Structural Engineering and Mechanics, *Vol. 71, No. 6 (2019) 669-682*
DOI: https://doi.org/10.12989/sem.2019.71.6.669

Nonlinear thermal buckling of bi-directional functionally graded nanobeams

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(Received January 15, 2019, Revised April 14, 2019, Accepted April 18, 2019)

Abstract. We in this article study nonlinear thermal buckling of bi-directional functionally graded beams in the theoretical frameworks of nonlocal strain graded theory. To begin with, it is assumed that the effective material properties of beams vary continuously in both the thickness and width directions. Then, we utilize a higher-order shear deformation theory that includes a physical neutral surface to derive the size-dependent governing equations combining with the Hamilton's principle and the von Kármán geometric nonlinearity. It should be pointed out that the established model, containing a nonlocal parameter and a strain gradient length scale parameter, can availably account for both the influence of nonlocal elastic stress field and the influence of strain gradient stress field. Subsequently, via using a easier group of initial asymptotic solutions, the corresponding analytical solution of thermal buckling of beams is obtained with the help of perturbation method. Finally, a parametric study is carried out in detail after validating the present analysis, especially for the effects of a nonlocal parameter, a strain gradient length scale parameter and the ratio of the two on the critical thermal buckling temperature of beams.

Keywords: Bi-directional FGMs; Nonlocal strain gradient theory; Nonlinear thermal buckling

1. Introduction

Over the past two decades, with the vigorous development of modern industry, the effective properties of traditional materials were confronted by the emerging challenges and new demands. Therefore, to satisfy the product design with special requirements, various composite materials have been designed, manufactured and improved, such as functionally graded materials (Gao *et al.* 2019a, Gao *et al.* 2019b, Gao *et al.* 2019c, Hosseini *et al.* 2017, Hosseini *et al.* 2016, Hosseini *et al.* 2018), carbon nanotubes reinforced composites(CNTRCs) (Yuping *et al.* 2018, Yengejeh *et al.* 2017) and fiber reinforced composites (Ahmed and Mamun 2017, Artioli 2018), to possess those desirable properties. Among those composites, functionally graded materials as a new kind of inhomogeneous composites whose properties can be varied continuously along a certain direction, exhibiting several mechanical and thermal remarkable advantages, such as eliminating stress concentrations and alleviate thermal stress, have long been widely applied in space projects, miscellaneous, communication field, aerospace and other fields (Barati 2017a, Ebrahimi *et al.* 2016, Yahia *et al.* 2015, Zemri *et al.* 2015, Khetir *et al.* 2017, Zidi *et al.* 2017, Bousahla *et al.* 2016). For instance, the team led by Karami respectively studied wave propagation in functionally graded nanoplates (Karami *et al.* 2017, Karami *et al.* 2018b), functionally graded porous nanoshells (Karami *et al.* 2019a) and functionally graded nanobeams (Karami *et al.* 2018a). Taati (2018) derived an exact solution for buckling and post-buckling of functionally graded nanobeams subjected to mechanical-thermal loading. Srividhya (2018) developed a first-order shear deformation theory to undertake comparative studies for showing the effect of the material homogenization on a functionally graded plate.

Beams, plates and shells which are used as structural elements in complex structures can be exposed to various types of loads. In order to effectively design such type structures on the nanometer scale, it is necessary to understand mechanical behaviors of the structural elements using respective methods. For one-dimensional components, Karami *et al.* (2019e) studied thermal buckling of smart porous functionally graded nanobeam; She *et al.* (2017b) analyzed buckling and postbuckling of functionally graded nanotubes; Karami *et al*. (2018e) provided a comprehensive analytical study on FG reinforced-nanotubes. Besides, for two-dimensional plate structure, the analyses of shear buckling of porous nanoplates and shear buckling of single layer graphene sheets under hygrothermal environment were respectively undertaken by Shahsavari *et al.* (2018b) and Shahsavari *et al.* (2017). More works can be found in references (Karami *et al.* 2018f, Karami and Shahsavari 2019, Karami *et al.* 2019e, Riccardo *et al*. 2018, Yazid *et al.* 2018, Bellifa *et al.* 2017b, Bakhadda *et al.* 2018, Mokhtar *et al.* 2018, Bouadi *et al.* 2018, Besseghier *et al.* 2017a, Bouafia *et al.* 2017, Youcef *et al*. 2018, Bounouara *et al.* 2016, Mouffoki *et al*. 2017, Larbi *et al.* 2015, Ahouel *et al.* 2016, Cherif *et al*. 2018, Besseghier *et al*. 2017b, Belkorissat *et al*. 2015, Kadari *et al*. 2018,Younsi *et al.* 2018, Bousahla *et al.* 2014, Bennoun *et al.* 2016, Bourada *et al.* 2015, Belabed *et al.* 2014, Draiche *et al.* 2016, Bouafia *et al.* 2017)

Owing to functionally graded materials related to

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http://www.techno-press.com/journals/sem&subpage=7
ISSN: 1225-4568 (Print), 1598-6217 (Online)

temperature variation, thermal effect on structural components made of functionally graded materials should be discussed in detail. Bouderba *et al*. (2016) and Bouderba *et al*. (2013) separately studied thermal buckling response of FG plates and thermal bending of FG plates using a simple first order shear deformation theory. El-Haina *et al*. (2017) attempted to present a simple analytical approach to analyze the thermal buckling behaviors of FG plates. Besides, for functionally graded plates subjected to uniform, linear and nonlinear thermal loads, Menasria *et al*. (2017) used a new displacement field that contains undetermined integral terms to study thermal buckling and Chikh *et al*. (2017) used a simplified HSDT to account for Thermal buckling of cross-ply laminated plates. Meanwhile, a series of researches relevant to linear and nonlinear bending of functionally graded plates were carried out, including Hamidi *et al*. (2015), Tounsi *et al*. (2013), Zidi *et al*. (2014), Mouffoki *et al*. (2017), Attia *et al*. (2018).

However, owing to the temperature field and the stress field in distributing in two or three directions, the conventional materials with one-directional functionally graded distribution are not well suitable for the complexity of changing circumstances. Furthermore, modern structures may require advanced materials whose properties vary continuously not only in one specified direction, but also different other directions. Thus, the study of bi-directional functionally graded materials in the frameworks of respective theories has been elevated from a fringe topic to a central analytic concern in recent years. Şimşek (2015) studied nonlinear free and forced vibration responses of bi-directional FG beams subjected to various boundary conditions, where the material properties vary exponentially along thickness and axial directions. Based on the expression of material properties proposed via Hao and Wei (2016) and Nguyen *et al.* (2017) respectively utilized the wittrick-william algorithm with a non-iterative algorithm and a finite element formulation in conjunction with the Newmark method to analyze the dynamic response and mid-span axial stress. Meanwhile, static analysis of bi-directional functionally graded beams has also been carried out, extensively. For instance, Pydah and Sabale (2016) with the help of the kinematical assumption of the Euler-Bernoulli theory conducted a parametric study for the variation of critical stresses as well as displacements with both gradation parameters, the results of which indicated that such design has the capacity to satisfy a range of structural constraints as widely and accurately as possible. Moreover, the team led by Nejad ulteriorly extended the nonlinear model of Euler-Bernoulli theory, making it possible to characterize the size-dependent effect on vibration of bi-directional FG beams (Nejad and Hadi 2016a), bending of bi-directional FG beams (Nejad and Hadi 2016b) and buckling of bi-directional FG beams (Nejad *et al.* 2016). Therefore, it is of salient theoretical and practical significance to further explore, and then to solve those essential mechanical problems concerning bi-directional functionally graded beams.

On the other hand, since the special structure of FGMs is absolutely different from that of homogeneous materials, the classic theories, like the Reddy beam model, the Timoshenko beam model and the Euler-Bernoulli beam model, all ought to be modified ulteriorly. To date, we have seen that more and more researchers put forward some modified displacement fields and novel theoretical analytical methods, the purpose of which is a step forward in acquiring the numerical solutions closely to the corresponding experiment results. Eltaher *et al.* (2013) proposed a formulas to determine the position of the neutral axis in FG beams, for which the finite element method was used to discretize the obtained approximate model. Huang and Li (2010) in analysis of transverse bending and vibration of circular shells made of FGMs presented a new high-order displacement field without acquiring a shear correction factor. More importantly, Zhang (2013a) was the first to put forward a high order shear deformation theory containing the physical neutral surface in studying nonlinear bending of FG beams. It should be noted that no stretching-bending couplings appears in the proposed constitutive equations due to the displacement components having the special forms. Subsequently, the researcher used the theory to analyze nonlinear bending of FG infinite cylindrical shallows (Zhang 2015) and vibration of FG rectangular plates (Zhang 2013b), and showed that the physical neutral surface theory has plenty of advantages in understanding mechanical properties of FGMs compared with the previous classic theories. Afterwards, Combined with the concept of physical neutral axis, Bousahla *et al.* (2014) proposed a new trigonometric higher-order theory to analyze the size-dependent bending of functionally graded plates and Al-Basyouni *et al.* (2015) developed a novel unified beam formulation to predict the size-dependent vibration of functionally graded beam. Recently, She *et al.* (2017) based on the above-mentioned theory derived a general higher-order shear deformation theory to study the difference among sixteen types of shear deformation model in predicting the critical thermal buckling temperature and the post-buckling thermal behaviors FG beams.

