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The influence of the rheological parameters on the dispersion of the flexural waves in a viscoelastic bi-layered hollow cylinder

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Abstract. The paper investigates the influence of the rheological parameters which characterize the creep time, the long-term values of the mechanical properties of viscoelastic materials and a form of the creep function around the initial state of a deformation of the materials of the hollow bi-layered cylinder on the dispersion of the flexural waves propagated in this cylinder. Constitutive relations for the cylinder's materials are given through the fractional exponential operators by Rabotnov. The dispersive attenuation case is considered and numerical results related to the dispersion curves are presented and discussed for the first and second modes under the first harmonic in the circumferential direction. According to these results, it is established that the viscosity of the materials of the constituents causes a decrease in the flexural wave propagation velocity in the bi-layered cylinder under consideration. At the same time, the character of the influence of the rheological parameters, as well as other problem parameters such as the thickness-radius ratio and the elastic modulus ratio of the layers' materials on the dispersion curves, are established.

Keywords: flexural waves; rheological materials; viscoelastic material; wave dispersion; fractional-exponential operator; bi-layered hollow cylinder

1. Introduction

The correct and useful applications of guided waves for non-destructive testing of the structural elements made of viscoelastic materials (for instance, made of polymer materials) and which are used in many branches of modern industries require sufficient knowledge obtained from the corresponding theoretical investigations. At the same time, this knowledge can also be used under weakening of the amplitudes of the propagated waves caused by earthquakes, explosions and other similar types of wave sources. As follows from the papers by Benjamin *et al*. (2016), Yasar *et al*. (2013a), and Yasar *et al*. (2013b) the aforementioned knowledge can also be used under nondestructive monitoring of the growth of an engineered tissue which can be used for the corresponding medical goals. The theoretical investigations of wave dispersion in the elements of constructions made of viscoelastic materials are also required for correct application of ultrasonic guided wave (UGW) defect detection methods. Note that the present level of these methods is discussed in the papers by Lowe *et al*. (2015), Lowe *et al*. (2016) and others listed therein. However, as noted in these papers, application of the UGW for defect inspection has high sensitivity in the cases where attenuation of the waves is very low and therefore this application also requires the corresponding theoretical investigations on the wave dispersion propagated in the elements of constructions made of viscoelastic materials.

Consequently, it can be concluded that investigations of the rule of dispersion of the flexural waves propagating in the bi-layered circular hollow cylinder, which is the subject of the study of the present paper, have not only theoretical but also practical significance. For illustration of the significance and contribution of the results of the present investigations we consider a brief review of the corresponding studies.

We begin this review with the papers by Weiss (1959) and Tamm and Weiss (1961) which study the Lamb wave propagation in a viscoelastic layer in the case where the elastic constants are complex and frequency independent. Coquin (1964) also studies the same problem within the scope of the assumption that the plate material has small losses, and a frequency dependent complex elastic modulus is used. The Lamb wave propagation in a viscoelastic plate is also studied in the paper by Chervinko and Senchenkov (1986) in which it is assumed that the material of the plate is a low-compressible one and its Poisson's ratio is constant, and the frequency dependent complex modulus is considered. Moreover, Lamb wave propagation in the bi-layered viscoelastic + elastic plate is studied in the paper by Simonetti (2004), the results of which are also detailed in the monograph by Rose (2004). The Lamb wave propagation for the viscoelastic plate is also studied in the paper by Manconi and Sorokin (2013) in which the non-dispersive attenuation case is considered and numerical results are presented and discussed for illustration of the role of the viscoelasticity in the cut-off phenomenon and in

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the phenomenon of veering for dispersion curves.

The paper by Barshinger and Rose (2004) investigates axisymmetric longitudinal guided wave dispersion and attenuation in a metal elastic hollow cylinder coated with a polymer viscoelastic layer. Through attenuation coefficients of the bulk and shear waves in the selected viscoelastic material, which are determined experimentally for the frequencies in the order 1-5 MHz, the viscoelasticity of the coated material is taken into consideration. Consequently, in this paper the frequency independent complex modulus of elasticity is also used and within this assumption, the axisymmetric longitudinal wave dispersion and its attenuation dispersion are studied. Note that the aforementioned approach on the accounting of the viscoelasticity of the coatings of the pipes is also used in the papers by Kirby *et al*. (2012, 2013) under investigations of the corresponding scattering problem of the longitudinal and torsional waves.

Experimental studies of the attenuation of the torsional and longitudinal axisymmetric wave propagation in pipes buried in sand are made in the paper by Leonov *et al*. (2015) in which it is assumed that the attenuation appears as a result of the energy leakage into the embedded soil. Moreover, the paper by Jiangong (2011) studies the dispersion of the viscoelastic SH waves in functionally graded material and laminated plates within the scope of the Kelvin-Voigt model. This model is also used in the papers by Bartoli *et al*. (2006), Mace and Manconi (2008), Manconi and Mace (2009), Mazotti *et al*. (2012), and Hernando Quintanilla *et al*. (2015) for investigation of wave dispersion and propagation in plates, roads and pipes made of viscoelastic materials. Note that in these works alongside the Kelvin-Voigt model, the hysteretic model, i.e. the model based on the frequency independent complex modulus, is also examined. Moreover, note that in these works the main focus is on the development of the numerical solution method based on discretization with finite elements. For instance, in the papers by Barotti *et al*. (2006), Mazotti *et al*. (2012) and others listed therein, the semi-analytical finite element method is developed and employed, according to which, the sought values are presented with multiplying $\exp i(kz-\omega t)$ and in this way the dimension of the problem with respect to the spatial coordinates is reduced. The other numerical method which is employed in these papers is the wave finite element method (see, the papers by Mace and Manconi (2008), Manconi and Mace (2009) etc.) which is based on the presentation of the sought values with multiplying $\exp i\omega t$ and, for obtaining the dispersion equations, the periodicity relations between the nodal displacements in the wave propagation direction are used. Here, we also note the spectral collocation method which is employed in the paper by Hernando Quintanilla *et al*. (2015) for investigation of the dispersion of the guided waves in an anisotropic viscoelastic layered medium, the viscoelasticity of which is modelled through the Kelvin-Voigt model.

In the papers by Meral *et al*. (2009, 2010) an attempt is made for the use of the very real and complicated model under investigation of the wave propagation and attenuation problems related to the viscoelastic materials. For such a model, the fractional order Voigt (or Kelvin-Voigt) model is selected for investigation of the corresponding 2D problems. Note that this model is obtained from the classical Voigt model by replacing the derivative $\partial/\partial t$, with respect to time, with the fractional order derivative $\partial^{\alpha}/\partial t^{\alpha}$ in the Weyl sense. Here, α is a new rheological parameter through which the description of the related experimental data is improved. Moreover, in the paper by Meral *et al*. (2010), by employing the aforementioned fractional order Voigt model, the Lamb wave dispersion and attenuation are studied theoretically and verified experimentally for a tissue mimicking phantom material.

This completes the review of the investigations on the dispersion of guided waves in the plates or cylinders made from viscoelastic materials which are carried out mainly within the scope of the following assumptions: i) the complex modulus of viscoelastic materials is taken as frequency independent (the hysteretic model); ii) the viscoelasticity of the materials is described by the simplest models such as the classical Kelvin-Voigt or simplest fractional Kelvin-Voigt models; and iii) the expression for the complex elasticity modulus is obtained experimentally for concrete polymer materials. It is evident that for a more real and sufficiently accurate description of the character of the influence of the material's viscoelasticity on the wave dispersion and attenuation propagated in the structural elements made of this material it is necessary to use the more complicated models for the corresponding constitutive relations. As an example of such a model, the fractional-exponential operator by Rabotnov (1980) can be taken which has many advantages, one of which is the describing, with the very high accuracy required, of the initial parts of the experimentally constructed creep and relaxation functions and their asymptotic values. The other advantage of this operator is having many simple rules for complicated mathematical transformations, for example, the Fourier and Laplace transformations. At the same time, the fractional exponential operator by Rabotnov can be employed successfully to describe the viscoelasticity of various polymer materials and epoxy-based composites with continuous fibers and layers.

One of the first attempts on the application the fractional-exponential operator by Rabotnov on the wave dispersion in the viscoelastic medium was made by Meshkov and Rossikhin (1968) in which the characteristics of acoustic waves propagating in an infinite viscoelastic medium is studied. More detail review of the related investigations was made in the paper by Rossikhin and Shitikova (1997). Note that in all these works the shear, longitudinal and Rayleigh waves in the viscoelastic medium are investigated. The dispersion the mentioned waves in the infinite viscoelastic medium the viscoelasticity properties of which is described by the fractional operator is studied in the paper by Usuki (2013). However, in this paper as the operator for differentiation of fractional order, the Caputo derivative is used. The discussions of the application of the fractional operators in mechanics of solids is made in the paper by Rossikhin (2010).

Apparently, the first attempt on the application of the fractional exponential operators for investigations of the guided wave dispersion propagating in elements of constructions made of viscoelastic materials was made in the paper by Akbarov and Kepceler (2015) for investigation of the torsional wave dispersion in the sandwich hollow cylinder made of the viscoelastic material.

At the same time, in the paper by Akbarov (2014) the Rabotnov's operator was employed under investigations of the forced vibration of the "viscoelastic layer + viscoelastic half-space" system. Note that the results obtained in the papers by Akbarov (2014) and Akbarov and Kepceler (2015) are also listed and detailed in the monograph by Akbarov (2015).

The further investigations on the wave dispersion and attenuation propagated in the layered cylinders made of viscoelastic materials with employing the aforementioned fractional-exponential operator are developed in the papers by Akbarov *et al*. (2016a, 2016b) and Kocal and Akbarov (2017) in which the axisymmetric longitudinal wave dispersion in the solid bi-layered (Akbarov *et al*. (2016a)) and in the hollow bi-layered (Akbarov *et al*. (2016b), Kocal and Akbarov (2017)) circular cylinders is studied. Note that in the investigations carried out in the papers by Akbarov *et al*. (2016a, 2016b), as in the paper by Akbarov and Kepceler (2015), it is assumed that the attenuation of the considered waves is given a priori. However, in the paper by Kocal and Akbarov (2017) the attenuation of the waves is determined under given possible dispersion curves.

These are the only studies in this field employing the fractional-exponential operator. Taking the significance and usefulness in the theoretical and application senses of the related results, in the present paper we continue the aforementioned investigation for flexural waves propagated in the bi-layered hollow cylinder made of viscoelastic material. Numerical results are presented and discussed for the first and second modes under the first harmonic in the circumferential direction of the cylinder. As in the papers by Akbarov *et al*. (2016a, 2016b), it is assumed that the attenuation of the wave under consideration is given a priori.

2. Formulation of the problem

We consider the hollow compound cylinder, the sketch of which is shown in Fig. 1 and assume that the radius of the cross section of the interface cylindrical surface between the cylinders is R. The thickness of the outer and inner hollow cylinders we denote through $h^{(1)}$ and $h^{(2)}$, respectively. We associate the cylindrical system of coordinates $Or\theta z$ (Fig. 1) with the central axis of the cylinder. The values related to the inner and outer hollow cylinders will be denoted by the upper indices (2) and (1), respectively. We assume that the materials of the constituents are isotropic, homogeneous and hereditary-viscoelastic. Moreover, we assume that the cylinders have infinite length in the direction of the Oz axis.

Thus, let us investigate the flexural wave propagation along the Oz axis in the considered compound cylinder with the use of the equations of motion of the linear theory for viscoelastic bodies. For this purpose we write the complete system of field equations of this theory in the cylindrical coordinate system $Or\theta z$.

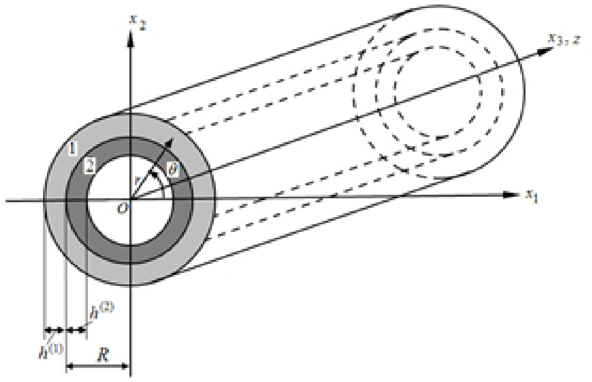

Fig. 1 The sketch of the bi-layered hollow cylinder

Equations of motion:

$$\frac{\partial T_{rr}^{(n)}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta r}^{(n)}}{\partial \theta} + \frac{\partial T_{zr}^{(n)}}{\partial z} + \frac{1}{r}(T_{rr}^{(n)} - T_{\theta\theta}^{(n)}) = \rho^{(n)}\frac{\partial^2 u_r^{(n)}}{\partial t^2},$$

$$\frac{\partial T_{r\theta}^{(n)}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}^{(n)}}{\partial \theta} + \frac{\partial T_{z\theta}^{(n)}}{\partial z} + \frac{2}{r}T_{r\theta}^{(n)} = \rho^{(n)}\frac{\partial^2 u_\theta^{(n)}}{\partial t^2}, \tag{1}$$

$$\frac{\partial T_{rz}^{(n)}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta z}^{(n)}}{\partial \theta} + \frac{\partial T_{zz}^{(n)}}{\partial z} + \frac{1}{r}T_{rz}^{(n)} = \rho^{(n)}\frac{\partial^2 u_z^{(n)}}{\partial t^2}.$$

Constitutive relations

$$T_{(ii)}^{(n)} = \lambda^{(n)*}\varepsilon^{(n)} + 2\mu^{(n)*}\varepsilon_{(ii)}^{(n)}, \quad (ii) = rr; \theta\theta; zz,$$

$$T_{r\theta}^{(n)} = 2\mu^{(n)*}\varepsilon_{r\theta}^{(n)},$$

$$T_{rz}^{(n)} = 2\mu^{(n)*}\varepsilon_{rz}^{(n)}, \quad T_{z\theta}^{(n)} = 2\mu^{(n)*}\varepsilon_{z\theta}^{(n)}, \tag{2}$$

$$\varepsilon^{(n)} = \varepsilon_{rr}^{(n)} + \varepsilon_{\theta\theta}^{(n)} + \varepsilon_{zz}^{(n)}$$

where $\lambda^{(n)*}$ and $\mu^{(n)*}$ are the following viscoelastic operators

$$\begin{Bmatrix} \lambda^{(n)*} \\ \mu^{(n)*} \end{Bmatrix}\varphi(t) = \begin{Bmatrix} \lambda_0^{(n)} \\ \mu_0^{(n)} \end{Bmatrix}\varphi(t) + \int_0^t \begin{Bmatrix} \lambda_1^{(n)} \\ \mu_1^{(n)} \end{Bmatrix}(t-\tau)\varphi(\tau)d\tau \tag{3}$$

In Eq. (3), $\lambda_0^{(n)}$ and $\mu_0^{(n)}$ are instantaneous values of Lame's constants as $t \to 0$, and $\lambda_1^{(n)}(t)$ and $\mu_1^{(n)}(t)$ are the corresponding kernel functions describing the hereditary properties of the materials of the cylinder's layers. Moreover, in Eqs. (1)-(3) the case where $n = 2$ ($n = 1$) relates to the inner layer (outer layer). This notation is also used below throughout the paper.

