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Free vibration analysis of beams with various interfaces by using a modified matched interface and boundary method

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Abstract. This paper proposes a modified matched interface and boundary (MMIB) method to analyze the free vibration of beams with various interfaces caused by steps, intermediate rigid and elastic supports, intermediate concentrated masses and spring-mass systems, etc. A new strategy is developed to determine the parameters in the iterative computation of MMIB. The MMIB procedures are established to deal with boundary conditions and various interface conditions, which overcomes the shortcoming of the traditional MIB. A number of examples are utilized to illustrate the performance of MMIB method. Numerical results indicate that the MMIB method is a highly accurate and convergent approach for solving interface problems.

Keywords: matched interface and boundary; interface problem; beam; free vibration

1. Introduction

Beams are very important structural elements which have been widely used in many engineering fields such as civil, bridge and mechanical engineering. Apart from continuous beams with invariable cross sections, beams with steps, internal supports, concentrate masses and spring-mass systems are also very popular in engineering practices. The free vibration of complicated beam systems such as stepped beams (Jang and Bert 1989 a, b, Naguleswaran 2002, Lin and Ng 2004, Lu *et al*. 2009, Mao and Pietrzko 2010), multi-span beams (Lin and Tsai 2007,Lin 2009, Farghaly and EI-Sayed 2016), beams with intermediate elastic supports (Maurizi and Bambill 1987, De Rosa *et al*. 1995, Lin 2008), beams with intermediate concentrated masses (Maiz *et al*. 2007, Lin 2008, 2009), and beams carrying multiple spring-mass systems (Lin and Tsai 2007, Lin 2009) has been studied by many researchers in the past few decades. It is worthy to note that steps, supports, concentrated masses and spring-mass systems may lead to interface problems. Although there are many kinds of numerical methods like finite element method (FEM) (Jang and Bert 1989 a, b), Smoothed FEM (Chai *et al*. 2018), central finite difference (CFD) (Li *et al*. 2019), generalized differential quadrature rule (GDQR) (Liu and 2001), differential quadrature element method (DQEM) (Wang and Wang 2013), discrete singular convolution (DSC) (Wei 1999, Duan and Wang 2013, Civalek 2013, 2017, Li *et al*. 2015, Song *et al*. 2012, 2016, Wang and Yuan 2017), and Chebyshev-tau method (Lee 2015), it can be found that interface problems can be solved easily by FEM and DQEM etc., while, due to the existence of interfaces, the CFD and DSC methods may not be applied near interfaces.

Recently, the method of matched interface and boundary (MIB) has been proposed by Zhao and Wei (2004) to solve material interface problems, and its interpolation formulation (Zhou and Wei 2006) has also been used to handle stepped interfaces in the method of DSC (Duan and Wang 2013, 2014). During the past few years, MIB method has been widely used to solve partial differential equations with complex interfaces (Zhou *et al*.2006, Yu *et al.* 2007, 2009, Zhao and Wei 2009). Many investigations reveal that MIB method has the ability to deal with arbitrarily complex interfaces and geometric singularities (Yu and Wei 2007, Wang *et al*.2015). Meanwhile, it is noted that a parameter L is required to determine the total number of grid nodes to approximate derivatives in the iterative computation of MIB (Zhao and Wei 2004, 2009), and the parameter L affects the accuracy of MIB. Up to date, only some empirical values or a small range of parameter L are given through some numerical tests (Zhao and Wei 2009). And there is no effective way to determine the parameter L, which limits

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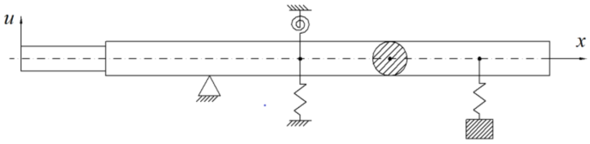


Fig. 1 Sketch of a beam with different interfaces

the application of MIB.

In this study, a new strategy is proposed to select the parameter L in the iterative computation of MIB. The low-order free vibration of beams with various interfaces is analyzed by using a modified MIB (MMIB). Free edges, elastic restraints and edges with tip masses and various interfaces caused by steps, supports, concentrated masses and spring-mass systems are dealt with. The accuracy and convergence of MMIB are examined through various numerical tests. Numerical results are compared with those available in the literature to illustrate the performance of MMIB. Some important conclusions are drawn at the end of this study.

2. Theory and algorithm

2.1 Governing equation

To analyze a beam with different interfaces, as shown in Fig.1, the beam is divided into six segments from each interface, so that the beam is uniform without any interfaces in each segment. The governing equation in each segment is (Mao and Pietrzko 2010)

$$EI_j \frac{\partial^4 u_j(x_j,t)}{\partial x_j^4} - \rho A_j \omega^2 u_j(x_j,t) = 0 \qquad (1)$$

$$0 \le x_j \le l_j \quad (j = 1,2,3, \ldots n)$$

where subscript j denotes the jth segment, n is the number of segments, E and ρ are the elastic modulus and density. u_j, I_j, A_j, l_j denote the displacement, sectional moment of inertia, sectional area, and the length of the jth segment, respectively. And l is the total length of the beam, and ω is the circular frequency.

For simplicity, the following non-dimensional quantities are introduced

$$U_j = u_j / l, \quad X_j = x_j / l, \quad \Omega_j = \sqrt[4]{\omega^2 \rho A_j l^4 / EI_j} \qquad (2)$$

$$(j = 1,2,3, \ldots n)$$

And then Eq. (1) can be rewritten as

$$\frac{\partial^4 U_j}{\partial X_j^4} - \Omega_j^4 U_j = 0 \qquad (3)$$

where Ω_j denotes non-dimensional frequency of the jth segment.

2.2 Interface and boundary conditions

Five different interfaces caused by steps, intermediate rigid and elastic supports, concentrated masses with rotary inertia, and spring-mass systems are shown in Fig. 2. All the interfaces considered here are located on nodes. And the non-dimensional equations of these interface conditions can be expressed as follows:

- The stepped interface (Duan and Wang 2013)

$$U^-(X_\xi) = U^+(X_\xi) \qquad (4a)$$

$$U^{(1)-}(X_\xi) = U^{(1)+}(X_\xi) \qquad (4b)$$

$$\frac{I^-}{I^+} U^{(2)-}(X_\xi) = U^{(2)+}(X_\xi) \qquad (4c)$$

$$\frac{I^-}{I^+} U^{(3)-}(X_\xi) = U^{(3)+}(X_\xi) \qquad (4d)$$

$$\frac{I^- A^+}{I^+ A^-} U^{(4)-}(X_\xi) = U^{(4)+}(X_\xi) \qquad (4e)$$

- The interface caused by single rigid support

$$U^-(X_\xi) = U^+(X_\xi) = 0 \qquad (5a)$$

$$U^{(1)-}(X_\xi) = U^{(1)+}(X_\xi) \qquad (5b)$$

$$U^{(2)-}(X_\xi) = U^{(2)+}(X_\xi) \qquad (5c)$$

- The interface caused by single elastic support with translational and rotational springs (Lin 2008)

$$U^-(X_\xi) = U^+(X_\xi) \qquad (6a)$$

$$U^{(1)-}(X_\xi) = U^{(1)+}(X_\xi) \qquad (6b)$$

$$U^{(2)-}(X_\xi) + K_r U^{(1)-}(X_\xi) = U^{(2)+}(X_\xi) \qquad (6c)$$

$$U^{(3)-}(X_\xi) - K_t U^-(X_\xi) = U^{(3)+}(X_\xi) \qquad (6d)$$

$$U^{(4)-}(X_\xi) = U^{(4)+}(X_\xi) \qquad (6e)$$

where

$$K_r = \frac{k_r l}{EI}, \quad K_t = \frac{k_t l^3}{EI} \qquad (7)$$

- The interface caused by single concentrated mass with rotary inertia (Lin 2008, Maiz 2007)

$$U^-(X_\xi) = U^+(X_\xi) \qquad (8a)$$

$$U^{(1)-}(X_\xi) = U^{(1)+}(X_\xi) \qquad (8b)$$

$$U^{(2)-}(X_\xi) - \Omega^4 J_c U^{(1)-}(X_\xi) = U^{(2)+}(X_\xi) \qquad (8c)$$

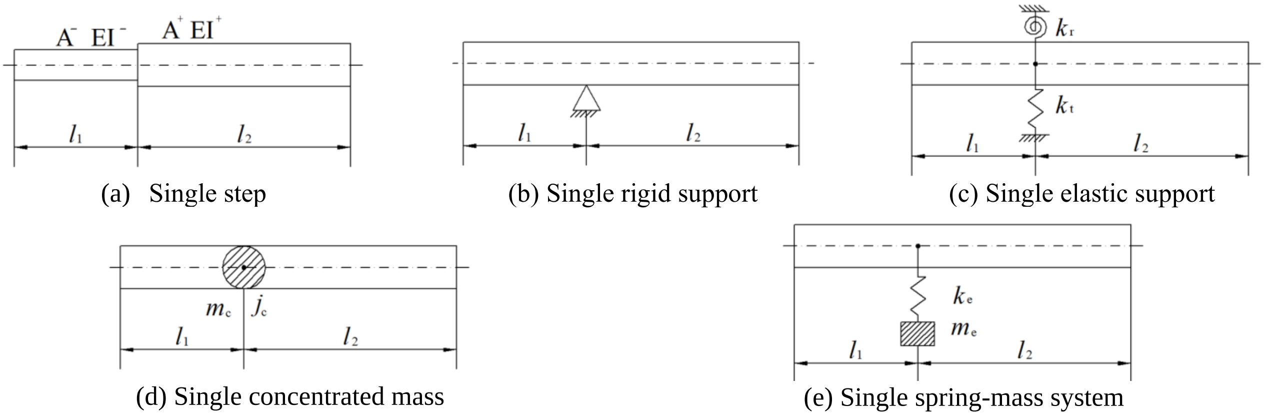


Fig. 2 Sketch of various interfaces

$$U^{(3)-}(X_\xi)+\Omega^4 M_c U^-(X_\xi)=U^{(3)+}(X_\xi) \tag{8d}$$

$$U^{(4)-}(X_\xi)=U^{(4)+}(X_\xi) \tag{8e}$$

where

$$J_c=\frac{j_c}{\rho A l^3},\quad M_c=\frac{m_c}{\rho A l} \tag{9}$$

- The interfaces caused by single spring-mass system (Lin et al. 2007)

$$U^-(X_\xi)=U^+(X_\xi) \tag{10a}$$

$$U^{(1)-}(X_\xi)=U^{(1)+}(X_\xi) \tag{10b}$$

$$U^{(2)-}(X_\xi)=U^{(2)+}(X_\xi) \tag{10c}$$

$$U^{(3)-}(X_\xi)-\frac{\Omega^4}{\lambda_e^2-1}M_e U^-(X_\xi)=U^{(3)+}(X_\xi) \tag{10d}$$

$$U^{(4)-}(X_\xi)=U^{(4)+}(X_\xi) \tag{10e}$$

where

$$M_e=\frac{m_e}{\rho A l},\quad K_e=\frac{k_e l^3}{EI},\quad \lambda_e^2=\frac{\Omega^4}{K_e/M_e} \tag{11}$$

where superscripts '-' and '+' denote the left and right sides of the interface, respectively. Although the above interface conditions are separated. In fact, these conditions can be combined together (Lin 2008).

The common edges of beams such as simply-supported (S) edge, clamped (C) edge, free (F) edge, free edge with a tip mass, and the edge elastically restrained against translation and rotation are considered. The boundary conditions of these edges are given in the previous work (Li *et al*. 2016) and other publications (Liu and Wu 2001, Maiz *et al*. 2007, Mao and Pietrzko 2010). It can be observed that these boundary conditions can be obtained by reducing some physical quantities in the interface conditions. Thus, boundary conditions can be considered as special cases of interface conditions.