For small-scale beams, the size-dependent effect on mechanical behaviors is growing in significance, which in different ways has been manifested via the experiments. It is not uncommon to utilize experimental methods, molecular dynamic simulations as well as continuum mechanical theories to perform the study of the size-dependent effect on nanostructures. Even if the experimental method is likely to reach a more exact result, it is almost impossible to provide accurate instruments and set specific requirements for each test (Hod *et al.* 2018). As for molecular dynamic simulations, it is no doubt to consume much time, specially, partly because the amount of atoms of nano-structures is huge and the current computational performance is poor (Zhen and Zhou 2017). At the present time, to overcome the difficulty, researchers attempt to explore the size-dependent effect on the mechanical behaviors of nanostructures using some promising continuum mechanical theories, such as gradient elasticity theories (Mindlin 1965, Lam *et al.* 2003, Yang *et al.* 2002), nonlocal elasticity theory (Eringen 1972, Ebrahimi and Barati 2016) and nonlocal strain gradient theory (Lim *et al.* 2015).

Through summarizing the published literature, we have

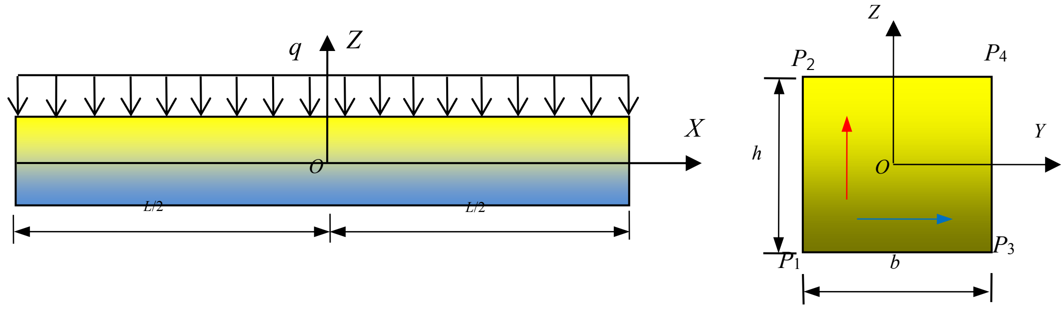

Fig. 1 Geometry and coordinate of functionally graded beam

learned that the effect of size-dependent on the stiffness of nanostructures can be classified into the stiffness-softening aligned with the stiffness-hardening effect. Nonlocal elasticity theory proposed via Eringen (1972) assumed that the stress is not only associated with a point but also a function of strains at all points in the body. Since then, researchers with the aid of the theory have studied the vibration (Ebrahimi and Salari 2015a, Ebrahimi and Salari 2015b), buckling (Ansarim *et al.* 2011, Şimşek and Yurtcu 2013) and static (Reddy 2007, John *et al.* 2003) of nanostructures in last decade, indicating that Eringen's theory merely describes the stiffness-softening effect. For the stiffness-enhancement effect, there are a variety of the strain gradient or couple stress theories to account for this size effect (Mindlin 1965, Lam *et al.* 2003, Yang *et al.* 2002) in which the stress at any point is closely related to high-order strain gradient terms. Based on these theories, numerous works with respect to the size-dependent mechanical behaviors of nanostructures have been carried out (Akgöz 2011, Bekir and Ömer 2013, Ansari *et al.* 2013, Zhang *et al.* 2015, Miandoab *et al.* 2015, Shojaeian *et al.* 2016, Thai *et al.* 2017). Differing from the above-mentioned theories, the nonlocal strain gradient theories proposed via Lim *et al.* (2015) build a bridge between the nonlocal elasticity theory and the strain gradient theory, which can synchronously characterize the stiffness-softening effect and the stiffness-hardening effect owing to taking into account for the influences of nonlocal stress field and strain gradient stress field. Malikan and Nguyen (2018) combining with the nonlocal strain gradient theory developed a new first-order shear deformation theory in terms of the in-plane stability of composite nanoplates so that the precision of results could be improved, greatly. She *et al.* used it to study nonlinear bending of FG curved beams subjected to uniform transverse shear (She *et al.* 2019a), linear vibrations of nanotubes with evenly distributed porosity (She *et al.* 2018a), wave propagation of porous nanotubes (She *et al.* 2018b) and snap-buckling of porous FG curved nanobeams (She *et al.* 2019b). Particularly in the last few years, researchers have witnessed frequent usage of such theory and significant achievements in further studying the size-dependent effect on nanostructures, such as dynamics of FG viscoelastic nanobeams (Ghayesh 2018), free vibration of even and uneven porous nanoshells (Barati *et al.* 2017b), exact solutions of vibration of nanorods (Xu *et al.* 2017), free vibration of bi-directional FG nanotubes (Shafiei and She 2018) as well as forced vibration of porous FG nanoshells (Faleh *et al.* 2018). For more information see the following papers (Karami and Janghorban 2016; Karami and Janghorban 2019; Karami and Maziar 2019; Karami *et al.* 2019b; Karami *et al.* 2019c; Karami *et al.* 2019d; Shahsavari *et al.* 2018a; Karami *et al.* 2018c; Karami *et al.* 2018d) Except for these, Lu *et al.* (2017) used a unified nonlocal strain gradient model to assess the effect of higher order terms on mechanical behaviors of nanobeams and Barati and Zenkour (2017) utilized a general bi-Helmholtz nonlocal strain-gradient model to explore the characteristics of wave propagation in FG double-nanobeams. Different from what were discussed above, Apuzzo *et al.* (2018) presented a modified nonlocal strain gradient model to seek new benchmarks for vibrations of beams, which can provide advantageous approaches for designing nanobeams. Nevertheless, to authors' knowledge, there is no study related to nonlinear thermal buckling of bi-directional FG beams in the framework of nonlocal strain gradient theory, especially for two types of size dependent effect on the critical thermal buckling temperature.

It should be pointed out that our study is motivated by the recent analysis of the effects of the nonlocal parameters, the strain gradient length scale and the ratio of the two on mechanical behaviors of nanostructures. Firstly, we put forward a bi-directional functionally graded beam model where the effective material properties are changed in the thickness and the width, and then give its expression. Secondly, we with the aid of perturbation method utilize a easier group of asymptotic solutions to obtain nonlinear thermal buckling approximate analytical solutions. Thirdly, a parametric study has been carried out in detail, especially for the effects of different parameters on the critical thermal buckling temperature of FG beams.

2. Analysis

2.1 Constitutive relation of bi-dimensional FG beam

As shown in Fig. 1, the functionally graded beam with thickness h, width b and length L is composed of four different materials, where the effective material properties, such as Young's modulus and thermal expansion, continuously vary along both thickness as well as width directions. The origin of Cartesian coordinates (X,Y, Z) is set at the middle of such nanoscale beam, while the axis of

Table 1 Temperature-independent coefficients of Young's modulus, thermal expansion efficient for Al, Si_3N_4, SUS304 and Al_2O_3 (Huang and Li 2010, She *et al.* 2017, Eltaher *et al.* 2012)

	Si_3N_4	SUS304	Al_2O_3	Al
E(pa)	322.27×10^9	207.79×10^9	390×10^9	70×10^9
α(1/K)	7.4746×10^{-6}	15.32×10^{-6}	7.7×10^{-6}	23.2×10^{-6}

X is along the mid-plane of the beam, and then the axis of Z is perpendicular to O-XY plane and directed upward.

From the above figure we can know that, the effective material properties of bi-directional functionally beams can be defined as

$$P_f = [P_1 + V_1(P_2 - P_1)](1 - V_2) + [P_3 + V_1(P_4 - P_3)]V_2;\ V_1 = (\tfrac{1}{2} + \tfrac{z}{h})^{N_1}\ and\ V_2 = (\tfrac{1}{2} + \tfrac{y}{b})^{N_2} \quad (1)$$

in which P_1, P_2, P_3 and P_4 respectively refer to Al, Si_3N_4, SUS304 and Al_2O_3. Moreover, N_1 and N_2 are both non-negative gradient indexes which can change material compositions in gradient directions. Table 1 presents temperature-independent coefficients of respective material components. It ought to be pointed out that a constant Poisson for functionally graded materials has no pronounced effect on the results, which in different ways demonstrates the assumption is true (Yang *et al.* 2014, Cao and Evans 1989). Thus, the Poisson's ratio v of four different materials are all set to be 0.3 based on the result from She *et al.* (2017). Besides, unless noted otherwise, both the thickness h and the width b are respectively equal to 1nm and the length of L is variable.

