism for these equations is employed with the approximate laminate or multi-layer technique.

The paper (Akbarov and Bagirov 2019) makes the first attempt to study the axisymmetric longitudinal wave dispersion in the bi-layered hollow cylinder with inhomogeneous initial stresses caused by the internal and external hydrostatic pressures. Under these studies, it is assumed that the materials of the constituents are sufficiently rigid and that the initial stress state is determined within the scope of the classical linear theory of elasticity. However, the wave propagation in the bi-layered hollow cylinder under consideration is described by utilizing the three-dimensional linearized theory of elastic waves in initially stressed bodies. Concrete numerical results are presented for concrete selected materials and it is established that the inhomogeneity of the initial stresses can influence the character of the dispersion curves not only in the quantitative, but also in the qualitative sense.

That is all that relates to wave propagation in the hollow cylinder with inhomogeneous initial stresses. Taking into account the great significance and necessity of such investigations in the sense of the safety and security of underground structures under seismic and geodynamical action, in the present work, an attempt is made to investigate the influence of the inhomogeneous initial stresses on the dispersion of the axisymmetric wave propagation on the "hollow cylinder + surrounding medium". It is assumed that the inhomogeneous initial stresses appear as a result of the initial loading (i.e. the loading which acts before the wave propagation) of the surrounding elastic medium with the radial compression forces acting at infinity. Note that these inhomogeneous initial stresses caused by the initial loading can be a model for taking into account the inhomogeneous initial stresses appearing around the underground structures as a result of the Earth's gravity. For the solution of the corresponding mathematical problem, the discrete-analytical solution method is employed. Numerical results on the influence of the inhomogeneous initial stresses on the dispersion curves are presented and discussed.

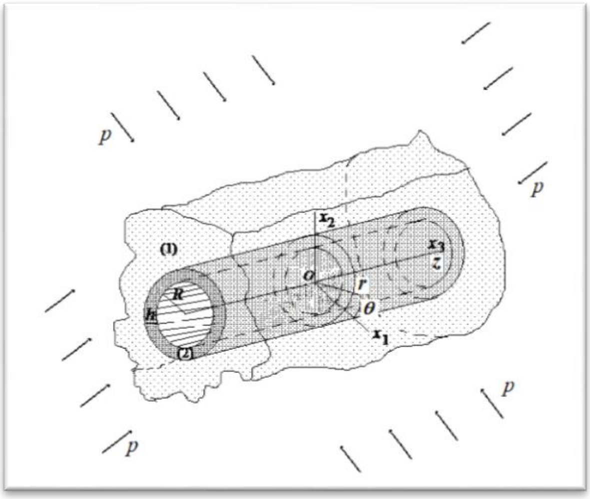


Fig. 1 The sketch of the "hollow cylinder + surrounding medium" system and the forces causing the initial stresses

2. Formulation of the problem

Consider the system consisting of the hollow cylinder with h thickness and R radius of the internal circle of the cross section and surrounding elastic medium, a sketch of which is shown in Fig. 1. The position of the points of this system we determine through the Lagrange coordinates in the cylindrical coordinate system $Or\theta z$ which is associated with the central axis of the cylinder. We suppose that the materials of the cylinder and surrounding elastic medium are homogeneous, isotropic and linear elastic.

Below we use the upper index (2) (upper index (1)) for indicating the values related to the cylinder (to the surrounding elastic medium).

Assume that in the initial state at infinity, the static radial compressional axisymmetric forces with intensity p act on the surrounding medium and as a result of this action the initial stress state appears in the constituents of the system. Moreover, assume that the materials of the constituents of the system are sufficiently rigid and the initial stress state can be determined within the scope of the classical linear theory of elasticity. Denoting the values related to the initial stress state by the additional upper index "0", we can write the following expressions for these stresses

$$\sigma_{rr}^{(1)0} = p - 2\mu^{(1)}B^{(1)}\frac{1}{r^2}, \quad \sigma_{\theta\theta}^{(1)0} = p + 2\mu^{(1)}B^{(1)}\frac{1}{r^2},$$

$$\sigma_{zz}^{(1)0} = \nu^{(1)}(\sigma_{rr}^{(1)0} + \sigma_{\theta\theta}^{(1)0}),$$

$$\sigma_{rr}^{(2)0} = 2(\lambda^{(2)} + \mu^{(2)})A^{(2)} - 2\mu^{(2)}B^{(2)}\frac{1}{r^2}, \tag{1}$$

$$\sigma_{\theta\theta}^{(2)0} = 2(\lambda^{(2)} + \mu^{(2)})A^{(2)} + 2\mu^{(2)}B^{(2)}\frac{1}{r^2},$$

$$\sigma_{zz}^{(2)0} = \nu^{(2)}(\sigma_{rr}^{(2)0} + \sigma_{\theta\theta}^{(2)0}),$$

where $\lambda^{(n)}$ and $\mu^{(n)}$ are the Lame constants and $\nu^{(n)}$ is Poisson's ratio of the n–th material. The unknown constants $B^{(1)}$, $A^{(2)}$ and $B^{(2)}$ are determined from the following system of algebraic equations

$$2\mu^{(1)}B^{(1)}\frac{1}{(R+h^{(1)})^2} + 2(\lambda^{(2)} + \mu^{(2)})A^{(2)} -$$

$$2\mu^{(2)}B^{(2)}\frac{1}{(R+h^{(1)})^2} = p,$$

$$\frac{p}{2(\lambda^{(1)} + \mu^{(1)})}(R+h^{(1)}) + B^{(1)}\frac{1}{(R+h^{(1)})} - \tag{2}$$

$$A^{(2)}(R+h^{(1)}) - B^{(2)}\frac{1}{(R+h^{(1)})} = 0,$$

$$2(\lambda^{(2)} + \mu^{(2)})A^{(2)} - 2\mu^{(2)}B^{(2)}\frac{1}{R^2} = 0.$$

Note that the equations in (2) are obtained from satisfaction of the boundary and contact conditions which relate to determination of the initial stress state, and the notation used in (1) and (2) is conventional.

In this way, the initial state in the system "hollow cylinder + surrounding elastic medium" is determined completely through the relations given in (1) and (2). It is required to investigate how these initial stresses affect the dispersion of the axisymmetric longitudinal waves in the bi-material infinite elastic system under consideration. Namely, this investigation is the subject of the present paper and will be made within the scope of the 3D linearized theory of elastic waves in bodies with initial stresses. Thus, according to (Guz 2004, Eringen and Suhubi 1975, Akbarov 2015) and others listed therein, the equations of motion and the elasticity relations of this theory for the case under consideration are obtained as follows.

The equations of motion

$$\frac{\partial t_{rr}^{(i)}}{\partial r} + \frac{\partial t_{zr}^{(i)}}{\partial z} + \frac{1}{r}(t_{rr}^{(i)} - t_{\theta\theta}^{(i)}) = \rho^{(i)}\frac{\partial^2 u_r^{(i)}}{\partial t^2},$$

$$\frac{\partial t_{rz}^{(i)}}{\partial r} + \frac{1}{r}t_{rz}^{(i)} + \frac{\partial t_{zz}^{(i)}}{\partial z} = \rho^{(i)}\frac{\partial^2 u_z^{(i)}}{\partial t^2}, \tag{3}$$

where i=1,2 and

$$t_{rr}^{(i)} = \sigma_{rr}^{(i)} + \sigma_{rr}^{(i)0}(r)\frac{\partial u_r^{(i)}}{\partial r}, \quad t_{rz}^{(i)} = \sigma_{rz}^{(i)} + \sigma_{rr}^{(i)0}(r)\frac{\partial u_z^{(i)}}{\partial r},$$

$$t_{\theta\theta}^{(i)} = \sigma_{\theta\theta}^{(i)} + \sigma_{\theta\theta}^{(i)0}(r)\frac{u_r^{(i)}}{r}, \quad t_{zr}^{(i)} = \sigma_{zr}^{(i)} + \sigma_{zz}^{(i)0}(r)\frac{\partial u_r^{(i)}}{\partial z}, \tag{4}$$

$$t_{zz}^{(i)} = \sigma_{zz}^{(i)} + \sigma_{zz}^{(i)0}(r)\frac{\partial u_z^{(i)}}{\partial z}$$

The elasticity and strain-displacement relations

$$\sigma_{(jj)}^{(i)} = \lambda^{(i)}(\varepsilon_{rr}^{(i)} + \varepsilon_{\theta\theta}^{(i)} + \varepsilon_{zz}^{(i)}) + 2\mu^{(i)}\varepsilon_{(jj)}^{(i)},$$

$$(jj) = rr; \theta\theta; zz, \ \sigma_{rz}^{(n)} = 2\mu^{(n)}\varepsilon_{rz}^{(n)}. \tag{5}$$

$$\varepsilon_{rr}^{(i)}=\frac{\partial u_r^{(i)}}{\partial r},\quad \varepsilon_{\theta\theta}^{(i)}=\frac{u_r^{(i)}}{r},\ \varepsilon_{zz}^{(i)}=\frac{\partial u_z^{(i)}}{\partial z},\quad \varepsilon_{rz}^{(i)}=\frac{1}{2}\left(\frac{\partial u_r^{(i)}}{\partial z}+\frac{\partial u_z^{(i)}}{\partial r}\right). \tag{6}$$

The foregoing equations (3)-(6) are the complete system of field equations within the framework of which the present investigations are made.

Consider also the corresponding boundary and contact conditions which must be added to these equations. These conditions are

$$\left\{\left|\sigma_{rr}^{(1)}\right|;\left|\sigma_{\theta\theta}^{(1)}\right|;\left|\sigma_{rz}^{(1)}\right|;\left|u_r^{(1)}\right|;\left|u_z^{(1)}\right|\right\}\to 0 \ \text{ as } \ r\to\infty,$$
$$\left.\sigma_{rr}^{(1)}\right|_{r=R+h}=\left.\sigma_{rr}^{(2)}\right|_{r=R+h},\quad \left.\sigma_{rz}^{(1)}\right|_{r=R+h}=\left.\sigma_{rz}^{(2)}\right|_{r=R+h},$$
$$\left.u_r^{(1)}\right|_{r=R+h}=\left.u_r^{(2)}\right|_{r=R+h},\quad \left.u_z^{(1)}\right|_{r=R+h}=\left.u_z^{(2)}\right|_{r=R+h}, \tag{7}$$
$$\left.\sigma_{rr}^{(2)}\right|_{r=R}=0,\quad \left.\sigma_{rz}^{(2)}\right|_{r=R}=0.$$

This completes the mathematical formulation of the problem.

3. Solution method

We use the discrete-analytical method to investigate the problem under consideration because it is impossible to obtain an analytical solution to this system of equations in the classical sense due to variability in the coefficients of the equations (3) and (4). According to the discrete-analytical method, the cylinder and surrounding elastic medium are divided into a certain number of sublayers within each of which the initial stresses determined through the expressions (1) and (2) are taken as constant, the values of which are determined through the expressions (1) and (2) by replacing r with r_m, where r_m is the middle radius of the sublayer's cross section. After this replacement, the attempt is made to obtain the analytical solution to the system of equations (3)-(6) and if it succeeds, then the discrete-analytical method also succeeds. Under satisfaction of the boundary conditions in (7), it is assumed that between the sublayers there exist perfect contact conditions.

Thus, we realize the above-described procedure.

3.1 Dividing regions into a certain number of sub-regions and contact conditions between these sub-regions

We divide the region [R, R+h] which is occupied by the hollow cylinder into the N_2 number of sub-regions $[R+(n_2-1)h/N_2], (R+n_2h/N_2)]$, where $n_2=1,2,\ldots N_2$. Moreover, we divide the region $[R+h, \infty]$ occupied by the surrounding elastic medium into the N_1 number of finite sub-regions $[R+h+(R_M-R-h)(n_1\text{-}1)N_1), R+h+(R_M-R-h)n_1N_1]$ and infinite sub-regions $[R_M, \infty]$, where $n_1=1,2,\ldots N_1$ and the numbers N_2 and N_1, and the value for R_M are determined in the solution procedure from the convergence requirement of the numerical results.