2.3 MMIB procedures

The *m*th-order derivative of a function *f*(x) at point x_i can be approximated by using high order central finite difference (HO-CFD) (Zhao and Wei 2009, Li *et al*. 2019)

$$f^{(m)}(x_i)=\sum_{k=-W}^{W} C_{ik}^{(m)} f(x_{i+k}) \qquad (i=0,1,\ldots,N) \tag{12}$$

where $2W+1$ is the computational bandwidth, N is the total number of grid points of the entire beam, and the weighting coefficients $C_{ik}^{(m)}$ can be computed by GDQ algorithm (Li *et al*. 2019) and the fast algorithm (Fornberg 1998) conveniently. And Eq. (3) can be discretized as

$$\sum_{k=-W}^{W} C_{ik}^{(4)} U_{i+k}=\Omega_j^4 U_i \qquad (i=0,1,\ldots,N) \tag{13}$$

where $C_{ik}^{(4)}$ $(i=0,1,\ldots,N)$ denote the HO-CFD coefficients of the four-order derivatives. For convenience, the factor η_j is introduced as

$$\eta_j=\Omega_j^4/\Omega_1^4=A_j I_1/A_1 I_j \qquad (j=2,3,\ldots) \tag{14}$$

and $\eta_j=1$ is set with j denoting the jth segment. Thus, Eq. (13) can be rewritten as

$$\frac{1}{\eta_j}\sum_{k=-W}^{W} C_{ik}^{(4)} U_{i+k}=\Omega_1^4 U_i \qquad (i=0,1,\ldots,N) \tag{15}$$

To illustrate MMIB procedures, the stepped beam is taken as an example. As shown in Fig.3, the grid nodes can be divided into two parts from the step, and the step is located on a node. In the MIB procedure, fictitious points (FPs) and known values of grid points on both sides of the interface are employed (Zhao and Wei 2004). And the values of FPs can be carried out by using the interface conditions repeatedly. Detail information on this procedure is not presented here, readers may refer to the publications (Zhao and Wei 2004, Zhou *et al* 2006). In the traditional MIB procedure, values of parameter L are the same in each iterative step. In the present MMIB procedures, values of parameter L are different in each iterative step and denoted as $L_1, L_2, \ldots L_n$ (n is the number of iterative steps) which can be determined by using a new strategy introduced in the following part.

In the first step, four FPs can be carried out (f_1, f_2, ... denote FPs) in which two FPs are gained in each subdomain, as shown in Fig. 4, and $2L_1$ -1 known values of grid points (g_1, g_2, ...) are employed. The stepped interface conditions in Eq. (4) can be discretized by the MIB scheme (Zhao and Wei 2004)

$$\sum_{i=1}^{L_1} C_{1,i}^{-} g_i + C_{1,L_1+1}^{-} f_3 + C_{1,L_1+2}^{-} f_4 = C_{1,1}^{+} f_2 + C_{1,2}^{+} f_1 + \sum_{i=3}^{L_1+2} C_{1,i}^{+} g_{L_1+i-3} \tag{16a}$$

$$\frac{I^-}{I^+}\Big(\sum_{i=1}^{L_1} C_{2,i}^{-} g_i + C_{2,L_1+1}^{-} f_3 + C_{2,L_1+2}^{-} f_4\Big) = C_{2,1}^{+} f_2 + C_{2,2}^{+} f_1 + \sum_{i=3}^{L_1+2} C_{2,i}^{+} g_{L_1+i-3} \tag{16b}$$

$$\frac{I^-}{I^+}\Big(\sum_{i=1}^{L_1} C_{3,i}^{-} g_i + C_{3,L_1+1}^{-} f_3 + C_{3,L_1+2}^{-} f_4\Big) = C_{3,1}^{+} f_2 + C_{3,2}^{+} f_1 + \sum_{i=3}^{L_1+2} C_{3,i}^{+} g_{L_1+i-3} \tag{16c}$$

$$\frac{I^-A^+}{I^+A^-}\Big(\sum_{i=1}^{L_1} C_{4,i}^{-} g_i + C_{4,L_1+1}^{-} f_3 + C_{4,L_1+2}^{-} f_4\Big) = C_{4,1}^{+} f_2 + C_{4,2}^{+} f_1 + \sum_{i=3}^{L_1+2} C_{4,i}^{+} g_{L_1+i-3} \tag{16d}$$

where only Eqs. (4b) - (4e) are used here. $C_{j,i}^{-}$ and $C_{j,i}^{+}$ (i=1, 2, ..., L_1+2, and j=1-4) are the coefficients of one-sided finite difference (FD) on the left and right sub-domains, which can be computed by using the fast algorithm (Fornberg 1998), and j represents jth order derivative approximation. Four FPs can be obtained from Eq. (16) in the first step. Through the iterative procedure in Fig.4, $2W$ FPs can be solved by using the stepped interface conditions in Eq. (4) repeatedly. Hence, in principle, 2(W-1) order accuracy can be achieved for fourth order derivatives. It is noted that four PFs can be solved by using the iterative step once, therefore, the iterative procedure is adapted to an even W. When W is odd, only two FPs are required in the final step, and thus, only Eqs. (16a) and (16b) are used to compute the two unknown FPs. The solutions of other FPs are the same as those for an even W. For other interfaces, similar iterative procedures can be constructed. When Eqs. (6), (8) and (10) are considered, Eqs. (6b), (8b), (10b) and (6e), (8e), (10e) can also be discretized as Eqs. (16a) and (16d), respectively. Eqs. (6c) and (6d) can be discretized as

$$\sum_{i=1}^{L_1} C_{2,i}^{-} g_i + C_{2,L_1+1}^{-} f_3 + C_{2,L_1+2}^{-} f_4 + 0.5K_r\Big(\sum_{i=1}^{L_1} C_{1,i}^{-} g_i + C_{1,L_1+1}^{-} f_3 + C_{1,L_1+2}^{-} f_4 + C_{1,1}^{+} f_2 + C_{1,2}^{+} f_1 + \sum_{i=3}^{L_1+2} C_{1,i}^{+} g_{L_1+i-3}\Big) = C_{2,1}^{+} f_2 + C_{2,2}^{+} f_1 + \sum_{i=3}^{L_1+2} C_{2,i}^{+} g_{L_1+i-3} \tag{17a}$$

$$\sum_{i=1}^{L_1} C_{3,i}^{-} g_i + C_{3,L_1+1}^{-} f_3 + C_{3,L_1+2}^{-} f_4 - K_t g_{L_1} = C_{3,1}^{+} f_2 + C_{3,2}^{+} f_1 + \sum_{i=3}^{L_1+2} C_{3,i}^{+} g_{L_1+i-3} \tag{17b}$$

where, $U^{(1)-}(X_\xi) = 0.5(U^{(1)-}(X_\xi) + U^{(1)+}(X_\xi))$ is considered to improve the computational accuracy. $C_{j,i}^{-}$ and $C_{j,i}^{+}$ are defined in Eq. (16). And similarly, Eqs. (8c) and (8d) can be discretized as

$$\sum_{i=1}^{L_1} C_{2,i}^{-} g_i + C_{2,L_1+1}^{-} f_3 + C_{2,L_1+2}^{-} f_4 - 0.5\Omega^4 J_c\Big(\sum_{i=1}^{L_1} C_{1,i}^{-} g_i + C_{1,L_1+1}^{-} f_3 + C_{1,L_1+2}^{-} f_4 + C_{1,1}^{+} f_2 + C_{1,2}^{+} f_1 + \sum_{i=3}^{L_1+2} C_{1,i}^{+} g_{L_1+i-3}\Big) = C_{2,1}^{+} f_2 + C_{2,2}^{+} f_1 + \sum_{i=3}^{L_1+2} C_{2,i}^{+} g_{L_1+i-3} \tag{18a}$$

$$\sum_{i=1}^{L_1} C_{3,i}^{-} g_i + C_{3,L_1+1}^{-} f_3 + C_{3,L_1+2}^{-} f_4 + \Omega^4 M_c g_{L_1} = C_{3,1}^{+} f_2 + C_{3,2}^{+} f_1 + \sum_{i=3}^{L_1+2} C_{3,i}^{+} g_{L_1+i-3} \tag{18b}$$

and Eq.(8c) can also be expressed as

$$\sum_{i=1}^{L_1} C_{2,i}^{-} g_i + C_{2,L_1+1}^{-} f_3 + C_{2,L_1+2}^{-} f_4 - 0.5\Omega^4 J_c\Big(\sum_{i=-1}^{L_1} C_{1,i}^{-} g_i + \sum_{i=1}^{L_1+2} C_{1,i}^{+} g_{L_1+i-1}\Big) = C_{2,1}^{+} f_2 + C_{2,2}^{+} f_1 + \sum_{i=3}^{L_1+2} C_{2,i}^{+} g_{L_1+i-3} \tag{19}$$

where, only L_1+2 known function values of g_i are used to approximate $U^{(1)-}$ and $U^{(1)+}$, while two FPs and L_1 known function values of g_i are involved in these approximations in Eq.(18a).

Eqs. (10c) and (10d) can be discretized as

$$\sum_{i=1}^{L_1} C_{2,i}^{-} g_i + C_{2,L_1+1}^{-} f_3 + C_{2,L_1+2}^{-} f_4 = C_{2,1}^{+} f_2 + C_{2,2}^{+} f_1 + \sum_{i=3}^{L_1+2} C_{2,i}^{+} g_{L_1+i-3} \tag{20a}$$

$$\sum_{i=1}^{L_1} C_{3,i}^{-} g_i + C_{3,L_1+1}^{-} f_3 + C_{3,L_1+2}^{-} f_4 - \frac{\Omega^4 M_e}{\lambda_e^2 - 1} g_{L_1} = C_{3,1}^{+} f_2 + C_{3,2}^{+} f_1 + \sum_{i=3}^{L_1+2} C_{3,i}^{+} g_{L_1+i-3} \tag{20b}$$

When the interface caused by rigid support is considered, it can be observed from Eq. (5) that Eqs. (5b) and (5c) can be discretized as Eqs. (16a) and (16b). It should be noted that only these two equations can be used to gain two FPs, in which only one FP can be obtained on the right and left sides of the interface, respectively. The iterative procedure is shown in Fig. 5. For treatments of different boundary conditions, procedures similar to the above can be taken (Li *et al*.2015).

In addition, in the traditional MIB (Zhao and Wei 2009, Li *et al*. 2019), when W is very large, CFD scheme is applied when the number of FPs is larger than L. Therefore, in MMIB, CFD scheme is also used for very large W (W>10). From the above analysis, the traditional MIB can be considered as a special case of MMIB.

In order to use MMIB procedures to solve eigenvalue problems, one may not obtain the function values of FPs,

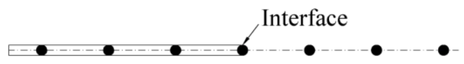


Fig 3 Sketch of gird nodes

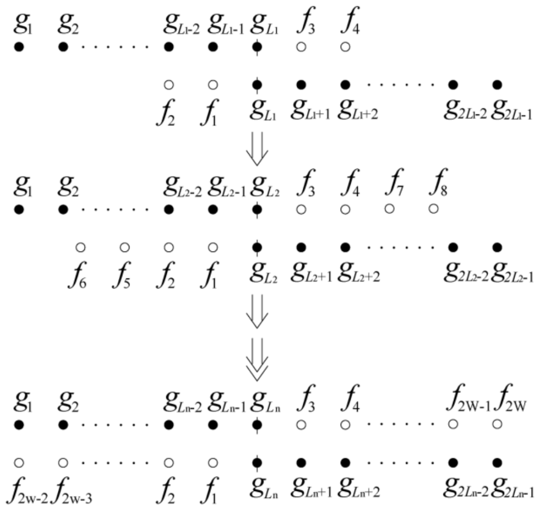


Fig 4 Sketch of the iterative procedure

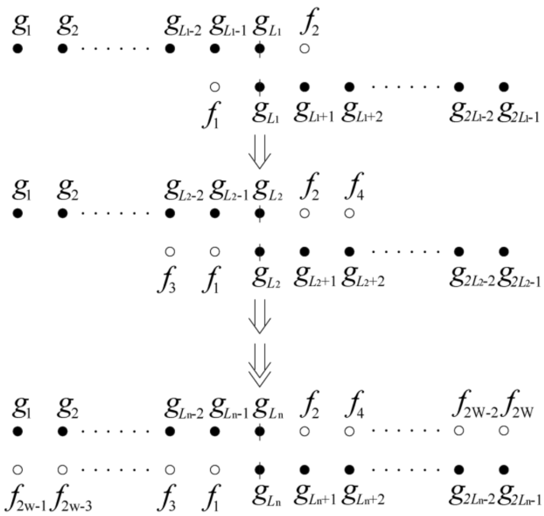


Fig. 5 Sketch of the iterative procedure for the interface caused by single rigid support

and just gain the relationship between f_i (i=1, 2, ... , $2W$) and U_j (j= 0,1, ... , N). When the stepped interface is analyzed, one can gain $2W$ algebraic equations by using the iterative procedure in Fig. 4, and these algebraic equations can be re-written in the form of matrix

$$\mathbf{AF}+\mathbf{BU}=\mathbf{0} \tag{21}$$

where vector $\mathbf{F}=[f_1, f_2, ..., f_{2W}]^T$, and vector $\mathbf{U}=[U_0, U_1, ... , U_N]^T$, and the elements of matrix $\mathbf{A}$ and $\mathbf{B}$ can be determined by the iterative procedure. Thus, one can have

$$\mathbf{F}=-\mathbf{A}^{-1}\mathbf{BU} \tag{22}$$

Considering Eq. (15), the following equation can be obtained

$$\mathbf{MU}=\Omega_1^4\mathbf{U} \tag{23}$$

Eq. (23) is a typical eigenvalue equation and can be solved directly.