2.2 Nonlocal strain gradient theory

Owing to the small scale effect exhibiting a stiffness-softening effect and a stiffness-hardening effect, Lim *et al.* (2015) put forward the nonlocal strain gradient elasticity theory interpreting both effects. Therefore, the total stress can be expressed as

$$\boldsymbol{t} = \boldsymbol{\sigma} - \nabla\boldsymbol{\sigma}^{(1)} \quad (2)$$

in which $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^{(1)}$ respectively represent the classical stress and the higher order stress tensor, and are given as

$$\boldsymbol{\sigma} = \int_V \alpha_0(\mathbf{x}, \mathbf{x}', e_0 a)\mathbf{C} : \boldsymbol{\varepsilon}' dV' \quad (3)$$

$$\boldsymbol{\sigma}^{(1)} = l^2 \int_V \alpha_1(\mathbf{x}, \mathbf{x}', e_1 a)\mathbf{C} : \nabla\boldsymbol{\varepsilon}' dV' \quad (4)$$

where $\alpha_0(\mathbf{x}, \mathbf{x}', e_0 a)$ and $\alpha_1(\mathbf{x}, \mathbf{x}', e_1 a)$ are two nonlocal kernel functions which satisfy those conditions defined by Eringen (1972). Owing to attenuation functions for classical stress and higher order stress being the same, we consider both nonlocal parameters e_0a and e_1a have the numerical relationship: e_0a=e_1a=ea, the purpose of which is to account for the effect of nonlocal elastic stress field as compactly and reasonably as possible. Besides, $\boldsymbol{\varepsilon}, \nabla\boldsymbol{\varepsilon}$ and $\boldsymbol{C}$ represent the strain tensor, the strain gradient tensor as well as the fourth-order elasticity tensor respectively, while $\boldsymbol{l}$ is the strain gradient length-scale parameter utilized to character the effect of higher-order strain gradient stress field. However, obtaining analytical solutions from nonlinear equations is difficult when using the above-derived formula. Thus, for the sake of simplification, the generalized nonlocal strain gradient constitutive relation can be defined in the differential form, again.

$$\left[1 - (ea)^2\right]\mathbf{t} = \mathbf{C} : \boldsymbol{\varepsilon} - l^2 \nabla\mathbf{C} : \nabla\boldsymbol{\varepsilon} \quad (5)$$

2.3 Nonlocal strain gradient FG beam model

In the remainder of this chapter, the nonlinear governing equations for nanobeams are derived in the framework of nonlocal strain gradient theory. For structures with rectangular cross section, some efficient and simplified models (Fourn *et al.* 2018, Bellifa *et al.* 2017a, Zine *et al.* 2018, Bourada *et al.* 2018, Meziane *et al.* 2014, Meksi *et al.* 2019, Boukhari *et al.* 2016) satisfying the stress boundary conditions on the surfaces have been developed. In these theories the transverse shear strains are assumed to be parabolically distributed across thickness. Aside from those theory, other theories, like trigonometric shear deformation theory (Bourada *et al.* 2019), hyperbolic shear deformation theory (Abdelaziz *et al.* 2017, Belabed *et al.* 2018), exponential shear deformation theory (Karama *et al.* 2003) are also widely used. In view of functionally graded beams with the special structure, the third-order shear deformation theory, including the physical neutral surface, further developed via Zhang (2013a) is used to establish the mathematical model. That is because that the physical neutral surface doesn't coincide with the mid-plane of a FG beam. The displacement fields can be expressed as

$$\begin{aligned} u_1(x, y, z) &= u_0(x) + (z - z_0)\theta - c_1(z^3 - c_0)\left(\frac{\partial w}{\partial x} + \theta\right) \\ u_2(x, y, z) &= 0 \\ u_3(x, y, z) &= w(x) \end{aligned} \quad (6)$$

in which u_0 and $w(x)$ stand for displacements of any point along X and Y directions, and θ represents rotation angle on the physical neutral surface. Besides, z_0 and c_0 in Eq. (6) can be calculated by the following formula.

$$z_0 = \frac{\int_A zE(z,T)\,\mathrm{d}\sigma}{\int_A E(z,T)\,\mathrm{d}\sigma};\ c_0 = \frac{\int_A z^3 E(z,T)\,\mathrm{d}\sigma}{\int_A E(z,T)\,\mathrm{d}\sigma} \quad (7)$$

It is obvious that Eq. (6) can automatically degenerate into the third-order shear deformation theory proposed via Reddy when both z_0 and c_0 are equivalent to zero, which indicates the nanoscale beam made of a homogeneous material.

By submitting the displacements into the von Kármán nonlinear strain-displacement relations, the strains can be written as

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x} + \frac{1}{2}\left(\frac{\partial u_3}{\partial x}\right)^2 = \varepsilon_x^{(0)} + (z - z_0)\varepsilon_x^{(1)} - c_1(z^3 - c_0)\varepsilon_x^{(3)} \quad (8)$$

$$\gamma_{xz} = \frac{\partial u_3}{\partial x} + \frac{\partial u_1}{\partial z} = \gamma_{xz}^{(0)} - c_2 z^2 \gamma_{xz}^{(2)}$$

where

$$\varepsilon_x^{(0)} = \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2; \varepsilon_x^{(1)} = \frac{\partial \theta}{\partial x}; \varepsilon_x^{(3)} = \frac{\partial \theta}{\partial x} + \frac{\partial^2 w}{\partial x^2}$$

$$\gamma_{xz}^{(0)} = \gamma_{xz}^{(2)} = \theta + \frac{\partial w}{\partial x}; c_1 = \frac{4}{3h^2}; c_2 = \frac{4}{h^2};$$

When a beam is subjected to a uniform temperature environment, based on Hooke's law, the stresses associated with strain components can be arrived at.

$$\left(\sigma_x, \tau_{xz}\right) = \left(E_f \varepsilon_{xx} - E_f \alpha_f \Delta T, G_f \gamma_{xz}\right) \tag{9}$$

in which

$$G_f = \frac{E_f}{2\left(1 + v_f\right)}$$

The variation of the virtual strain energy of the beam can be expressed as

$$\delta \Pi_s = \int_\Omega \left(\sigma_x \delta \varepsilon_{xx} + \tau_{xy} \delta \gamma_{xy}\right) \mathrm{d}\Omega \tag{10}$$

where Ω is the volume of the beam. The submission of Eq. (8) into Eq. (10) yields the result.

$$\delta\Pi_s = \int_\Omega \left(\sigma_x \delta \left[\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \left(z - z_0\right)\frac{\partial \theta}{\partial x} - c_1\left(z^3 - c_0\right)\left(\frac{\partial \theta}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right)\right] + \tau_{xy}\delta\left[\left(1 - c_2 z^2\right)\left(\theta + \frac{\partial w}{\partial x}\right)\right]\right) \mathrm{d}\Omega \tag{11}$$

Meanwhile, the variation of virtual work done by the external force is given by

$$\delta \Pi_W = -\int_{-L/2}^{L/2} \left(q \delta w\right) \mathrm{d}x \tag{12}$$

Then, the Hamilton principle denotes that

$$\int_{t_1}^{t_2} \delta \Pi \mathrm{d}t = \int_{t_1}^{t_2} \delta\left(\Pi_s + \Pi_w\right) \mathrm{d}t = 0 \tag{13}$$

When substituting Eq. (12) and Eq. (13) into Eq. (13), the governing equations can be obtained as

$$\begin{aligned}
&\delta u: \frac{\mathrm{d}N_x}{\mathrm{d}x} = 0 \\
&\delta \theta: \frac{\mathrm{d}M_x}{\mathrm{d}x} - c_1 \frac{\mathrm{d}p_x}{\mathrm{d}x} - Q_x + c_2 R_x = 0 \\
&\delta w: N_x \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + c_1 \frac{\mathrm{d}^2 p_x}{\mathrm{d}x^2} + \frac{\mathrm{d}Q_x}{\mathrm{d}x} - c_2 \frac{\mathrm{d}R_x}{\mathrm{d}x} + q = 0
\end{aligned} \tag{14}$$

where

$$\left(N_x, Q_x\right) = \iint_A \left(\sigma_{xx}, \sigma_{xz}\right) \mathrm{d}y\mathrm{d}z; M_x$$

$$= \iint_A \sigma_{xx}\left(z - z_0\right) \mathrm{d}y\mathrm{d}z; P_x = \iint_A \sigma_{xx}\left(z^3 - c_0\right) \mathrm{d}y\mathrm{d}z$$