Strain-displacements relations:

$$\varepsilon_{rr}^{(n)} = \frac{\partial u_r^{(n)}}{\partial r}, \quad \varepsilon_{rr}^{(n)} = \frac{1}{r}(\frac{\partial u_\theta^{(n)}}{\partial \theta} + u_r^{(n)}), \quad \varepsilon_{zz}^{(n)} = \frac{\partial u_z^{(n)}}{\partial z},$$

$$\varepsilon_{r\theta}^{(n)} = \frac{1}{2}\left(\frac{\partial u_\theta^{(n)}}{\partial r} + \frac{1}{r}\frac{\partial u_r^{(n)}}{\partial \theta} - \frac{1}{r}u_\theta^{(n)}\right), \tag{4}$$

$$\varepsilon_{z\theta}^{(n)} = \frac{1}{2}\left(\frac{\partial u_\theta^{(n)}}{\partial z} + \frac{\partial u_z^{(n)}}{r\partial \theta}\right), \quad \varepsilon_{rz}^{(n)} = \frac{1}{2}\left(\frac{\partial u_r^{(n)}}{\partial z} + \frac{\partial u_z^{(n)}}{\partial r}\right),$$

In (1)-(4) the conventional notation is used.

According to expressions in (2) the "standard linear solid body" model is used for formulation of the constitutive relations.

Now we formulate the boundary and contact conditions which, according to Fig. 1, can be written as follows

$$
\begin{gathered}
T_{rr}^{(1)}\Big|_{r=R+h^{(1)}} = 0,\ T_{r\theta}^{(1)}\Big|_{r=R+h^{(1)}} = 0, \\
T_{rz}^{(1)}\Big|_{r=R+h^{(1)}} = 0\ ,\ T_{rr}^{(1)}\Big|_{r=R} = T_{rr}^{(2)}\Big|_{r=R}\ , \\
T_{r\theta}^{(1)}\Big|_{r=R} = T_{r\theta}^{(2)}\Big|_{r=R}\ ,\ T_{rz}^{(1)}\Big|_{r=R} = T_{rz}^{(2)}\Big|_{r=R}\ , \\
u_{r}^{(1)}\Big|_{r=R} = u_{r}^{(2)}\Big|_{r=R}\ ,\ u_{\theta}^{(1)}\Big|_{r=R} = u_{\theta}^{(2)}\Big|_{r=R}\ , \\
u_{z}^{(1)}\Big|_{r=R} = u_{z}^{(2)}\Big|_{r=R}\ , \\
T_{rr}^{(2)}\Big|_{r=R-h^{(2)}} = 0,\quad T_{r\theta}^{(2)}\Big|_{r=R-h^{(2)}} = 0,\ T_{rz}^{(2)}\Big|_{r=R-h^{(2)}} = 0
\end{gathered} \tag{5}
$$

This completes the formulation of the problem on the flexural wave dispersion in the bi-layered hollow cylinder made of viscoelastic materials with arbitrary kernel functions $\lambda_1^{(n)}(t)$ and $\mu_1^{(n)}(t)$ through which the constitutive relations in (2) and (3) are written.

3. Method of solution

As we consider the flexural wave dispersion which propagates in the direction of the *Oz* axis, we can represent the displacements and strains through multiplying $e^{i(kz-\omega t)}$ where k is the wave number and ω is the circular frequency. In other words, we can write the following representations.

$$
\begin{gathered}
u_{(i)}^{(n)}(r,\theta,z,t) = v_{(i)}^{(n)}(r,\theta)e^{i(kz-\omega t)}, (i) = r;\theta;z, \\
\varepsilon_{r\theta}^{(n)}(r,\theta,z,t) = \gamma_{r\theta}^{(n)}(r,\theta)e^{i(kz-\omega t)}, \\
\varepsilon_{rz}^{(n)}(r,\theta,z,t) = \gamma_{rz}^{(n)}(r,\theta)e^{i(kz-\omega t)}, \\
\varepsilon_{z\theta}^{(n)}(r,\theta,z,t) = \gamma_{z\theta}^{(n)}(r,\theta)e^{i(kz-\omega t)}, \\
\varepsilon_{(ii)}^{(n)}(r,\theta,z,t) = \gamma_{(ii)}^{(n)}(r,\theta)e^{i(kz-\omega t)},\ (ii) = rr;\theta\theta;zz, \\
\varepsilon^{(n)} = s^{(n)}e^{i(kz-\omega t)}
\end{gathered} \tag{6}
$$

In (6) the following notation is used.

$$
\begin{gathered}
\gamma_{rr}^{(n)} = \frac{\partial v_r^{(n)}}{\partial r},\ \gamma_{rr}^{(n)} = \frac{1}{r}\left(\frac{\partial v_\theta^{(n)}}{\partial \theta} + v_r^{(n)}\right),\ \gamma_{zz}^{(n)} = \frac{\partial v_z^{(n)}}{\partial z}, \\
\gamma_{r\theta}^{(n)} = \frac{1}{2}\left(\frac{\partial v_\theta^{(n)}}{\partial r} + \frac{1}{r}\frac{\partial v_r^{(n)}}{\partial \theta} - \frac{1}{r}v_\theta^{(n)}\right), \\
\gamma_{z\theta}^{(n)} = \frac{1}{2}\left(\frac{\partial v_\theta^{(n)}}{\partial z} + \frac{\partial v_z^{(n)}}{r\partial \theta}\right),\quad \gamma_{rz}^{(n)} = \frac{1}{2}\left(\frac{\partial v_r^{(n)}}{\partial z} + \frac{\partial v_z^{(n)}}{\partial r}\right),
\end{gathered} \tag{7}
$$

Taking into account the approximate equality

$$
\int_0^t f_1(t-\tau)f_2(\tau)d\tau \approx \int_{-\infty}^t f_1(t-\tau)f_2(\tau)d\tau \tag{8}
$$

in the mechanical relations (2) and (3), the meaning of which is described in Appendix A, and doing related mathematical manipulations as made, for instance, in the papers by Akbarov (2014) and Akbarov and Kepceler (2015), the following representations for the stresses are obtained

$$
\begin{gathered}
T_{(ii)}^{(n)} = \left(\Lambda^{(n)}s^{(n)} + 2M^{(n)}\gamma_{(ii)}^{(n)}\right)e^{i(kz-\omega t)} = \sigma_{(ii)}^{(n)}e^{i(kz-\omega t)} \\
(ii) = rr;\theta\theta;zz, \\
T_{r\theta}^{(n)} = 2M^{(n)}\gamma_{r\theta}^{(n)}e^{i(kz-\omega t)} = \sigma_{r\theta}^{(n)}e^{i(kz-\omega t)}, \\
T_{rz}^{(n)} = 2M^{(n)}\gamma_{rz}^{(n)}e^{i(kz-\omega t)} = \sigma_{rz}^{(n)}e^{i(kz-\omega t)}, \\
T_{z\theta}^{(n)} = 2M^{(n)}\gamma_{z\theta}^{(n)}e^{i(kz-\omega t)} = \sigma_{z\theta}^{(n)}e^{i(kz-\omega t)},
\end{gathered} \tag{9}
$$

where

$$
\begin{gathered}
\Lambda^{(n)} = \lambda_0^{(n)} + \lambda_{1c}^{(n)} + i\lambda_{1s}^{(n)}, \\
M^{(n)} = \mu_0^{(n)} + \mu_{1c}^{(n)} + i\mu_{1s}^{(n)},
\end{gathered} \tag{10}
$$

$$
\begin{gathered}
\begin{Bmatrix} \lambda_{1c}^{(n)} \\ \mu_{1c}^{(n)} \end{Bmatrix} = \int_0^\infty \begin{Bmatrix} \lambda_1^{(n)} \\ \mu_1^{(n)} \end{Bmatrix}(\xi)\cos(\omega\xi)d\xi, \\
\begin{Bmatrix} \lambda_{1s}^{(n)} \\ \mu_{1s}^{(n)} \end{Bmatrix} = \int_0^\infty \begin{Bmatrix} \lambda_1^{(n)} \\ \mu_1^{(n)} \end{Bmatrix}(\xi)\sin(\omega\xi)d\xi
\end{gathered} \tag{11}
$$

Thus, according to relations in (9)-(11), it can be concluded that for the case under consideration (i.e. for the study-state wave propagation problems) the complete system of field equations for the viscoelastic medium can be obtained from the related complete system of field equations for the corresponding elastic medium through replacing of the elastic moduli with corresponding complex elastic moduli. We recall that this statement is called the dynamic correspondence principle (see, for instance, Fung (1965)), which is also proved and used in the present investigations.

Thus, as a result of the representations given in Eqs. (6)-(9), we obtain the equations for the amplitude of the displacements from equations of motion in (1) and, according to the representation by Guz (1999, 2004), the solution of these equations can be represented as follows

$$
\begin{gathered}
v_r^{(n)} = \frac{1}{r}\frac{\partial}{\partial \theta}\Psi^{(n)} - ik\frac{\partial}{\partial r}X^{(n)}, \\
v_\theta^{(n)} = -\frac{\partial}{\partial r}\Psi^{(n)} - ik\frac{1}{r}\frac{\partial^2}{\partial \theta}X^{(n)}, \\
v_z^{(n)} = \frac{\left((\Lambda^{(n)} + 2M^{(n)})\Delta_1 - k^2M^{(n)} + \omega^2\rho^{(n)}\right)}{(\Lambda^{(n)} + M^{(n)})}X^{(n)}, \\
\Delta_1 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},\quad n = 1,2
\end{gathered} \tag{12}
$$

where the functions $\psi^{(n)}$ and $X^{(n)}$ in (12) are the solutions of the equations

$$\left(\varDelta_1 - k^2 + \omega^2 \frac{\rho^{(n)}}{\mu^{(n)}}\right)\varPsi^{(n)} = 0,$$

$$\left[\left(\varDelta_1 - k^2\right)\left(\varDelta_1 - k^2\right) + \rho^{(n)} \frac{\varLambda^{(n)} + 3M^{(n)}}{M^{(n)}(\varLambda^{(n)} + 2M^{(n)})}(\varDelta_1 - k^2)\omega^2 + \frac{(\rho^{(n)})^2\omega^4}{\mathrm{M}^{(n)}(\Lambda^{(n)} + 2\mathrm{M}^{(n)})}\right]X^{(n)} = 0 \tag{13}$$

Note that first, the decompositions (12) and (13) were proposed in the paper by Guz (1970) and developed and applied under investigations of numerous concrete problems detailed in the monographs by Guz (1999, 2004) and in many others listed therein.

Instead of Guz's foregoing decomposition (12) and (13) it can also be used the well-known, classical Lame (or Helmholtz) decompositions described, for instance, in the monograph by Eringen and Suhubi (1975). It must be recalled that the classical Lame (or Helmholtz) decomposition is given for the cases where the material of the considered body is isotropic. These decompositions contains one scalar and one vector potential divergence of which is equal to zero and each of them satisfies the well-known Helmholtz equation. Moreover, an additional equation for the components of the vector potential is obtained by equating to zero the divergence of this vector potential. For instance, if the materials of the cylinders is transversally-isotropic with the *Oz* symmetry axis, then the Lame decomposition is not applicable. Namely, for such cases and for the cases where the cylinders have homogeneous initial stresses such as $\sigma_{zz}^{0(n)} = const$ (here the upper index 0 denotes that these quantities regard the initial state), then as in the monographs by Guz (1999, 2004) and other related works listed therein, the corresponding decompositions, the simplest expressions of which for the isotropic materials under absent of the aforementioned initial stress, are expressions (12) and (13), are proposed for the solution to the equations of the three-dimensional linearized theory of wave propagation in transversal isotropic elastic bodies with initial stresses. Consequently, the Guz's decompositions, the expressions of which for the cases under considerations, are (12) and (13), are more general and available, and as it is planned by the authors to continue the present investigations in future to the cases where the materials of the cylinders are transversely-isotropic and to the cases where the cylinders have homogeneous initial stresses, and in order to use the unified decomposition in all the listed above cases, in the present investigation, it is used the decompositions (12) and (13). Note that the Gus's decompositions have also been used under investigations of many dynamical problems, examples of which can be found in the monograph by Guz (2004), Akbarov (2015) and others listed therein.