Similarly, when the interface caused by single elastic support is analyzed by using Eq. (17), and Eq. (23) can also be obtained and solved directly. And when the interface is caused by single rigid support, one can obtain the eigenvalue equation like Eq. (23) and solve it directly.

When the interface caused by single concentrated mass with rotary inertia is analyzed. Considering Eq. (18), one can obtain

$$(\mathbf{A}_1+\Omega^4\mathbf{A}_2)\mathbf{F}+\mathbf{BU}+\Omega^4\mathbf{CU}=\mathbf{0} \tag{24}$$

Then, yields

$$\mathbf{F}=-(\mathbf{A}_1+\Omega^4\mathbf{A}_2)^{-1}(\mathbf{B}+\Omega^4\mathbf{C})\mathbf{U} \tag{25}$$

and the final eigenvalue equation can be given as

$$\mathbf{MU}+\mathbf{H}(\Omega^4)\mathbf{U}=\Omega^4\mathbf{U} \tag{26}$$

where $\mathbf{H}(\Omega^4)$ is a function of Ω^4. Obviously, Eq. (26) cannot be solved directly, and the iterative solution is needed to compute Ω. And when another approximation in Eq. (19) is employed, one can obtain

$$\mathbf{AF}+\mathbf{BU}+\Omega^4\mathbf{CU}=\mathbf{0} \tag{27}$$

and

$$\mathbf{F}=-\mathbf{A}^{-1}\mathbf{BU}-\Omega^4\mathbf{A}^{-1}\mathbf{CU} \tag{28}$$

The eigenvalue equation of this case can be expressed as

$$\mathbf{MU}+\Omega^4\mathbf{GU}=\Omega^4\mathbf{U} \tag{29}$$

and by transformation, Eq. (29) can be re-written as

$$\hat{\mathbf{M}}\mathbf{U}=\Omega^4\mathbf{U} \tag{30}$$

where $\hat{\mathbf{M}}=(\mathbf{I}-\mathbf{G})^{-1}\mathbf{M}$, and this equation can also be solved directly, which is non-iterative solution. However, it is noted that this solution is a little different from that for the stepped interface. From the above analysis, it is found that both the iterative solution in Eq. (26) and non-iterative solution in Eq. (30) can be employed in this case. In addition, when only the concentrated mass is considered, the final eigenvalue equation can be expressed as Eq. (30) and solved directly.

When the interface caused by single spring-mass system is considered, considering Eq. (20), the final eigenvalue equation can be obtained as Eq. (26) and solved by using the iterative manner. From the above analysis, it can be found that when the interface is caused by one spring-mass system, the iterative solution is necessary. For other interfaces, the eigenvalue equations can be solved directly, and also, the iterative manner can be applied. Therefore, when there are various mixed interfaces caused by many factors, the iterative solution is recommended.

2.4 New strategy

To determine values of parameter L_k (k=1, 2, ..., n, n is the total number of steps) in MMIB procedures, a new strategy is proposed. Details are as follows: for fourth order derivatives discretized by FD method, in order to

Table 1 Convergence tests of relative errors of the first ten non-dimensional frequencies Ω for a C-F beam

N	W=2		W=3		W=4		W=5		W=6		W=7	
	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate
20	4.79		1.19		4.33E-1		1.98E-1		3.65E-2		8.35E-2	
30	2.16	1.96	3.03E-1	3.39	6.32E-2	4.75	8.72E-3	7.70	4.24E-4	11.0	2.54E-3	8.61
40	1.22	1.98	1.09E-1	3.55	1.58E-2	4.81	2.09E-3	4.98	3.38E-4	0.79	1.60E-3	1.61
50	0.786	1.99	4.89E-2	3.60	5.82E-3	4.48	9.00E-4	3.76	2.45E-4	1.44	8.35E-4	2.91
60	0.547	1.99	2.53E-2	3.60	2.76E-3	4.10	4.82E-4	3.42	1.55E-4	2.51	4.85E-4	2.98

Table 2 The first ten non-dimensional frequencies Ω with W=6 by MMIB for a C-F beam

Exact	N=20	N=30	N=40	N=50	N=60
1.875104	1.875104	1.875104	1.875104	1.875104	1.875104
4.694091	4.694102	4.694094	4.694093	4.694092	4.694092
7.854757	7.854805	7.854772	7.854764	7.854761	7.854759
10.995541	10.995657	10.995580	10.995557	10.995549	10.995546
14.137168	14.137317	14.137248	14.137203	14.137186	14.137179
17.278760	17.278771	17.278887	17.278822	17.278792	17.278779
20.420352	20.420967	20.420499	20.420449	20.420405	20.420383
23.561945	23.568392	23.562012	23.562076	23.562024	23.561992
26.703538	26.725051	26.703424	26.703680	26.703645	26.703605
29.845130	29.868421	29.845111	29.845226	29.845262	29.845221
$\tilde{L}_2$(%)	3.65E-2	4.24E-4	3.38E-4	2.45E-4	1.55E-4
$\tilde{L}_\infty$(%)	8.06E-2	7.34E-4	5.56E-4	4.44E-4	3.04E-4

gain 2(W-1) order accuracy, the number of discrete points should be at least 2(W+1). To this end, in each iterative step, the total number of FPs and known grid points should be at least 2(W+1). In the iterative procedure in Fig.4, the number of FPs and known grid points in k step is L_k+2k and in the last step is L_n+W. In this strategy, the total number is set to2(W+1). Therefore, values of parameter L in each iterative step can be chosen as

$$L_k = 2(W+1)-2k \tag{31}$$

where k denotes the kth iterative step ($k < n$), and

$$L_n = W+2 \tag{32}$$

in the final step. For example, when W=4, values of L_1, L_2 are 8, 6, respectively, and when W=3, values of L_1 and L_2 are 6 and 5, respectively. When the interface is caused by single rigid support, the iterative procedure in Fig.5 is taken. It is noted that only one FP on the right and left sides can be obtained in each iteration. Values of L_k can be selected as

$$L_k = 2(W+1)-k \tag{33}$$

where k denotes the kth iterative step ($k \leq n$). For example, when W=4, values of L_1-L_4 are 9-6, respectively, and when W=3, values of L_1 - L_3 are 7-5, respectively. Obviously, more iterative steps are needed in the iterative procedure in Fig.5 to gain the same PFs compared to the iterative procedure in Fig. 4.

3. Numerical results for beams with single interfaces

In this section, at first, several different boundary conditions are handled by using MMIB method. And then, various examples of beams with single interfaces caused by different factors are analyzed. To estimate the errors of results by using MMIB, the relative $\tilde{L}_2$ and $\tilde{L}_\infty$ errors are employed (Li *et al*. 2019)

$$\tilde{L}_2 = \sqrt{\frac{1}{M}\sum_{k=1}^{M}(|\omega_k - \hat{\omega}_k|/\hat{\omega}_k)^2} \times 100\% \tag{34}$$

and

$$\tilde{L}_\infty = \max(|\omega_k - \hat{\omega}_k|/\hat{\omega}_k) \times 100\% \tag{35}$$

where ω_k and $\hat{\omega}_k$ are the numerical and reference results, respectively, and the first M (10 or 5) frequencies are considered in Eq. (34).

3.1 Case 1: beams with different boundary conditions

Due to the fact that anti-symmetric and symmetric extensions have been applied to treatments of simply-supported and clamped boundary conditions in the previous work (Li *et al*. 2015), free edge, elastic edge and free edge with a tip mass are considered here. The first example is a C-F beam. The convergence tests for W=2-7 are given in Table 1, and these results are also shown in Fig. 6. It can be observed that second- and fourth- order convergence rates can be achieved for W=2 and 3, respectively, and the results are convergent for other W. The accuracy of MMIB results increases with the increase of W such as W=2-6, but is degraded for W=7. The first ten non-dimensional frequencies with W=6 are listed in Table 2. It can be found that MMIB results agree well with the exact solutions, and the maximum relative error is only 7×10^{-4} % with N=30, which illustrates that MMIB results are highly accurate. In addition, comparisons between MMIB and the traditional MIB are presented in Fig.7. It can be seen that on the

Table 3 Convergence tests of relative errors of the first ten non-dimensional frequencies Ω for beams with different boundaries

Example*	N	W=2		W=3		W=4		W=5		W=6		W=7	
		$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate
C-E	20	4.77		1.24		4.65E-1		2.02E-1		6.49E-2		7.91E-2	
	30	2.17	1.94	3.13E-1	3.40	6.55E-2	4.83	8.14E-3	7.92	8.45E-4	10.7	2.62E-3	8.40
	40	1.24	1.95	1.13E-1	3.54	1.59E-2	4.92	2.13E-3	4.67	3.25E-4	3.32	1.63E-3	1.65
	50	7.96E-1	1.99	5.04E-2	3.62	5.79E-3	4.53	9.25E-4	3.74	2.46E-4	1.25	8.48E-4	2.93
	60	5.55E-1	1.98	2.61E-2	3.61	2.74E-3	4.10	4.93E-4	3.45	1.57E-4	2.46	4.93E-4	2.97
C-M	20	4.60		1.12		3.51E-1		8.72E-2		4.55E-2		5.94E-2	
	30	2.10	1.93	2.80E-1	3.42	4.78E-2	4.92	9.82E-3	5.39	1.08E-3	9.23	3.48E-3	7.00
	40	1.19	1.97	1.01E-1	3.54	1.28E-2	4.58	2.19E-3	5.22	4.01E-4	3.44	1.51E-3	2.90
	50	7.66E-1	1.97	4.50E-2	3.62	4.97E-3	4.24	8.67E-4	4.15	2.45E-4	2.21	7.84E-4	2.94
	60	5.33E-1	1.99	2.34E-2	3.59	2.44E-3	3.90	4.54E-4	3.55	1.48E-4	2.76	4.57E-4	2.96

*The first ten exact frequencies of these two examples are: 3.788815, 5.756179, 8.488866, 11.487601, 14.552369, 17.642182, 20.745025, 23.855959, 26.972456, 30.093027; 1.419964, 4.111133, 7.190335, 10.298445, 13.421002, 16.550279, 19.683265, 22.818506, 25.955221, 29.092950, respectively.

whole, errors of MMIB are very close to the minimum errors of the traditional MIB, although there are some differences. Due to the fact that there is no effective way to select parameter L in MIB computation, MMIB is more convenient than the traditional MIB.