Thus, according to the foregoing discussions, the inhomogeneous initial stresses determined through the expressions (1) and (2) in each sub-region are taken as constants, the values of which are defined by the following relations:

In the n_2-th sub-region occupied by the hollow cylinder

$$\sigma_{rr}^{(2)0}(r)\approx\sigma_{rr}^{(2)0}(r_{n_2}),\quad \sigma_{\theta\theta}^{(2)0}(r)\approx\sigma_{\theta\theta}^{(2)0}(r_{n_2}),$$
$$\sigma_{zz}^{(2)0}(r)\approx\sigma_{zz}^{(2)0}(r_{n_2}), \tag{8}$$
$$r_{n_2}=R+(n_2-1)h/N_2+h/(2N_2).$$

In the n_1-th finite sub-region occupied by the surrounding medium

$$\sigma_{rr}^{(1)0}(r)\approx\sigma_{rr}^{(1)0}(r_{n_1}),$$
$$\sigma_{\theta\theta}^{(1)0}(r)\approx\sigma_{\theta\theta}^{(1)0}(r_{n_1}),\quad \sigma_{zz}^{(1)0}(r)\approx\sigma_{zz}^{(1)0}(r_{n_1}), \tag{9}$$
$$r_{n_1}=R+h+(R_M-R-h)(n_1-1)/N_1+(R_M-R-h)/(2N_1),$$

and in the infinite sub-region $[R_M, \infty]$ also occupied by the surrounding medium

$$\sigma_{rr}^{(1)0}(r)\approx\sigma_{rr}^{(1)0}(R_M),\quad \sigma_{\theta\theta}^{(1)0}(r)\approx\sigma_{\theta\theta}^{(1)0}(R_M),\quad \sigma_{zz}^{(1)0}(r)\approx\sigma_{zz}^{(1)0}(R_M). \tag{10}$$

Using the upper index notation $(2)n_2$ (the upper index $(1)n_1$) for the values related to the hollow cylinder (surrounding medium) and the upper index notation $(1)\infty$ for the values related to the infinite sub-region $[R_M, \infty]$ we formulate the contact conditions between the sub-regions and corresponding boundary conditions. Thus, taking the boundary condition in (7) and the assumption of perfect contact conditions between the sub-regions into consideration we can write the following relations:

$$\left.\sigma_{rr}^{(2)1}\right|_{r=R+h}=0,\quad \left.\sigma_{rz}^{(2)1}\right|_{r=R+h}=0,$$
$$\left.\sigma_{rr}^{(2)1}\right|_{r=R+h/N_2}=\left.\sigma_{rr}^{(2)2}\right|_{r=R+h/N_2},$$
$$\left.\sigma_{rz}^{(2)1}\right|_{r=R+h/N_2}=\left.\sigma_{rz}^{(2)2}\right|_{r=R+h/N_2},$$
$$\left.u_r^{(2)1}\right|_{r=R+h/N_2}=\left.u_r^{(2)2}\right|_{r=R+h/N_2},$$
$$\left.u_z^{(2)1}\right|_{r=R+h/N_2}=\left.u_z^{(2)2}\right|_{r=R+h/N_2},\ \ldots, \tag{11}$$
$$\left.\sigma_{rr}^{(2)n_2-1}\right|_{r=R+(n_2-1)h/N_2}=\left.\sigma_{rr}^{(2)n_2}\right|_{r=R+(n_2-1)h/N_2},$$
$$\left.\sigma_{rz}^{(2)n_2-1}\right|_{r=R+(n_2-1)h/N_2}=\left.\sigma_{rz}^{(2)n_2}\right|_{r=R+(n_2-1)h/N_2},$$
$$\left.u_r^{(2)n_2-1}\right|_{r=R+(n_2-1)h/N_2}=\left.u_r^{(2)n_2}\right|_{r=R+(n_2-1)h/N_2},$$

$$u_z^{(2)n_2-1}\Big|_{r=R+(n_2-1)h/N_2} = u_z^{(2)n_2}\Big|_{r=R+(n_2-1)h/N_2}, \dots,$$

$$\sigma_{rr}^{(2)N_2}\Big|_{r=R+h} = \sigma_{rr}^{(1)1}\Big|_{r=R+h},$$

$$\sigma_{rz}^{(2)N_2}\Big|_{r=R+h} = \sigma_{rz}^{(1)1}\Big|_{r=R+h},$$

$$u_r^{(2)N_2}\Big|_{r=R+h} = u_r^{(1)1}\Big|_{r=R+h},$$

$$u_z^{(2)N_2}\Big|_{r=R+h} = u_z^{(1)1}\Big|_{r=R+h}, \dots,$$

$$\sigma_{rr}^{(1)n_1-1}\Big|_{r=R+h+(n_1-1)H/N_1} = \sigma_{rr}^{(1)n_1}\Big|_{r=R+h+(n_1-1)H/N_1},$$

$$\sigma_{rz}^{(1)n_1-1}\Big|_{r=R+h+H(n_1-1)/N_1} = \sigma_{rz}^{(1)n_1}\Big|_{r=R+h+(n_1-1)H/N_1},$$

$$u_r^{(1)n_1-1}\Big|_{r=R+h+(n_1-1)H/N_1} = u_r^{(1)n_1}\Big|_{r=R+h+(n_1-1)H/N_1},$$

$$u_z^{(1)n_1-1}\Big|_{r=R+h+(n_1-1)H/N_1} = u_z^{(1)n_1}\Big|_{r=R+h+(n_1-1)H/N_1}, \dots,$$

$$\sigma_{rr}^{(1)N_1}\Big|_{r=R+h+R_M} = \sigma_{rr}^{(1)\infty}\Big|_{r=R+h+R_M},$$

$$\sigma_{rz}^{(1)N_1}\Big|_{r=R+h+R_M} = \sigma_{rz}^{(1)\infty}\Big|_{r=R+h+R_M},$$

$$u_r^{(1)N_1}\Big|_{r=R+h+R_M} = u_r^{(1)\infty}\Big|_{r=R+h+R_M},$$

$$u_z^{(1)N_1}\Big|_{r=R+h+R_M} = u_z^{(1)\infty}\Big|_{r=R+h+R_M},$$

$$\left\{\left|\sigma_{rr}^{(1)\infty}\right|;\left|\sigma_{\theta\theta}^{(1)\infty}\right|;\left|\sigma_{rz}^{(1)\infty}\right|;\left|\sigma_{zz}^{(1)\infty}\right|;\left|u_r^{(1)\infty}\right|;\left|u_z^{(1)\infty}\right|\right\} \to 0 \quad \text{as} \quad r \to \infty,$$

where $H = R_M - R - h$.

This completes the consideration of the formulation of the contact conditions between the sub-regions and boundary conditions for the values related to the face sub-regions.

3.2 Solution procedure to the system of equations (3)-(6)

Now we consider the equations which are obtained from the equations (3)-(6) within the scope of the foregoing assumptions. Under this consideration, we will take into account that the expressions in (5) and (6) remain valid in each sub-region as are, and under mathematical transformations, the upper indices (1) and (2) in these expressions are replaced with the upper indices $(1)n_1$, $(2)n_2$ and $(1)\infty$, respectively.

Thus, taking the relations (8)-(10) into consideration, we obtain the following equations of motion from the equations (3) and (4).

$$\begin{gathered}\frac{\partial\sigma_{rr}^{(i)n_i}}{\partial r} + \sigma_{rr}^{(i)0}(r_{n_i})\frac{\partial^2 u_r^{(i)n_i}}{\partial r^2} + \frac{\partial\sigma_{zr}^{(i)n_i}}{\partial z} + \\ \sigma_{zz}^{(i)0}(r_{n_i})\frac{\partial^2 u_r^{(i)n_i}}{\partial z^2} + \frac{1}{r}(\sigma_{rr}^{(i)n_i} - \sigma_{\theta\theta}^{(i)n_i}) + \\ \sigma_{rr}^{(i)0}(r_{n_i})\frac{1}{r}\frac{\partial u_r^{(i)n_i}}{\partial r} - \sigma_{\theta\theta}^{(i)0}(r_{n_i})\frac{u_r^{(i)n_i}}{r^2} = \rho^{(i)}\frac{\partial^2 u_r^{(i)n_i}}{\partial t^2}, \\ \frac{\partial\sigma_{rz}^{(i)n_i}}{\partial r} + \sigma_{rr}^{(i)0}(r_{n_i})\frac{\partial^2 u_z^{(i)n_i}}{\partial r^2} + \frac{1}{r}\sigma_{rz}^{(i)n_i} \\ + \sigma_{rr}^{(i)0}(r_{n_i})\frac{1}{r}\frac{\partial u_z^{(i)n_i}}{\partial r} + \frac{\partial\sigma_{zz}^{(i)n_i}}{\partial z} + \\ \sigma_{zz}^{(i)0}(r_{n_i})\frac{\partial^2 u_z^{(i)n_i}}{\partial z^2} = \rho^{(i)}\frac{\partial^2 u_z^{(i)n_i}}{\partial t^2}, \quad i = 1, 2.\end{gathered} \tag{12}$$

Note that by replacing the upper index $(1)n_i$ with the upper index $(1)\infty$, we can write the corresponding system of equations for the infinite sub-region $[R_M, \infty]$. However, here we do not consider separately these equations for this infinite sub-region because these solutions can be obtained automatically from the solution to the equations in (12) taking the last condition in (11) into consideration.

Thus, the equations in (12) with the equations in (5) and (6) comprise the complete system of equations with respect to the sought values. We attempt to solve this system of equations by the use of the classical Lame decomposition (see, for instance, the monograph (Eringen and Suhubi 1975) which for the case under consideration can be presented as follows:

$$\begin{aligned}u_r^{(i)n_i} &= \frac{\partial\Phi^{(i)n_i}}{\partial r} + \frac{\partial^2\Psi^{(i)n_i}}{\partial r\partial z}, \\ u_z^{(i)n_i} &= \frac{\partial\Phi^{(i)n_i}}{\partial z} - \frac{\partial^2\Psi^{(i)n_i}}{\partial r^2} - \frac{\partial\Psi^{(i)n_i}}{r\partial r},\end{aligned} \tag{13}$$

Substituting the expression into the equations (5), (6) and (12), and doing some cumbersome mathematical manipulations, it is established that the potentials $\Phi^{(i)n_i}$ and $\Psi^{(i)n_i}$ must satisfy the following equations.

$$\begin{gathered}(1 + \frac{\sigma_{rr}^{(i)0}(r_{n_i})}{\lambda^{(i)} + 2\mu^{(i)}})\frac{\partial^2\Phi^{(i)n_i}}{\partial r^2} + (1 + \frac{\sigma_{\theta\theta}^{(i)0}(r_{n_i})}{\lambda^{(i)} + 2\mu^{(i)}})\frac{\partial\Phi^{(i)n_i}}{r\partial r} + \\ (1 + \frac{\sigma_{zz}^{(i)0}(r_{n_i})}{\lambda^{(i)} + 2\mu^{(i)}})\frac{\partial^2\Phi^{(i)n_i}}{\partial z^2} = \frac{1}{(c_1^{(i)})^2}\frac{\partial^2\Phi^{(i)n_i}}{\partial t^2}, \\ (1 + \frac{\sigma_{rr}^{(i)0}(r_{n_i})}{\mu^{(i)}})\frac{\partial^2\Psi^{(i)n_i}}{\partial r^2} + (1 + \frac{\sigma_{\theta\theta}^{(i)0}(r_{n_i})}{\mu^{(i)}})\frac{\partial\Psi^{(i)n_i}}{r\partial r} + \\ (1 + \frac{\sigma_{zz}^{(i)0}(r_{n_i})}{\mu^{(i)}})\frac{\partial^2\Psi^{(i)n_i}}{\partial z^2} = \frac{1}{(c_2^{(i)})^2}\frac{\partial^2\Psi^{(i)n_i}}{\partial t^2}.\end{gathered} \tag{14}$$

As can be predicted, in the cases where $\sigma_{zz}^{(i)0}(r_{n_i})=0$, $\sigma_{rr}^{(i)0}(r_{n_i})=0$ and $\sigma_{\theta\theta}^{(i)0}(r_{n_i})=0$, the equations in (14) coincide with the corresponding ones in classical elastodynamics (see, for instance Eringen and Suhubi 1975).