By using the generalized nonlocal strain gradient differential constitutive equation, the stress result appearing in Eq. (14) can be reappraised as

$$\begin{aligned}
&N_x - \mu^2 \frac{\partial^2 N_x}{\partial x^2} = \left(1 - l^2 \nabla^2\right)\left[A_{11} \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + B_{11}\frac{\partial \theta}{\partial x} - c_1 E_{11}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial \theta}{\partial x}\right) - N_T\right] \\
&M_x - \mu^2 \frac{\partial^2 M_x}{\partial x^2} = \left(1 - l^2 \nabla^2\right)\left[B_{11} \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + D_{11}\frac{\partial \theta}{\partial x} - c_1 F_{11}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial \theta}{\partial x}\right) - M_T\right] \\
&P_x - \mu^2 \frac{\partial^2 P_x}{\partial x^2} = \left(1 - l^2 \nabla^2\right)\left[E_{11} \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + F_{11}\frac{\partial \theta}{\partial x} - c_1 H_{11}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial \theta}{\partial x}\right) - P_T\right] \\
&Q_x - \mu^2 \frac{\partial^2 Q_x}{\partial x^2} = \left(1 - l^2 \nabla^2\right)\left(A_{55} - c_2 D_{55}\right)\left(\frac{\partial w}{\partial x} + \theta\right) \\
&R_x - \mu^2 \frac{\partial^2 R_x}{\partial x^2} = \left(1 - l^2 \nabla^2\right)\left(D_{55} - c_2 F_{55}\right)\left(\frac{\partial w}{\partial x} + \theta\right)
\end{aligned} \tag{15}$$

in which

$$\left(A_{11}, B_{11}, E_{11}, D_{11}, F_{11}, H_{11}\right) =$$

$$\iint_A E_f \left[1, \left(z - z_0\right), \left(z^3 - c_0\right), \left(z - z_0\right)^2, \left(z - z_0\right)\left(z^3 - c_0\right), \left(z^3 - c_0\right)^2\right] \mathrm{d}z\mathrm{d}y$$

$$\left(A_{55}, D_{55}, F_{55}\right) =$$

$$\iint_A G_f\left(1, z^2, z^4\right) \mathrm{d}z\mathrm{d}y; \left(N_T, M_T, P_T\right) = \iint_A E_f \alpha_f \Delta T\left(1, z - z_0, z^3 - c_0\right) \mathrm{d}z\mathrm{d}y$$

Substituting Eq. (15) into Eq. (14), we have

$$\begin{aligned}
&r_1\left(\theta + \frac{\partial w}{\partial x}\right) + r_2 \frac{\partial^3 w}{\partial x^3} + r_3 \frac{\partial^2 \theta}{\partial x^2} + r_8 \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} \\
&-l^2 \left[r_1\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^3 w}{\partial x^3}\right) + r_2 \frac{\partial^5 w}{\partial x^5} + r_3 \frac{\partial^4 \theta}{\partial x^4} + r_8\left(3\frac{\partial^2 w}{\partial x^2}\frac{\partial^3 w}{\partial x^3} + \frac{\partial w}{\partial x}\frac{\partial^4 w}{\partial x^4}\right)\right] = 0
\end{aligned} \tag{16}$$

$$\begin{aligned}
&r_6\left(\frac{\partial \theta}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + r_7 \frac{\partial^3 \theta}{\partial x^3} - c_1^2 H_{11}\frac{\partial^4 w}{\partial x^4} + q + c_1 E_{11}\left[\frac{\partial w}{\partial x}\cdot\frac{\partial^3 w}{\partial x^3} + \left(\frac{\partial^2 w}{\partial x^2}\right)^2\right] + N_x\left(\frac{\partial^2 w}{\partial x^2} - \mu^2 \frac{\partial^4 w}{\partial x^4}\right) \\
&-l^2\left[r_6\left(\frac{\partial^3 \theta}{\partial x^3} + \frac{\partial^4 w}{\partial x^4}\right) + r_7\frac{\partial^5 \theta}{\partial x^5} - c_1^2 H_{11}\frac{\partial^6 w}{\partial x^6} + c_1 E_{11}\left(3\frac{\partial^3 w}{\partial x^3}\cdot\frac{\partial^3 w}{\partial x^3} + 4\frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^4 w}{\partial x^4} + \frac{\partial w}{\partial x}\cdot\frac{\partial^5 w}{\partial x^5}\right)\right] \\
&-N_T\left(\frac{\partial^2 w}{\partial x^2} - \mu^2\frac{\partial^4 w}{\partial x^4}\right) = 0
\end{aligned} \tag{17}$$

where N_x in Eq. (17) is a constant and calculated by

$$N_x = \frac{1}{L}\int_{-L/2}^{L/2}\left\{A_{11}\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + B_{11}\frac{\partial \theta}{\partial x} - c_1 E_{11}\left(\frac{\partial \theta}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) - l^2\left[A_{11}\left(\frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x^3} + \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x^2}\right) + B_{11}\frac{\partial^3 \theta}{\partial x^3} - c_1 E_{11}\left(\frac{\partial^3 \theta}{\partial x^3} + \frac{\partial^4 w}{\partial x^4}\right)\right]\right\}\mathrm{d}x$$

The coefficients appearing Eq. (16) and Eq. (17) are given by

$$r_1 = 2c_2 D_{55} - c_2^2 F_{55} - A_{55}$$

$$r_2 = c_1^2 H_{11} - c_1 F_{11}$$

$$r_3 = D_{11} - 2c_1 F_{11} + c_1^2 H_{11};$$

$$r_7 = c_1 F_{11} - c_1^2 H_{11}; r_8 = \left(B_{11} - c_1 E_{11}\right);$$

$$r_7 = c_1 F_{11} - c_1^2 H_{11}; r_8 = \left(B_{11} - c_1 E_{11}\right);$$

For a beam subjected to immovable clamped ends, it is essential to satisfy some boundary conditions.

$$X = -L/2, L/2; u = 0, w = 0, \theta = 0. \tag{18}$$

For the sake of simplicity and generality, we introduce the following dimensionless parameters.

$$\xi = \frac{x}{L}\pi, \bar{w} = \frac{w}{L}, \bar{\theta} = \frac{\theta}{\pi}, \bar{\mu} = \frac{\mu}{L}\pi, (\bar{A}_{11}, \bar{B}_{11}, \bar{E}_{11}, \bar{D}_{11}, \bar{F}_{11}, \bar{H}_{11})$$

$$= \left(\frac{A_{11}}{S_o}, \frac{B_{11}\pi}{S_o L}, \frac{E_{11}\pi^3}{S_o L^3}, \frac{D_{11}\pi^2}{S_o L^2}, \frac{F_{11}\pi^4}{S_o L^4}, \frac{H_{11}\pi^6}{S_o L^6}\right)$$

$$(\bar{A}_{55}, \bar{D}_{55}, \bar{F}_{55}) = \left(\frac{A_{55}}{S_o}, \frac{D_{55}\pi^2}{S_o L^2}, \frac{F_{55}\pi^4}{S_o L^4}\right)$$

$$(\bar{c}_1, \bar{c}_2) = (c_1, c_2)\frac{L^2}{\pi^2}, \lambda_T = \Delta T, \lambda_q = \frac{qL^2}{S_o \pi^2}, S_n = \frac{r_T}{S_o}, \bar{l} = \frac{l\pi}{L},$$

$$r_T = \iint_A E_f \alpha_f \mathrm{d}z\mathrm{d}y, S_0 = \iint_A E_f \mathrm{d}z\mathrm{d}y;$$

After some mathematical operations, the governing equations of FG beams can be reappraised in the following form.