MMIB is also used to solve free vibration of a clamped-elastically restrained (C-E) beam and a cantilever beam with a tip mass (C-M) at free edge. K_t=100, K_r=10 are set for the translational and rotational springs of the C-E beam, M_c=0.5 is set for the tip mass of the C-M beam. Errors of the first ten non- dimensional frequencies of C-E and C-M beams are listed in Table 3. It can be observed that similar conclusions obtained from Table 1 can also be drawn from Table 3.

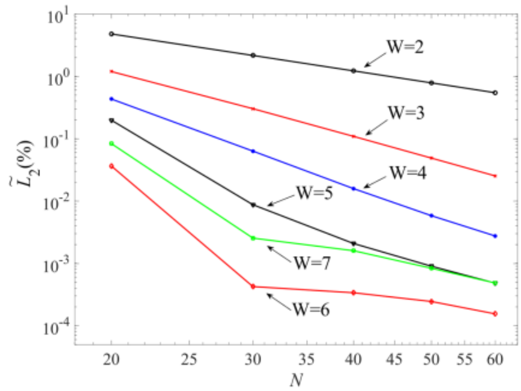


Fig. 6 Log-log plot of errors with different W by MMIB or a C-F beam

3.2 Case 2: beams with single steps

At first, it is assumed that continuous beams have single virtual steps in their midpoints. Thus, MMIB procedures can be used to solve these problems. S-S, C-C and C-F beams with single virtual steps are considered. Errors of the first ten non-dimensional frequencies of the S-S beam by MMIB with W=2-8 are presented in Table 4. It can be observed that the second-, fourth-, sixth-, eighth- and tenth-order convergence rates can be achieved for W=2-6, respectively, and the results for W=7 are convergent. The accuracy of results increases with the increase of W such as W=2-7, but is degraded for W=8. Obviously, the rates of convergence for W=4-6 are higher than those in Case 1. The first ten non-dimensional frequencies by MMIB with W=7 are given in Table 5. It can be clearly seen that $\tilde{L}_\infty$ error decreases rapidly with the increase of N, and high accuracy can be obtained. Comparisons between MMIB and the traditional MIB are shown in Fig.8. It can be observed that on the whole, $\tilde{L}_2$ errors of MMIB are very close to the minimum errors of the traditional MIB, although there are some differences, especially for W=6 and 7. Meanwhile, it is noted that the errors of MMIB are very small for W=6 and 7 with large N. Due to the fact that there is no effective way to select parameter L in the traditional MIB, only some empirical values of parameter L are recommended for some numerical tests (Zhao and Wei 2009), MMIB is a convenient and effective approach to deal with interfaces.

Then, the C-C and C-F beams with single virtual steps are also analyzed. Errors of the first ten non-dimensional frequencies are listed in Table 6. It can be seen that similar findings as described in Table 1 can be obtained from Table 6. Obviously, the rates of convergence for W=4-6 in Table 6 are lower than those of the S-S beam in Table 4. In addition, it can also be observed from Tables 1 and 6 that errors of the C-F beam with single virtual step are extremely close to those of the C-F beam without virtual step, which implies that the treatment of steps may not cause a loss of accuracy. Other examples are S-S, C-C and C-F beams with single steps, which are selected from Mao and Pietrzkothe (2010). As shown in Fig.9, the geometric parameters are: b_2/b_1=0.5, l_1:l_2=3:5, thus, $A_2/A_1=I_2/I_1$=0.5. The convergence tests of results of the S-S beam with single step are presented in Table 7. It can be clearly observed that the results are convergent for all W, and second-, fourth-, sixth- and eighth- order convergence rates can be achieved for W=2-5, respectively, which is very similar to the conclusion obtained from Table 4. Comparisons of MMIB and the traditional MIB are shown in Fig.10. It can also be found that on the whole, errors of MMIB are close to the minimum errors by the traditional MIB, which is similar to the conclusion from Fig. 8. In addition, errors of C-C and

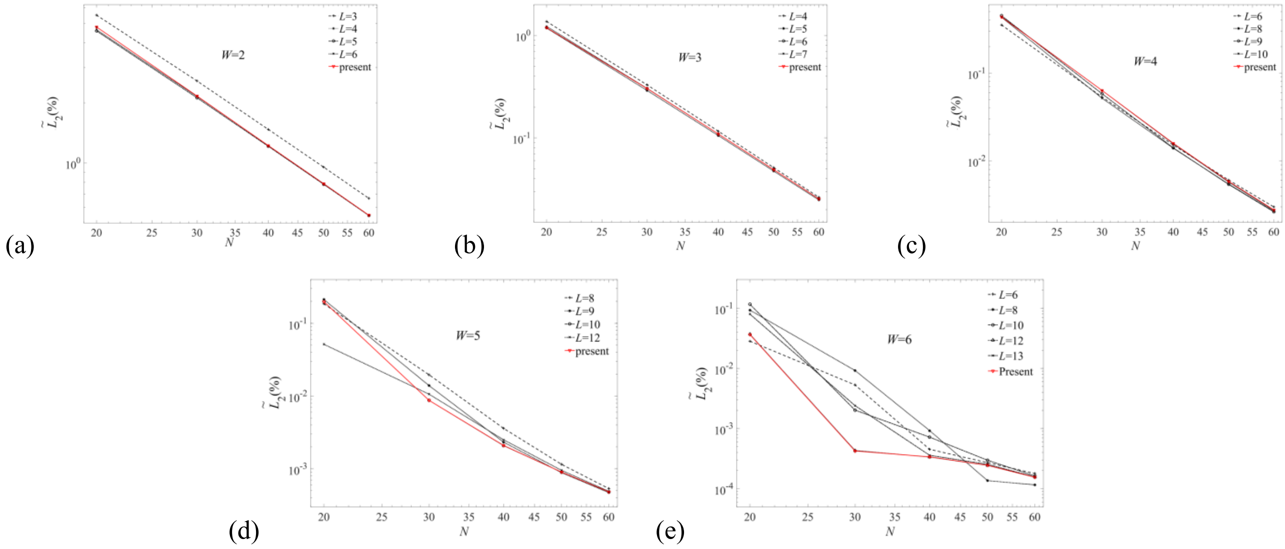


Fig. 7 Log-log plots of $\tilde{L}_2$ errors by MMIB and the traditional MIB with different W for a C-F beam

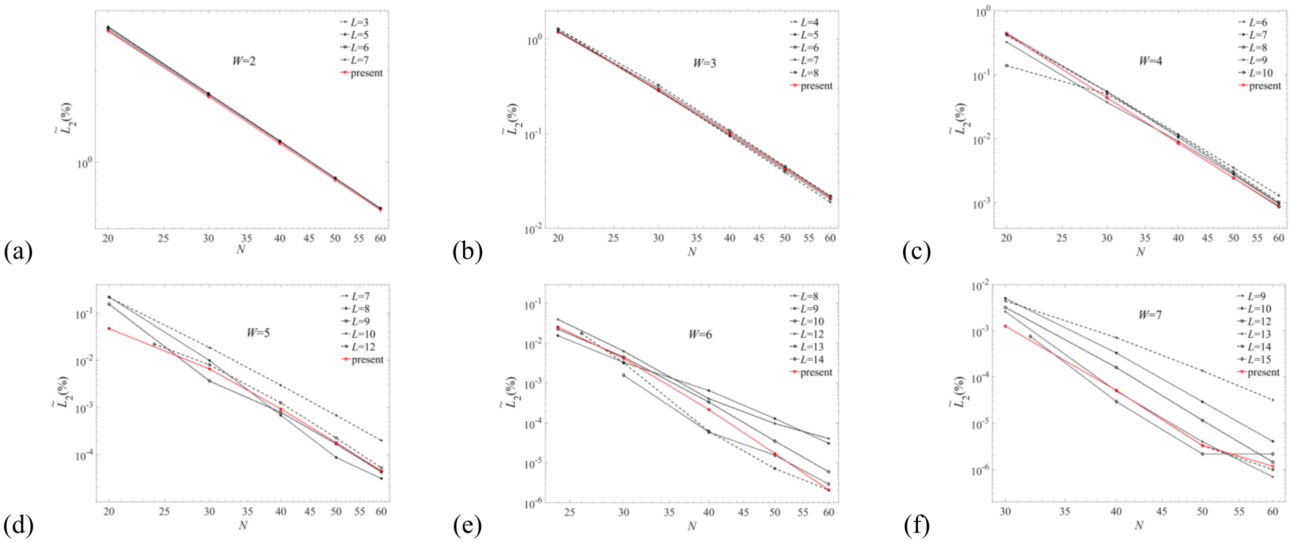


Fig. 8 Log-log plots of errors by MMIB and the traditional MIB with different W for a S-S beam with single virtual step

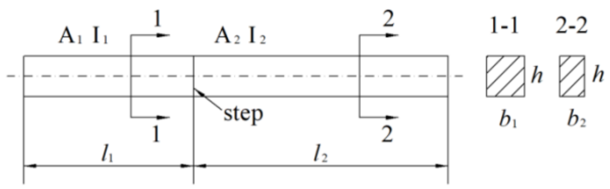


Fig. 9 A beam with single step

C-F beams with single steps are presented in Table 8. It can be seen that similar findings as described in Table 6 can be obtained.

3.3 Case 3: beams with single supports

Examples of beams with single rigid supports are analyzed, and S-S and C-C boundary conditions, different locations of supports are considered. The convergence tests are listed in Table 9. It can be clearly observed that on the whole, similar findings obtained from Tables 7 and 8 can be observed from Table 9. However, it can be seen that the convergence rates are lower than those in Tables 7 and 8 for small values of N. The possible reason is that the procedure in Fig. 5 to treat rigid supports is different from that in Fig. 4 for other interfaces, and more iterative steps are required to gain same fictitious points, which causes a loss of accuracy.

The examples of beams with single elastic supports in Fig. 2(c) are studied next. The convergence tests of results are listed in Table 10. It can be found that similar conclusions from Table 8 can be drawn from Table 10.

3.4 Case 4: beams carrying single concentrated masses

In this case, examples of beams carrying single concentrated masses in Fig.2(d) are solved by MMIB. First, the convergence tests of results of beams carrying single masses are presented in Table 11. It can be found that the results are convergent for all W in the solutions of S-S and

Table 4 Convergence tests of relative errors of the first ten non-dimensional frequencies Ω for a S-S beam with single virtual step (l_1:l_2=1:1)

N	W=2		W=3		W=4		W=5		W=6		W=7		W=8	
	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate
20	4.84		1.19		4.33E-1		4.66E-2							
30	2.20	1.94	2.88E-1	3.50	4.36E-2	5.66	6.50E-3	4.86	4.17E-3		1.26E-3			
40	1.25	1.97	9.96E-2	3.69	8.50E-3	5.68	9.26E-4	6.77	2.17E-4	10.3	5.13E-5	11.1	2.29E-5	
50	0.807	1.96	4.27E-2	3.80	2.43E-3	5.61	1.78E-4	7.39	1.71E-5	11.4	3.30E-6	12.3	4.09E-6	7.72
60	0.563	1.97	2.11E-2	3.87	8.83E-4	5.55	4.49E-5	7.55	2.08E-6	11.6	1.16E-6	5.73	7.41E-6	—

Table 5 The first ten non-dimensional frequencies Ω with W=7 by MMIB for a S-S beam

Exact	N=30	N=40	N=50	N=60
3.14159265	3.14159267	3.14159269	3.14159272	3.14159276
6.28318531	6.28318531	6.28318531	6.28318531	6.28318531
9.42477796	9.42477796	9.42477796	9.42477796	9.42477796
12.56637061	12.56637061	12.56637061	12.56637061	12.56637061
15.70796327	15.70796432	15.70796328	15.70796327	15.70796327
18.84955592	18.84955588	18.84955588	18.84955592	18.84955592
21.99114858	21.99120475	21.99115058	21.99114865	21.99114858
25.13274123	25.13282491	25.13274097	25.13274113	25.13274122
28.27433388	28.27379012	28.27437313	28.27433674	28.27433410
31.41592654	31.41701054	31.41595283	31.41592606	31.41592637
$\tilde{L}_2$(%)	1.26E-3	5.13E-5	3.30E-6	1.16E-6
$\tilde{L}_\infty$(%)	3.45E-3	1.39E-4	1.01E-5	3.57E-6