Thus, as it is considered that the guided wave propagates along the *Oz* axis, all the sought quantities can be presented with multiplying cos(*kz–ωt)* (for the functions $\Phi^{(i)n_i}$, $u_r^{(i)n_i}$, $\sigma_{rr}^{(i)n_i}$, $\sigma_{\theta\theta}^{(i)n_i}$ and $\sigma_{zz}^{(i)n_i}$) and with multiplying sin(*kz–ωt)* (for the functions $\Psi^{(i)n_i}$, $u_z^{(i)n_i}$ and $\sigma_{rz}^{(i)n_i}$). Taking this into consideration and denoting the amplitudes of these quantities with the same symbols, we obtain the following equations for the amplitudes of the potentials $\Phi^{(i)n_i}$ and $\Psi^{(i)n_i}$.

$$(1+\frac{\sigma_{rr}^{(i)0}(r_{n_i})}{\lambda^{(i)}+2\mu^{(i)}})\frac{d^2\Phi^{(i)n_i}}{d(kr)^2}+$$

$$(1+\frac{\sigma_{\theta\theta}^{(i)0}(r_{n_i})}{\lambda^{(i)}+2\mu^{(i)}})\frac{d\Phi^{(i)n_i}}{krd(kr)}+$$

$$(\frac{1}{(c_1^{(i)})^2}\frac{\omega^2}{k^2}-1-\frac{\sigma_{zz}^{(i)0}(r_{n_i})}{\lambda^{(i)}+2\mu^{(i)}})\Phi^{(i)n_i}=0, \tag{15}$$

$$(1+\frac{\sigma_{rr}^{(i)0}(r_{n_i})}{\mu^{(i)}})\frac{d^2\Psi^{(i)n_i}}{d(kr)^2}+(1+\frac{\sigma_{\theta\theta}^{(i)0}(r_{n_i})}{\mu^{(i)}})\frac{d\Psi^{(i)n_i}}{krd(kr)}+$$

$$(\frac{1}{(c_2^{(i)})^2}\frac{\omega^2}{k^2}-1-\frac{\sigma_{zz}^{(i)0}(r_{n_i})}{\mu^{(i)}})\Psi^{(i)n_i}=0.$$

To simplify the equations in (15) we introduce the following notation:

$$\alpha^{(i)}(r_{n_i})=\frac{1+\sigma_{\theta\theta}^{(i)0}(r_{n_i})/\mu^{(i)}}{1+\sigma_{rr}^{(i)0}(r_{n_i})/\mu^{(i)}},$$

$$\beta^{(i)}(r_{n_i})=\frac{1+\sigma_{zz}^{(i)0}(r_{n_i})/\mu^{(i)}}{1+\sigma_{rr}^{(i)0}(r_{n_i})/\mu^{(i)}},$$

$$r_1^{(i)n_i}=kr\sqrt{\frac{c^2}{(c_2^{(i)})^2(1+\sigma_{rr}^{(i)0}(r_{n_i})/\mu^{(i)})}-(\beta^{(i)}(r_{n_i}))^2}$$

$$c=\omega/\kappa, \tag{16}$$

$$\alpha_1^{(1)}(r_{n_i})=\frac{1+\sigma_{\theta\theta}^{(i)0}(r_{n_i})/(\lambda^{(i)}+2\mu^{(i)})}{1+\sigma_{rr}^{(i)0}(r_{n_i})/(\lambda^{(i)}+2\mu^{(i)})},$$

$$\beta_1^{(1)}(r_{n_i})=\frac{1+\sigma_{zz}^{(i)0}(r_{n_i})/(\lambda^{(i)}+2\mu^{(i)})}{1+\sigma_{rr}^{(i)0}(r_{n_i})/(\lambda^{(i)}+2\mu^{(i)})}$$

$$r_2^{(i)n_i}=kr\times$$

$$\sqrt{\frac{c^2}{(c_1^{(i)})^2(1+\sigma_{rr}^{(i)0}(r_{n_i})/(\lambda^{(i)}+2\mu^{(i)}))}-(\beta_1^{(i)}(r_{n_i}))^2}.$$

3.3 Solution to the equations in (15)

Using the notation (16), we can rewrite the equations in (15) as follows

$$\frac{d^2\Phi^{(i)n_i}}{d(r_2^{(i)})^2}+\frac{\alpha_1^{(i)}(r_{n_i})}{r_2^{(i)}}\frac{d\Phi^{(i)n_i}}{dr_2^{(i)}}+\Phi^{(i)n_i}=0,$$

$$\frac{d^2\Psi^{(i)n_i}}{d(r_1^{(i)})^2}+\frac{\alpha^{(i)}(r_{n_i})}{r_1^{(i)}}\frac{d\Psi^{(i)n_i}}{dr_1^{(i)}}+\Psi^{(i)n_i}=0. \tag{17}$$

We represent the equations in (17) in the following form:

$$\frac{d^2y}{dx^2}+\frac{\alpha}{x}\frac{dy}{dx}+y=0. \tag{18}$$

According to (Watson 1966), using the substitution

$$y=x^{(1-\alpha)/2}y_1(x). \tag{19}$$

the following Bessel equation for the function $y_1(x)$ is obtained from the equation (18):

$$\frac{d^2y_1}{dx^2}+\frac{1}{x}\frac{dy_1}{dx}+(1-\frac{(1-\alpha)^2/4}{x^2})y_1=0. \tag{20}$$

Thus, according to (19) and (20), the following expressions and equations can be written:

$$\Phi^{(i)n_i}(r_2^{(i)})=(r_2^{(i)})^{(1-\alpha_1^{(i)}(r_{n_i}))/2}\Phi_1^{(i)n_i}(r_2^{(i)}). \tag{21}$$

$$\Psi^{(i)n_i}(r_1^{(i)})=(r_1^{(i)})^{(1-\alpha^{(i)}(r_{n_i}))/2}\Psi_1^{(i)n_i}(r_1^{(i)}), \tag{22}$$

$$\frac{d^2\Phi_1^{(i)n_i}}{d(r_2^{(i)})^2}+\frac{1}{r_2^{(i)}}\frac{d\Phi_1^{(i)n_i}}{dr_2^{(i)}}+$$

$$(1-\frac{(1-(\alpha_1^{(i)}(r_{n_i})))^2/4}{(r_2^{(i)})^2})\Phi_1^{(i)n_i}=0, \tag{23}$$

$$\frac{d^2\Psi_1^{(i)n_i}}{d(r_1^{(i)})^2}+\frac{1}{r_1^{(i)}}\frac{d\Psi_1^{(i)n_i}}{dr_1^{(i)}}+$$

$$(1-\frac{(1-(\alpha^{(i)}(r_{n_i})))^2/4}{(r_1^{(i)})^2})\Psi_1^{(i)n_i}=0, \tag{24}$$

from which we obtain the following expressions for the functions $\Phi^{(i)n_i}$ and $\Psi^{(i)n_i}$.

$$\Phi^{(i)n_i} = A_1^{(i)n_i}(r_2^{(i)})^{\gamma_1^{(i)}(r_{n_i})} E_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i}) + A_2^{(i)n_i}(r_2^{(i)})^{\gamma_1^{(i)}(r_{n_i})} F_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i}), \quad (25)$$

$$\Phi^{(1)\infty} = A_2^{(1)\infty}(r_2^{(1)\infty})^{\gamma_1^{(1)\infty}(R_M)} F_{\gamma_1^{(1)\infty}(R_M)}(r_2^{(1)\infty}),$$

$$\Psi^{(i)n_i} = B_1^{(i)n_i}(r_1^{(i)})^{\gamma^{(i)}(r_{n_i})} E_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i}) +, \quad B_2^{(i)n_i}(r_1^{(i)})^{\gamma^{(i)}(r_{n_i})} F_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i}), \quad (26)$$

$$\Psi^{(1)\infty} = B_2^{(1)\infty}(r_1^{(1)\infty})^{\gamma^{(1)\infty}(R_m)} F_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty}).$$

where

$$\gamma_1^{(i)}(r_{n_i}) = (1-\alpha_1^{(i)}(r_{n_i}))/2, \quad \gamma^{(i)}(r_{n_i}) = (1-\alpha^{(i)}(r_{n_i}))/2,$$

$$\gamma_1^{(1)\infty}(R_M) = \frac{1-\alpha_1^{(1)\infty}(R_M)}{2}, \quad \gamma^{(1)\infty}(R_M) = \frac{1-\alpha^{(1)\infty}(R_M)}{2},$$

$$\alpha^{(1)\infty}(R_M) = \frac{1+\sigma_{\theta\theta}^{(1)0}(R_M)/\mu^{(1)}}{1+\sigma_{rr}^{(1)0}(R_M)/\mu^{(1)}},$$

$$\alpha_1^{(1)\infty}(R_M) = \frac{1+\sigma_{\theta\theta}^{(1)0}(R_M)/(\lambda^{(1)}+2\mu^{(1)})}{1+\sigma_{rr}^{(1)0}(R_M)/(\lambda^{(1)}+2\mu^{(1)})},$$

$$\beta^{(1)\infty}(R_M) = \frac{1+\sigma_{zz}^{(1)0}(R_M)/\mu^{(1)}}{1+\sigma_{rr}^{(1)0}(R_M)/\mu^{(1)}},$$

$$\beta_1^{(1)\infty}(R_M) = \frac{1+\sigma_{zz}^{(1)0}(R_M)/(\lambda^{(1)}+2\mu^{(1)})}{1+\sigma_{rr}^{(1)0}(R_M)/(\lambda^{(1)}+2\mu^{(1)})},$$

$$r_1^{(1)\infty} = kr \times \sqrt{\frac{c^2}{(c_2^{(1)})^2(1+\sigma_{rr}^{(1)0}(R_M)/\mu^{(i)})} - (\beta^{(1)\infty}(R_M))^2}, \quad (27)$$

$$r_2^{(1)\infty} = kr \times \sqrt{\frac{c^2(c_1^{(1)})^{-2}}{(1+\sigma_{rr}^{(1)0}(R_M)/(\lambda^{(1)}+2\mu^{(1)}))} - (\beta_1^{(1)\infty}(R_M))^2},$$

$$E_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i}) = \begin{cases} J_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i}) \text{ if } (r_2^{(i)n_i})^2/r^2 > 0 \\ I_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i}) \text{ if } (r_2^{(i)n_i})^2/r^2 < 0 \end{cases},$$

$$F_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i}) = \begin{cases} Y_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i}) \text{ if } (r_2^{(i)n_i})^2/r^2 > 0 \\ K_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i}) \text{ if } (r_2^{(i)n_i})^2/r^2 < 0 \end{cases},$$

$$E_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i}) = \begin{cases} J_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i}) \text{ if } (r_1^{(i)n_i})^2/r^2 > 0 \\ I_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i}) \text{ if } (r_1^{(i)n_i})^2/r^2 < 0 \end{cases},$$

$$F_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i}) = \begin{cases} Y_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i}) \text{ if } (r_1^{(i)n_i})^2/r^2 > 0 \\ K_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i}) \text{ if } (r_1^{(i)n_i})^2/r^2 < 0 \end{cases},$$

$$F_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty}) = \begin{cases} Y_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty}) \text{ if } (r_1^{(1)\infty})^2/r^2 > 0 \\ K_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty}) \text{ if } (r_1^{(1)\infty})^2/r^2 < 0 \end{cases},$$

$$F_{\gamma_1^{(1)\infty}(R_M)}(r_2^{(1)\infty}) = \begin{cases} Y_{\gamma_1^{(1)\infty}(R_M)}(r_2^{(1)\infty}) \text{ if } (r_2^{(1)\infty})^2/r^2 > 0 \\ K_{\gamma_1^{(1)\infty}(R_M)}(r_2^{(1)\infty}) \text{ if } (r_2^{(1)\infty})^2/r^2 < 0 \end{cases},$$

where $J_\delta(x)$ and $I_\delta(x)$ ($Y_\delta(x)$ and $K_\delta(x)$) are the Bessel and modified Bessel functions of the first (second) kind.

3.4 The expressions for the displacements and stresses and construction of the dispersion equation

Thus, using the expressions (21)-(27) we determine completely the exact analytical solution to the equations in (15), and substituting this solution into the foregoing relations, we determine the expressions for the amplitude of the displacements and stresses which enter the contact and boundary conditions. These expressions are given in Appendix A with formulae in (A1).