$$\begin{aligned} &\bar{r}_1(\bar{\theta} + \frac{\partial \bar{w}}{\partial \xi}) + \bar{r}_2 \frac{\partial^3 \bar{w}}{\partial \xi^3} + \bar{r}_3 \frac{\partial^2 \bar{\theta}}{\partial \xi^2} + \pi \bar{r}_8 \frac{\partial \bar{w}}{\partial \xi}\frac{\partial^2 \bar{w}}{\partial \xi^2} \\ &-\bar{l}^2\left[\bar{r}_1(\frac{\partial^2 \bar{\theta}}{\partial \xi^2} + \frac{\partial^3 \bar{w}}{\partial \xi^3}) + \bar{r}_2 \frac{\partial^5 \bar{w}}{\partial \xi^5} + \bar{r}_3 \frac{\partial^4 \bar{\theta}}{\partial \xi^4} + \pi \bar{r}_8 \left(3\frac{\partial^2 \bar{w}}{\partial \xi^2}\frac{\partial^3 \bar{w}}{\partial \xi^3} + \frac{\partial \bar{w}}{\partial \xi}\frac{\partial^4 \bar{w}}{\partial \xi^4}\right)\right] = 0 \end{aligned} \tag{19}$$

$$\begin{aligned} &\bar{r}_6\left(\frac{\partial \bar{\theta}}{\partial \xi} + \frac{\partial^2 \bar{w}}{\partial \xi^2}\right) + \bar{r}_7 \frac{\partial^3 \bar{\theta}}{\partial \xi^3} - \bar{c}_1^2 \bar{H}_{11}\frac{\partial^4 \bar{w}}{\partial \xi^4} + \lambda_q + \pi \bar{c}_1 \bar{E}_{11}\left[\frac{\partial \bar{w}}{\partial \xi}\cdot\frac{\partial^3 \bar{w}}{\partial \xi^3} + \left(\frac{\partial^2 \bar{w}}{\partial \xi^2}\right)^2\right] + \bar{N}_x\left(\frac{\partial^2 \bar{w}}{\partial \xi^2} - \mu^2\frac{\partial^4 \bar{w}}{\partial \xi^4}\right) \\ &-\bar{l}^2\left[\bar{r}_6\left(\frac{\partial^3 \bar{\theta}}{\partial \xi^3} + \frac{\partial^4 \bar{w}}{\partial \xi^4}\right) + \bar{r}_7\frac{\partial^5 \bar{\theta}}{\partial \xi^5} - \bar{c}_1^2 \bar{H}_{11}\frac{\partial^6 \bar{w}}{\partial \xi^6} + \pi \bar{c}_1 \bar{E}_{11}\left(3\frac{\partial^3 \bar{w}}{\partial \xi^3}\cdot\frac{\partial^3 \bar{w}}{\partial \xi^3} + 4\frac{\partial^2 \bar{w}}{\partial \xi^2}\cdot\frac{\partial^4 \bar{w}}{\partial \xi^4} + \frac{\partial \bar{w}}{\partial \xi}\cdot\frac{\partial^5 \bar{w}}{\partial \xi^5}\right)\right] \\ &-S_n \lambda_T\left(\frac{\partial^2 \bar{w}}{\partial \xi^2} - \bar{\mu}^2\frac{\partial^4 \bar{w}}{\partial \xi^4}\right) = 0 \end{aligned} \tag{20}$$

where

$$\bar{N}_x = \int_0^\pi \left\{\frac{\pi \bar{A}_{11}}{2}\left(\frac{\partial \bar{w}}{\partial \xi}\right)^2 + \bar{B}_{11}\frac{\partial \bar{\theta}}{\partial \xi} - \bar{c}_1 \bar{E}_{11}\left(\frac{\partial \bar{\theta}}{\partial \xi} + \frac{\partial^2 \bar{w}}{\partial \xi^2}\right) - \bar{l}^2\left[\pi \bar{A}_{11}\left(\frac{\partial \bar{w}}{\partial \xi}\frac{\partial^3 \bar{w}}{\partial \xi^3} + \frac{\partial^2 \bar{w}}{\partial \xi^2}\frac{\partial^2 \bar{w}}{\partial \xi^2}\right) + \bar{B}_{11}\frac{\partial^3 \bar{\theta}}{\partial \xi^3} - \bar{c}_1 \bar{E}_{11}\left(\frac{\partial^3 \bar{\theta}}{\partial \xi^3} + \frac{\partial^4 \bar{w}}{\partial \xi^4}\right)\right]\right\} d\xi$$

$$\bar{r}_1 = 2\bar{c}_2 \bar{D}_{55} - \bar{c}_2^2 \bar{F}_{55} - \bar{A}_{55}; \bar{r}_2 = \bar{c}_1^2 \bar{H}_{11} - \bar{c}_1 \bar{F}_{11}$$

$$\bar{r}_3 = \bar{D}_{11} - 2\bar{c}_1 \bar{F}_{11} + \bar{c}_1^2 \bar{H}_{11}; \bar{r}_6 = \bar{A}_{55} - 2\bar{c}_2 \bar{D}_{55} + \bar{c}_2^2 \bar{F}_{55};$$

$$\bar{r}_7 = \bar{c}_1 \bar{F}_{11} - \bar{c}_1^2 \bar{H}_{11}; r_8 = \left(\bar{B}_{11} - \bar{c}_1 \bar{E}_{11}\right);$$

Moreover, the dimensionless boundary conditions can be rewritten as

$$\bar{u} = 0, \bar{w} = 0, \bar{\theta} = 0; \ at \ \xi = -\pi/2, \pi/2 \tag{21}$$

3. Solution methodology

In this section, a two-step perturbation method is used to solve the governing equations, and then to obtain the analytical solutions. At the beginning, it should be noted that λ_q is equal to zero in the case of nonlinear thermal buckling. Subsequently, we assume that the dimensionless displacement, dimensionless rotation angle and dimensionless temperature can be expanded as

$$\begin{aligned} \bar{w}(\xi, \varepsilon) &= \sum_{n=1} \varepsilon^n w_n(\xi); \\ \bar{\theta}(\xi, \varepsilon) &= \sum_{n=1} \varepsilon^n \bar{\theta}_n(\xi); \\ \lambda_T(\xi, \varepsilon) &= \sum_{n=1} \varepsilon^n \lambda_n(\xi); \end{aligned} \tag{22}$$

in which ε is only a perturbation parameter, but has no physical meaning. Via substituting Eq. (22) into Eqs, (19-20), then collecting the same order ε, we acquire a set of perturbation equations

$O(\varepsilon^1)$

$$\begin{aligned} &\bar{r}_1(\bar{\theta}_1 + \frac{\partial \bar{w}_1}{\partial \xi}) + \bar{r}_2\frac{\partial^3 \bar{w}_1}{\partial \xi^3} + \bar{r}_3\frac{\partial^2 \bar{\theta}_1}{\partial \xi^2} \\ &-\bar{l}^2\left[\bar{r}_1(\frac{\partial^2 \bar{\theta}_1}{\partial \xi^2} + \frac{\partial^3 \bar{w}_1}{\partial \xi^3}) + \bar{r}_2\frac{\partial^5 \bar{w}_1}{\partial \xi^5} + \bar{r}_3\frac{\partial^4 \bar{\theta}_1}{\partial \xi^4}\right] = 0 \end{aligned} \tag{23}$$

$$\begin{aligned} &\bar{r}_6\left(\frac{\partial \bar{\theta}_1}{\partial \xi} + \frac{\partial^2 \bar{w}_1}{\partial \xi^2}\right) + \bar{r}_7\frac{\partial^3 \bar{\theta}_1}{\partial \xi^3} - \bar{c}_1^2 \bar{H}_{11}\frac{\partial^4 \bar{w}_1}{\partial \xi^4} \\ &-\bar{l}^2\left[\bar{r}_6\left(\frac{\partial^3 \bar{\theta}_1}{\partial \xi^3} + \frac{\partial^4 \bar{w}_1}{\partial \xi^4}\right) + \bar{r}_7\frac{\partial^5 \bar{\theta}_1}{\partial \xi^5} - \bar{c}_1^2 \bar{H}_{11}\frac{\partial^6 \bar{w}_1}{\partial \xi^6}\right] \\ &-S_n \lambda_T^0\left(\frac{\partial^2 \bar{w}_1}{\partial \xi^2} - \bar{\mu}^2\frac{\partial^4 \bar{w}_1}{\partial \xi^4}\right) = 0 \end{aligned} \tag{24}$$

$O(\varepsilon^2)$

$$\begin{aligned} &\bar{r}_1(\bar{\theta}_2 + \frac{\partial \bar{w}_2}{\partial \xi}) + \bar{r}_2\frac{\partial^3 \bar{w}_2}{\partial \xi^3} + \bar{r}_3\frac{\partial^2 \bar{\theta}_2}{\partial \xi^2} + \pi \bar{r}_8\frac{\partial \bar{w}_1}{\partial \xi}\frac{\partial^2 \bar{w}_1}{\partial \xi^2} \\ &-\bar{l}^2\left[\bar{r}_1(\frac{\partial^2 \bar{\theta}_2}{\partial \xi^2} + \frac{\partial^3 \bar{w}_2}{\partial \xi^3}) + \bar{r}_2\frac{\partial^5 \bar{w}_2}{\partial \xi^5} + \bar{r}_3\frac{\partial^4 \bar{\theta}_2}{\partial \xi^4} - \pi \bar{r}_8\left(3\frac{\partial^2 \bar{w}_1}{\partial \xi^2}\frac{\partial^3 \bar{w}_1}{\partial \xi^3} + \frac{\partial \bar{w}_1}{\partial \xi}\frac{\partial^4 \bar{w}_1}{\partial \xi^4}\right)\right] = 0 \end{aligned} \tag{25}$$