Table 6 Convergence tests of relative errors of the first ten non-dimensional frequencies Ω for beams with single virtual steps

Example*	N	W=2		W=3		W=4		W=5		W=6		W=7	
		$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate
C-C l_1:l_2=1:1	20	6.29		1.89		7.93E-1		1.89E-1					
	30	2.94	1.88	5.08E-1	3.24	1.13E-1	4.81	2.27E-2	5.23	7.04E-3		8.61E-3	
	40	1.69	1.92	1.92E-1	3.38	3.11E-2	4.48	6.02E-3	4.61	8.17E-4	7.49	4.08E-3	2.60
	50	1.10	1.92	9.00E-2	3.40	1.25E-2	4.08	2.41E-3	4.10	6.22E-4	1.22	2.11E-3	2.96
	60	0.767	1.98	4.74E-2	3.52	6.29E-3	3.77	1.25E-3	3.60	3.92E-4	2.53	1.23E-3	2.96
C-F l_1:l_2=1:1	20	4.51		1.04		4.50E-1		1.79E-1					
	30	2.06	1.93	2.83E-1	3.21	6.32E-2	4.84	4.06E-3	9.34	1.84E-3		3.21E-3	
	40	1.18	1.94	1.04E-1	3.48	1.49E-2	5.02	1.65E-3	3.13	3.45E-4	5.82	1.65E-3	2.31
	50	0.761	1.97	4.73E-2	3.53	5.43E-3	4.52	8.63E-4	2.90	2.35E-4	1.72	8.40E-4	3.03
	60	0.532	1.96	2.47E-2	3.56	2.60E-3	4.04	4.79E-4	3.23	1.54E-4	2.32	4.86E-4	3.00

*The first ten exact frequencies of these two examples are: 4.730041, 7.853205, 10.995608, 14.137165, 17.278760, 20.420352, 23.561945, 26.703538, 29.845130, 32.986723; 1.875104, 4.694091, 7.854757, 10.995541, 14.137168, 17.278760, 20.420352, 23.561945, 26.703538, 29.845130, respectively.

Table 7 Convergence tests of relative errors of the first five non-dimensional frequencies Ω_1 for a S-S beam with single step

N	W=2		W=3		W=4		W=5		W=6	
	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate
24	9.48E-1		5.76E-2		3.15E-3					
32	5.39E-1	1.96	1.93E-2	3.81	6.71E-4	5.38	3.47E-5		6.54E-7	
40	3.48E-1	1.96	8.09E-3	3.90	1.90E-4	5.65	6.35E-6	7.61	1.56E-7	6.42
48	2.43E-1	1.97	3.96E-3	3.92	6.77E-5	5.66	1.51E-6	7.88	2.98E-7	—
56	1.79E-1	1.98	2.15E-3	3.96	2.76E-5	5.82	4.42E-7	7.97	3.00E-7	—

The first five exact frequencies are: 3.08902397, 6.28314030, 9.46673623 ,12.50751459, 15.74913892.

Table 8 Convergence tests of relative errors of the first five non-dimensional frequencies Ω_1 for beams with single steps

Example*	N	W=2		W=3		W=4		W=5		W=6		W=7	
		$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate
C-C	24	1.63		1.85E-1		3.26E-2							
	32	9.34E-1	1.94	7.16E-2	3.30	1.16E-2	3.59	2.48E-3		7.84E-4			
	40	6.05E-1	1.95	3.43E-2	3.30	5.50E-3	3.34	1.21E-3	3.22	4.21E-4	2.79	1.32E-3	
	48	4.23E-1	1.96	1.89E-2	3.27	3.06E-3	3.22	6.84E-4	3.13	2.47E-4	2.92	7.42E-4	3.16
	56	3.12E-1	1.97	1.14E-2	3.28	1.89E-3	3.13	4.27E-4	3.06	1.57E-4	2.94	4.68E-4	2.99
C-F	24	8.39E-1		6.60E-2		1.03E-2							
	32	4.82E-1	1.93	2.52E-2	3.34	3.61E-3	3.64	7.66E-4		2.81E-4			
	40	3.12E-1	1.95	1.20E-2	3.32	1.73E-3	3.30	3.85E-4	3.08	1.38E-4	3.19	4.78E-4	
	48	2.19E-1	1.94	6.56E-3	3.31	9.74E-4	3.15	2.21E-4	3.04	8.04E-5	2.96	2.41E-4	3.76
	56	1.62E-1	1.96	3.95E-3	3.29	6.04E-4	3.10	1.39E-4	3.01	4.93E-5	3.17	1.52E-4	2.99

*The first five exact frequencies of these two examples are: 4.816633, 7.842811, 11.040688, 14.079494, 17.301176; 2.152629, 4.794976, 7.834272, 11.041393, 14.079613, respectively.

Table 9 Convergence tests of relative errors of the first five non-dimensional frequencies Ω for beams with single rigid supports

Example*	N	W=2		W=3		W=4		W=5		W=6	
		$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate	$\tilde{L}_2$(%)	Rate
S-S l_1:l_2=1:1	20	1.90		8.59E-2		3.62E-2					
	30	8.72E-1	1.92	3.32E-2	2.34	6.08E-3	4.40	5.25E-4		6.45E-5	
	40	4.96E-1	1.96	1.36E-2	3.10	9.41E-4	6.49	6.18E-5	7.43	1.23E-5	5.50
	50	3.18E-1	1.99	6.18E-3	3.53	2.19E-4	6.53	7.79E-6	7.20	3.95E-6	5.33
	60	2.22E-1	1.97	3.13E-3	3.73	6.85E-5	6.37	3.16E-6	4.95	3.16E-6	—
S-S l_1:l_2=3:5	24	1.27		6.47E-2		1.38E-2					
	32	7.22E-1	1.96	2.82E-2	2.89	2.41E-3	6.07	1.71E-4		3.11E-5	
	40	4.64E-1	1.98	1.31E-2	3.44	5.86E-4	6.34	1.95E-5	9.73	5.00E-6	8.19
	48	3.23E-1	1.99	6.68E-3	3.69	1.85E-4	6.32	2.50E-6	11.3	0	—
	56	2.37E-1	2.00	3.72E-3	3.80	7.14E-5	6.18	0	—	0	—
C-C l_1:l_2=1:1	20	3.03		3.61E-1		1.06E-1					
	30	1.41	1.89	1.19E-1	2.74	2.79E-2	3.29	4.06E-3		1.62E-3	
	40	8.05E-1	1.38	5.00E-2	3.01	8.86E-3	3.99	1.62E-3	3.19	5.61E-4	3.69
	50	5.19E-1	1.97	2.47E-2	3.16	4.02E-3	3.54	8.40E-4	2.94	3.01E-4	2.79
	60	3.62E-1	1.98	1.37E-2	3.23	2.21E-3	3.28	4.85E-4	3.01	1.77E-4	2.91

*The first five exact frequencies of these three examples are: 6.283185, 7.853205, 12.566371, 14.137166, 18.849556; 5.609582, 9.117249, 11.062918, 15.478609,17.896480; 7.853205, 9.460081, 14.137165, 15.706409, 20.420352, respectively.

C-C beams, and the second-, fourth- and sixth-order convergence rates can be achieved for W=2-4 in the solution of S-S beam, which is similar to that observed from Table 9. While only second-, and fourth-order convergence rates can be obtained for W=2, 3 in the solution of C-C beam, which is similar to that obtained from Tables 6 and 8. Then, the rotary inertia of concentrated mass is considered. The convergence tests of results of a S-S beam carrying single concentrated mass with rotary inertia are given in Table 12. It can be observed that results by using iterative and non-iterative manners are very close to each other, although there are some differences, which illustrates that both iterative and non-iterative manners can be used in this case.

3.5 Case 5: beams carrying single spring-mass systems

Examples of beams carrying single spring-mass systems with S-S and C-C boundary conditions are analyzed by using MMIB. As shown in Fig.2(e), the non-dimensional mass and spring coefficient are M_e=0.2, K_e=3, respectively. The convergence tests of results are given in Table 13. It can be observed that similar findings as described in Table 11 can be obtained from Table 13.

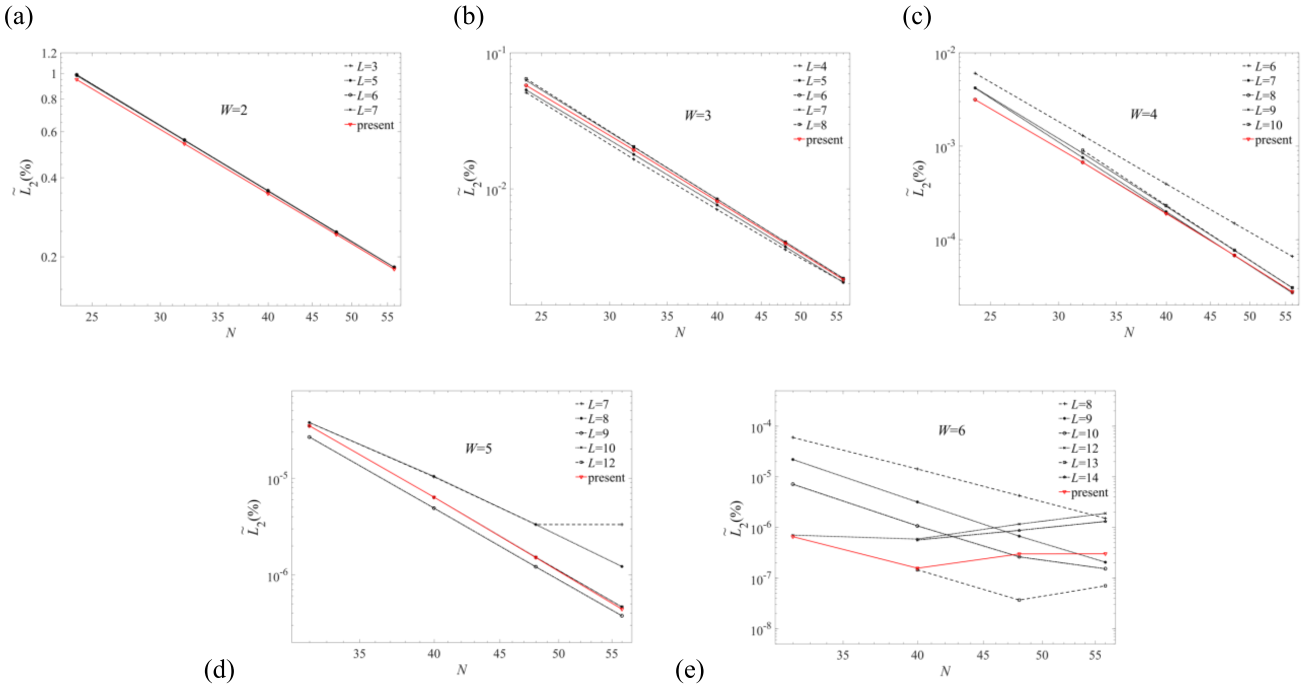


Fig. 10 Log-log plots of errors by MMIB and the traditional MIB with different W for a S-S beam with single step

Table 10 Convergence tests of relative errors of the first five non-dimensional frequencies Ω for beams with single elastic supports

Example*	N	W=2 $\tilde{L}_2$(%)	Rate	W=3 $\tilde{L}_2$(%)	Rate	W=4 $\tilde{L}_2$(%)	Rate	W=5 $\tilde{L}_2$(%)	Rate	W=6 $\tilde{L}_2$(%)	Rate
C-C K_t=100 K_r=0 l_1:l_2=1:1	20	2.26		3.34E-1		7.21E-2		1.07E-2			
	30	1.05	1.89	8.80E-2	3.29	1.46E-2	3.94	3.06E-3	3.09	9.26E-4	
	40	6.00E-1	1.95	3.42E-2	3.29	5.50E-3	3.39	1.21E-3	3.23	4.20E-4	2.75
	50	3.88E-1	1.95	1.65E-2	3.27	2.69E-3	3.21	6.09E-4	3.08	2.18E-4	2.94
	60	2.71E-1	1.97	9.12E-3	3.25	1.53E-3	3.09	3.50E-4	3.04	1.26E-4	3.01
C-C K_t=100 K_r=10 l_1:l_2=1:1	20	2.26		3.38E-1		7.28E-2		1.09E-2			
	30	1.05	1.89	8.92E-2	3.29	1.47E-2	3.95	3.09E-3	3.11	9.34E-4	
	40	6.00E-1	1.95	3.46E-2	3.29	5.55E-3	3.39	1.22E-3	3.23	4.25E-4	2.74
	50	3.88E-1	1.95	1.67E-2	3.26	2.72E-3	3.20	6.13E-4	3.08	2.20E-4	2.95
	60	2.71E-1	1.97	9.22E-3	3.26	1.54E-3	3.12	3.52E-4	3.04	1.28E-4	2.97

*The first five exact frequencies of these two examples are: 5.230375, 7.853205, 11.033107, 14.137165, 17.288488; 5.230375, 8.327341, 11.033107, 14.429777, 17.288487, respectively.