Substituting these expressions into the contact and boundary conditions in (11) we obtain the system of homogeneous linear algebraic equations with respect to the unknown constants $A_1^{(i)n_i}$, $A_2^{(i)n_i}$, $B_1^{(i)n_i}$, $B_2^{(i)n_i}$, $A_2^{(i)\infty}$ and $B_2^{(1)\infty}$ where i =1,2, n_1=1,2,…N_1 and n_2=1,2,…N_2. Equating to zero the determinant of the coefficient matrix of these equations we obtain the following dispersion equation:

$$\det(a_{nm}(c, kR, p/\mu^{(1)}, \mu^{(1)}/\mu^{(2)}, h/R)) = 0, \quad n;m = 1,2,\ldots,4(N_1+N_2)+2. \quad (28)$$

As the explicit expressions of the components of the matrix $\{a_{nm}\}$ can be easily determined from the formulae (A1) given in Appendix A, here we do not give these expressions.

This completes the consideration of the solution method and obtaining the dispersion equation. Note that the method used in the present paper was also used in the papers

(Akbarov *et al*. 2017, 2018 and Akbarov 2006), the results of which are also detailed in the monograph by Akbarov (2015)

4. Numerical results and discussions

In the present section we consider the numerical results, i.e. the dispersion curves constructed as a result of the solution of the dispersion equation (28). This solution is made numerically with employing the well-known "bi-section" method, the PC programs for this calculation having been constructed by the authors of the paper. Testing of these programs will be made with comparison of the present results obtained in certain particular cases with the corresponding results obtained by other researchers.

Under obtaining and discussing of the numerical results for characterization of the magnitude of the initial stresses we will use the ratio $p/\mu^{(1)}$ and we intend to consider only the first two lower dispersion modes. Despite this intent, as will be shown below, additional dispersion curves which appear due to the inhomogeneity of the initial stresses are obtained.

Numerical results, which will be discussed below, are obtained for the following two cases for the values of the mechanical constants of the constituents of the bi-material elastic system under consideration.

$$\text{Case 1:} \quad E^{(1)}/E^{(2)} = 0.05, \quad \rho^{(1)}/\rho^{(2)} = 0.01, \quad \nu^{(1)} = \nu^{(2)} = 0.25. \tag{29}$$

$$\text{Case 2:} \quad E^{(1)}/E^{(2)} = 0.35, \quad \rho^{(1)}/\rho^{(2)} = 0.1, \quad \nu^{(1)} = \nu^{(2)} = 0.25. \tag{30}$$

Note that these materials were also selected for consideration in the paper (Abdulkadirov 1981). However, in that paper in Case 2 the case was considered where full slipping conditions are satisfied on the interface between the cylinder and surrounding medium.

Therefore, below, comparison of the present results with the corresponding ones obtained in the paper (Abdulkadirov 1981) will be made only for Case 1.

Moreover, note that all the results which will be discussed below are obtained in the case where N_1=15 and N_2=30, and R_M is selected as $5R$, i.e. $R_M = 5R$. The validity of such selection of the number of sub-regions and the convergence of the numeric results with respect to these numbers will be considered after analysis of the dispersion curves obtained in the foregoing three (29)-(30) cases and under this analysis, we consider separately the dispersion curves obtained in these cases.

4.1 Dispersion curves obtained for Case 1

Thus, we consider the graphs given in Fig. 2 which illustrate the dispersion curves for the "first" mode obtained in Case 1 under h/R = 0.15 (Fig. 2a), 0.25 (Fig. 2b) and 0.5 (Fig. 2c). Here, under "first" mode we understand the dispersion curves which are near the dispersion curves related to the corresponding first mode obtained in the case where $p/\mu^{(1)}$ = 0.00 which are drawn with dashed lines in the figures. Note that the considered curves are obtained for various values of $p/\mu^{(1)}$, namely, for the values $p/\mu^{(1)}$ = 0.0001; 0.0003; 0.0005; 0.0007 and 0.001.

Analysis of the graphs given in Fig. 2 shows that the character of the influence of the inhomogeneous initial stresses on the dispersion curves depends not only on the values of the ratio $p/\mu^{(1)}$ but also on the values of the ratio h/R. This is because in the case where h/R = 0.5, the dispersion curves obtained for all values of $p/\mu^{(1)}$ are similar to those which are obtained in the case where $p/\mu^{(1)}$ =0. Consequently, in the case where h/R = 0.5 in Case 1, the influence of the inhomogeneous initial stresses on the dispersion curves has only quantitative character and, according to the zoomed parts Z1, Z2, Z3, Z4 and Z5, there exists such a value of the dimensionless wavenumber kR (denote it by kR_1^*) before which (after which) the initial stresses cause to increase (to decrease) the wave propagation velocity.

Note that in the paper by Abdulkadirov (1981), this case was also considered, i.e. the case where h/R = 0.5. Comparison of the dispersion curve constructed under $p/\mu^{(1)}$ =0 and illustrated in Fig. 2c with the dashed line, with the corresponding one obtained in that paper, shows that the present curve coincides with it completely. Moreover, note that in the present paper, the results obtained for the case where $p/\mu^{(1)}$ = 0.00 are calculated with the same PC programs which are also used for calculating the results related to the cases where $p/\mu^{(1)} > 0$. Thus, we can conclude the validity of the PC programs used in the present investigation. Unfortunately, we have not found any related results of other authors to compare with the present results.

Thus, the main conclusion which follows from the results given in Fig. 2c is the following: in Case 1 under h/R = 0.5 the influence of the inhomogeneous initial stresses on the dispersion curves has only quantitative character. However, the results given in Fig. 2a and Fig. 2b show that in the cases where h/R = 0.15 and h/R = 0.25 this conclusion is violated in the relatively greater values of $p/\mu^{(1)}$. In other words, in the cases where $p/\mu^{(1)} \geq 0.0003$ under h/R = 0.15 and in the cases where $p/\mu^{(1)} \geq 0.0007$ under h/R = 0.25 this conclusion is violated and in these cases the inhomogeneous initial stresses on the dispersioncurves have not only quantitative but also qualitative character. Thus, in these cases, new types of dispersion curves are obtained, each of which has some branches. The first branch appears before a certain value of kR (denote it by $(kR)_1^{"}$ under which $d(c/c_2^{(2)})/\mathrm{d}(kR)=\infty$, however, the second branch appears after a certain value of kR(denote it by $(kR)_2^{"}(>(kR)_1^{"})$) under which the relation $d(c/c_2^{(2)})/\mathrm{d}(kR)=\infty$ also takes place and the values of $(kR)_1^{"}$ (of $(kR)_2^{"}$) decrease (increase) with $|p/\mu^{(1)}|$. Consequently, in the frequency interval $(\omega R)_1^{"}/c_2 < \omega R/c_2 < (\omega R)_2^{"}/c_2$ is the band gap interval, where $(\omega R)_1^{"}$ and $(\omega R)_2^{"}$ are wave frequencies under $kR = (kR)_1^{"}$ and$kR = (kR)_2^{"}$, respectively.

Analyses of the zoomed parts Z1–Z5 in Figs. 2a and 2b allow us to conclude that before a certain value of kR (denote it by $(kR)^*$) the initial stresses cause to increase the wave propagation velocity, however, in the cases where $kR>(kR)^*$, the initial stresses cause to decrease significantly the wave propagation velocity for all the selected values of $p/\mu^{(1)}$.

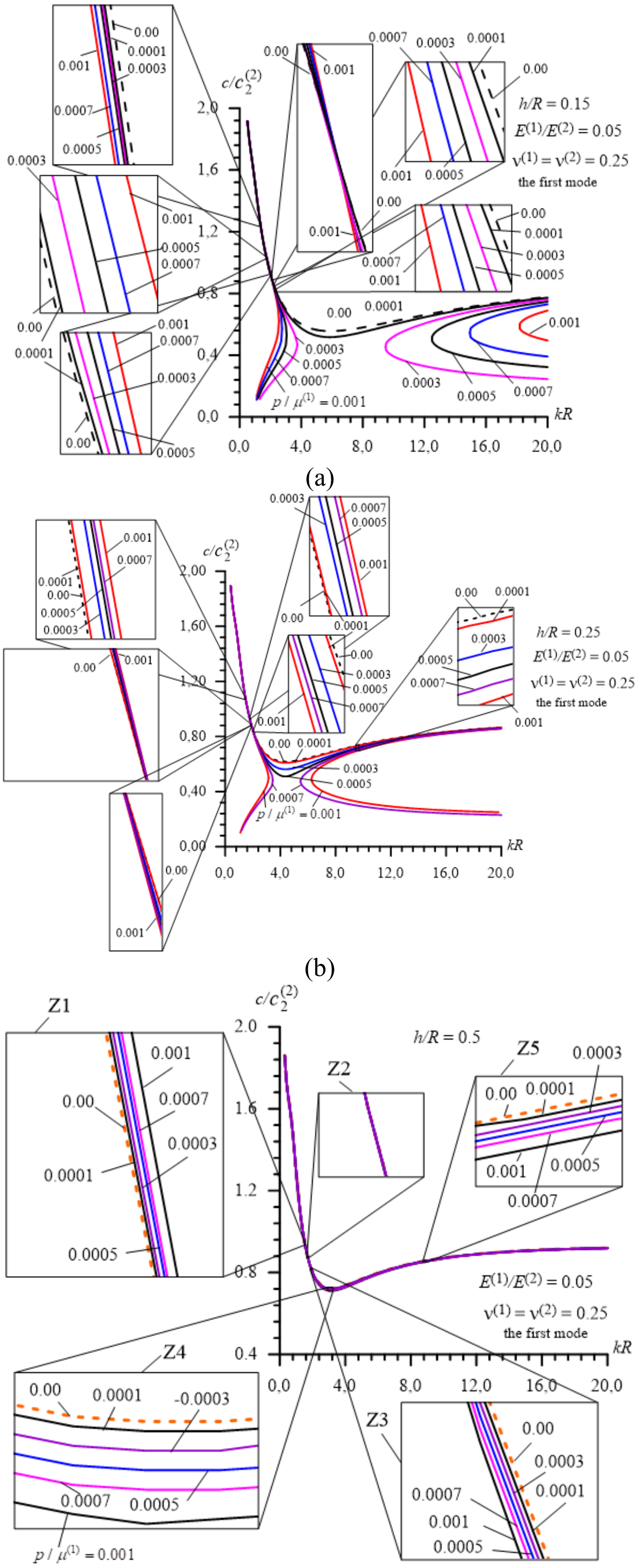


Fig. 2 Dispersion curves for the first modes in Case 1 under $h/R = 0.15$ (a), 0.25 (b) and 0.50 (c)

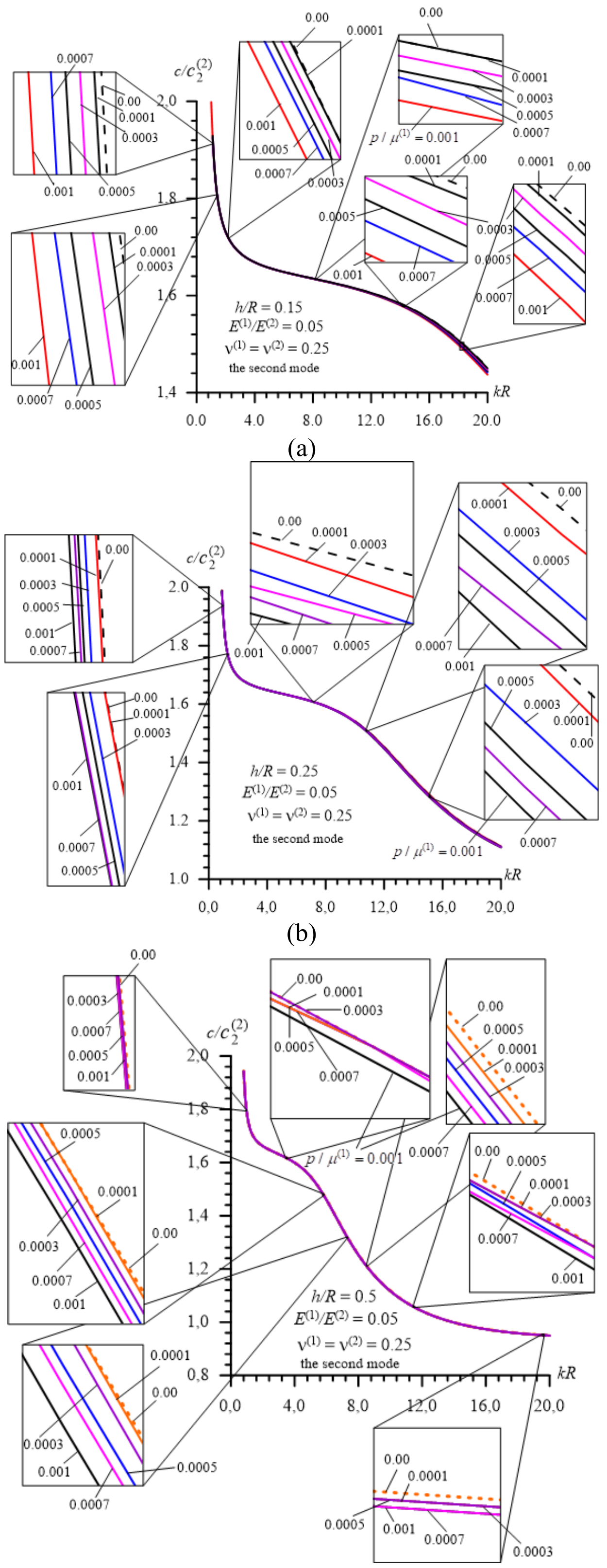


Fig. 3 Dispersion curves for the second modes in Case 1 under $h/R = 0.15$ (a), 0.25 (b) and 0.50 (c)

Also, we consider the dispersion curves related to the second mode and illustrated in Fig. 3 which are constructed for the cases where $h/R = 0.15$ (Fig. 3a), 0.25 (Fig. 3b) and 0.5 (Fig. 3c). It follows from the zoomed parts in these figures that the initial stresses cause to decrease the wave propagation velocity for all the considered values of the dimensionless wavenumber kR. Moreover, it follows from the results that the magnitude of the "decrease" increases with $p/\mu^{(1)}$ and with the decrease of h/R. Consequently, in the second mode, the influence of the initial inhomogeneous stresses on the dispersion curves has only quantitative character.