$$\begin{aligned} &\bar{r}_6\left(\frac{\partial \bar{\theta}_2}{\partial \xi} + \frac{\partial^2 \bar{w}_2}{\partial \xi^2}\right) + \bar{r}_7\frac{\partial^3 \bar{\theta}_2}{\partial \xi^3} - \bar{c}_1^2 \bar{H}_{11}\frac{\partial^4 \bar{w}_2}{\partial \xi^4} + \pi \bar{c}_1 \bar{E}_{11}\left[\frac{\partial \bar{w}_1}{\partial \xi}\cdot\frac{\partial^3 \bar{w}_1}{\partial \xi^3} + \left(\frac{\partial^2 \bar{w}_1}{\partial \xi^2}\right)^2\right] \\ &-\bar{l}^2\left[\bar{r}_6\left(\frac{\partial^3 \bar{\theta}_2}{\partial \xi^3} + \frac{\partial^4 \bar{w}_2}{\partial \xi^4}\right) + \bar{r}_7\frac{\partial^5 \bar{\theta}_2}{\partial \xi^5} - \bar{c}_1^2 \bar{H}_{11}\frac{\partial^6 \bar{w}_2}{\partial \xi^6} + \pi \bar{c}_1 \bar{E}_{11}\left(3\frac{\partial^3 \bar{w}_1}{\partial \xi^3}\cdot\frac{\partial^3 \bar{w}_1}{\partial \xi^3} + 4\frac{\partial^2 \bar{w}_1}{\partial \xi^2}\cdot\frac{\partial^4 \bar{w}_1}{\partial \xi^4} + \frac{\partial \bar{w}_1}{\partial \xi}\cdot\frac{\partial^5 \bar{w}_1}{\partial \xi^5}\right)\right] \\ &+\int_0^\pi\left\{\bar{B}_{11}\frac{\partial \bar{\theta}_1}{\partial \xi} - \bar{c}_1 \bar{E}_{11}\left(\frac{\partial \bar{\theta}_1}{\partial \xi} + \frac{\partial^2 \bar{w}_1}{\partial \xi^2}\right) - \bar{l}^2\left[\bar{B}_{11}\frac{\partial^3 \bar{\theta}_1}{\partial \xi^3} - \bar{c}_1 \bar{E}_{11}\left(\frac{\partial^3 \bar{\theta}_1}{\partial \xi^3} + \frac{\partial^4 \bar{w}_1}{\partial \xi^4}\right)\right]\right\}\mathrm{d}\xi\left(\frac{\partial^2 \bar{w}_1}{\partial \xi^2} - \mu^2\frac{\partial^4 \bar{w}_1}{\partial \xi^4}\right) \\ &-S_n \lambda_T^1\left(\frac{\partial^2 \bar{w}_1}{\partial \xi^2} - \bar{\mu}^2\frac{\partial^4 \bar{w}_1}{\partial \xi^4}\right) = 0 \end{aligned} \tag{26}$$

$O(\varepsilon^3)$

$$\begin{aligned} &\bar{r}_1(\bar{\theta}_3 + \frac{\partial \bar{w}_3}{\partial \xi}) + \bar{r}_2\frac{\partial^3 \bar{w}_3}{\partial \xi^3} + \bar{r}_3\frac{\partial^2 \bar{\theta}_3}{\partial \xi^2} \\ &-\bar{l}^2\left[\bar{r}_1(\frac{\partial^2 \bar{\theta}_3}{\partial \xi^2} + \frac{\partial^3 \bar{w}_3}{\partial \xi^3}) + \bar{r}_2\frac{\partial^5 \bar{w}_3}{\partial \xi^5} + \bar{r}_3\frac{\partial^4 \bar{\theta}_3}{\partial \xi^4}\right] + \pi \bar{r}_8\frac{\partial \bar{w}_1}{\partial \xi}\frac{\partial^2 \bar{w}_2}{\partial \xi^2} \\ &+\pi \bar{r}_8\frac{\partial \bar{w}_2}{\partial \xi}\frac{\partial^2 \bar{w}_1}{\partial \xi^2} = 0 \end{aligned} \tag{27}$$

$$\begin{aligned} &\bar{r}_6\left(\frac{\partial \bar{\theta}_3}{\partial \xi} + \frac{\partial^2 \bar{w}_3}{\partial \xi^2}\right) + \bar{r}_7\frac{\partial^3 \bar{\theta}_3}{\partial \xi^3} - \bar{c}_1^2 \bar{H}_{11}\frac{\partial^4 \bar{w}_3}{\partial \xi^4} + \pi \bar{c}_1 \bar{E}_{11}\left[\frac{\partial \bar{w}_2}{\partial \xi}\cdot\frac{\partial^3 \bar{w}_1}{\partial \xi^3} + \frac{\partial \bar{w}_1}{\partial \xi}\cdot\frac{\partial^3 \bar{w}_2}{\partial \xi^3} + 2\frac{\partial^2 \bar{w}_1}{\partial \xi^2}\cdot\frac{\partial^2 \bar{w}_2}{\partial \xi^2}\right] \\ &-\bar{l}^2\left[\begin{aligned} &\pi \bar{c}_1 \bar{E}_{11}\left(6\frac{\partial^3 \bar{w}_2}{\partial \xi^3}\cdot\frac{\partial^3 \bar{w}_1}{\partial \xi^3} + 4\frac{\partial^2 \bar{w}_2}{\partial \xi^2}\cdot\frac{\partial^4 \bar{w}_1}{\partial \xi^4} + 4\frac{\partial^2 \bar{w}_1}{\partial \xi^2}\cdot\frac{\partial^4 \bar{w}_2}{\partial \xi^4} + \frac{\partial \bar{w}_1}{\partial \xi}\cdot\frac{\partial^5 \bar{w}_2}{\partial \xi^5} + \frac{\partial \bar{w}_2}{\partial \xi}\cdot\frac{\partial^5 \bar{w}_1}{\partial \xi^5}\right) \\ &+\bar{r}_6\left(\frac{\partial^3 \bar{\theta}_3}{\partial \xi^3} + \frac{\partial^4 \bar{w}_3}{\partial \xi^4}\right) + \bar{r}_7\frac{\partial^5 \bar{\theta}_3}{\partial \xi^5} - \bar{c}_1^2 \bar{H}_{11}\frac{\partial^6 \bar{w}_3}{\partial \xi^6} \end{aligned}\right] \\ &\int_0^\pi\left\{\bar{B}_{11}\frac{\partial \bar{\theta}_1}{\partial \xi} - \bar{c}_1 \bar{E}_{11}\left(\frac{\partial \bar{\theta}_1}{\partial \xi} + \frac{\partial^2 \bar{w}_1}{\partial \xi^2}\right) - \bar{l}^2\left[\bar{B}_{11}\frac{\partial^3 \bar{\theta}_1}{\partial \xi^3} - \bar{c}_1 \bar{E}_{11}\left(\frac{\partial^3 \bar{\theta}_1}{\partial \xi^3} + \frac{\partial^4 \bar{w}_1}{\partial \xi^4}\right)\right]\right\}\mathrm{d}\xi\left(\frac{\partial^2 \bar{w}_2}{\partial \xi^2} - \mu^2\frac{\partial^4 \bar{w}_2}{\partial \xi^4}\right) \\ &\int_0^\pi\left\{\begin{aligned} &\frac{\pi \bar{A}_{11}}{2}\left(\frac{\partial \bar{w}_1}{\partial \xi}\right)^2 + \bar{B}_{11}\frac{\partial \bar{\theta}_2}{\partial \xi} - \bar{c}_1 \bar{E}_{11}\left(\frac{\partial \bar{\theta}_2}{\partial \xi} + \frac{\partial^2 \bar{w}_2}{\partial \xi^2}\right) \\ &-\bar{l}^2\left[\pi \bar{A}_{11}\left(\frac{\partial \bar{w}_1}{\partial \xi}\frac{\partial^3 \bar{w}_1}{\partial \xi^3} + \frac{\partial^2 \bar{w}_1}{\partial \xi^2}\frac{\partial^2 \bar{w}_1}{\partial \xi^2}\right) + \bar{B}_{11}\frac{\partial^3 \bar{\theta}_2}{\partial \xi^3} - \bar{c}_1 \bar{E}_{11}\left(\frac{\partial^3 \bar{\theta}_2}{\partial \xi^3} + \frac{\partial^4 \bar{w}_2}{\partial \xi^4}\right)\right] \end{aligned}\right\}\mathrm{d}\xi\left(\frac{\partial^2 \bar{w}_1}{\partial \xi^2} - \mu^2\frac{\partial^4 \bar{w}_1}{\partial \xi^4}\right) \\ &-S_n \lambda_T^2\left(\frac{\partial^2 \bar{w}_1}{\partial \xi^2} - \bar{\mu}^2\frac{\partial^4 \bar{w}_1}{\partial \xi^4}\right) = 0 \end{aligned} \tag{28}$$