Table 11 Convergence tests of relative errors of the first five non-dimensional frequencies Ω for beams with single concentrated masses

Example*	N	W=2 $\tilde{L}_2$(%)	Rate	W=3 $\tilde{L}_2$(%)	Rate	W=4 $\tilde{L}_2$(%)	Rate	W=5 $\tilde{L}_2$(%)	Rate	W=6 $\tilde{L}_2$(%)	Rate
S-S l_1:l_2=1:1 M_c=0.5 J_c=0	20	1.14		8.89E-2		7.27E-3		2.13E-4			
	30	5.13E-1	1.97	1.86E-2	3.86	6.81E-4	5.84	2.17E-5	5.60	2.46E-6	
	40	2.90E-1	1.98	6.00E-3	3.93	1.22E-4	5.98	2.68E-6	7.27	3.07E-7	7.23
	50	1.86E-1	1.99	2.48E-3	3.96	3.21E-5	5.98	3.07E-7	9.71	3.07E-7	—
	60	1.30E-1	1.96	1.20E-3	3.98	1.06E-5	6.08	3.07E-7	—	3.07E-7	—
C-C l_1:l_2=1:1 M_c=0.5 J_c=0	20	2.06		2.89E-1		5.67E-2		1.13E-2			
	30	9.39E-1	1.94	7.61E-2	3.29	1.27E-2	3.69	2.68E-3	3.55	8.67E-4	
	40	5.34E-1	1.96	2.95E-2	3.29	4.86E-3	3.34	1.07E-3	3.19	3.84E-4	2.83
	50	3.44E-1	1.97	1.43E-2	3.25	2.38E-3	3.20	5.35E-4	3.11	2.02E-4	2.88
	60	2.39E-1	2.00	7.91E-3	3.25	1.35E-3	3.11	3.05E-4	3.08	1.20E-4	2.86

*The first five exact frequencies of these two examples are: 2.6393143, 6.2831853, 8.4744038, 12.5663706, 14.5616702; 3.847071, 7.853205, 9.999906, 14.137165, 16.099838, respectively.

Table 12 Convergence tests of relative errors of the first five non-dimensional frequencies Ω for S-S beams carrying single concentrated masses with rotary inertia

Example*	N	W=2 $\tilde{L}_2$(%)	Rate	W=3 $\tilde{L}_2$(%)	Rate	W=4 $\tilde{L}_2$(%)	Rate	W=5 $\tilde{L}_2$(%)	Rate	W=6 $\tilde{L}_2$(%)	Rate
Iterative solution	20	8.31E-1		7.55E-2		5.74E-3					
	30	3.73E-1	1.98	1.56E-2	3.89	5.69E-4	5.70	2.57E-5		1.33E-6	
	40	2.10E-1	1.99	5.01E-3	3.95	1.02E-4	5.98	3.13E-6	7.32	5.69E-7	2.95
	50	1.35E-1	1.98	2.06E-3	3.98	2.76E-5	5.86	7.82E-7	6.22	5.69E-7	—
	60	9.36E-2	2.00	9.98E-4	3.97	9.34E-6	5.94	5.69E-7	1.74	5.69E-7	—
Non-Iterative solution	20	8.89E-1		2.31E-2		2.25E-2					
	30	3.70E-1	2.16	1.33E-2	1.36	9.97E-4	7.69	9.45E-5		2.01E-5	
	40	2.08E-1	2.00	4,81E-3	3.54	1.17E-4	7.45	1.62E-6	14.1	1.11E-6	10.1
	50	1.34E-1	1.97	2.04E-3	3.84	2.76E-5	6.47	7.82E-7	3.26	9.69E-7	0.61
	60	9.31E-2	2.00	9.93E-4	3.95	9.03E-6	6.13	5.69E-7	1.74	5.69E-7	2.92

*Note: the relevant parameters are l_1:l_2=1:1, M_c=0.5, J_c=0.005. The first five exact frequencies are : 2.6393143, 5.6977598, 8.4744038, 9.3696109,14.4411245.

Table 13 Convergence tests of relative errors of the first five non-dimensional frequencies Ω for beams with single spring-mass systems

Example*	N	W=2 $\tilde{L}_2$(%)	Rate	W=3 $\tilde{L}_2$(%)	Rate	W=4 $\tilde{L}_2$(%)	Rate	W=5 $\tilde{L}_2$(%)	Rate	W=6 $\tilde{L}_2$(%)	Rate
S-S l_1:l_2=1:1	20	8.34E-1		4.66E-2		2.20E-3		2.10E-4			
	30	3.75E-1	1.97	9.97E-3	3.80	2.71E-4	5.16	9.71E-6	7.58		
	40	2.12E-1	1.98	3.24E-3	3.91	5.33E-5	5.65	1.07E-6	7.67		
	50	1.36E-1	1.99	1.34E-3	3.96	1.47E-5	5.77	3.56E-7	4.93		
	60	9.50E-2	1.97	6.50E-4	3.97	5.00E-6	5.91	0	—		
C-C l_1:l_2=1:1	20	1.58		1.85E-1		3.37E-2		7.54E-3			
	30	7.23E-1	1.93	4.96E-2	3.25	8.42E-3	3.42	1.84E-3	3.48	6.25E-4	
	40	4.11E-1	1.96	1.96E-2	3.27	3.34E-3	3.21	7.51E-4	3.11	2.68E-4	2.94
	50	2.65E-1	1.97	9.56E-3	3.22	1.66E-3	3.13	3.81E-4	3.04	1.41E-4	2.88
	60	1.84E-1	2.00	5.34E-3	3.19	9.50E-4	3.06	2.19E-4	3.04	8.16E-5	3.00

*The first five exact frequencies of these two examples are: 1.9337069, 3.1965661, 6.2831853, 9.4265737, 12.5663706 ; 1.960155, 4.748343, 7.853205, 10.996725, 14.137165, respectively.

Table 14 The first ten frequencies (Hz) of a simply-supported beam with three steps

MMIB					DSC*		DQEM*	FEM*
W=3	W=4	W=5	W=6	W=7	W=3	W=7		
0.43369	0.43369	0.43369	0.43369	0.43369	0.43369	0.43369	0.43369	0.43369
1.80275	1.80276	1.80276	1.80276	1.80276	1.80276	1.80276	1.80276	1.80276
4.41469	4.41470	4.41470	4.41470	4.41470	4.41471	4.41470	4.41470	4.41470
9.54111	9.54133	9.54133	9.54133	9.54133	9.54118	9.54133	9.54133	9.54133
13.26560	13.26609	13.26609	13.26609	13.26609	13.26588	13.26609	13.26609	13.26609
19.35734	19.35884	19.35885	19.35885	19.35885	19.35805	19.35885	19.35885	19.35885
25.75681	25.76028	25.76032	25.76032	25.76032	25.76000	25.76031	25.76032	25.76032
34.99384	35.00395	35.00419	35.00419	35.00419	34.99914	35.00414	35.00420	35.00420
43.20179	43.21838	43.21880	43.21882	43.21882	43.21409	43.21867	43.21882	43.21882
55.62708	55.66141	55.66232	55.66242	55.66242	55.63908	55.66238	55.66242	55.66242

*Cited from Duan and Wang (2013)

Table 15 The natural frequencies (Hz) of a cantilever beam with twelve steps

Type	Mode	MMIB				DSC*	DQEM*	FEM*
		W=2	W=3	W=4	W=5	W=4		
Flap-wise	1	10.745	10.745	10.745	10.745	10.745	10.746	10.745
	2	67.467	67.473	67.473	67.473	67.470	67.473	67.473
	3	189.515	189.559	189.559	189.559	189.546	189.559	189.559
	4	373.295	373.460	373.461	373.461	373.426	373.461	373.460
	5	621.820	622.271	622.274	622.274	622.198	622.274	622.271
	10	2858.614	2867.510	2867.621	2867.626	2867.061	2867.629	2867.583
	120	2.597E+5	3.344E+5	3.731E+5	3.940E+5	3.009E+5	8.966E+5	4.562E+5
	140	3.161E+5	5.170E+5	6.479E+5	6.510E+5	5.531E+5	2.450E+6	6.222E+5
Chord-wise	1	54.4963	54.4965	54.4965	54.4965	54.496	54.495	54.499
	2	344.789	344.808	344.808	344.808	344.793	344.808	344.807
	3	977.660	977.812	977.813	977.813	977.740	977.812	977.809
	4	1950.780	1951.407	1951.410	1951.409	1951.199	1951.409	1951.398
	5	3299.763	3301.632	3301.639	3301.639	3301.141	3301.639	3301.606
	10	17415.488	17463.515	17464.085	17464.096	17460.834	17464.100	17463.810
	120	1.821E+6	2.403E+6	2.643E+6	2.773E+6	2.538E+6	4.569E+6	2.819E+6
	140	2.149E+6	3.077E+6	3.796E+6	3.853E+6	3.352E+6	1.300E+7	3.838E+6

* Cited from Duan and Wang (2013)

Table 16 The first five non-dimensional frequencies Ω of S-S beams with multiple supports

Example	MMIB								Maiz *et al.* (2007)
	W=3		W=4		W=5		W=6		
	N=40	N=60	N=40	N=60	N=40	N=60	N=40	N=60	
l_1=l_2=0.25l, l_3=0.5l	7.1710	7.1711	7.1711	7.1711	7.1711	7.1711	7.1711	7.1711	7.1711
	12.5657	12.5662	12.5663	12.5664	12.5664	12.5664	12.5664	12.5664	12.5664
	13.7729	13.7738	13.7740	13.7741	13.7741	13.7741	13.7741	13.7741	13.7741
	16.6390	16.6412	16.6417	16.6419	16.6419	16.6419	16.6419	16.6419	16.6419
	19.8465	19.8523	19.8535	19.8539	19.8540	19.8539	19.8540	19.8539	19.8539
l_1=l_3=0.25l, l_2=0.5l	7.8531	7.8532	7.8532	7.8532	7.8532	7.8532	7.8532	7.8532	7.8532
	12.5657	12.5662	12.5663	12.5664	12.5664	12.5664	12.5664	12.5664	12.5664
	14.1358	14.1369	14.1371	14.1372	14.1372	14.1372	14.1372	14.1372	14.1372
	15.7038	15.7059	15.7063	15.7064	15.7064	15.7064	15.7064	15.7064	15.7064
	20.4132	20.4186	20.4197	20.4203	20.4204	20.4204	20.4204	20.4204	20.4204

Table 17 The first five non-dimensional frequencies Ω of S-S beams with two masses

Example	MMIB						Maiz *et al.* (2007)
	W=3		W=4		W=5		
	N=40	N=60	N=40	N=60	N=40	N=60	
l_1= l_2=0.25l, l_3=0.5l, M_{c1}=M_{c2}=0.5, J_{c1}=J_{c2}=0	2.4946	2.4946	2.4946	2.4946	2.4946	2.4946	2.4946
	5.3428	5.3428	5.3428	5.3428	5.3428	5.3428	5.3428
	7.9643	7.9643	7.9643	7.9643	7.9643	7.9643	7.9643
	12.5655	12.5662	12.5664	12.5664	12.5664	12.5664	12.5664
	14.2156	14.2168	14.2170	14.2171	14.2171	14.2171	14.2171
l_1= l_2=0.25l, l_3=0.5l, M_{c1}=M_{c2}=0.5, J_{c1}=J_{c2}=0.005	2.4824	2.4824	2.4824	2.4824	2.4824	2.4824	2.4824
	5.0881	5.0881	5.0881	5.0881	5.0881	5.0881	5.0881
	7.2183	7.2183	7.2183	7.2183	7.2183	7.2183	7.2183
	8.8187	8.8187	8.8187	8.8187	8.8187	8.8187	8.8187
	9.6265	9.6262	9.6262	9.6262	9.6262	9.6262	9.6262

4. Numerical results for beams with multiple interfaces

In Section 3, free vibration of beams with single interfaces has been discussed and the accuracy and convergence of MMIB have been validated. In this section MMIB is applied to the free vibration analysis of beams with various multiple interfaces.