Thus, we turn to the solution procedure and for the case under consideration, we select the functions $\psi^{(n)}$ and $X^{(n)}$ as follows

$$\varPsi_p^{(n)} = \psi_p^{(n)}(r)\sin p\theta, \quad X_p^{(n)} = \chi_p^{(p)}(r)\cos p\theta, \tag{14}$$

where p indicates the order of the harmonic of the flexural waves in the circumferential direction in the cylinder.

Substituting the expressions (14) into equations (13) we obtain

$$\left(\Delta_{1p} + (\zeta_1^{(n)})^2\right)\psi_p^{(n)} = 0,$$
$$\left(\Delta_{1p} + (\zeta_2^{(n)})^2\right)\left(\Delta_{1p} + (\zeta_3^{(n)})^2\right)\chi_p^{(n)} = 0, \tag{15}$$
$$\Delta_{1p} = \frac{d^2}{dr^2} + \frac{d}{rdr} - \frac{p^2}{r^2},$$

where

$$(\zeta_1^{(n)})^2 = k^2\left(\frac{\rho^{(n)}}{M^{(n)}}\left(\frac{\omega}{k}\right)^2 - 1\right) \tag{16}$$

but $(\zeta_2^{(n)})^2$ and $(\zeta_3^{(n)})^2$ in equation (15) are determined as solutions of the equation

$$M^{(n)}(\zeta^{(n)})^4 - k^2(\zeta^{(n)})^2\left[\rho^{(n)}\left(\frac{\omega}{k}\right)^2 - (\varLambda^{(n)} + 2M^{(n)}) + \frac{M^{(n)}}{\varLambda^{(n)} + 2M^{(n)}}\left(\rho^{(n)}\left(\frac{\omega}{k}\right)^2 - M^{(n)}\right) + \frac{(\varLambda^{(n)} + M^{(n)})^2}{\varLambda^{(m)} + 2M^{(m)}}\right] + k^4\left(\frac{\rho^{(m)}}{\varLambda^{(m)} + 2M^{(m)}}\left(\frac{\omega}{k}\right)^2 - 1\right)\left(\rho^{(m)}\left(\frac{\omega}{k}\right)^2 - M^{(m)}\right) = 0 \tag{17}$$

Note that the ratio *ω/k* in (16) and (17) is the complex phase velocity of the wave propagation.

Thus, we find the following expressions for the functions $\varPsi_p^{(n)}$ and $\chi_p^{(n)}$ from Eqs. (15)-(17).

$$\psi_p^{(n)} = A_{1p}^{(n)}J_p(\zeta_1^{(n)}kr) + B_{1p}^{(n)}Y_p(\zeta_1^{(n)}kr), \quad n = 1,2,$$
$$\chi_p^{(n)} = A_{2p}^{(n)}J_p(\zeta_2^{(n)}kr) + A_{3p}^{(n)}J_p(\zeta_3^{(n)}kr) + B_{2p}^{(n)}Y_p(\zeta_2^{(n)}kr) + B_{3p}^{(n)}Y_p(\zeta_3^{(n)}kr), \tag{18}$$

where $A_{1p}^{(n)}$, $A_{2p}^{(n)}$, $A_{3p}^{(n)}$, $B_{1p}^{(n)}$, $B_{2p}^{(n)}$ and $B_{3p}^{(n)}$ are unknown constants, and $J_p(x)$ and $Y_p(x)$ are Bessel functions of the first and second kind of the *p-th* order.

Substituting the expressions given in (18) into the expressions given in (14) and using the relations (12), (9) and (7) we obtain explicit analytical expressions for the amplitude of the stresses and displacements which contain the unknown constants $A_{1p}^{(n)}$, $A_{2p}^{(n)}$, $A_{3p}^{(n)}$, $B_{1p}^{(n)}$, $B_{2p}^{(n)}$ and $B_{3p}^{(n)}$. After this determination, we satisfy the boundary and contact conditions in (5) rewritten for the amplitudes for the corresponding quantities and, according to the usual procedure, we obtain the dispersion equation

$$\det\|\beta_{lm}(p)\| = 0, \; l; m = 1, 2, ..., 12. \tag{19}$$

The explicit expressions of the components of the matrix $(\beta_{lm}(p))$ are given in Appendix B through expressions in (B1) and (B2). In the case where the viscosity of the materials of the constituents of the cylinder is completely absent, i.e. in the case where $\lambda_{1c}^{(n)} = \lambda_{1s}^{(n)} = \mu_{1c}^{(n)} = \mu_{1s}^{(n)} = 0$ in equation (10), the foregoing dispersion equation coincides with the corresponding one given in the monograph by Akbarov (2015).

Note in the case where $p = 0$, the equation in (19) can be presented as follows

$$\det\|\beta_{lm}(0)\| = \det\left\|\beta_{l_1 m_1}^{T}\right\| \det\left\|\beta_{l_2 m_2}^{L}\right\| = 0, \quad l_1; m_1 = 1, 2, 3, 4, \quad l_2; m_2 = 1, 2, ..., 8. \tag{20}$$

According to equation (20), under $p = 0$ we obtain two types of dispersion equation:

$$\det\left\|\beta_{l_1 m_1}^{T}\right\| = 0 \tag{21}$$

$$\det\left\|\beta_{l_2 m_2}^{L}\right\| = 0 \tag{22}$$

Note that equation (21) corresponds to the dispersion of the axisymmetric torsional waves propagated in the compound cylinder under consideration of the related problems, which is studied in the paper by Akbarov and Kepceler (2015). However, equation (22) relates to the dispersion of the axisymmetric longitudinal waves propagated in the bi-layered hollow cylinder, which is studied in the papers by Akbarov *et al*. (2016b) and Kocal and Akbarov (2017).

For the cases where $p \geq 1$, equation (19) is the dispersion equation of the flexural waves of the *p-th* harmonic and in these cases determinant in (19) cannot be presented as a product of two corresponding determinants.

4. Numerical results and discussions

4.1 Selection of the viscoelastic operators and dimensionless rheological parameters

According to the well-known consideration, under studying the wave propagation in a viscoelastic medium the wave-number k is selected as a complex one which can be presented as follows.

$$k = k_1 + ik_2 = k_1(1+\beta), \quad \beta = \frac{k_2}{k_1}. \tag{23}$$

Here k_2, which is an imaginary part of the complex wave number k, defines the attenuation of the wave amplitude with the propagating distance and, as usual, β is called the coefficient of attenuation. Under phase velocity of the wave propagation, we will understand the following ratio

$$c = \frac{\omega}{k_1} \tag{24}$$

We also introduce the notation $c_2^{(n)} = \sqrt{\mu_0^{(n)} / \rho^{(n)}}$. Thus, according to the foregoing discussions, the expression of the dispersion equation in (19) contains the parameters

$$\frac{c}{c_2^{(2)}}, \quad \mu_0^{(2)} / \mu_0^{(1)}, \quad \rho^{(2)} / \rho^{(1)}, \quad k_1 R, \quad \frac{h^{(1)}}{R} \text{ and } \frac{h^{(2)}}{R} \tag{25}$$

Apart from these parameters, the expression of the dispersion equation (19) contains the parameters $\lambda_{1c}^{(n)}$, $\lambda_{1s}^{(n)}$, $\mu_{1c}^{(n)}$ and $\mu_{1s}^{(n)}$ which are determined through the formula in equation (11) by the kernel functions $\mu_1^{(n)}(t)$ and $\lambda_1^{(n)}(t)$ which enter into the operators in (3). Note that, these operators determine the viscoelastic properties of the materials of the constituents of the cylinder, and for determination of the quantities of the parameters $\lambda_{1c}^{(n)}$, $\lambda_{1s}^{(n)}$, $\mu_{1c}^{(n)}$ and $\mu_{1s}^{(n)}$ it is necessary to give the explicit expressions for the functions $\mu_1^{(n)}(t)$ and $\lambda_1^{(n)}(t)$. For this purpose, we use the fractional-exponential operators by Rabotnov (1980) related to the case where the volumetric expansion of the related material is purely elastic, i.e. to the case where

$$(\lambda^{(n)*} + \frac{2}{3}\mu^{(n)*})\varphi(t) = (\lambda_0^{(n)} + \frac{2}{3}\mu_0^{(n)})\varphi(t)$$

Thus, we select Young's operator as

$$E^{(n)*}\varphi(t) = \mu_0^{(n)}\left[\varphi(t) - \beta_0^{(n)} \Pi_{\alpha^{(n)}}^{(n)*}\left(\beta_0^{(n)} - \beta_\infty^{(n)}\right)\varphi(t)\right]$$

and taking the above assumption on the volumetric expansion into consideration and employing the algebra of the fractional-exponential operators by Rabotnov, it is determined the Lame operators (see, Rabotnov (1980), Rossikhin and Shitikova (2015), Akbarov (2015) and others listed therein). Here we present only the expression for the fractional-exponential operator $\mu^{(n)*}$ which is determined as follows

$$\mu^{(n)*}\varphi(t) = \mu_0^{(n)}\left[\varphi(t) - \frac{3\beta_0^{(n)}}{2(1+\nu_0^{(n)})} \Pi_{\alpha^{(n)}}^{(n)*}\left(\frac{3\beta_0^{(n)}}{2(1+\nu_0^{(n)})} - \beta_\infty^{(n)}\right)\varphi(t)\right] \tag{26}$$

where

$$\Pi_{\alpha^{(n)}}^{(n)*}(x^{(n)})\varphi(t) = \int_0^\infty \Pi_{\alpha^{(n)}}^{(n)}(x^{(n)}, (t-\tau)\varphi(\tau)d\tau,$$
$$\Pi_{\alpha^{(n)}}^{(n)}(x^{(n)}, (t) = t^{-\alpha^{(n)}} \sum_{q=0}^{\infty} \frac{(x^{(n)})^q t^{q(1-\alpha^{(n)})}}{\Gamma((1+q)(1-\alpha^{(n)}))}, \tag{27}$$
$$0 \leq \alpha^{(n)} < 1,$$

In (27), $\Gamma(x)$ is the gamma function and $\alpha^{(n)}$, $\beta_0^{(n)}$ and $\beta_\infty^{(n)}$ in (26) and (27) are the rheological parameters of the *n-th* material.

Note that the expression for the operator $\lambda^{(n)*}$ can be easily determined from the condition on the pure elasticity of volumetric expansion and from expression (26).

As in the monograph by Rabotnov (1980) and in the papers by Akbarov (2014), Akbarov and Kepceler (2015), Akbarov *et al*. (2016a, 2016b) and Kocal and Akbarov (2017) we introduce the following dimensionless rheological parameters.

$$d^{(n)} = \frac{\beta_{\infty}^{(n)}}{\beta_0^{(n)}}, \quad Q^{(n)} = \frac{c_{20}^{(n)}}{R(\beta_{01}^{(n)} + \beta_{\infty}^{(n)})}, \tag{28}$$

where

$$\beta_{01}^{(n)} = 3\beta_0^{(n)} / (2(1+\nu_0^{(n)}))$$

As detailed in the papers by Akbarov (2014), Meshkov and Rossikhin (1968) and in others listed therein, the dimensionless parameter $d^{(n)}$ characterizes the long-term values of the elastic constants, however, the dimensionless parameter $Q^{(n)}$ characterizes the creep time. At the same time, the dimensionless parameter $\alpha^{(n)}$ which enter into the foregoing expressions is called as divisibility parameter through which the order of the fractional derivatives is determined.

Thus, using notation in (28) and doing the corresponding mathematical manipulations we obtain the following expressions for the long-term value of the shear modulus (denote it by $\mu_{\infty}^{(n)}$) and for $\mu_{1c}^{(n)}$ and $\mu_{1s}^{(n)}$ which enter into Eq. (10).

$$\mu_{\infty}^{(n)} = \lim_{t\to\infty} (\mu^{(n)*}.1) =$$
$$\mu_0^{(n)} \left(1 - \frac{3}{2(1+\nu_0^{(n)})} \frac{1}{(3/(2(1-\nu_0^{(n)})) + d^{(n)})} \right),$$

$$\mu_{1c}^{(n)} = \mu_0^{(n)} \begin{pmatrix} 1 - \dfrac{3}{2(1+\nu_0^{(n)})} (d^{(n)} + \beta_{01}^{(n)})^{-1} \times \\ \varPi_{\alpha^{(n)}c}^{(n)} (-\beta_{01}^{(n)} - \beta_{\infty}^{(n)}, k_1 Rc) \end{pmatrix}, \tag{29}$$

$$\mu_{1s}^{(n)} = \mu_0^{(n)} \frac{3}{2(1+\nu_0^{(n)})} (d^{(n)} + \beta_{01}^{(n)})^{-1} \times$$
$$\varPi_{\alpha^{(n)}s}^{(n)} (-\beta_{01}^{(n)} - \beta_{\infty}^{(n)}, k_1 Rc),$$

where

$$\varPi_{\alpha^{(n)}c}^{(n)} (-\beta_{01}^{(n)} - \beta_{\infty}^{(n)}, k_1 Rc) = \frac{(\varsigma^{(n)})^2 + \varsigma^{(n)} \sin\dfrac{\pi\alpha^{(n)}}{2}}{(\varsigma^{(n)})^2 + 2\varsigma^{(n)} \sin\dfrac{\pi\alpha^{(n)}}{2} + 1},$$

$$\varPi_{\alpha^{(n)}s}^{(n)} (-\beta_{01}^{(n)} - \beta_{\infty}^{(n)}, k_1 Rc) = \frac{\varsigma^{(n)} \cos\dfrac{\pi\alpha^{(n)}}{2}}{(\varsigma^{(n)})^2 + 2\varsigma^{(n)} \sin\dfrac{\pi\alpha^{(n)}}{2} + 1} \tag{30}$$

$$\varsigma^{(n)} = (Q^{(n)}\varOmega)^{\alpha^{(n)}-1}, \qquad \varOmega = k_1 R \frac{c}{c_{20}^{(2)}} \tag{31}$$

Note that the rheological parameter $d^{(n)}$, determined by expression in (28), characterizes the long-term values of the mechanical properties, however, the parameter $Q^{(n)}$ also determined by the corresponding expression in (28), characterizes the creep time of the *n-th* material. Moreover, we note that the rheological parameter $\alpha^{(n)}$ which mainly characterizes the deformation of the *n-th* material in its initial state and the successful selection of this parameter improve with sufficiently high order, the accuracy of the mathematical modeling of the experimentally determined creep (or relaxation) functions. At the same time, it should be noted that for the time-harmonic problems, the fractional order operators are distinguished from the ordinary operator namely through this parameter.