To resolve these perturbation equations as easily and reasonably as possible, we in this study propose a group of easier asymptotic solutions of dimensionless displacement and dimensionless rotation angle that can satisfy necessary boundary conditions.

$$\begin{aligned}\bar{w}(\xi) &= \varepsilon A_{10}^{1}\left[\cos(2m\xi)+1\right]+O(\varepsilon^{4});\\ \bar{\theta}(\xi) &= -\varepsilon B_{10}^{1}\sin(2m\xi)+O(\varepsilon^{4});\end{aligned} \tag{29}$$

We submit Eq. (29) into Eq. (23) obtaining

$$B_{10}^{1} = -\frac{2m\left(\bar{r}_1 - 4m^2\bar{r}_2\right)}{\bar{r}_1 - 4m^2\bar{r}_3}A_{10}^{1} \tag{30}$$

Then, the submission of Eq. (30) and Eq. (29) into Eq. (24) yields:

$$\lambda_T^0 = \frac{(1+4m^2\bar{l}^2)}{(1+4m^2\bar{\mu}^2)}\left[\bar{r}_6 + 4m^2\bar{c}_1^2\bar{H}_{11} - \frac{\left(\bar{r}_6 - 4m^2\bar{r}_7\right)\left(\bar{r}_1 - 4m^2\bar{r}_2\right)}{\bar{r}_1 - 4m^2\bar{r}_3}\right] \tag{31}$$

By submitting Eq. (29) into Eq. (26), λ_T^1 is obtained as

$$\lambda_T^1 = 0 \tag{32}$$

After substituting Eq. (29) and Eq. (30) into Eq. (28), one has

$$\lambda_T^2 = \frac{m^2\pi^2\bar{A}_{11}}{S_n}(A_{10}^1)^2 \tag{33}$$

Finally, the asymptotic solution of dimensionless temperature can be arrived at

$$\lambda_T = \lambda_T^0 + \lambda_T^1 + \lambda_T^2 + O(\varepsilon^4) \tag{34}$$

A_{10}^1 is another perturbation parameter that can be determined, submitting $\xi=0$ into the first expression of Eq. (29).

$$A_{10}^1 = \bar{W}_m = \frac{W_m}{2L} \tag{35}$$

Obviously, the perturbation parameter is closely connected with the dimensional maximum deflection W_m. Thus, the asymptotic solution of dimensionless temperature can be rewritten as

$$\begin{aligned}\lambda_T &= \frac{(1+4m^2\bar{l}^2)}{(1+4m^2\bar{\mu}^2)}\left[\bar{r}_6 + 4m^2\bar{c}_1^2\bar{H}_{11} - \frac{\left(\bar{r}_6 - 4m^2\bar{r}_7\right)\left(\bar{r}_1 - 4m^2\bar{r}_2\right)}{\bar{r}_1 - 4m^2\bar{r}_3}\right]\left(\frac{W_m}{2L}\right)^0\\ &+0\bullet\left(\frac{W_m}{2L}\right)^1 + \frac{m^2\pi^2\bar{A}_{11}}{S_n}\left(\frac{W_m}{2L}\right)^2 + \ldots\end{aligned} \tag{36}$$

Here, it should be pointed out that the result of λ_T^0 is equal to the critical buckling temperature of beams, partly because the buckling of beams always occurs in $W_m = 0$. Moreover, m is always equivalent to 1 in analysis of static mechanical behaviors (Shen *et al.* 2013).

4. Numerical results and discussions

The main objective in this section is to study the effect of respective physical parameters on the static thermal buckling of bi-directional FG beams, especially for the relation between both size-dependent effects.

Table 2 Comparisons of the critical thermal buckling temperature λ_T of Si_3N_4 /SUS304 beams with two clamped ends (L/h = 25)

L/h	Sources	*N*=1	*N*=2	*N*=3	*N*=4	*N*=5
20	She *et al.* (2017a)	710.491	655.616	633.376	619.886	610.086
	Present	710.519	655.648	633.409	619.92	610.12
	Differences	0.0039%	0.0049%	0.0052%	0.0055%	0.0056%
30	She *et al.* (2017a)	320.154	295.608	285.698	279.673	275.276
	Present	320.167	295.623	285.713	279.688	275.292
	Differences	0.0041%	0.0051%	0.0053%	0.0054%	0.0058%
40	She *et al.* (2017a)	180.966	167.128	161.548	158.153	155.672
	Present	180.973	167.136	161.557	158.162	155.681
	Differences	0.0039%	0.0048%	0.0056%	0.0057%	0.0058%

Table 3 Effect of both volume index N_1 and N_2 on critical thermal buckling temperature of bi-directional FG nanobeams. (μ = 1nm, l = 0.5nm, h = b = 1nm)

N_2	N_1	L=15h	L=20h	L=25h	L=30h	L=35h	L=40h	L=45h	L=50h
0	1	1083.37	652.052	431.287	305.053	226.652	174.812	138.826	112.86
	2	1013.54	610.721	404.169	285.958	212.504	163.919	130.186	105.842
	3	989.216	596.584	394.978	279.519	207.748	160.265	127.292	103.494
	4	974.428	587.959	389.359	275.579	204.836	158.027	125.519	102.055
	5	962.8	581.083	384.851	272.406	202.485	156.217	124.084	100.889
1	1	935.328	562.571	371.981	263.059	195.43	150.721	119.688	97.2984
	2	861.258	518.76	343.246	242.829	180.442	139.182	110.537	89.8653
	3	843.671	508.877	336.932	238.45	177.227	136.723	108.594	88.2924
	4	835.753	504.577	334.235	236.6	175.879	135.696	107.786	87.6396
	5	829.448	501.047	331.984	235.041	174.736	134.821	107.096	87.0809
2	1	889.002	534.517	353.372	249.876	185.626	143.154	113.677	92.4096
	2	813.455	489.812	324.044	229.225	170.325	131.374	104.333	84.8205
	3	799.332	482.091	319.183	225.883	167.885	129.514	102.868	83.6365
	4	795.3	480.225	318.127	225.206	167.413	129.166	102.601	83.4241
	5	792.261	478.744	317.258	224.635	167.009	128.864	102.366	83.2365
3	1	865.192	520.093	343.803	243.096	180.583	139.262	110.585	89.8951
	2	788.302	474.569	313.929	222.058	164.994	127.26	101.064	82.162
	3	775.852	467.893	309.771	219.218	162.929	125.69	99.8301	81.1661
	4	773.937	467.357	309.612	219.182	162.937	125.713	99.8584	81.1947
	5	772.764	467.051	309.537	219.179	162.957	125.74	99.886	81.2207

4.1 Validation and comparison

Before conducting the parametric studies, we should perform a validation research to check the accuracy and reliability of the present solutions for thermal buckling problems. Table 2 shows a comparison of the critical thermal buckling temperature of beams with two clamped ends, where the beam without the size-dependent effect is made of Si_3N_4/SUS304 functionally graded materials and L=25h. As shown in Table 2, the results obtained from the present solution are much more consistent with ones form She *et al.* (2017), indicating that the present analysis is reliable and reasonable.

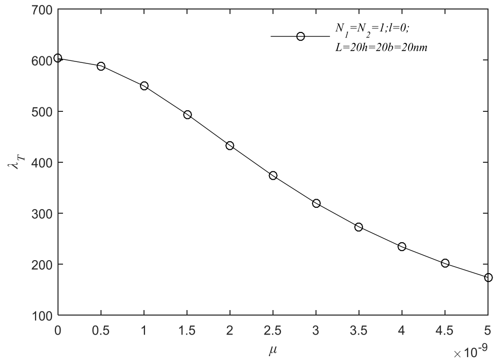

Fig. 2 Effect of the nonlocal parameter μ on critical thermal buckling temperature of beams with bi-directional functionally graded distribution

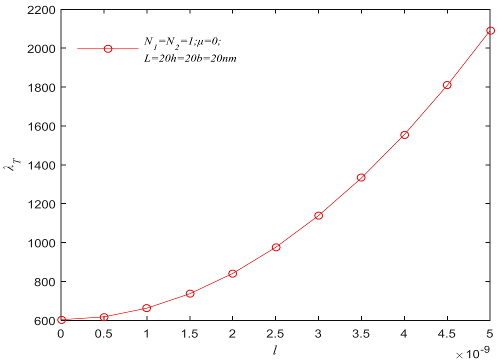

Fig. 3 Effect of the strain gradient parameter l on critical thermal temperature buckling with bi-directional functionally graded distribution