Finally, note that the dashed line in Fig. 3c which relates to the case where $p/\mu^{(1)}$ = 0.00 coincides completely with that obtained in the paper by Abdulkadirov (1981) and this statement again illustrates the validity of the PC programs and algorithm used in the present investigation.

This completes the results related to Case 1 (29).

4.2 *Dispersion curves obtained for Case 2*

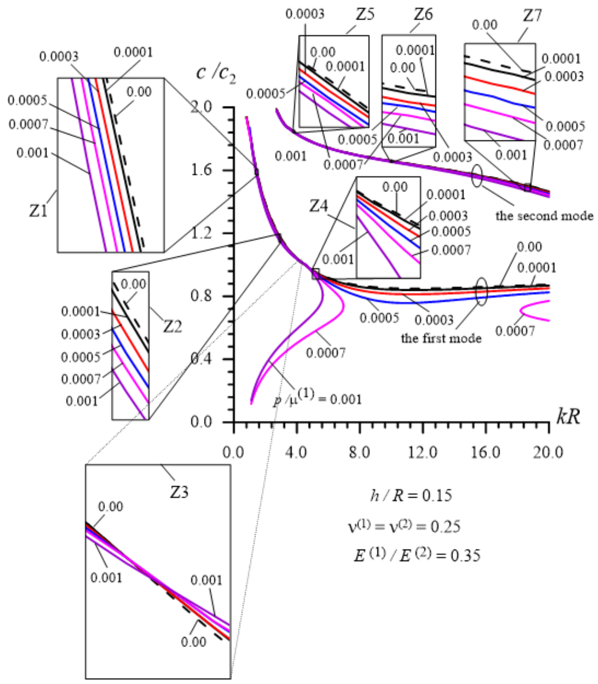
The dispersion curves related to this case are given in Fig. 4 and these curves are constructed in the cases where h/R = 0.15 (Fig. 4a), 0.25 (Fig. 4b) and 0.50 (Fig. 4c). Note, unlike the previous case in Fig. 4, the dispersion curves related to the first and second modes are given together.

Thus, we analyze these dispersion curves and begin this analysis with the ones given in Fig. 4a for the first mode, according to which, before a certain value of the ratio $p/\mu^{(1)}$ these curves in the qualitative sense are similar to those obtained in the case where $p/\mu^{(1)}$ = 0.00. However, after this "certain" value of $p/\mu^{(1)}$ (i.e. for the case under consideration under $p/\mu^{(1)} \geq 0.0007$) the character of the dispersion curves changes and as in the previous case, the band gap for the wave frequencies appears. Moreover, according to the zoomed parts Z1–Z4 in Fig. 4a, before the zoomed part Z3 (to say more precisely, the cases where $kR < (kR)_1^*$, and $(kR)_1^*$, is in the zoomed part Z3), the existence of the initial stresses causes to decrease the wave propagation velocity, however, in the cases where $(kR)_1^* < kR < (kR)_2^*$, where $(kR)_2^*$ is between the zoomed parts Z3 and Z4, the initial stresses cause to increase the wave propagation velocity. According to both the zoomed part Z4 and direct observation of the dispersion curves, it can be concluded that in the cases where $kR > (kR)_2^*$, the wave propagation velocity decreases again as a result of the existence of the initial stresses and the magnitude of this "decrease" grows with $p/\mu^{(1)}$. It follows from the analysis of the zoomed parts Z1–Z7 in Fig. 4b and the zoomed parts Z1–Z10 in Fig. 4c that the character of the initial stresses on the dispersion curves obtained under h/R = 0.25 (Fig. 4b) and 0.50 (Fig. 4c) is similar to that which is described above for the case where h/R = 0.15 However, in the latter two cases, unlike the case where h/R = 0.15, in all the considered values of $p/\mu^{(1)}$, the influence of the initial stresses on the dispersion curves has qualitative character. At the same time, under h/R = 0.5 and $p/\mu^{(1)} \geq 0.0005$, as a result of the initial stresses, new additional dispersion curves appear which are illustrated in Fig. 5.

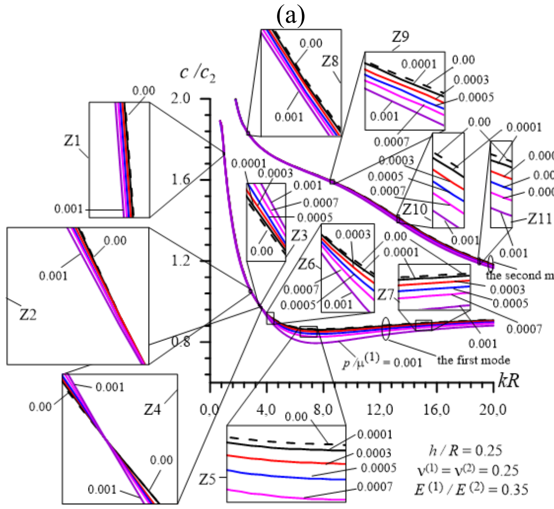
Besides all of these, it follows from the zoomed parts Z5–Z7 in Fig. 4a, Z8–Z11 in Fig. 4b and Z11–Z14 in Fig. 4c that in Case 2 for all the selected values of the problem parameters, the initial stresses cause to decrease the wave propagation velocity in the second mode.

4.3 *Convergence of the numerical results*

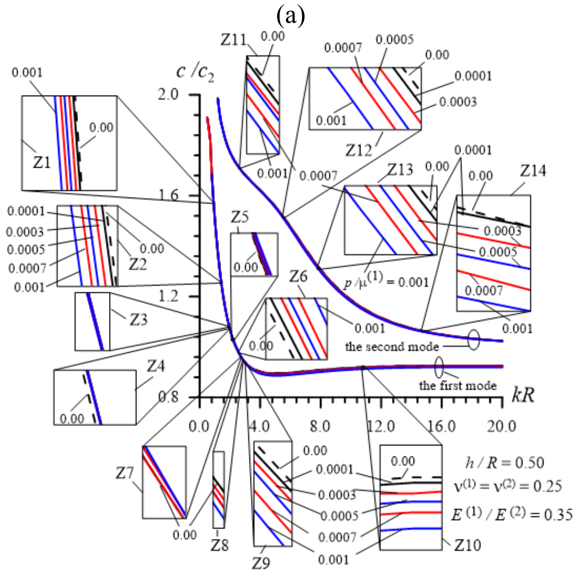
We consider the convergence of the numerical results with respect to the numbers N_1, N_2 and R_M/R, the meaning



(a)



(a)



(c)

Fig. 4 Dispersion curves obtained in Case 2 under h/R = 0.15 (a), 0.25 (b) and 0.50 (c)

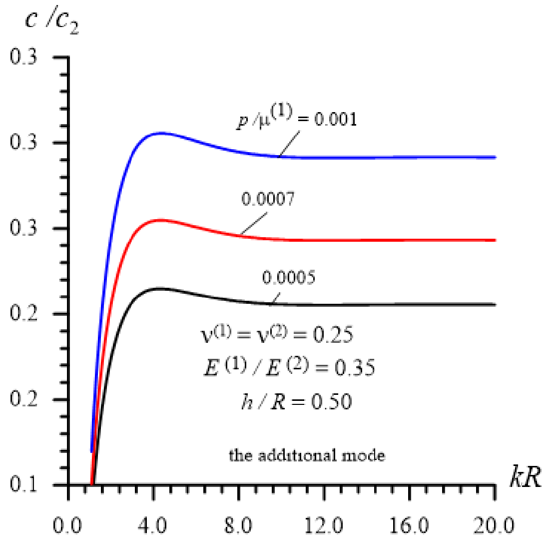


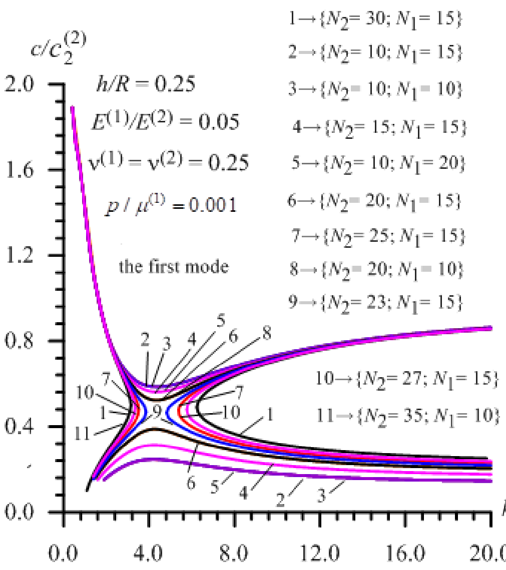
Fig. 5 Additional dispersion curves obtained in Case 2 under $h/R = 0.50$

of which is given in sub-section 3.1. First we assume that $R_M/R = 5$ and examine the dispersion curves obtained in Case 1 for various values of the numbers N_1 and N_2 under $h/R = 0.25$ and $h/R = 0.001$. These dispersion curves are given in Fig. 6 for the first (Fig. 6a) and second (Fig. 6b) modes and it follows from these curves that an increase in the values of the numbers N_1 and N_2 (especially in the values of the number N_2) causes to change the dispersion curves related to the first mode not only qualitatively but also quantitatively. However, the dispersion curves related to the second mode move slightly downward with the aforementioned numbers and under this movement the curves conserve their initial form. At the same time, it follows from these curves that the influence of the increase of the numbers N_1 and N_2 on the dispersion curves decays significantly after certain values. The presented results and others which are not given here, allow us to conclude that it is sufficient to take $N_2 = 30$ and $N_1 = 15$ in order to obtain numerical results with $10^{-3} - 10^{-4}$ accuracy.

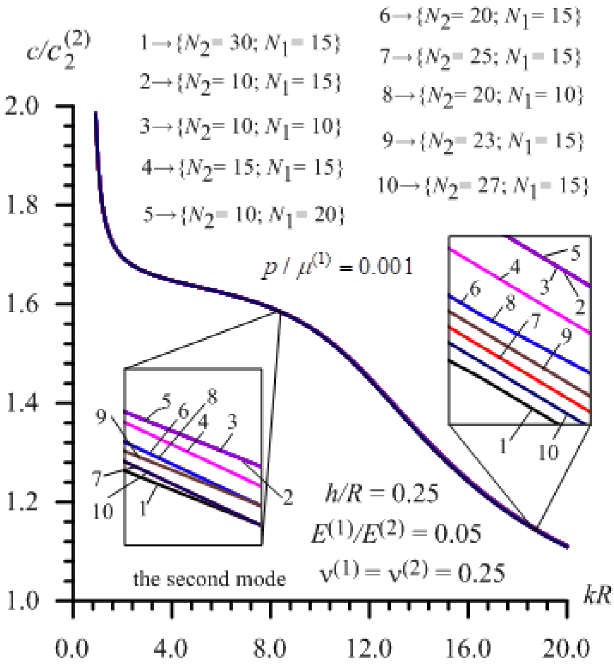
Now we consider the results given in Fig. 7 which illustrate the convergence of the numerical results obtained for the first (Fig. 7a) and second (Fig. 7b) modes with respect to the number R_M. Note that these results are also obtained in Case 1 under $N_2 = 30$ $N_1 = 15$, $h/R = 0.25$ and $p/\mu^{(1)} = 0.001$ It follows from the comparison of the results obtained for various values of R_M, that for obtaining numerical results with $10^{-3} - 10^{-4}$ accuracy, it is sufficient to take $R_M/R = 5$.