So the influence of the viscoelasticity of the *n-th* material on the dispersion curves will be estimated through the dimensionless rheological parameters $d^{(n)}$, $Q^{(n)}$and $\alpha^{(n)}$.

4.2 Remarks on the algorithm for numerical solution of the dispersion equation

According to the nature of the considered problem, the components of the matrix (β_{lm}), the expressions of which are given through the formulae (B1) and (B2) in Appendix B, are complex and therefore the values of the determinant $\det\|\beta_{lm}(p)\|$ which enter into dispersion equation (19) are also complex. In connection with this, the dispersion equation can be reduced to the following one

$$\left|\det\|\beta_{lm}(p)\|\right| = 0 \tag{32}$$

where $\left|\det\|\beta_{lm}(p)\|\right|$ is the modulus of the complex number $\det\|\beta_{lm}(p)\|$. In this way, the solution of the dispersion equation is reduced to the solution of equation (32) for which we use the algorithm which is based on direct calculation of the values of the moduli of the dispersion determinant $\left|\det\|\beta_{lm}(p)\|\right|$. Under the solution procedure the sought roots are determined from the criterion $\left|\det\|\beta_{lm}(p)\|\right| \le 10^{-12}$. Note that this algorithm is also used in the previous papers by the authors and in the paper by Barshinger and Rose (2004). Moreover, note that while employing this algorithm a certain value of one of the unknowns c, k_1R or β must be given in advance. The selection of this "one" parameter depends on the aim of the investigation. If the aim of the investigation is the study of the wave attenuation dispersion, then the selected parameter is the wave propagation velocity c (for instance, such an approach is used in the papers by Barshinger and Rose (2004), and Kocal and Akbarov (2017)). However, if the aim of the investigation is the study of the wave velocity dispersion, then the selected parameter is the coefficient of the attenuation β (for instance, such an approach is used in the papers by Akbarov and Kepceler (2015), and Akbarov *et al*. (2016a, 2016b)). Thus, under investigation of the dispersion of the wave propagation velocity, the values for the attenuation coefficient β are given in advance and for each possible value of k_1R, equation (32) is solved with respect to the unknown c and as a result of this solution the wave velocity dispersion curves are constructed. However, under investigation of the dispersion of the wave attenuation, the roles of the parameters β and c in the foregoing solution algorithm are replaced. As in the present paper, the aim of the investigations is the study of the wave velocity dispersion, therefore the values of the coefficient of attenuation are given in advance. According to Ewing *et al*. (1957) and Kolsky (1963), it is assumed that

$$\beta = \frac{1}{2}\frac{\mu_{1s}^{(1)}(\omega)}{\mu_0^{(1)}+\mu_{1s}^{(1)}(\omega)} \text{ or } \beta = \frac{1}{2}\frac{\mu_{1s}^{(2)}(\omega)}{\mu_0^{(2)}+\mu_{1s}^{(2)}(\omega)} \quad (33)$$

4.3 Numerical results related to the influence of the rheological parameters $Q^{(n)}$ and $d^{(n)}$ on dispersion curves

We assume that $v_0^{(1)} = v_0^{(2)} = 0.3$ and $\rho^{(1)}/\rho^{(2)} = 1$, and consider only the case where $p = 1$. We recall that the number p enters into the dispersion equation (19) (or 32) and into the expressions (B1) and (B2) given in Appendix B. Note that in Appendix B instead of p it is used the symbol n. Moreover, we recall that the number p shows the order of the harmonic of the flexural waves in the circumferential direction in the cylinder. Consequently, here we consider only the numerical results related to the first harmonic of the flexural waves in the circumferential direction.

Note that in the present subsection we assume that $\alpha^{(1)} = \alpha^{(2)} = 0.5$ and consider the cases where $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$ and $\mu_0^{(2)}/\mu_0^{(1)} = 2.0$. Moreover, we assume that $h^{(1)}/R = h^{(2)}/R = 0.1$, if otherwise not specified. First, we analyze the results obtained in the case where the viscoelasticity properties of the cylinder's layers are the same, i.e. first, we consider the case where $Q^{(1)} = Q^{(2)} (=Q)$ and $d^{(1)} = d^{(2)} (= d)$. We denote this case as the "V.V. case" and recall that the results discussed below are obtained within the scope of the attenuation relation (33). Note that in the present subsection we consider the dispersion curves related to the first and second lowest modes.

Thus, we consider the graphs given in Figs. 2-5 which illustrate the dispersion curves related to the first (fundamental) mode. Note that the graphs given in Figs. 2 and 3 (in Figs. 4 and 5) relate to the case where $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$ (where $\mu_0^{(2)}/\mu_0^{(1)} = 2.0$). Moreover, note that the results given in Figs. 2 and 4 (in Figs. 3 and 5) illustrate the influence of the rheological parameter d (parameter Q) under fixed $Q(=10)$ (under fixed d(=10)). It follows from these results that a decrease in the values of the rheological parameters d and Q causes a decrease in the wave propagation velocity. However, the dispersion curves in all the cases are limited to those obtained for the purely elastic cases with the instantaneous values of elastic constants at

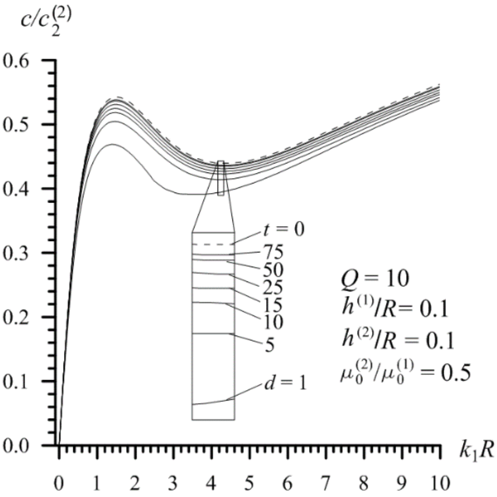

Fig. 2 The influence of the rheological parameter d on the dispersion curves related to the first (fundamental) mode constructed in the case where $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$

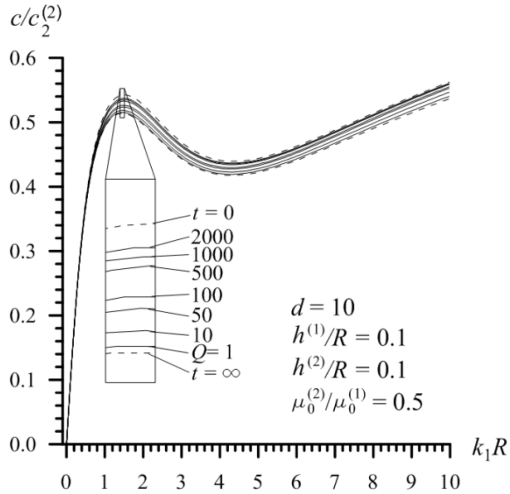

Fig. 3 The influence of the rheological parameter Q on the dispersion curves related to the first (fundamental) mode constructed in the case where $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$

$t = 0$ (upper limit) and with the long-term values of the elastic constants at $t = \infty$ (lower limit).

Now, we analyze the low and high wavenumber limit values of the wave propagation velocity regarding the first mode. According to Figs. 2-5, it can be concluded that in the considered dispersive attenuation case, i.e. in the case where

$$\beta \to 0 \text{ as } k_1R \to 0 \quad (34)$$

which follows directly from the relations (29)-(31) and (33), we get the following low wavenumber limit value for the wave propagation velocity

$$c \to 0 \text{ as } k_1R \to 0 \quad (35)$$

Note that the relation (35) occurs for all the values of the rheological parameters and does not depend on these parameters. Consequently, the relation (35) occurs also for the purely elastic cases. However, according to the well-

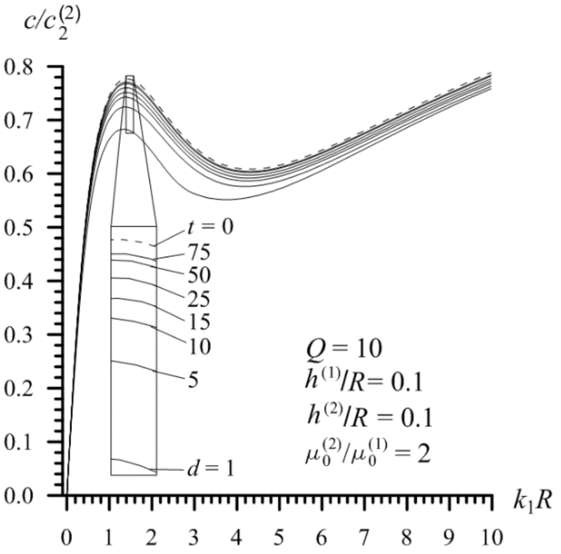

Fig. 4 The influence of the rheological parameter d on the dispersion curves related to the first (fundamental) mode constructed in the case where $\mu_0^{(2)}/\mu_0^{(1)}=2$

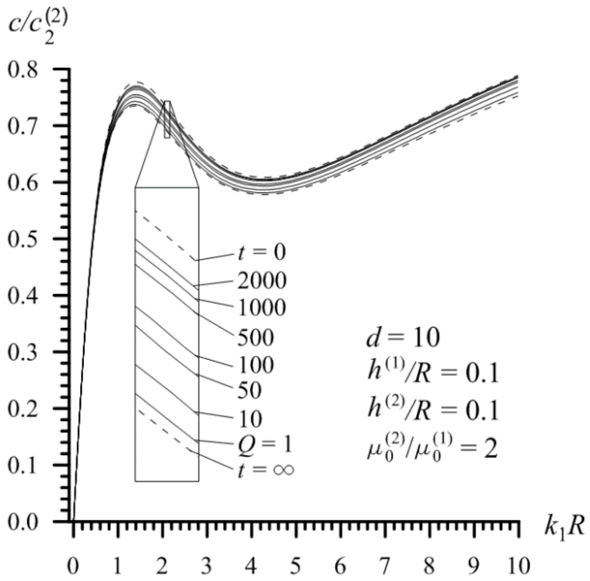

Fig. 5 The influence of the rheological parameter Q on the dispersion curves related to the first (fundamental) mode constructed in the case where $\mu_0^{(2)}/\mu_0^{(1)}=2$

known physico-mechanical consideration, the high wavenumber limit value of the wave propagation velocity in the first mode can be determined through the following relation.

$$c \to \min\left\{c_R^{(1)};c_R^{(2)};c_S\right\} \quad \text{as} \quad k_1R \to \infty, \tag{36}$$

where $c_R^{(1)}$ $(c_R^{(2)})$ is the Rayleigh wave propagation velocity in the outer (inner) layer material and c_S is the Stoneley wave propagation velocity for the pair of materials of the cylinder's layers. It is known that, the Stoneley waves exist for the pair of materials with close acoustic properties and under the "close acoustic properties" it is understood that the ratio $c_2^{(1)}/c_2^{(2)}$ is very near to one, such $\left|c_2^{(1)}/c_2^{(2)}-1\right| = O(10^{-2})$. Note that this statement is indicated almost in all the references related to elastdynamics (see, for instance, the monographs by Eringen and Suhubi (1975), Guz (2004) and other listed therein). As in the numerical investigations considered here it is assumed that $\nu_0^{(1)}=\nu_0^{(2)}$, $\rho^{(1)}/\rho^{(2)}=1$ and $\mu_0^{(2)}/\mu_0^{(1)}=0.5$ and 2.0 for which $\left|c_2^{(1)}/c_2^{(2)}-1\right| \gg O(10^{-2})$, and as the Rayleigh wave propagation velocity of the $n-th$ material depends only on the Poisson ratio, therefore in the cases under consideration the Stoneley waves do not exist and we therefore obtain that $c_R^{(1)}=c_R^{(2)}$ $(=c_R)$ and

$$c \to c_R \quad \text{as} \quad k_1R \to \infty \tag{37}$$

It is evident that the limit cases indicated through the relations (35)-(37) cannot be taken as the flexural wave propagation velocity, however these limits agree with the well-known physico-mechanical considerations and it can also be taken as one of the factors which proves the trustiness of the calculation algorithms and PC programs used in the present investigations. Consequently, the velocities obtained in the cases where $0<k_1R<\infty$ relate to the flexural (bending) waves in the bi-layered hollow cylinder under consideration. It should be also noted that, as it follows from the foregoing numerical results, the flexural character of the waves is observed for the relatively low wavenumber cases under which the influence of the viscoelasticity of the cylinders' materials on the dispersion curves becomes significantly.