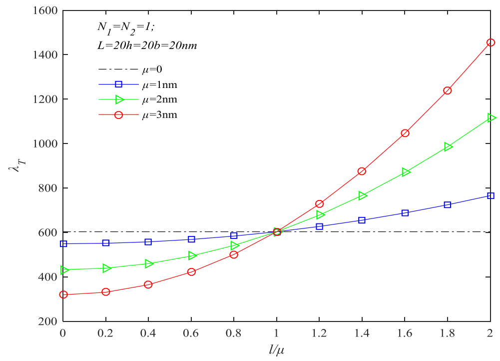

Fig. 4 Variation of the critical thermal buckling temperature relevant to l/μ for the nanobeam

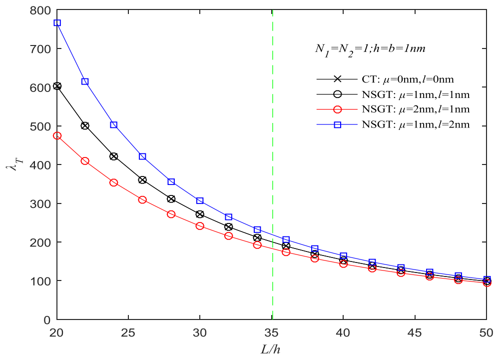

Fig. 5 Effect of the aspect ratio on critical thermal buckling temperature of beams using the nonlocal strain gradient theory

4.2 Parametric studies

Table 3 presents the effect of both volume indexes N_1 and N_2 on critical thermal buckling temperature of bi-directional FG nanobeams, where μ=1nm, l=0.5nm, h=b=1nm. It is indicated from the figure that the critical buckling temperature of beams are decreased greatly and continuously with the increment of N_1 and N_2. That is due to the fact that the increase of two volume indexes can improve portion of metal, SUS304 and Al, but reduce portion of ceramic, Al_2O_3 and Si_3N_4, and thus leading to a pronounced reduction in the overall stiffness of functionally graded beams. Furthermore, it also can be seen from the table that the critical buckling temperature of beams can be significantly reduced via the increase of aspect ratio, partly because the aspect ratio has a prominent influence on the overall stiffness of beam. To be specific, the effective stiffness of beams is weakened by the growing ratio of L/h.

Fig. 2 and Fig. 3 respectively exhibit the effects of the nonlocal parameter μ and the strain gradient parameter l on the critical thermal buckling temperature of beams. As shown in Fig. 2, the temperature is reduced continuously with the rise of nonlocal parameter μ, indicating that the effective stiffness of beams becomes smaller and smaller. The reason is that the nonlocal stress field plays a great influence on reducing the overall stiffness of beams. Thus, the size-dependent effect of nano-structures is interpreted as the effect of stiffness-softening via previous scholars. Unlike for the effect of the nonlocal parameter μ, the curve of Fig. 3 shows that the critical thermal buckling temperature is prominently improved with an increase in the value of the strain gradient parameter l, which indicates that the strain gradient parameter l is endowed with the larger value, the bigger the effective stiffness of beams will be. Thus, the strain gradient theory exerts a stiffness-hardening role in analyzing mechanical behaviors of nano-structures.

Fig. 4 describes the variation of the critical thermal buckling temperature with respect to the small-scale ratio of l/μ for bi-directional functionally graded beams. It can be seen from the figure that when l/μ<1, the critical thermal buckling temperature computed by the nonlocal strain gradient theory are all smaller than those computed by the classical elasticity theory, and the temperature becomes lower and lower with the rise of the nonlocal parameter μ at the same ratio of l/μ; when l/μ>1, the critical thermal buckling temperature computed by the nonlocal strain gradient theory are all bigger than those computed by the classical elasticity theory, and the temperature becomes higher and higher with the rise of the strain gradient parameter l at the same ratio of l/μ; when l/μ=1, the critical thermal buckling temperature obtained by the nonlocal strain gradient theory are eventually identical with ones obtained by the classical elasticity theory. The main reason is that the nonlocal effect is much more dominant than the microstructure effect at l/μ <1, and thus making that the

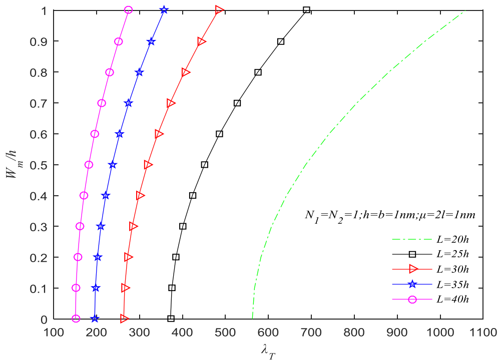

Fig. 6 The effect of aspect ratio on the thermal post-buckling load-deflection curves of beams

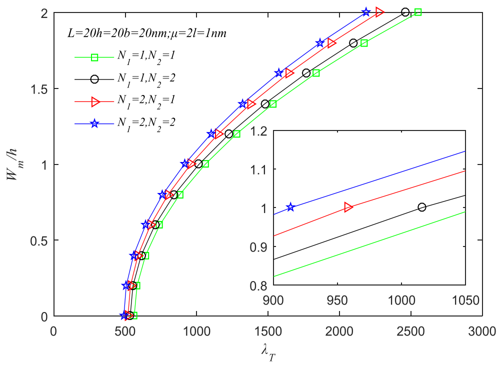

Fig. 7 The influence of two volume indexes relevant to the thermal post-buckling load-deflection of beams

nanostructures exert the effect of the stiffness-softening, but the microstructure effect is much more dominant than the nonlocal effect at $l/\mu>1$, and thus making that the nanostructures exert the effect of the stiffness-hardening. As for $l/\mu=1$, two kinds of size-dependent effect cancel each other out, as shown in the analytical solution Eq. (31), just like neither the stiffness-softening effect nor the stiffness-hardening effect can be captured via the classical elasticity theory.

Fig. 5 displays the effect of the aspect ratio on critical thermal buckling temperature of beams in the framework of the nonlocal strain gradient theory. As can be seen in the figure, both the nonlocal effect and the microstructure effect are gradually decreased with the aspect ratio of L/h becoming large. Furthermore, it should be mentioned out that the difference between two kinds of the size-dependent effect is insignificant after a specific aspect ratio. In other words, when $L>35h$, the primary influence on the effective stiffness of beams at the very small nano-scale is the slenderness ratio L/h rather than the size-dependent effect. From a physical standpoint, the reason is that the wavelength becomes large with an augment in the side length, thus weakening the size-dependent effects.

Fig. 6 shows the effect of aspect ratio L/h with respect to the thermal post-buckling load-deflection curves of beams. These curves from this figure indicates that an increment in the ratio L/h results in an overall reduction in the results of the critical thermal buckling temperature as well as thermal post-bucking strength. That is due to the fact that the effective stiffness of beams is reduced by the rise of the ratio of L/h. For every curve, the nonlinear thermal post-bucking temperature continuously varies along the maximum dimensionless buckling amplitude, quite differing from the linear thermal buckling temperature, the attribute of which is similar to enhancing spring behaviors.

Fig.7 presents the effect of two volume indexes N_1 and N_2 on the thermal post-buckling of bi-directional functionally graded beams. From the figure, we can know that the thermal post-buckling strength of bi-directional FG beams is declined gradually by the increase of N_1 and N_2, partly because the elastic modulus of bi-directional FG beams can be changed via the volume indexes, the result of which has a pronounced influence on the thermal post-buckling strength. Besides, it is clear that the effect of index N_1 on the thermal post-buckling response is more significant than the effect of index N_2. So, it is advised to choose both volume indexes to design those engineering structures subjected to the thermal environment so that the thermal post-buckling response with such design is much more flexible and controlled more accurately compared with that with one volume index.

5. Conclusions

Authors in this work study nonlinear thermal bucking of nanobeams subjected to two-directional functionally graded distributions in the framework of the nonlocal strain gradient theory. Based on the assumption and approximate mathematical model established by us, the effective material properties of the bi-directional functionally graded beams are defined, and then using it and the perturbation method, together, to investigate the critical thermal buckling and post-buckling mechanical behaviors of bi-directional functionally graded beams. Finally, several important conclusions are listed as follows.

- The nonlocal elasticity theory and the strain gradient theory have the opposite effect on the critical thermal buckling temperature of bi-directional FG nanobeams.
- Both the nonlocal effect and the microstructure effect on the critical thermal buckling temperature will be insignificant when the slenderness ratio $L/h>35$.
- The critical thermal buckling temperature of nanobeams predicted by the nonlocal strain gradient theory may be equal to that predicted by the classical elasticity theory, but also may be higher and lower, depending on the strain gradient parameter, the nonlocal parameter and the ratio of two small-scale parameters.
- Both the critical thermal buckling temperature and the post-buckling thermal strength are prominently influenced via the change of double volume indexes.
- The thermal buckling response of functionally graded beams with two volume indexes is much more flexible and controlled more accurately than that with one volume index.

The present results may be helpful to the design of 2D FGM in engineering applications. Furthermore, the method appearing in this article is also used to investigate other kinds of materials on the nano-meter length.

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