4.1 Case 6: beams with multiple steps

The first example is a S-S beam with three steps in Fig.11, which is selected from the literature (Duan and Wang 2013, Lin and Ng 2014, Lee 2015). The geometric dimensions are: l_1=5000 mm, h_1=200mm, h_2=100 mm, and b=200 mm. The elastic modulus and the density are: E=34GPa and ρ=2830 kg/m^3. The first ten frequencies by MMIB with W=3-7 and N=72 are listed in Table 14, and the

Table 18 The first five frequencies (rad/s) of beams with multiple spring -mass systems

Example	MMIB *W*=3 *N*=40	MMIB *W*=3 *N*=60	MMIB *W*=3 *N*=80	MMIB *W*=4 *N*=40	MMIB *W*=4 *N*=60	MMIB *W*=4 *N*=80	Wu and Chou (1999)	Lin and Tsai (2007)
l_1=0.1l	152.7341	152.7341	152.7341	152.7341	152.7341	152.7341	152.7341	152.7339
l_2=0.3l	185.0950	185.0950	185.0950	185.0950	185.0950	185.0950	185.0950	185.0949
l_3=0.4l	247.8314	247.8314	247.8314	247.8314	247.8314	247.8314	247.8314	247.8313
l_4=0.2l	677.5958	677.5960	677.5960	677.5961	677.5961	677.5961	677.5961	677.5959
	2548.6419	2548.6536	2548.6564	2548.6613	2548.6573	2548.6577	2548.6577	2548.6572
l_1=0.1l	150.9571	150.9571	150.9571	150.9571	150.9571	150.9571	150.9571	150.9571
l_2=0.1l	169.4729	169.4729	169.4729	169.4729	169.4729	169.4729	169.4729	169.4728
l_3=l_4=	187.9147	187.9147	187.9147	187.9147	187.9147	187.9147	187.9147	187.9146
l_5=l_6=	217.1279	217.1279	217.1279	217.1279	217.1279	217.1279	217.1279	217.1278
0.2l	247.9868	247.9868	247.9868	247.9868	247.9868	247.9868	247.9868	247.9867

Table 19 The first five non-dimensional frequencies Ω of beams with multiple various interfaces

Example	MMIB *W*=3 *N*=40	MMIB *W*=3 *N*=60	MMIB *W*=3 *N*=80	MMIB *W*=4 *N*=40	MMIB *W*=4 *N*=60	MMIB *W*=4 *N*=80	Lin (2008)
S-S	6.613064	6.613083	6.613084	6.613091	6.613084	6.613084	6.613083
	8.214093	8.214074	8.214077	8.214129	8.214078	8.214078	8.214078
	9.236036	9.235986	9.235991	9.236134	9.235994	9.235994	9.235993
	11.506351	11.506636	11.506639	11.506463	11.506642	11.506641	11.506641
	13.353904	13.353652	13.353663	13.354285	13.353669	13.353669	13.353669
C-F	4.089862	4.089876	4.089878	4.089869	4.089876	4.089880	4.089879
	7.633775	7.633886	7.633904	7.633846	7.633888	7.633915	7.633916
	10.002353	10.002601	10.002639	10.002602	10.002621	10.002663	10.002664
	10.943865	10.947141	10.947672	10.945893	10.947269	10.948018	10.948062
	11.986449	11.986838	11.986868	11.986651	11.986865	11.986886	11.986887

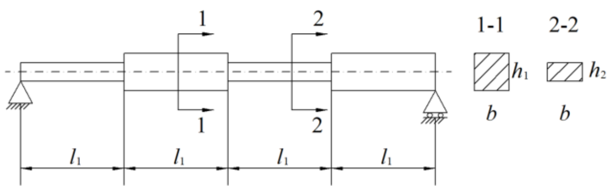


Fig. 11 A S-S beam with three steps

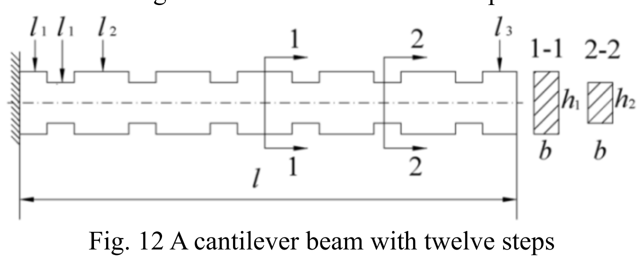


Fig. 12 A cantilever beam with twelve steps

existing results by DSC, DQEM and FEM are given for comparisons. It can be observed that the accuracy of MMIB results increases with the increase of *W*, and the MMIB results agree well with those highly accurate results by DSC, DQEM and FEM with fine meshes. In addition, it can be found that results of MMIB with *W*=7 and *N*=72 are more accurate than those of DSC with *W*=7 and *N*=70 (Duan and Wang 2013), while, the accuracy of results of MMIB is lower than that of DSC for *W*=3.

Another example is a cantilever beam with twelve steps selected from Duan and Wang (2013), as shown in Fig. 12 The geometric dimensions and material properties are: l=463.55mm, l_1=25.4mm, l_2=50.8mm, l_3=31.75mm, h_1=25.4mm, h_2=12.7mm, b=3.175mm, E=60.6GPa, ρ=2664 kg/m^3 Two types of bending vibrations, i.e. chord-wise and flap-wise, are taken into account. The natural frequencies by MMIB with *W*=2-5 and *N*=146 are presented in Table 15, and compared with the existing results by DSC, DQEM and FEM. Due to the limit of grid nodes, parameters L_1=L_2=8 and L_3=7 are chosen for *W*=5. For other cases, selections of parameter L_k in each iterative step are based on the proposed strategy. It can be found that MMIB results coincide well with those by DSC and DQEM for low order frequencies and agree well with those by FEM with fine meshes for higher order frequencies. In addition, it can also be observed that results of MMIB with *W*=4 and *N*=146 are more accurate than those of DSC with *W*=4 and *N*=150 (Duan and Wang 2013) for low and high order frequencies. Of course, more comparisons of MMIB and DSC associated with the interpolation polynomial need further investigation.

4.2 Case 7: beams with multiple intermediate supports

A continuous beam with two intermediate supports in Fig.13 is considered, and two different locations of supports are analyzed. The first five non-dimensional frequencies Ω

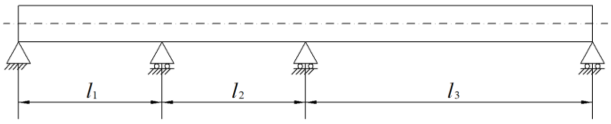


Fig. 13 A continuous beam with two internal supports

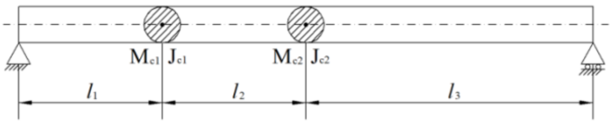
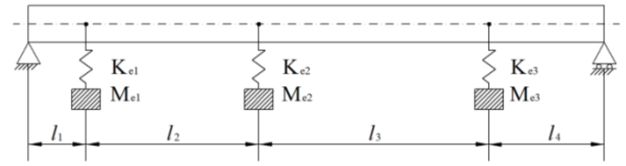
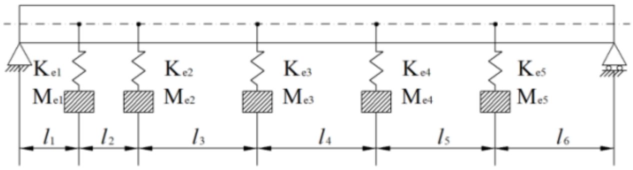


Fig. 14 A S-S beam with two concentrated masses



(a) A beam with three spring-mass systems



(b) A beam with five spring-mass systems

Fig. 15 S-S beams with multiple spring-mass systems

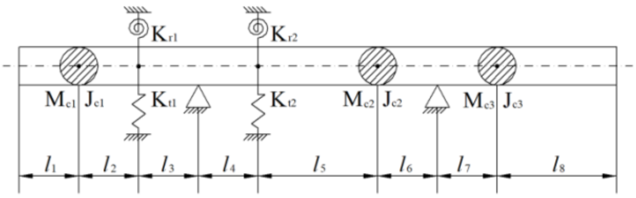


Fig. 16 A beam with multiple various interfaces

of these continuous beams by MMIB with W=3-6 are listed in Table 16, in which due to the limit of grid nodes, parameters L_1=L_2=L_3=11, L_4=10, L_5=9, and L_6=8 for W=6 and N=40 are selected in iterative procedure in Fig.5. For other cases, choices of parameter L_k are based on the proposed strategy. It can be seen that MMIB results show good agreements with the solutions of Maiz *et al*. (2007), and the accuracy of MMIB results increases with the increase of W.

4.3 Case 8: beams carrying multiple concentrated masses

A S-S beam carrying two concentrated masses in Fig.14 is studied, and the effect of rotary inertia is also considered. The first five non-dimensional frequencies Ω of beams with two concentrated masses by MMIB method with W=3-5 is listed in Table 17. For convenience, non-iterative solution is taken in the analysis of rotary inertia. Due to the limit of grid nodes, the total number of grid nodes to approximate the first order derivatives is set to 10 in Eq. (19) in each step for W=5 and N=40, and the proposed strategy is applied in other cases. It can be seen that MMIB results agree well with the solutions of Maiz *et al*. (2007), and MMIB yields more accurate results with large W.

4.4 Case 9: beams carrying multiple spring-mass systems

A S-S beam carrying multiple spring-mass systems is considered. As shown in Fig.15, cited from Lin and Tsai (2007), beams carrying three and five spring-mass systems are taken into account. The geometric dimensions and material properties are: elastic modulus 2.069×10^{11}N/m^2, sectional moment of inertia 3.06796×10^7m^4, mass per unit length 15.3875kg/m, and total length l=1m. The parameters of non-dimensional masses and spring coefficients are: M_{e1}=0.2, K_{e1}=3, M_{e2}=0.5, K_{e2}=4.5, M_{e3}=1.0, K_{e3}=6.0, as shown in Fig.15(a); M_{e1}=0.2, K_{e1}=3, M_{e2}=0.3, K_{e2}=3.5, M_{e3}=0.5 , K_{e3}=4.5, M_{e4}=0.65, K_{e4}=5.0, M_{e5}=1.0, K_{e5}=6.0, as shown in Fig.15(b). The first five frequencies of beams by MMIB with W=3 and 4 are given in Table 18, and the iterative solution is applied to these two examples. Due to the limit of gird nodes, parameters L_1=L_2=5 for W=3 and 4, N=40, and L_1=L_2=6 for W=4, N=60 are selected in the iterative computation. And the proposed strategy is used in other cases. It can be seen that the MMIB results agree well with the existing results (Wu and Chou 1999; Lin and Tsai 2007), and MMIB results show excellent convergence.