At the same, it follows from the foregoing discussions that the limit velocities (35) and (37) do not depend not only on the ratio the shear modulus $\mu_0^{(2)}/\mu_0^{(1)}$ but also on the ratios $h^{(1)}/R$ and $h^{(2)}/R$. However, in the cases where $\left|c_2^{(1)}/c_2^{(2)}-1\right| = O(10^{-2})$, i.e. in the cases where the Stoneley waves in the system exists the limit velocity $c_S$also does not depend on the ratios $h^{(1)}/R$ and $h^{(2)}/R$, however depends on the ratio of the shear modulus $\mu_0^{(2)}/\mu_0^{(1)}$. Thus, we obtain that not only the low wavenumber limit values but also the high wavenumber limit values of the flexural wave propagation velocity in the first mode do not depend on the rheological parameters. The low wavenumber limit relation (35) can be explained with the character of the flexural waves. We recall that for the axisymmetric torsional (see, the paper by Akbarov and Kepceler (2015)) and longitudinal (see the paper by Akbarov *et al*. (2016b)) waves in the hollow layered cylinders, the low wavenumber limit values of the wave propagation velocity depend significantly on the rheological parameter d and the main effects of the rheological parameters d and Q on the wave propagation velocities appear under $k_1R \le 1.5$. However, as follows from Figs. 2-5, the influence of the rheological parameters on the wave propagation velocity of the flexural waves in the first mode disappears as $k_1R \to 0$ and this effect is immediately observed for the cases where $k_1R \ge 0.5$. The numerical results also show that the effect for the flexural waves is considerable for the cases where $0.5 \le k_1R \le 10$.

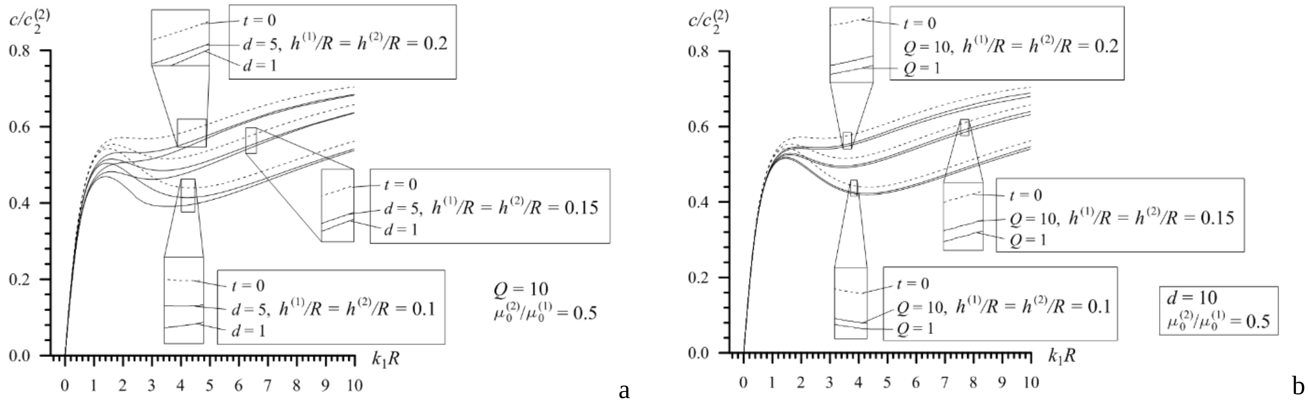

Fig. 6 The effect of the cylinder's thickness on the magnitude of the influence of the rheological parameters d (a) and Q (b) on the dispersion curves in the case where $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$

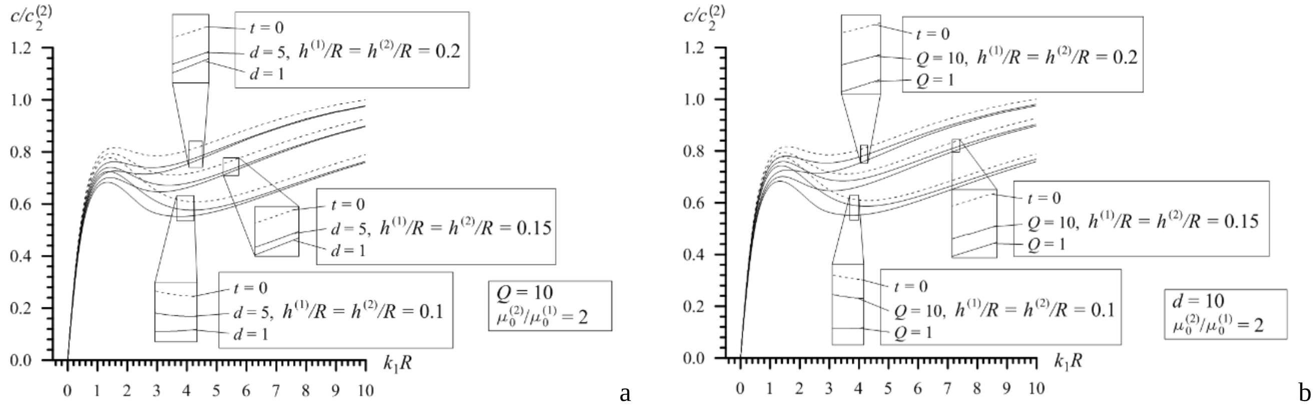

Fig. 7 The effect of the cylinder's thickness on the magnitude of the influence of the rheological parameters d (a) and Q (b) on the dispersion curves in the case where $\mu_0^{(2)}/\mu_0^{(1)} = 2$

Nonetheless, the maximal effect of the influence of the rheological parameters d and Q on the flexural wave propagation velocity appears in relatively small values of the dimensionless wavenumber k_1R. This statement can be explained with the well-known fact, according to which, the influence of the material viscosity on the vibration of the elements of construction made of this material increases (decreases) with decreasing (increasing) of the frequency of this vibration. Consequently, the explanation of the relation (37) (or (36)) is also based on this fact.

The analyses of the results given in Figs. 2-5 also show that the influence of the rheological parameter d on the dispersion curves of the flexural waves is more considerable than that of the rheological parameter Q.

We remind that the foregoing results are obtained only for the case where $h^{(1)}/R = h^{(2)}/R = 0.1$. For illustration of the effect of the ratios $h^{(1)}/R = h^{(2)}/R$ on the magnitude of the influence of the rheological parameters on the dispersion curves we consider the graphs given in Figs. 6 and 7 and constructed in the cases where $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$ and 2.0, respectively. Note that in these figures the dispersion curves obtained for various values of $h^{(1)}/R = h^{(2)}/R$ and for the same values of the rheological parameter d under fixed Q (=10) (Figs. 6a and 7a) and for the same values of the rheological parameter Q under fixed d (=10) (Figs. 6b and 7b), are given together. It follows from these graphs that the influence of the ratio $h^{(1)}/R = h^{(2)}/R$ on the magnitude of the influence of the rheological parameters d and Q on the propagation velocities of the flexural waves is insignificant. However, an increase in the values of the ratio $h^{(1)}/R = h^{(2)}/R$ causes the character of the dispersion curves to change.

Now we consider the numerical results related to the influence of the rheological parameters d and Q on the dispersion curves of the second mode and for a clearer illustration this influence we consider the graphs of the dependencies between $(c - c|_{t=\infty})/c_2^{(2)}$ and k_1R instead of the graphs of the dependencies between $c/c_2^{(2)}$ and k_1R. These graphs are given in Figs. 8-9 which are constructed in the cases for $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$ (Figs. 8a and 8b) and 2.0 (Figs. 9a and 9b) for various values of the parameter d under fixed Q (=10) (Figs. 8a and 9a) and for various values of the parameter Q under fixed d (=10) (Figs. 8b and 9b).

Thus, it follows from the foregoing graphs that a decrease in the values of the rheological parameters d and Q also causes a decrease in the values of the wave propagation velocity of the second mode. Moreover, these results show that the considerable effect of the influence of the rheological parameters d and Q on the wave propagation velocity in the second mode appears in the cases where $k_1R \le 0.2$.

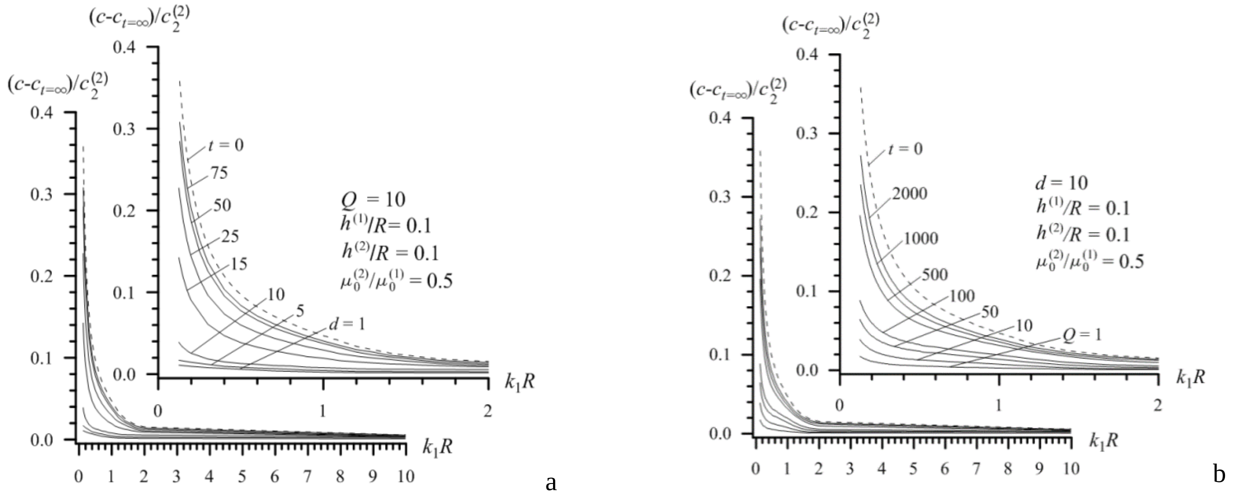

Fig. 8 The influence of the rheological parameters d (a) and Q (b) on the dispersion curves related to the second mode constructed in the case where $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$

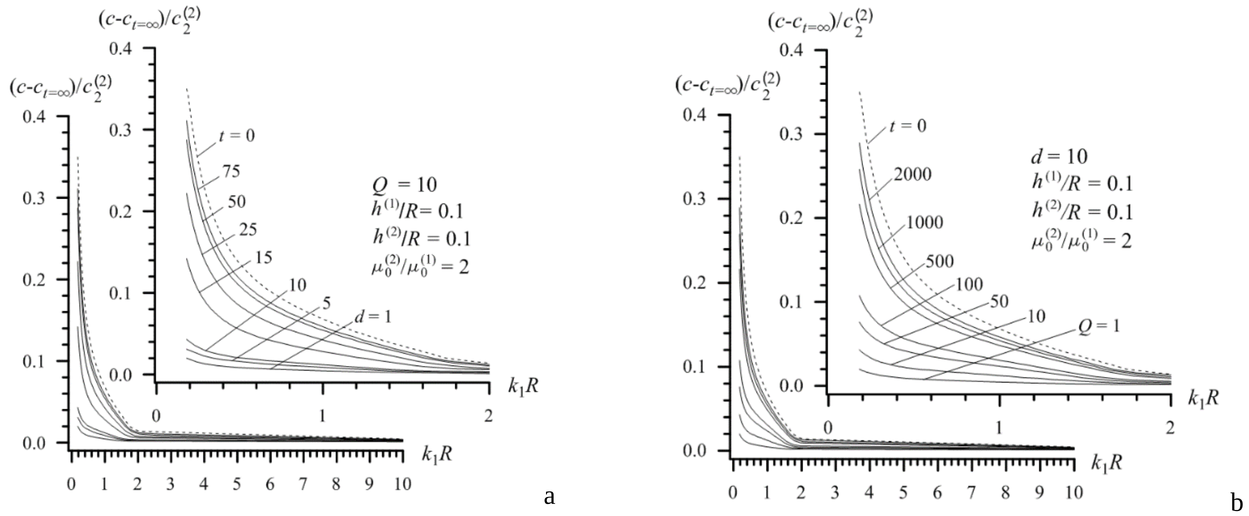

Fig. 9 The influence of the rheological parameters d (a) and Q (b) on the dispersion curves related to the second mode constructed in the case where $\mu_0^{(2)}/\mu_0^{(1)} = 2$