5. Conclusions

Thus, the present paper studies the influence of the inhomogeneous initial (or residual) stresses in the "hollow cylinder + surrounding medium" system on the propagation of the axisymmetric longitudinal waves in this system. It is assumed that the inhomogeneous initial stresses are caused by the uniformly distributed normal compressional forces



(a)

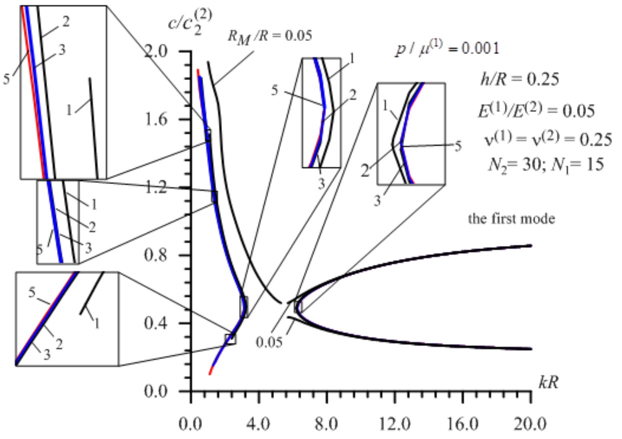


(b)

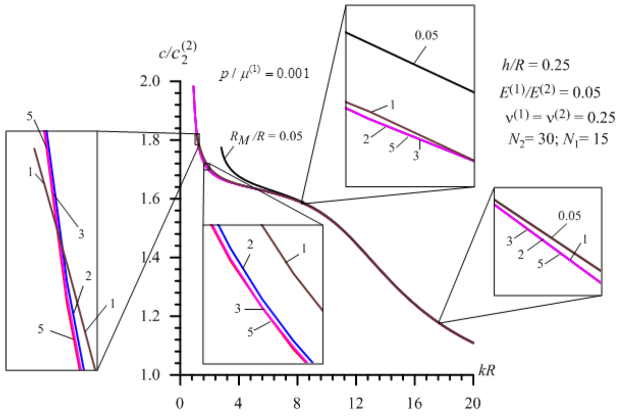
Fig. 6 Convergence of the numerical results obtained in Case 1 for the first (a) and second (b) modes with respect to the numbers N_1 and N_2

acting at infinity and in this manner the inhomogeneous initial stresses around the underground structures caused by the Earth's gravity are modelled. The wave propagation in the system is considered within the scope of the three-dimensional linearized theory of elastic waves in bodies with initial stresses. The discrete analytical solution method is developed for the solution to the formulated problem and the dispersion equation, which is solved numerically, is obtained.

As a result of this solution, the dispersion curves related to the first and second modes are constructed and the influence of the initial stresses on these curves under



(a)



(b)

Fig. 7 Convergence of the numerical results obtained in Case 1 for the first (a) and second (b) modes with respect to the ratio R_M/R

various values of the mechanical and geometrical parameters of the constituents of the system is analyzed. According to these analyses, the following concrete conclusions can be drawn:

- The inhomogeneous initial stresses under consideration not only qualitatively but also quantitatively change the character of the dispersion curves obtained for the "first modes". Under the "first modes" dispersion curves are those which are near to the dispersion curve which is obtained under absence of the initial stresses (i.e. the classical dispersion curve for the first mode) or the dispersion curve where the wave propagation velocity is less than that on the classical dispersion curve of the first mode;
- The character of the influence of the inhomogeneous initial stresses on the wave propagation velocity depends on the dimensionless wavenumber *kR*, nevertheless, it can be concluded that, in general, the inhomogeneous compressional initial stresses cause to decrease the wave propagation velocity;
- Under relatively great values of the intensity of the forces which cause the inhomogeneous initial stresses, the band gap appears for the wave frequencies related to the "first modes";
- As a result of the inhomogeneous initial stresses, additional dispersion curves may appear, the wave propagation velocity on which is less than that on the classical dispersion curve for the first mode;
- The influence of the inhomogeneous initial stresses under consideration on the dispersion curves related to the second mode has only qualitative character and this influence causes to decrease the wave propagation velocity. The magnitude of this "decrease" increases with the absolute values of the compressional forces in the initial state;
- The character of the influence of the inhomogeneous initial stresses on the dispersion curves related to the "first modes" depends significantly of the ratio h/R, where h is the cylinder's thickness and R is the internal radius of the cylinder's cross section. Especially in relatively small values of this ratio, the influence of the inhomogeneous initial stresses on the dispersion curves becomes more complicated.

Note that more concrete conclusions regarding the studied problem can be found in the text of the paper and the solution method developed herein allows the authors to investigate many other problems on wave dispersion in the elements of construction with inhomogeneous initial stresses. The obtained results can be used in underground structural engineering as oriented theoretical benchmark information on the influence of the inhomogeneous initial stresses on the wave dispersion in underground structures.

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Appendix A

Here we give the explicit expressions for amplitude of the sought values and the notation used in writing these expressions is described in the text with the relations given in (16) and (27).

$$u_r^{(i)n_i}(r) = A_1^{(i)n_i}\frac{dr_2^{(i)n_i}}{dr}\Big[\gamma_1^{(i)}(r_{n_i})(r_2^{(i)n_i})^{(\gamma_1^{(i)}(r_{n_i})-1)}\times$$

$$E_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i}) + (r_2^{(i)n_i})^{\gamma_1^{(i)}(r_{n_i})}\frac{dE_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i})}{dr_2^{(i)n_i}}\Big]+$$

$$A_2^{(i)n_i}\frac{dr_2^{(i)n_i}}{dr}\Big[\gamma_1^{(i)}(r_{n_i})(r_2^{(i)n_i})^{(\gamma_1^{(i)}(r_{n_i})-1)}\times$$

$$F_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i}) + (r_2^{(i)n_i})^{\gamma_1^{(i)}(r_{n_i})}\frac{dF_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i})}{dr_2^{(i)n_i}}\Big]$$

$$+B_1^{(i)n_i}\frac{dr_1^{(i)n_i}}{dr}\Big[\gamma^{(i)}(r_{n_i})(r_1^{(i)n_i})^{(\gamma^{(i)}(r_{n_i})-1)}\times$$

$$E_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i}) + (r_1^{(i)n_i})^{\gamma^{(i)}(r_{n_i})}\frac{dE_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})}\Big]+$$

$$B_2^{(i)n_i}\frac{dr_1^{(i)n_i}}{dr}\Big[\gamma^{(i)}(r_{n_i})(r_1^{(i)n_i})^{(\gamma^{(i)}(r_{n_i})-1)}\times \tag{A1}$$

$$F_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i}) + (r_1^{(i)n_i})^{\gamma^{(i)}(r_{n_i})}\frac{dF_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})}\Big],$$

$$u_z^{(i)n_i}(r) = A_1^{(i)n_i}(r_2^{(i)n_i})^{\gamma_1^{(i)}(r_{n_i})}E_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i})+$$

$$A_2^{(i)n_i}(r_2^{(i)n_i})^{\gamma_1^{(i)}(r_{n_i})}F_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i})-$$

$$B_1^{(i)n_i}\Big[\gamma^{(i)}(r_{n_i})(\gamma^{(i)}(r_{n_i})-1)\left(\frac{dr_2^{(i)n_i}}{dr}\right)^2\times$$

$$E_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i}) + \gamma^{(i)}(r_{n_i})(r_1^{(i)n_i})^{(\gamma^{(i)}(r_{n_i})-1)}\times$$

$$\frac{1}{r}E_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i}) + 2\gamma^{(i)}(r_{n_i})\left(\frac{dr_1^{(i)n_i}}{dr}\right)^2\times$$

$$\frac{dE_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})} + \frac{1}{r}(r_1^{(i)n_i})^{\gamma^{(i)}(r_{n_i})}\times$$

$$\frac{dr_1^{(i)n_i}}{dr}\frac{dE_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})}+$$

$$(r_1^{(i)n_i})^{\gamma^{(i)}(r_{n_i})}\left(\frac{dr_1^{(i)n_i}}{dr}\right)^2\frac{d^2E_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})^2}\Big]-$$

$$B_2^{(i)n_i}\Big[\gamma^{(i)}(r_{n_i})(\gamma^{(i)}(r_{n_i})-1)\left(\frac{dr_2^{(i)n_i}}{dr}\right)^2\times$$

$$F_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i}) + \gamma^{(i)}(r_{n_i})(r_1^{(i)n_i})^{(\gamma^{(i)}(r_{n_i})-1)}\times$$

$$\frac{1}{r}F_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i}) + 2\gamma^{(i)}(r_{n_i})\left(\frac{dr_1^{(i)n_i}}{dr}\right)^2\times$$

$$\frac{dF_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})} + \frac{1}{r}(r_1^{(i)n_i})^{\gamma^{(i)}(r_{n_i})}\times$$

$$\frac{dr_1^{(i)n_i}}{dr}\frac{dF_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})}+$$

$$(r_1^{(i)n_i})^{\gamma^{(i)}(r_{n_i})}\left(\frac{dr_1^{(i)n_i}}{dr}\right)^2\frac{d^2F_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})^2}\Big],$$

$$\frac{\sigma_{rr}^{(i)n_i}(r)}{\mu^{(i)}} = A_1^{(i)n_i}\Bigg\{\left(\frac{dr_2^{(i)n_i}}{dr}\right)^2 2\left(1+\frac{\lambda^{(i)}}{2\mu^{(i)}}\right)\times$$

$$\Big[\gamma_1^{(i)}(r_{n_i})(\gamma_1^{(i)}(r_{n_i})-1)(r_2^{(i)n_i})^{(\gamma_1^{(i)}(r_{n_i})-2)}\times$$

$$E_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i})+$$

$$2\gamma_1^{(i)}(r_{n_i})(r_2^{(i)n_i})^{(\gamma_1^{(i)}(r_{n_i})-1)}\frac{dE_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i})}{d(r_2^{(i)n_i})}+$$

$$\left.(r_2^{(i)n_i})^{\gamma_1^{(i)}(r_{n_i})}\frac{d^2E_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i})}{d(r_2^{(i)n_i})^2}\right]+$$

$$\frac{\lambda^{(i)}}{\mu^{(i)}}\frac{1}{r_2^{(i)n_i}}\left(\frac{dr_2^{(i)n_i}}{dr}\right)^2\times$$

$$\left[\gamma_1^{(i)}(r_{n_i})(r_2^{(i)n_i})^{(\gamma_1^{(i)}(r_{n_i})-1)}E_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i})+r_2^{(i)n_i}\times\right.$$

$$\left.\frac{dE_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i})}{d(r_2^{(i)n_i})}\right]+$$

$$\left.\frac{\lambda^{(i)}}{\mu^{(i)}}(r_2^{(i)n_i})^{\gamma_1^{(i)}(r_{n_i})}E_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i})\right\}+$$

$$A_2^{(i)n_i}\left\{\left(\frac{dr_2^{(i)n_i}}{dr}\right)^2 2\left(1+\frac{\lambda^{(i)}}{2\mu^{(i)}}\right)\times\right.$$

$$\left[\gamma_1^{(i)}(r_{n_i})(\gamma_1^{(i)}(r_{n_i})-1)(r_2^{(i)n_i})^{(\gamma_1^{(i)}(r_{n_i})-2)}\times\right.$$

$$F_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i})+$$

$$2\gamma_1^{(i)}(r_{n_i})(r_2^{(i)n_i})^{(\gamma_1^{(i)}(r_{n_i})-1)}\frac{dF_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i})}{d(r_2^{(i)n_i})}$$

$$\left.+(r_2^{(i)n_i})^{\gamma_1^{(i)}(r_{n_i})}\frac{d^2F_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i})}{d(r_2^{(i)n_i})^2}\right]+$$