4.5 Case 10: beams carrying multiple various interfaces

In this case, beams carrying multiple various interfaces are considered. As shown in Fig.16, cited from Lin (2008) and Farghaly and EI-Sayed (2016), multiple concentrated masses with rotary inertia, intermediate rigid and elastic supports are taken into account. The parameters of geometric dimensions, concentrated masses with rotary inertia and spring coefficients are (Lin 2008): l_1=l_2=l_3=l_4 =l_6=l_7=0.1l, l_5=l_8=0.2l, l is the total length, M_{c1}=0.3, J_{c1}= 0.001, M_{c2}=0.6, J_{c2}=0.002, M_{c3}=0.9, J_{c3}=0.003, K_{t1}=10, K_{r1} =3, K_{t2}=20, and K_{r2}=4. Due to mixed interfaces, iterative solution is employed here. The first five non-dimensional frequencies of beams with two different boundaries conditions are given in Table 19. Due to the limit of grid nodes, small values of parameter L_k in the iterative procedures should be selected in some cases. Parameter L=5 for N=40, W=3 and 4 is chosen in each iterative step; L_1=L_2=L_3=7, L_4=6 for N=60 and W=4 are taken in the iterative computation of beams with rigid supports; L_1=7 and L_2=6 for N=60 and W=4 are chosen in the iterative solutions of beams with concentrated masses and elastic supports. And the proposed strategy is applied in other cases. It can be observed that MMIB results are in good agreements with the solutions of Lin (2008) for S-S and C-F beams, respectively, and MMIB results show excellent convergence.

5. Conclusions

In this study, a new strategy is proposed to detemine the parameter L in each iterative step of MMIB computation, so that MMIB can be applied as if there would be no selection of parameter L. Various single and multiple interfaces caused by steps, intermediate rigid and elastic supports, concentrated masses and spring-mass systems are analyzed by using MMIB. The numerical results of MMIB are compared with the existing highly accurate solutions. A number of examples show that for beams with single interfaces, high order convergence and high accuracy can be achieved, and the accuracy of results increases with the increase of W, but too large W may cause the decrease of accuracy. For beams with multiple interfaces, MMIB also shows high accuracy and excellent convergence. Therefore, MMIB is considered as a highly accurate and convergent approach for solving various interfaces on beams.

In addition, the comparison between MMIB and the traditional MIB is conducted in some case studies, it can be observed that on the whole, errors of MMIB are very close to the minimum errors by the traditional MIB. Due to the fact that only some empirical values of parameter L are recommended (Zhao and Wei 2009), and there is no effective way to select parameter L. Therefore, MMIB is very convenient and effective for interface treatments, especially for interface problems without reference solutions.

Finally, it can be seen that MMIB can deal with arbitrary interfaces on beams, not just the stepped interfaces (Duan and Wang 2013), which extends applications of MIB method. In addition, the comparison analysis shows that MMIB works better than the interpolation formulation in some cases.

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References

Chai, Y., Gong, Z., Li, W., Li, T., Zhang, Q., Zou, Z. and Sun, Y. (2018), "Application of smoothed finite element method to two-dimensional exterior problems of acoustic radiation", *Int. J. Comput. Methods*, **15**, 1850029-1-35. https://doi.org/10.1142/S0219876218500299.

Civalek Ö. (2013), "Nonlinear dynamic response of laminated plates resting on nonlinear elastic foundations by the discrete singular convolution-differential quadrature coupled approaches", *Compos. Part B-Eng.,* **50**,171-179. https://doi.org/10.1016/j.compositesb.2013.01.027.

Civalek Ö. (2017), "Free vibration of carbon nanotubes reinforced (CNTR) and functionally graded shells and plates based on FSDT via discrete singular convolution method", *Compos. Part B-Eng.*, **111**, 45-59. https://doi.org/10.1016/j.compositesb.2016.11.030.

De Rosa, M.A., Bellés, P.M. and Maurizi, M.J. (1995), "Free vibration of stepped beams with intermediate elastic supports", *J. Sound Vib.*, **181**, 905-910.

Duan, G.H. and Wang, X.W. (2013), "Free vibration analysis of multiple-stepped beams by the discrete singular convolution", *Appl. Math. Comput.*, **219**, 11096-11109. https://doi.org/10.1016/j.amc.2013.05.023.

Duan, G.H. and Wang, X.W. (2014), "Vibration analysis of stepped rectangular plates by the discrete singular convolution algorithm", *Int. J. Mech. Sci.*, **82**, 100-109. https://doi.org/10.1016/j.ijmecsci.2014.03.004.

Farghaly, S.H. and EI-Sayed, T.A. (2016), "Exact free vibration of multi-step Timoshenko beam system with several attachments", *Mech. Syst. Signal Pr.*, **72-73**, 525-546. https://doi.org/10.1016/j.ymssp.2015.11.025.

Fornberg, B. (1998), "Calculation of weights in finite difference formulas", *SIAM Rev.*, **40**, 685-691. https://doi.org/10.1137/S0036144596322507.

Jang, S. K. and Bert, C.W. (1989a), "Free vibrations of stepped beams: exact and numerical solutions", *J. Sound Vib.*, **130**(2), 342-346.

Jang, S.K. and Bert, C. W. (1989b), "Free vibrations of stepped beams: higher mode frequencies and effects of steps on frequency", *J Sound Vib.*, **132**(1), 164-168.

Lee, J.H. (2015), "Application of Chebyshev-tau method to the free vibration analysis of stepped beams", *Int. J. Mech. Sci.*, **101-102**, 411-410. https://doi.org/10.1016/j.ijmecsci.2015.08.012.

Li, W., Song, Z. W. and Chai, Y. B. (2015), "Discrete singular convolution method for dynamic stability analysis of beams under periodic axial forces", *J. Eng. Mech.*, **141**, 04015033-1-13. https://doi.org/10.1061/(ASCE)EM.1943-7889.0000931.

Li, W., Song, Z. W., He, X, Q. and Xie, D. (2019), "A comparison study of HO-CFD and DSC-RSK for solving some classes of boundary-value and eigenvalue problems", *Int. J. Comput. Methods*. https://doi.org/10.1142/S0219876219500117.

Lin, H. Y. (2008), "Dynamic analysis of a multi-span beam carrying a number of various concentrated elements", *J. Sound Vib.*, **309**, 262-275. https://doi.org/10.1016/j.jsv.2007.07.015.

Lin, H. Y. (2009), "On the natural frequencies and mode shapes of a multi-span Timoshenko beam carrying a number of various concentrated elements", *J. Sound Vib.*, **319**, 593-605. https://doi.org/10.1016/j.jsv.2008.05.022.

Lin, H. Y. and Tsai, Y. C. (2007), "Free vibration analysis of a uniform multi-span beam carrying multiple spring-mass systems", *J. Sound Vib.*, **302**, 442-456. https://doi.org/10.1016/j.jsv.2006.06.080.

Lin, R. M. and Ng, T. Y. (2004), "Exact vibration modes of multiple-stepped beams with arbitrary steps and supports using elemental impedance method", *Eng. Struct.*, **152**, 24-34. https://doi.org/10.1016/j.engstruct.2017.07.095.

Liu, G. R. and Wu, T. Y. (2001), "Vibration analysis of beams using the generalized differential quadrature rule and domain decomposition", *J. Sound Vib.*, **246**(3), 461-481. https://doi.org/10.1006/jsvi.2001.3667.

Lu, Z. R., Huang, M., Liu, J. K., Chen, W. H. and Liao, W. Y. (2009), "Vibration analysis of multiple -stepped beams with the composite element model", *J. Sound Vib.*, **322**, 1070-1080. https://doi.org/10.1016/j.jsv.2008.11.041.

Maiz, S., Bambill, D.V., Rossit, C. A. and Laura, P.A. A. (2007), "Transverse vibration of Bernoulli-Euler beams carrying point masses and taking into account their rotatory inertia: exact solution", *J. Sound Vib.*, **303**, 895-908. https://doi.org/10.1016/j.jsv.2006.12.028.

Mao, Q. and Pietrzko, S. (2010), "Free vibration analysis of stepped beams by using a domian decomposition method", *Appl. Math. Comput.*, **217**, 3429-3441. https://doi.org/10.1016/j.amc.2010.09.010.

Maurizi, M. J. and Bambill, D. V. (1987), "Free vibration of a clamped-clamped beam with an intermediate elastic support", *J. Sound Vib*., **119**, 173-176.

Naguleswaran, S. (2002), "Vibration of an Euler–Bernoulli beam on elastic end supports and with up to three step changes in cross-section", *Int. J. Mech. Sci*., **44**, 2541-2555. https://doi.org/10.1016/S0020-7403(02)00190-X.

Song, Z. W., Li, W. and Liu G. R. (2012), "Stability and non-stationary vibration analysis of beams subjected to periodic axial forces using discrete singular convolution", *Stru. Eng. Mech*., **44**, 487-499.

Song, Z. W., Chen, Z. G., Li, W. and Chai, Y. B. (2016), "Dynamic stability analysis of beams with shear deformation and rotary inertia subjected to periodic axial forces by using discrete singular convolution method", *J. Eng. Mech* ., **142**, 04015099-1-18. https://doi.org/10.1061/(ASCE)EM.1943-7889.0001023.

Wang, B., Xia, K. L. and Wei, G. W. (2015), "Second order solving 3D elasticity equations with complex interfaces", *J. Comput. Phys*., **294**, 405-438. https://doi.org/10.1016/j.jcp.2015.03.053.

Wang, X. W. and Wang, Y. L. (2013), "Free vibration analysis of multiple-stepped beams by the differential quadrature element method", *Appl. Math. Comput*., **219**, 5802-5810. https://doi.org/10.1016/j.amc.2012.12.037.

Wang, X.W. and Yuan, Z.X. (2017), "Discrete singular convolution and Taylor series expansion method for free vibration analysis of beams and rectangular plates with free boundaries", *Int. J. Mech. Sci*., **122**,184-191. https://doi.org/10.1016/j.ijmecsci.2017.01.023.

Wei, G.W. (1999), "Discrete singular convolution for the solution of the Fokker-Planck equations", *J. Chem. Phys*., **110**, 8930-8942. https://doi.org/10.1063/1.478812.

Wu, J. S. and Chou, H. M. (1999), "A new approach for determining the natural frequencies and mode shapes of a uniform beam carrying any number of sprung masses", *J. Sound Vib*., **220**, 451-468. https://doi.org/10.1006/jsvi.1998.1958.

Yu, S.N. and Wei, G.W. (2007), "Three-dimensional matched interface and boundary (MIB) method for treating geometric singularities", *J. Comput. Phy*s., **227**, 602-632. https://doi.org/10.1016/j.jcp.2007.08.003.

Yu, S. N., Zhou, Y. C. and Wei, G. W. (2007), "Matched interface and boundary (MIB) method for elliptic problems with sharp-edged interfaces", *J. Comput. Phys*., **224**, 729-756. https://doi.org/10.1016/j.jcp.2006.10.030.

Yu, S. N., Xiang, Y. and Wei, G. W. (2009), "Matched interface and boundary (MIB) method for the vibration analysis of plates", *Commun. Numer. Methods Eng*., **25**, 923-950. https://doi.org/10.1002/cnm.1130.

Zhao, S. and Wei, G. W. (2004), "High order FDTD methods via derivative matching for Maxwell's equations with material interfaces", *J. Comput. Phys*., **200**, 60-103. https://doi.org/10.1016/j.jcp.2004.03.008.

Zhao, S. and Wei, G. W. (2009), "Matched interface and boundary (MIB) for the implementation of boundary conditions in high-order central finite differences", *Int. J. Numer. Meth. Eng*., **77**, 1690-1730. https://doi.org/10.1002/nme.2473.

Zhou, Y. C. and Wei, G. W. (2006), "On the fictitious-domain and interpolation formulations of the matched interface and boundary (MIB) method", *J. Comput. Phys*., **219**, 228-246. https://doi.org/10.1016/j.jcp.2006.03.027.

Zhou, Y. C., Zhao, S., Feig, M. and Wei, G. W. (2006), "High order matched interface and boundary (MIB) schemes for elliptic equations with discontinuous coefficients and singular sources", *J. Comput. Phys*., **213**, 1-30. https://doi.org/10.1016/j.jcp.2005.07.022.

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