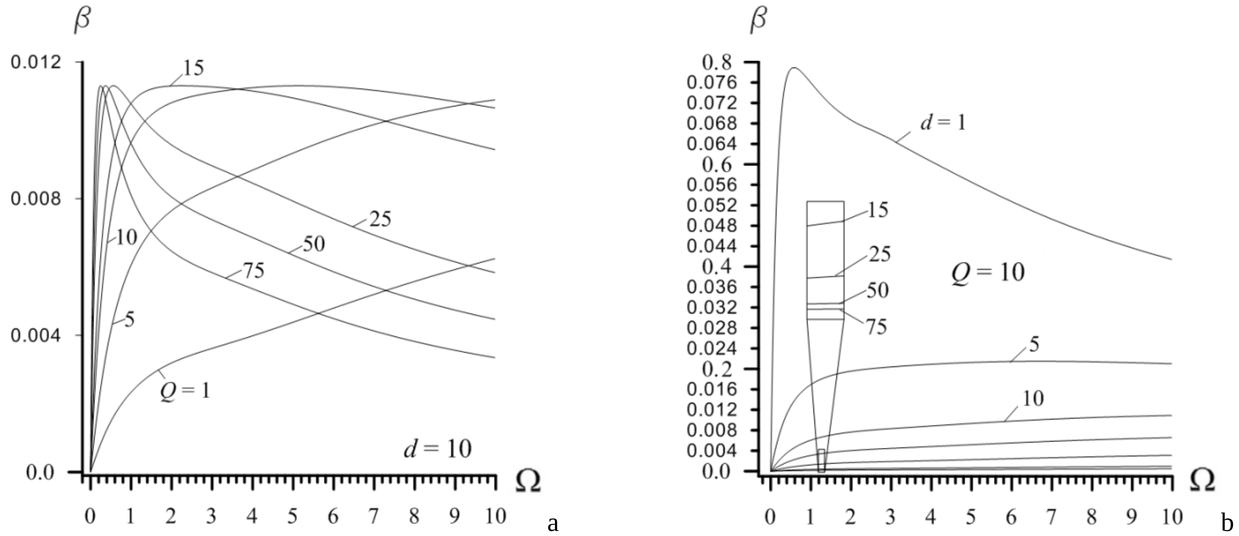

Fig. 10 The graphs illustrated the attenuation dispersion or various values of the rheological parameters Q(a) and d(b); the Ω is determined through the expression given in equation (31)

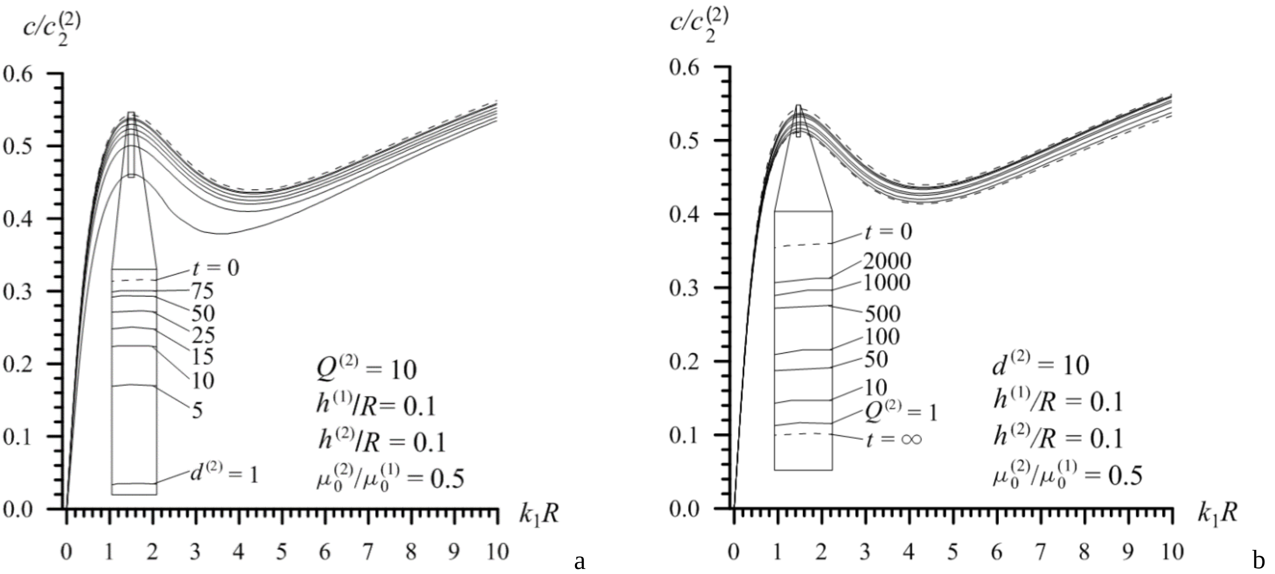

Fig. 11 The influence of the rheological parameters *d* (a) and *Q* (b) in the dispersion curves related to the first (fundamental) mode constructed in the case where $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$ in the E.V. case

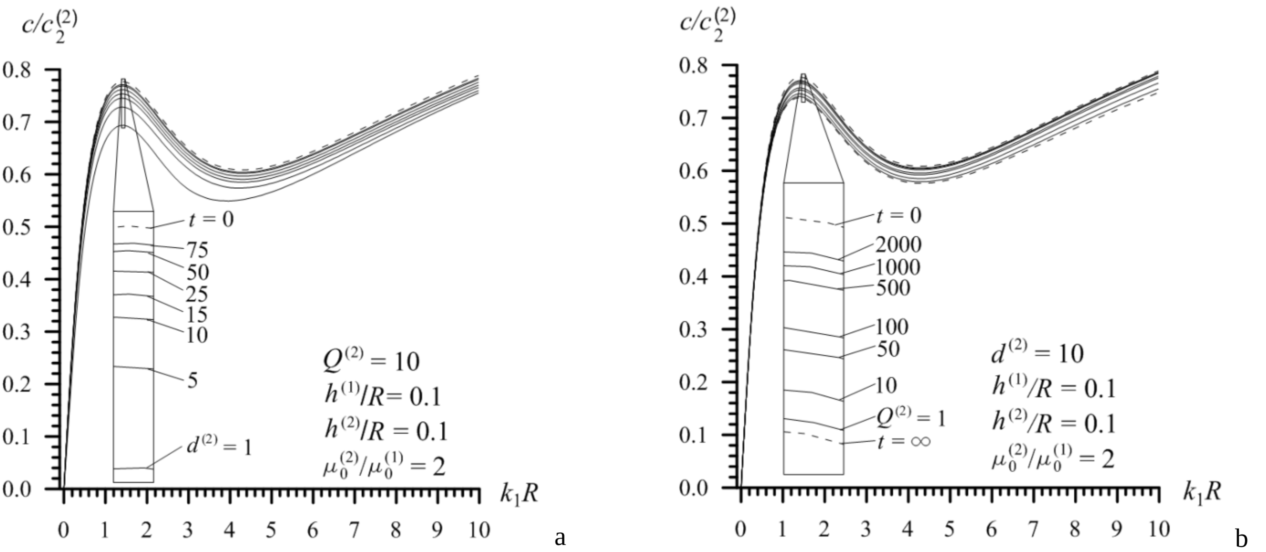

Fig. 12 The influence of the rheological parameter *d* (a) and *Q* (b) on the dispersion curves related to the first (fundamental) mode constructed in the case where $\mu_0^{(2)}/\mu_0^{(1)} = 2$ in the E.V. case

At the same time, comparison of the results obtained for various values of the ratios $\mu_0^{(2)}/\mu_0^{(1)}$ with each other, shows that the effect of this ratio on the magnitude of the influence of the rheological parameters *d* and *Q* on the wave propagation velocity in the second mode is insignificant. Moreover, the comparison of the results obtained for various $h^{(1)}/R$ (= $h^{(2)}/R$) (these results are not given here) shows that the effect of the change of $h^{(1)}/R$ on the aforementioned magnitude is also insignificant.

Note that all the foregoing numerical results are obtained for the dispersion attenuation determined by the relations given in equation (33). The graphs of these dispersion curves are given in Fig. 10 which are constructed for various values of the rheological parameter *Q* under fixed *d* (=10) (Fig. 10a) and for various values of the rheological parameter *d* under fixed *Q* (=10) (Fig. 10b). Note that the selected attenuation curves agree in the quantitative sense with the corresponding ones determined in the papers by Barshinger and Rose (2004), Hernando Quintanilla *et al*. (2015), Mazoti *et al*. (2012) and others listed therein.

We recall that the results analyzed above relate to the case where the layers' materials of the cylinder have the same rheological properties. Now we consider the numerical results related to the case where the material of the inner layer is viscoelastic, however the material of the outer layer is purely elastic and denote this case as the "E.V. case", and examine the first mode only. The dispersion curves regarding the E.V. case are given in Figs. 11-12 which are constructed in the cases where $\mu_0^{(2)}/\mu_0^{(1)} =$ 0.5 (Figs. 11a and 11b) and $\mu_0^{(2)}/\mu_0^{(1)} = 2.0$ (Figs. 12a and 12b) for various values of the parameter $d^{(2)}$ under fixed

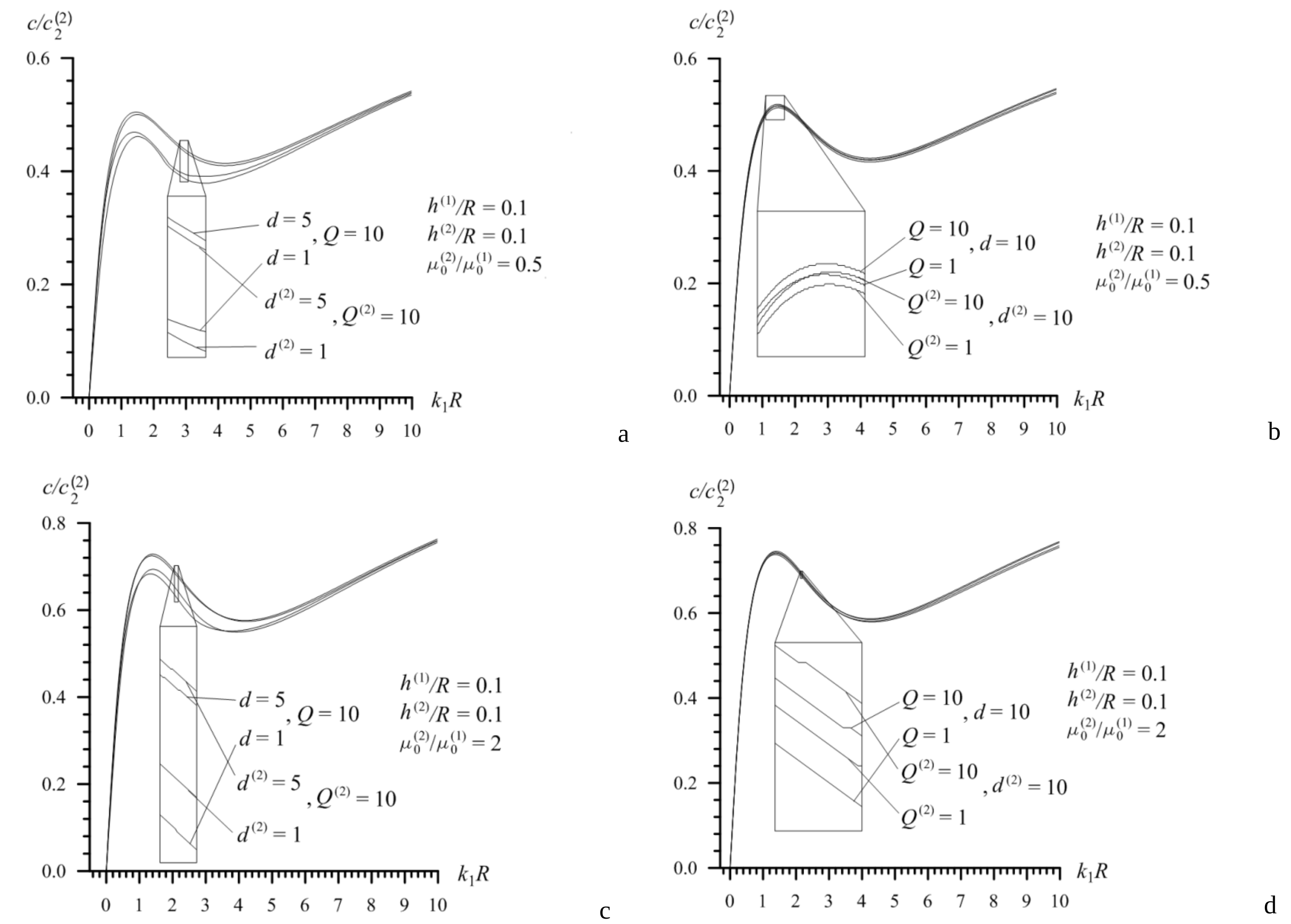

Fig. 13 Comparison of the results obtained in the V.V. case with the corresponding ones obtained in the E.V. case for various values of rheological parameters *d* (a, c) and *Q*(b, d) in the cases where $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$ (a, b) and 2 (c, d)

$Q^{(2)}$ (=10) (Figs. 11a and 12a) and for various values of the parameter $Q^{(2)}$ under fixed $d^{(2)}$ (=10) (Figs. 11b and 12b).

Analysis of the graphs given in Figs. 11-12 shows that the character and properties of the dispersion curves obtained in the E.V. case are the same as those obtained in the V.V. case which are detailed and analyzed above. However, comparison of the results obtained in the E.V. case with the corresponding ones obtained for the V.V. case shows that the magnitude of the influence of the rheological parameters $d^{(2)}$ and $Q^{(2)}$ on the wave propagation velocity in the E.V. case is greater than that in the V.V. case. For a clearer illustration of this conclusion, the dispersion curves obtained in the V.V. and E.V. cases are given together in Fig. 13 and these curves are constructed for $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$ (Figs. 13a and 13b) and 2.0 (Figs. 13c and 13d). Note that in Fig. 13 the dispersion curves constructed in the V.V. case for d=1 and 5 under fixed Q (=10) are compared with the dispersion curves constructed in the E.V. case for $d^{(2)}$ =1 and 5 under fixed $Q^{(2)}$ (=10) (Figs. 13a and 13c). The dispersion curves constructed in the V.V. case for Q = 1 and 10 under fixed d (=10) are also compared with the dispersion curves constructed in the E.V. case for $Q^{(2)}$ 1 and 10 under fixed $d^{(2)}$ = 10 (Figs. 13b and 13d). It follows from the results given in Fig. 13 that the wave propagation velocity obtained for the E.V. case is less than the corresponding one obtained for the V.V. case. The difference between the wave propagation velocities obtained in the E.V. and V.V. cases decreases with increasing of the rheological parameters and has zeroth limit. This is because the upper limits for the dispersion curves in the E.V. and V.V. cases are the same. However, the lower limits for the dispersion curves obtained for the E.V. and V.V. cases differ from each other and the limit wave propagation velocities obtained for the E.V. cases are less than the corresponding ones obtained for the V.V. cases.