$$\frac{\lambda^{(i)}}{\mu^{(i)}}\frac{1}{r_2^{(i)n_i}}\left(\frac{dr_2^{(i)n_i}}{dr}\right)^2\times$$

$$\left[\gamma_1^{(i)}(r_{n_i})(r_2^{(i)n_i})^{(\gamma_1^{(i)}(r_{n_i})-1)}F_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i})+r_2^{(i)n_i}\times\right.$$

$$\left.\frac{dF_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i})}{d(r_2^{(i)n_i})}\right]+$$

$$\left.\frac{\lambda^{(i)}}{\mu^{(i)}}(r_2^{(i)n_i})^{\gamma_1^{(i)}(r_{n_i})}F_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i})\right\}+$$

$$B_1^{(i)n_i}\left\{\left(\frac{dr_1^{(i)n_i}}{dr}\right)^2 2\left(1+\frac{\lambda^{(i)}}{2\mu^{(i)}}\right)\times\right.$$

$$\left[\gamma^{(i)}(r_{n_i})(\gamma^{(i)}(r_{n_i})-1)(r_1^{(i)n_i})^{(\gamma^{(i)}(r_{n_i})-2)}\times\right.$$

$$E_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})+$$

$$2\gamma^{(i)}(r_{n_i})(r_1^{(i)n_i})^{(\gamma^{(i)}(r_{n_i})-1)}\frac{dE_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})}$$

$$\left.+(r_1^{(i)n_i})^{\gamma^{(i)}(r_{n_i})}\frac{d^2E_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})^2}\right]+$$

$$\frac{\lambda^{(i)}}{\mu^{(i)}}\frac{1}{r_1^{(i)n_i}}\left(\frac{dr_1^{(i)n_i}}{dr}\right)^2\times$$

$$\left[\gamma^{(i)}(r_{n_i})(r_1^{(i)n_i})^{(\gamma^{(i)}(r_{n_i})-1)}E_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})+r_1^{(i)n_i}\times\right.$$

$$\left.\frac{dE_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})}\right]+\frac{\lambda^{(i)}}{\mu^{(i)}}\times$$

$$\left[\gamma^{(i)}(r_{n_i})(\gamma^{(i)}(r_{n_i})-1)(r_1^{(i)n_i})^{(\gamma^{(i)}(r_{n_i})-2)}\times\right.$$

$$E_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})+$$

$$\gamma^{(i)}(r_{n_i})(r_1^{(i)n_i})^{(\gamma^{(i)}(r_{n_i})-1)}\frac{1}{r_1^{(i)n_i}}\left(\frac{dr_1^{(i)n_i}}{dr}\right)^2\times$$

$$E_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})+2\gamma^{(i)}(r_{n_i})(r_1^{(i)n_i})^{(\gamma^{(i)}(r_{n_i})-1)}\times$$

$$\left(\frac{dr_1^{(i)n_i}}{dr}\right)^2\frac{dE_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})}+$$

$$\frac{1}{r_1^{(i)n_i}}\left(\frac{dr_1^{(i)n_i}}{dr}\right)^2 (r_1^{(i)n_i})^{\gamma^{(i)}(r_{n_i})} \frac{dE_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})} +$$

$$\left.\left.\left(\frac{dr_1^{(i)n_i}}{dr}\right)^2 (r_1^{(i)n_i})^{\gamma^{(i)}(r_{n_i})} \frac{d^2E_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})^2}\right]\right\}+$$

$$B_2^{(i)n_i}\left\{\left(\frac{dr_1^{(i)n_i}}{dr}\right)^2 2\left(1+\frac{\lambda^{(i)}}{2\mu^{(i)}}\right)\times\right.$$

$$\left[\gamma^{(i)}(r_{n_i})(\gamma^{(i)}(r_{n_i})-1)(r_1^{(i)n_i})^{(\gamma^{(i)}(r_{n_i})-2)}\times\right.$$

$$F_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})+$$

$$2\gamma^{(i)}(r_{n_i})(r_1^{(i)n_i})^{(\gamma^{(i)}(r_{n_i})-1)} \frac{dF_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})}$$

$$\left.+(r_1^{(i)n_i})^{\gamma^{(i)}(r_{n_i})} \frac{d^2F_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})^2}\right]+$$

$$\frac{\lambda^{(i)}}{\mu^{(i)}}\frac{1}{r_1^{(i)n_i}}\left(\frac{dr_1^{(i)n_i}}{dr}\right)^2\times$$

$$\left[\gamma^{(i)}(r_{n_i})(r_1^{(i)n_i})^{(\gamma^{(i)}(r_{n_i})-1)}F_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})+r_1^{(i)n_i}\times\right.$$

$$\left.\frac{dF_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})}\right]+\frac{\lambda^{(i)}}{\mu^{(i)}}\times$$

$$\left[\gamma^{(i)}(r_{n_i})(\gamma^{(i)}(r_{n_i})-1)(r_1^{(i)n_i})^{(\gamma^{(i)}(r_{n_i})-2)}\times\right.$$

$$F_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})+$$

$$\gamma^{(i)}(r_{n_i})(r_1^{(i)n_i})^{(\gamma^{(i)}(r_{n_i})-1)} \frac{1}{r_1^{(i)n_i}}\left(\frac{dr_1^{(i)n_i}}{dr}\right)^2\times$$

$$F_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})+2\gamma^{(i)}(r_{n_i})(r_1^{(i)n_i})^{(\gamma^{(i)}(r_{n_i})-1)}\times$$

$$\left(\frac{dr_1^{(i)n_i}}{dr}\right)^2 \frac{dF_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})}+$$

$$\frac{1}{r_1^{(i)n_i}}\left(\frac{dr_1^{(i)n_i}}{dr}\right)^2 (r_1^{(i)n_i})^{\gamma^{(i)}(r_{n_i})} \frac{dF_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})}+$$

$$\left.\left.\left(\frac{dr_1^{(i)n_i}}{dr}\right)^2 (r_1^{(i)n_i})^{\gamma^{(i)}(r_{n_i})} \frac{d^2F_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})^2}\right]\right\},$$

$$\frac{\sigma_{rz}^{(i)n_i}(r)}{\mu^{(i)}} = A_1^{(i)n_i}\,2\frac{dr_2^{(i)n_i}}{dr}\left[\gamma_1^{(i)}(r_{n_i})(r_2^{(i)n_i})^{(\gamma_1^{(i)}(r_{n_i})-1)}\times\right.$$

$$E_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i})+(r_2^{(i)n_i})^{\gamma_1^{(i)}(r_{n_i})}\times$$

$$\left.\frac{dE_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i})}{d(r_2^{(i)n_i})}\right]+A_2^{(i)n_i}\,2\frac{dr_2^{(i)n_i}}{dr}\times$$

$$\left[\gamma_1^{(i)}(r_{n_i})(r_2^{(i)n_i})^{(\gamma_1^{(i)}(r_{n_i})-1)}F_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i})+\right.$$

$$\left.(r_2^{(i)n_i})^{\gamma_1^{(i)}(r_{n_i})}\frac{dF_{\gamma_1^{(i)}(r_{n_i})}(r_2^{(i)n_i})}{d(r_2^{(i)n_i})}\right]+$$

$$B_1^{(i)n_i}\left\{-\left[\gamma^{(i)}(r_{n_i})(\gamma^{(i)}(r_{n_i})-1)(\gamma^{(i)}(r_{n_i})-2)\times\right.\right.$$

$$E_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})+3\gamma^{(i)}(r_{n_i})(\gamma^{(i)}(r_{n_i})-1)\times$$

$$(r_1^{(i)n_i})^{(\gamma^{(i)}(r_{n_i})-2)}\left(\frac{dr_1^{(i)n_i}}{dr}\right)^3\frac{dE_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})}+$$

$$3\gamma^{(i)}(r_{n_i})(r_1^{(i)n_i})^{(\gamma^{(i)}(r_{n_i})-1)}\left(\frac{dr_1^{(i)n_i}}{dr}\right)^3\times$$

$$\frac{d^2E_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})^2}+(r_1^{(i)n_i})^{\gamma^{(i)}(r_{n_i})}\left(\frac{dr_1^{(i)n_i}}{dr}\right)^3\times$$

$$\frac{d^3E_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})^3}-\frac{1}{(r_1^{(i)n_i})^2}\times$$

$$\gamma^{(i)}(r_{n_i})(r_1^{(i)n_i})^{(\gamma^{(i)}(r_{n_i})-1)}\left(\frac{dr_1^{(i)n_i}}{dr}\right)^3 E_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})-$$

$$\frac{1}{(r_1^{(i)n_i})^2}\gamma^{(i)}(r_{n_i})(r_1^{(i)n_i})^{\gamma^{(i)}(r_{n_i})}\left(\frac{dr_1^{(i)n_i}}{dr}\right)^3\times$$

$$\frac{dE_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})}+\gamma^{(i)}(r_{n_i})(\gamma^{(i)}(r_{n_i})-1)\times$$

$$\frac{1}{r_1^{(i)n_i}}(r_1^{(i)n_i})^{(\gamma^{(i)}(r_{n_i})-2)}\left(\frac{dr_1^{(i)n_i}}{dr}\right)^3\times$$

$$E_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})+2\gamma^{(i)}(r_{n_i})\left(\frac{dr_1^{(i)n_i}}{dr}\right)^3\frac{1}{r_1^{(i)n_i}}\times$$

$$\frac{dE_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})}+$$

$$\left.\frac{1}{r_1^{(i)n_i}}(r_1^{(i)n_i})^{\gamma^{(i)}(r_{n_i})}\left(\frac{dr_1^{(i)n_i}}{dr}\right)^3\frac{d^2E_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})^2}\right]+$$

$$\frac{dr_1^{(i)n_i}}{dr}\left[\gamma^{(i)}(r_{n_i})(r_1^{(i)n_i})^{(\gamma_1^{(i)}(r_{n_i})-1)}E_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})\right.$$

$$\left.\left.+(r_1^{(i)n_i})^{(\gamma_1^{(i)}(r_{n_i})-1)}\frac{dE_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})}\right]\right\}+$$

$$B_2^{(i)n_i}\left\{-\left[\gamma^{(i)}(r_{n_i})(\gamma^{(i)}(r_{n_i})-1)(\gamma^{(i)}(r_{n_i})-2)\times\right.\right.$$

$$F_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})+3\gamma^{(i)}(r_{n_i})(\gamma^{(i)}(r_{n_i})-1)\times$$

$$(r_1^{(i)n_i})^{(\gamma^{(i)}(r_{n_i})-2)}\left(\frac{dr_1^{(i)n_i}}{dr}\right)^3\frac{dF_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})}+$$

$$3\gamma^{(i)}(r_{n_i})(r_1^{(i)n_i})^{(\gamma^{(i)}(r_{n_i})-1)}\left(\frac{dr_1^{(i)n_i}}{dr}\right)^3\times$$

$$\frac{d^2F_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})^2}+(r_1^{(i)n_i})^{\gamma^{(i)}(r_{n_i})}\left(\frac{dr_1^{(i)n_i}}{dr}\right)^3\times$$

$$\frac{d^3F_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})^3}-\frac{1}{(r_1^{(i)n_i})^2}\times$$

$$\gamma^{(i)}(r_{n_i})(r_1^{(i)n_i})^{(\gamma^{(i)}(r_{n_i})-1)}\left(\frac{dr_1^{(i)n_i}}{dr}\right)^3 F_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})-$$

$$\frac{1}{(r_1^{(i)n_i})^2}\gamma^{(i)}(r_{n_i})(r_1^{(i)n_i})^{\gamma^{(i)}(r_{n_i})}\times$$

$$\left(\frac{dr_1^{(i)n_i}}{dr}\right)^3\frac{dF_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})}+\gamma^{(i)}(r_{n_i})(\gamma^{(i)}(r_{n_i})-1)\times$$

$$\frac{1}{r_1^{(i)n_i}}(r_1^{(i)n_i})^{(\gamma^{(i)}(r_{n_i})-2)}\left(\frac{dr_1^{(i)n_i}}{dr}\right)^3\times$$

$$F_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})+2\gamma^{(i)}(r_{n_i})\left(\frac{dr_1^{(i)n_i}}{dr}\right)^3\frac{1}{r_1^{(i)n_i}}\times$$

$$\frac{dF_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})}+$$

$$\left.\frac{1}{r_1^{(i)n_i}}(r_1^{(i)n_i})^{\gamma^{(i)}(r_{n_i})}\left(\frac{dr_1^{(i)n_i}}{dr}\right)^3\frac{d^2F_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})^2}\right]+$$