This statement can be explained with the fact that in the cases where the materials of the layers are purely elastic ones and the Poisson's ratios of the layers' materials are equal to each other, an increase in the ratio of the shear modulus of the inner layer material to that of the outer layer material causes an increase in the values of the dimensionless wave propagation velocity $c/c_2^{(2)}$. This fact is proven with a comparison of the results obtained for the case where $\mu_0^{(2)}/\mu_0^{(1)} = 2.0$ with those obtained for the case where $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$. Moreover, this fact is proven with the other similar results obtained for the purely elastic cases under other values of the ratio $\mu_0^{(2)}/\mu_0^{(1)}$ and detailed in the monograph by Akbarov (2015) and in the papers listed therein. Moreover, the wave propagation velocity in the bi-layered cylinder in the purely elastic cases depends also on the ratios of the Lame constants, i.e. on $\lambda^{(m)}/\mu^{(m)}$ (m=1,2) and an increase in the values of these ratios causes a decrease in the wave propagation velocity under purely elastic cases. In the V.V. case, the equality $\mu_0^{(2)}/\mu_0^{(1)} = \mu_\infty^{(2)}/\mu_\infty^{(1)}$ takes place and the lower limit dispersion curves appear as a result of the differences $\lambda_0^{(1)}/\mu_0^{(1)} \neq \lambda_\infty^{(1)}/\mu_\infty^{(1)}$ and $\lambda_0^{(2)}/\mu_0^{(2)} \neq \lambda_\infty^{(2)}/\mu_\infty^{(2)}$. However, in the E.V. case, the lower limit dispersion curves are obtained for the cases where the ratio of the shear modulus of the constituents is $\mu_\infty^{(2)}/\mu_0^{(1)}$ which is less than $\mu_0^{(2)}/\mu_0^{(1)} = \mu_\infty^{(2)}/\mu_\infty^{(1)}$ and therefore the lower limit wave propagation velocities obtained for the E.V. case are less than those obtained for the V.V. case. Thus, the results illustrated in Fig. 13, according to which, the magnitude of the influence of the rheological parameters on the wave propagation velocity in the E.V. case is greater than that in the V.V. case, are explained.

We do not present here the results regarding the case where the outer layer material is viscoelastic but the material of the inner layer is purely elastic (denote this case as the "V.E. case"). This is because for the range change of the problem parameters considered here in the V.E. case the influence of the rheological parameters on the wave propagation dispersions is insignificant. The reason for this is the fact that the lower limit wave propagation velocities obtained for the V.E. case are greater than those obtained for the V.V. case. According to the foregoing discussions, this statement can be also explained with the inequality $\mu_0^{(2)}/\mu_\infty^{(1)} > \mu_\infty^{(2)}/\mu_\infty^{(1)} = \mu_0^{(2)}/\mu_0^{(1)}$. Consequently, the increment of the lower limit wave propagation velocities in the V.E. case causes the lower limit dispersion curve to become closer to the upper limit dispersion curve and as a result of this 'closing', the effect of the rheological parameters on the dispersion curves decreases.

We recall that under consideration of the axisymmetric longitudinal waves in the bi-layered hollow cylinder made of viscoelastic material the results of which are detailed in the paper by Akbarov *et al*. (2016b), it is established that in the corresponding purely elastic case an increase in the ratio of the shear modulus of the inner layer material to that of the outer layer material causes a decrease of the wave propagation velocities. Consequently, the effect of the ratio of the shear modulus on the wave propagation velocities of the flexural waves in the first mode is the reverse of that on the axisymmetric longitudinal wave propagation velocities. Therefore, analyses obtained for the axisymmetric longitudinal waves in the bi-layered hollow cylinder and given in the paper by Akbarov *et al*. (2016b) show that the influence of the rheological parameters of the outer cylinder material in the V.E. case is more significant than those obtained in the V.V. case and the effect of the viscoelasticity in the E.V. cases is insignificant. In other words, the effect of the viscoelasticity in the V.E. and E.V. cases on the wave propagation velocities obtained for the first mode of the flexural waves is the reverse of that for the axisymmetric longitudinal waves. It is evident that the aforementioned 'reverse' statements can be explained with the difference of the particularities of the flexural and axisymmetric longitudinal waves.

With this we restrict ourselves to consideration of the rheological parameters $d^{(m)}$ and $Q^{(m)}$ on the dispersion curves.

4.4 Numerical results related to the influence of the rheological parameter $\alpha^{(m)}$ on the dispersion curves

As noted above, the parameter $\alpha^{(m)}$ is the main rheological parameter which distinguishes the fractional exponential operators from the ordinary ones. Sometimes the parameter $\alpha^{(m)}$ is called the fractional order or the order of the singularity of the operators. We again recall that in the case where $\alpha^{(m)} = 0$, the operators given through the expressions in equations (26) and (27) become the operators which relate to the "standard liner solid body model" with ordinary derivatives.

Thus, we consider the numerical results illustrating the influence of the parameter $\alpha^{(m)}$ on the dispersion curves. We consider here only the V.V. case under which $\alpha^{(1)} = \alpha^{(2)}$ (= α) and analyze the dispersion curves regarding the first mode. These curves are given in Fig. 14a (Fig. 14b) which are constructed in the case where $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$ (in the case where $\mu_0^{(2)}/\mu_0^{(1)} = 2.0$) for various values of the parameter α under $\{d = 1;\ Q = 10\}$.

It follows from the foregoing graphs that an increase in the values of the rheological parameter α causes a decrease in the values of the wave propagation velocity and a change in the values of α does not influence the character of the dispersion curves.

This completes the analyses of the numerical results. This completes the analyses of the numerical results. We recall that these results are obtained within the scope of the exact three-dimensional equations and relations of viscoelastodynamics. It is evident that these investigations can be also made within the scope of the various approximate shell theories (for instance, within the scope of the Kirchhoff– Love shell theory), according to which, the radial coordinate r and the derivatives with respect to this coordinate disappear in the governing field equations. Moreover, under consideration within the scope of the approximate shell theories the boundary conditions with respect to coordinate r at $r = R-h^{(1)}$ and at $r = R-h^{(2)}$, and the contact conditions at $r = R$ formulated in equation (5), disappear also. However, these conditions are entered into the field equations of the mentioned approximate shell theories.

It follows from the physico-mechanical considerations that the results obtained within the scope of the approximate shell theories acceptable for the low wavenumber, i.e. for the cases where $1/(k_1R) \gg \max\{h^{(1)}/R; h^{(2)}/R\}$. Consequently, the difference between the results obtained within the scope of the approximate and exact approaches increases with increasing the dimensionless wavenumber k_1R (or with decreasing of the

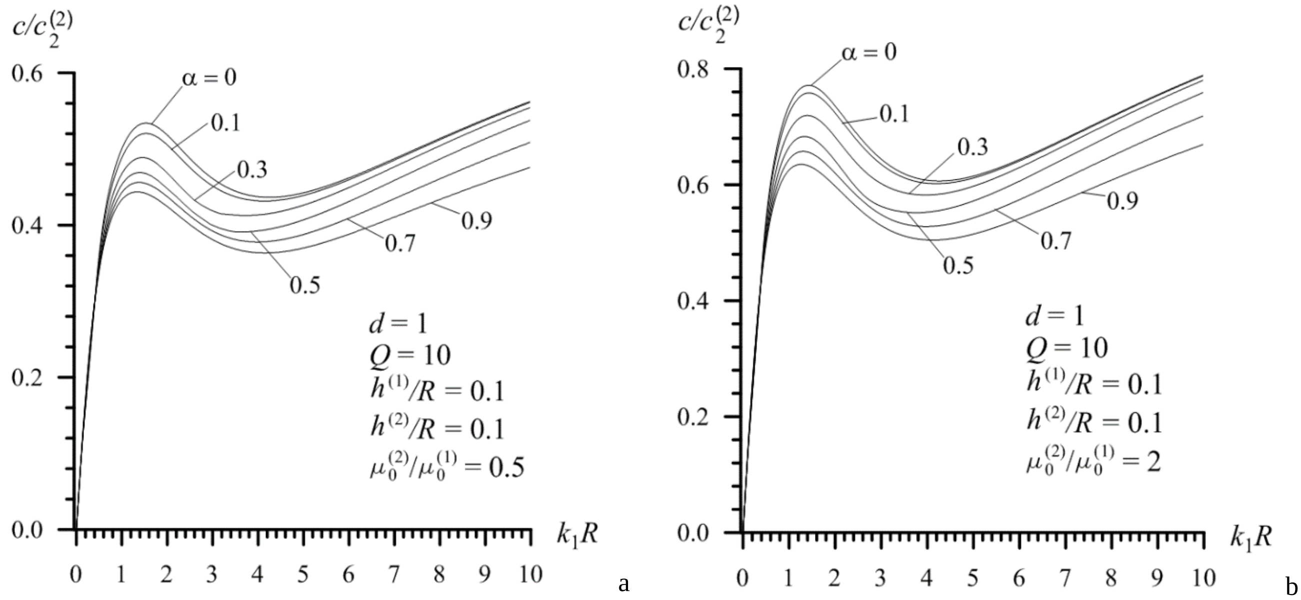

Fig. 14 The influence of the rheological parameter $\alpha (= \alpha^{(1)} = \alpha^{(2)})$ on the dispersion curves of the first mode in the cases where $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$ (a) and 2 (b)

length of the flexural (bending) waves) and after a certain value of the wavenumber becomes more considerable not only in the quantitative sense but also in the qualitative sense. Such conclusions are made in the many references (see, for instance, the monograph by Eringen and Suhubi (1975) and others listed therein) for the purely elastic cases. As the results obtained for the viscoelastic cases are limited with the corresponding results obtained for the purely elastic cases with the instantaneous and long-term values of the elastic constants, therefore the mentioned conclusion remain also valid for the problems considered in the present paper. Consequently, the results obtained and discussed in the present paper can be taken as the benchmark ones for the results and discussions obtained within the scope of the various approximate shell theories for the layered cylindrical shells.

4.5 The difference between the present results from those obtained for the corresponding axisymmetric waves

According to the comparison of the present results with corresponding results obtained earlier in the papers by Akbarov and Kepceler (2015) and Akbarov *et al*. (2016a, 2016b) for the corresponding axisymmetric waves, it can be made the following main difference between them in the qualitative sense.

1. In the present case the low wavenumber limit values of the wave propagation velocity in the first mode do not depend on the rheological parameters, however, in the axisymmetric wave propagation case (see, the paper by Akbarov and Kepceler (2015) for tortional waves and the papers by Akbarov *et al*. (2016a, 2016b) for longitudinal waves) the low wavenumber limit values of the wave propagation velocity depends significantly on the rheological parameter d.
2. In the axisymmetric wave propagation case the main effects (in the quantitative sense) of the rheological d and Q on the wave propagation velocities appear under $k_1R \leq 1.5$, however, in the present case this effect is immediately observed for the cases where $0.5 \leq k_1R \leq 10$.
3. In the axisymmetric wave propagation case the influence of the rheological parameters on wave propagation velocity in the second mode is more significant in the quantitative sense than that in the present case.
4. In the present case, the effect of the geometrical parameter $h^{(1)}/R$ ($= h^{(2)}/R$) on the wave propagation velocity in the first mode is more considerable not only in the quantitative sense but also in the qualitative sense than that in the axisymmetric wave propagation cases.
5. Besides all these, in the present paper, it is also studied the influence of the dimensionless parameter
6. $\alpha(=\alpha^{(1)}=\alpha^{(2)})$ on the dispersion curves, however, in the previous works this influence does not study.

Namely, the foregoing differences determine the originality and significance of the present results.

4.6 On the influence of the viscoelasticity on the wave dispersion in the higher order of the harmonic of the flexural waves in the circumferential direction

We recall that above it has been considered dispersion curves related to the so-called modes $F(1,1)$ and $F(1,2)$, i.e. the first two modes of the first harmonic in the circumferential direction of the cylinder.