$$\frac{dr_1^{(i)n_i}}{dr}\left[\gamma^{(i)}(r_{n_i})(r_1^{(i)n_i})^{(\gamma_1^{(i)}(r_{n_i})-1)}F_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})+\right.$$

$$\left.\left.(r_1^{(i)n_i})^{(\gamma_1^{(i)}(r_{n_i})-1)}\frac{dF_{\gamma^{(i)}(r_{n_i})}(r_1^{(i)n_i})}{d(r_1^{(i)n_i})}\right]\right\},$$

$$u_r^{(1)\infty}(r)=A_2^{(1)\infty}\frac{dr_2^{(1)\infty}}{dr}\times$$

$$\left[\gamma_1^{(1)\infty}(R_M)(r_2^{(1)\infty})^{(\gamma_1^{(1)\infty}(R_M)-1)}F_{\gamma_1^{(i)}(R_M)}(r_2^{(i)\infty})+\right.$$

$$(r_2^{(1)\infty})^{\gamma_1^{(1)\infty}(R_M)} \frac{dF_{\gamma_1^{(1)\infty}(R_M)}(r_2^{(1)\infty})}{dr_2^{(1)\infty}} \Bigg] +$$

$$B_2^{(1)\infty} \frac{dr_1^{(1)\infty}}{dr}\Big[\gamma^{(1)\infty}(R_M)\times$$

$$(r_1^{(1)\infty})^{(\gamma^{(1)\infty}(R_M)-1)} F_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty}) +$$

$$(r_1^{(1)\infty})^{\gamma^{(1)\infty}(R_M)} \frac{dF_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty})}{d(r_1^{(1)\infty})}\Bigg],$$

$$u_z^{(1)\infty}(r) = A_2^{(1)\infty}(r_2^{(1)\infty})^{\gamma_1^{(1)\infty}(R_M)} F_{\gamma_1^{(1)\infty}(R_M)}(r_2^{(1)\infty}) -$$

$$B_2^{(1)\infty}\Big[\gamma^{(1)\infty}(R_M)(\gamma^{(1)\infty}(R_M)-1)\left(\frac{dr_2^{(1)\infty}}{dr}\right)^2\times$$

$$F_{\gamma^{(1)}\infty(R_M)}(r_1^{(1)\infty}) + \gamma^{(1)\infty}(R_M)(r_1^{(1)\infty})^{(\gamma^{(1)\infty}(R_M)-1)}\times$$

$$\frac{1}{r}F_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty}) + 2\gamma^{(1)\infty}(R_M)\times$$

$$\left(\frac{dr_1^{(1)\infty}}{dr}\right)^2 \frac{dF_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty})}{d(r_1^{(1)\infty})} + \frac{1}{r}(r_1^{(1)\infty})^{\gamma^{(1)\infty}(R_M)}\times$$

$$\frac{dr_1^{(1)\infty}}{dr}\frac{dF_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty})}{d(r_1^{(1)\infty})}+$$

$$(r_1^{(1)\infty})^{\gamma^{(1)\infty}(R_M)}\left(\frac{dr_1^{(1)\infty}}{dr}\right)^2 \frac{d^2F_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty})}{d(r_1^{(1)\infty})^2}\Bigg],$$

$$\frac{\sigma_{rr}^{(1)\infty}(r)}{\mu^{(1)}} = A_2^{(1)\infty}\Bigg\{\left(\frac{dr_2^{(1)\infty}}{dr}\right)^2 2\left(1+\frac{\lambda^{(1)}}{2\mu^{(1)}}\right)\times$$

$$\Big[\gamma_1^{(1)\infty}(R_M)(\gamma_1^{(1)}(R_M)-1)(r_2^{(1)\infty})^{(\gamma_1^{(1)\infty}(R_M)-2)}\times$$

$$F_{\gamma_1^{(1)\infty}(R_M)}(r_2^{(1)\infty})+$$

$$2\gamma_1^{(1)\infty}(R_M)(r_2^{(1)\infty})^{(\gamma_1^{(1)\infty}(R_M)-1)}\frac{dF_{\gamma_1^{(1)\infty}(R_M)}(r_2^{(i)n_i})}{d(r_2^{(1)\infty})}$$

$$+(r_2^{(1)\infty})^{\gamma_1^{(1)}(R_M)} \frac{d^2F_{\gamma_1^{(1)\infty}(R_M)}(r_2^{(1)\infty})}{d(r_2^{(1)\infty})^2}\Bigg]+$$

$$\frac{\lambda^{(1)}}{\mu^{(1)}}\frac{1}{r_2^{(1)\infty}}\left(\frac{dr_2^{(1)\infty}}{dr}\right)^2\Big[\gamma_1^{(1)\infty}(R_M)\times$$

$$(r_2^{(1)\infty})^{(\gamma_1^{(1)\infty}(R_M)-1)}F_{\gamma_1^{(1)\infty}(R_M)}(r_2^{(1)\infty})+r_2^{(1)\infty}\times$$

$$\frac{dF_{\gamma_1^{(1)\infty}(R_M)}(r_2^{(1)\infty})}{d(r_2^{(1)\infty})}\Bigg]+$$

$$\frac{\lambda^{(1)}}{\mu^{(1)}}(r_2^{(1)\infty})^{\gamma_1^{(1)}(R_M)}F_{\gamma_1^{(1)\infty}(R_M)}(r_2^{(1)\infty})\Bigg\}+$$

$$B_2^{(1)\infty}\Bigg\{\left(\frac{dr_1^{(1)\infty}}{dr}\right)^2 2\left(1+\frac{\lambda^{(1)}}{2\mu^{(1)}}\right)\Big[\gamma^{(1)\infty}(R_M)\times$$

$$(\gamma^{(1)\infty}(R_M)-1)(r_1^{(1)\infty})^{(\gamma^{(1)}(R_M)-2)}F_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty})+$$

$$2\gamma^{(1)\infty}(R_M)(r_1^{(1)\infty})^{(\gamma^{(1)\infty}(R_M)-1)}\frac{dF_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty})}{d(r_1^{(1)\infty})}$$

$$+(r_1^{(1)\infty})^{\gamma^{(1)\infty}(R_M)}\frac{d^2F_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty})}{d(r_1^{(1)\infty})^2}\Bigg]+$$

$$\frac{\lambda^{(1)}}{\mu^{(1)}}\frac{1}{r_1^{(1)\infty}}\left(\frac{dr_1^{(1)\infty}}{dr}\right)^2\Big[\gamma^{(1)\infty}(R_M)\times$$

$$(r_1^{(1)\infty})^{(\gamma^{(1)\infty}(R_M)-1)}F_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty})+r_1^{(1)\infty}\times$$

$$\frac{dF_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty})}{d(r_1^{(1)\infty})}\Bigg]+\frac{\lambda^{(1)}}{\mu^{(1)}}\Big[\gamma^{(1)\infty}(R_M)\times$$

$$(\gamma^{(1)\infty}(R_M)-1)(r_1^{(1)\infty})^{(\gamma^{(1)\infty}(R_M)-2)}F_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty})$$

$$+\gamma^{(1)\infty}(R_M)(r_1^{(1)\infty})^{(\gamma^{(1)\infty}(R_M)-1)}\frac{1}{r_1^{(1)\infty}}\left(\frac{dr_1^{(1)\infty}}{dr}\right)^2\times$$

$$F_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty})+$$

$$2\gamma^{(1)\infty}(R_M)(r_1^{(1)\infty})^{(\gamma^{(1)\infty}(R_M)-1)}\times$$

$$\left(\frac{dr_1^{(1)\infty}}{dr}\right)^2\frac{dF_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty})}{d(r_1^{(1)\infty})}+$$

$$\frac{1}{r_1^{(1)\infty}}\left(\frac{dr_1^{(1)\infty}}{dr}\right)^2(r_1^{(1)\infty})^{\gamma^{(1)}(R_M)}\frac{dF_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty})}{d(r_1^{(1)\infty})}+$$

$$\left.\left.\left(\frac{dr_1^{(1)\infty}}{dr}\right)^2(r_1^{(1)\infty})^{\gamma^{(1)\infty}(R_M)}\frac{d^2F_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty})}{d(r_1^{(1)\infty})^2}\right]\right\},$$

$$\frac{\sigma_{rz}^{(1)\infty}(r)}{\mu^{(1)}}=A_2^{(1)\infty}2\frac{dr_2^{(1)\infty}}{dr}\times$$

$$\left[\gamma_1^{(1)\infty}(R_M)(r_2^{(1)\infty})^{(\gamma_1^{(1)\infty}(R_M)-1)}F_{\gamma_1^{(1)}(R_M)}(r_2^{(1)\infty})+\right.$$

$$\left.(r_2^{(1)\infty})^{\gamma_1^{(1)\infty}(R_M)}\frac{dF_{\gamma_1^{(1)\infty}(R_M)}(r_2^{(1)\infty})}{d(r_2^{(1)\infty})}\right]+$$

$$B_2^{(1)\infty}\left\{-\left[\gamma^{(1)\infty}(R_M)(\gamma^{(1)\infty}(R_M)-1)(\gamma^{(1)\infty}(R_M)-2)\right.\right.$$

$$\times F_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty})+3\gamma^{(1)\infty}(R_M)(\gamma^{(1)\infty}(R_M)-1)\times$$

$$(r_1^{(1)\infty})^{(\gamma^{(1)\infty}(R_M)-2)}\left(\frac{dr_1^{(1)\infty}}{dr}\right)^3\frac{dF_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty})}{d(r_1^{(1)\infty})}$$

$$+3\gamma^{(1)\infty}(R_M)(r_1^{(1)\infty})^{(\gamma^{(1)\infty}(R_M)-1)}\left(\frac{dr_1^{(1)\infty}}{dr}\right)^3\times$$

$$\frac{d^2F_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty})}{d(r_1^{(1)\infty})^2}+(r_1^{(1)\infty})^{\gamma^{(1)\infty}(R_M)}\left(\frac{dr_1^{(1)\infty}}{dr}\right)^3$$

$$\times\frac{d^3F_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty})}{d(r_1^{(1)\infty})^3}-\frac{1}{(r_1^{(1)\infty})^2}\gamma^{(1)\infty}(R_M)\times$$

$$(r_1^{(1)\infty})^{(\gamma^{(1)\infty}(R_M)-1)}\left(\frac{dr_1^{(1)\infty}}{dr}\right)^3F_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty})-$$

$$\frac{1}{(r_1^{(1)\infty})^2}\gamma^{(1)\infty}(R_M)(r_1^{(1)\infty})^{\gamma^{(1)\infty}(R_M)}\left(\frac{dr_1^{(1)\infty}}{dr}\right)^3\times$$

$$\frac{dF_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty})}{d(r_1^{(1)n_i}\infty)}+\gamma^{(1)\infty}(R_M)(\gamma^{(1)\infty}(R_M)-1)\times$$

$$\frac{1}{r_1^{(1)\infty}}(r_1^{(1)\infty})^{(\gamma^{(1)\infty}(R_M)-2)}\left(\frac{dr_1^{(1)\infty}}{dr}\right)^3\times$$

$$F_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty})+2\gamma^{(1)\infty}(R_M)\left(\frac{dr_1^{(1)\infty}}{dr}\right)^3\frac{1}{r_1^{(1)\infty}}\times$$

$$\frac{dF_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty})}{d(r_1^{(1)\infty})}+\frac{1}{r_1^{(1)\infty}}(r_1^{(1)\infty})^{\gamma^{(1)\infty}(R_M)}\times$$

$$\left.\left(\frac{dr_1^{(1)\infty}}{dr}\right)^3\frac{d^2F_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty})}{d(r_1^{(1)\infty})^2}\right]+$$

$$\frac{dr_1^{(1)\infty}}{dr}\left[\gamma^{(1)\infty}(R_M)(r_1^{(1)\infty})^{(\gamma_1^{(1)\infty}(R_M)-1)}\times\right.$$

$$F_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty})+$$

$$\left.\left.(r_1^{(1)\infty})^{(\gamma_1^{(1)\infty}(R_M)-1)}\frac{dF_{\gamma^{(1)\infty}(R_M)}(r_1^{(1)\infty})}{d(r_1^{(1)\infty})}\right]\right\}.$$

$$i=1,2$$