It is also evident that the consideration of the influence of the rheological parameters of the cylinders materials on the dispersion curves in the higher harmonics in the circumferential direction of the cylinder, i.e. the modes $F(n,1)$, $F(n,2)$, ...,where $n \geq 2$ also represents a great significance. However, within the scope of the one paper, it is impossible to consider and analyze all these dispersion

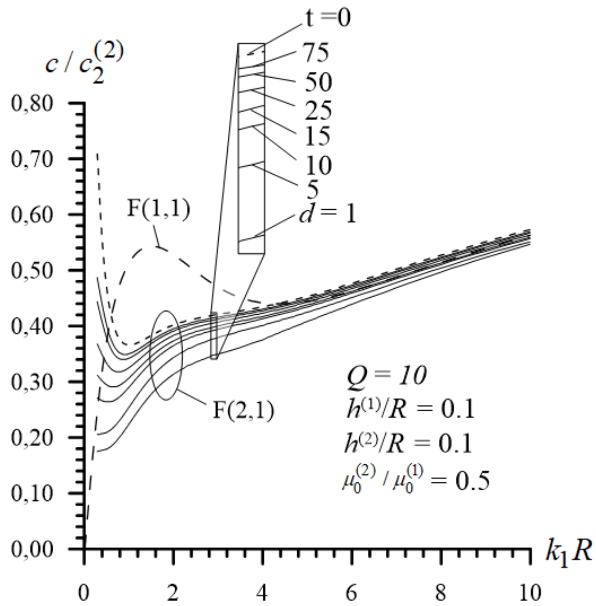

Fig. 15 The influence of the rheological parameter d on the dispersion curves obtained for the mode F(2,1) in the case where $\alpha^{(1)} = \alpha^{(2)} = 0.5$

curves related all these modes. Nevertheless, as an example of the illustration of the mentioned significance, we consider the graphs given in Fig. 15 which show the dispersion curves for the mode $F(2,1)$ in the V.V. case under d = 1, 5, 10, 15, 25, 50 and 75. Note that under construction of these graphs it is assumed that $h^{(1)}/R = h^{(2)}/R = 0.1$, $Q = 10$ and $\mu_0^{(2)}/\mu_0^{(1)} = 0.5$ In this figure, it is also shown the dispersion curve related to mode $F(2,1)$ the case where the materials of the layers of the cylinder are the purely elastic (in Fig. 15 this curve is indicated by the t=0) and the dispersion curve related to the mode $F(1,1)$.

It follows from the comparison the dispersion curve of the mode $F(1,1)$ with the dispersion curve of the mode $F(2,1)$ constructed for the purely elastic case that there is a certain change interval of the k_1R under which the wave propagation velocity in the mode $F(2,1)$ is less than that in the mode $F(1,1)$. Note that such situation is characteristic one for the $F(2,1)$ mode in the hollow cylinder and it is also observed from the corresponding results obtained by the other authors (see, for instance, the paper by Nishido *et al*. (2001)). Moreover, it follows from the graphs given in Fig. 15 that the influence of the rheological parameter d on the dispersion curves obtained for the $F(2,1)$ modes is considerable with the quantitative sense and causes to decrease the wave propagation velocity in this mode. At the same time, Fig. 15 shows that the magnitude of the mentioned influence increases with the decreasing of the k_1R.

The foregoing results show that the study of the influence of the rheological parameters of the cylinder's layers materials on the wave dispersion in the modes $F(n,1)$ $F(n,2)$..., where $n \geq 2$ may have considerable significance not only in the theoretical but also in the application sense. Therefore, the study of dispersion curves in these cases will be made in further works by the authors.

5. Conclusions

Thus, in the present paper the flexural wave dispersion in the bi-layered circular hollow cylinder made of viscoelastic materials is investigated by utilizing the exact 3D equations and relations of elastodynamics. It is assumed that the materials of the layers of the cylinder are homogeneous and isotropic. The viscoelasticity of the layers' materials of the cylinder is described through the fractional exponential operators by Rabotnov (1980), according to which, the rheological parameters $d^{(m)}$, $Q^{(m)}$ and $\alpha^{(m)}$\ are introduced, through which the long term values of the mechanical properties, the characteristic creep times and the improving of the mathematical approximation of the experimental creep or relaxation functions in the initial state of the deformations of the *m-th* material, respectively, are estimated.

Numerical results are presented for the first harmonic of the flexural waves in the circumferential direction and dispersion curves related to the lowest first and second modes are presented and analyzed. The following cases are considered: the V.V. case for which the viscoelasticity properties of the layers' materials are the same (i.e. the equalities $d^{(1)}=d^{(2)}=d$, $Q^{(1)}=Q^{(2)}=Q$, $\alpha^{(1)}=\alpha^{(2)}=\alpha$ take place) and the E.V. case for which the outer layer material is purely elastic but the material of the inner layer is viscoelastic. The particularities of the influence of the viscoelasticity of the cylinder's layers' materials on the flexural wave dispersion are established.

Analysis of the numerical results allows us to draw the following concrete conclusions:

- Dispersion curves obtained for all the viscoelastic cases are limited with the corresponding ones obtained for the purely elastic cases with the instantaneous values of the elastic constant at t = 0 (upper limit) and with the long-term values of the elastic constants at t = ∞ and the "distance" between these limit cases increase with decreasing of the parameter $d^{(m)}$;
- All the dispersion curves "shift" up with increasing of the ratio of the shear modulus of the inner layer to that of the outer layer and as a result of this shifting the "distance" between the upper and lower limit dispersion curves obtained for the V.V. case is less (is more) than that obtained for the E.V. case (for the V.E. case);
- As a result of the foregoing statement, the magnitude of the influence of the rheological parameters $d^{(2)}$ and $Q^{(2)}$ on the dispersion curves in the E.V. case is more significant than that of the rheological parameters d and Q in the V.V. case;
- In the V.E. case, the influence of the viscoelasticity of the outer layer material on the dispersion curves of the first mode is insignificant for the problem parameters considered in the present investigation;
- An increase in the values of the rheological parameter α in the V.V. case causes a decrease in the values of the wave propagation velocities in the first mode;
- The change in the values of the ratio $h^{(1)}/R = h^{(2)}/R$ causes the character of the dispersion curves to change;

Besides all the foregoing conclusions, more detailed ones can be made from the numerical results discussed, some of which can be found in the text of the paper.

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Appendix A

In the present appendix we attempt to justify the approximate equality (8). For this purpose, first, we note that in the theoretical sense the operator for the linear theory of viscoelasticity of the non-aging materials is selected as

$$y(t)=x(t)+\int_{-\infty}^{t}K(t-\tau)x(\tau)d\tau \qquad (A1)$$

where the function $x(t)$ is named the "action" to the system, however the function $y(t)$ is named the "reaction" of this system to this action.
However, as all the pfysico-mechanical processes take place within the finite interval of time and the beginning of each process can be taken as $t=0$, therefore, as usual, it is assumed that the functions $x(t)$ and $y(t)$ are the Heaviside type functions, i.e. it is supposed that $x(t) = y(t) = 0$ for $t\in(-\infty,0]$. This mathematical formalism based on the fact that in the real cases the first term in the right side of the expression

$$\int_{-\infty}^{t}K(t-\tau)x(\tau)d\tau=\int_{-\infty}^{0}K(t-\tau)x(\tau)d\tau+\int_{0}^{t}K(t-\tau)x(\tau)d\tau \qquad (A2)$$

is so small that it can be neglected with very high accuracy. In other words, it is assumed that

$$y(t)=x(t)+\int_{0}^{t}K(t-\tau)x(\tau)d\tau \qquad (A3)$$

In this way, the relation (A1) is transformed into the following one

$$y(t+T)=x(t+T)+\int_{0}^{t+T}K(t+T-\tau)x(\tau)d\tau \qquad (A4)$$

which, as in many references, is also used under writing the expressions in equation (3).

However, the relation (A4) does not satisfy the "closed cycle conditions", i.e. the (A4) type relation doesn't maintain the periodicity of the function $y(t)$ in the cases where the function $x(t)$ is periodic one, i.e. $x(t+T)=x(t)$ where T is a period. This fact can be proven as follows.

We rewrite the relation (A4) by replacing t with $t+T$

$$y(t+T)=x(t+T)+\int_{-T}^{t}K(t+T-\tau-T)x(\tau+T)d\tau \qquad (A5)$$

Doing the transform $\tau=\tau'+T$ in equation (A5) and omitting the upper prime over the τ', we obtain that

$$y(t+T)=x(t+T)+\int_{-T}^{t}K(t+T-\tau-T)x(\tau+T)d\tau \qquad (A6)$$

Using the equality $x(t+T)=x(t)$ and $\int_{-T}^{t}(\bullet)d\tau=\int_{-T}^{0}(\bullet)d\tau+\int_{0}^{t}(\bullet)d\tau$ we can write $y(t+T)=x(t)+\int_{0}^{t}K(t-\tau)x(\tau)d\tau+\int_{-T}^{0}K(t-\tau)x(\tau)d\tau$ from (A6),

In other words we obtain that

$$y(t+T)=y(t)+\int_{-T}^{0}K(t-\tau)x(\tau)d\tau \qquad (A7)$$

which proves the non-periodicity of the function $y(t)$ if this function is determined through the relation (A4).

However, if we remake the foregoing mathematical manipulations for the relation (A1), we obtain

$$\begin{aligned} y(t+T)&=x(t+T)+\int_{-\infty}^{t+T}K(t+T-\tau)x(\tau)d\tau= \\ &x(t+T)+\int_{-\infty}^{t}K(t+T-\tau-T)x(\tau+T)d\tau= \\ &x(t)+\int_{-\infty}^{t}K(t-\tau)x(\tau)d\tau=y(t), \quad\Rightarrow\quad y(t+T)=y(t), \end{aligned} \qquad (A8)$$

from which follows that $y(t)$ is periodic with the period T if the function $x(t)$ is periodic with the same period and if the function $y(t)$ is determined through the relation (A1).

As the nature of the problem considered in the present paper requires the periodicity of the stresses under periodicity of the displacements therefore we return to (A1) type relations assuming the satisfaction of the relation (8), the acceptability of which is discussed above.

Appendix B

In the present appendix we give the explicit expressions for the components of the matrix $(\beta_{lm}(p))$ under p=1. These expressions are

$$\beta_{11}\left(\zeta_1^{(2)},\Psi_{2h^{(2)}}^{(2)}\right)=\left(\Lambda^{(2)}+2M^{(2)}\right)\left(\frac{\zeta_1^{(2)}}{\eta_1}J_1^I\left(\Psi_{2h^{(2)}}^{(2)}\right)-\frac{1}{\eta_1^2}J_1\left(\Psi_{2h^{(2)}}^{(2)}\right)\right)+$$
$$\frac{\Lambda^{(2)}}{\eta_1}\left(-\zeta_1^{(2)}J_1^I\left(\Psi_{2h^{(2)}}^{(2)}\right)+\frac{1}{\eta_1}J_1\left(\Psi_{2h^{(2)}}^{(2)}\right)\right),$$

$$\beta_{12}\left(\zeta_1^{(2)},\Psi_{2h^{(2)}}^{(2)}\right)=\left(\Lambda^{(2)}+2M^{(2)}\right)\left(\frac{\zeta_1^{(2)}}{\eta_1}Y_1^I\left(\Psi_{2h^{(2)}}^{(2)}\right)-\frac{1}{\eta_1^2}Y_1\left(\Psi_{2h^{(2)}}^{(2)}\right)\right)$$
$$+\frac{\Lambda^{(2)}}{\eta_1}\left(-\zeta_1^{(2)}Y_1^I\left(\Psi_{2h^{(2)}}^{(2)}\right)+\frac{1}{\eta_1}Y_1\left(\Psi_{2h^{(2)}}^{(2)}\right)\right),$$

$$\beta_{21}\left(\zeta_1^{(2)},\Psi_{2h^{(2)}}^{(2)}\right)=M^{(2)}\left[\begin{array}{l}\left(-\left(\zeta_1^{(2)}\right)^2J_1^{II}\left(\Psi_{2h^{(2)}}^{(2)}\right)\right)\\ +\left(-\frac{1}{\eta_1^2}J_1\left(\Psi_{2h^{(2)}}^{(2)}\right)+\frac{\zeta_1^{(2)}}{\eta_1}J_1^I\left(\Psi_{2h^{(2)}}^{(2)}\right)\right)\end{array}\right],$$

$$\beta_{22}\left(\zeta_1^{(2)},\Psi_{2h^{(2)}}^{(2)}\right)=M^{(2)}\left[\begin{array}{l}\left(-\left(\zeta_1^{(2)}\right)^2 Y_1^{II}\left(\Psi_{2h^{(2)}}^{(2)}\right)\right)\\ +\left(-\dfrac{1}{\eta_1^2}Y_1\left(\Psi_{2h^{(2)}}^{(2)}\right)+\dfrac{\zeta_1^{(2)}}{\eta_1}Y_1^I\left(\Psi_{2h^{(2)}}^{(2)}\right)\right)\end{array}\right],$$

$$\beta_{31}\left(\zeta_1^{(2)},\Psi_{2h^{(2)}}^{(2)}\right)=\frac{M^{(2)}}{\eta_1}J_1\left(\Psi_{2h^{(2)}}^{(2)}\right)$$

$$\beta_{32}\left(\zeta_1^{(2)},\Psi_{2h^{(2)}}^{(2)}\right)=\frac{M^{(2)}}{\eta_1}Y_1\left(\Psi_{2h^{(2)}}^{(2)}\right)$$

$$\beta_{41}\left(\zeta_1^{(2)},\Psi_1^{(2)}\right)=\left(\Lambda^{(2)}+2M^{(2)}\right)\left(\frac{\zeta_1^{(2)}}{\eta_0}J_1^I\left(\Psi_1^{(2)}\right)-\frac{1}{\eta_0^2}J_1\left(\Psi_1^{(2)}\right)\right)+$$
$$\frac{\Lambda^{(2)}}{\eta_0}\left(-\zeta_1^{(2)}J_1^I\left(\Psi_1^{(2)}\right)+\frac{1}{\eta_0}J_1\left(\Psi_1^{(2)}\right)\right),$$

$$\beta_{42}\left(\zeta_1^{(2)},\Psi_1^{(2)}\right)=\left(\Lambda^{(2)}+2M^{(2)}\right)\left(\frac{\zeta_1^{(2)}}{\eta_0}Y_n^I\left(\Psi_1^{(2)}\right)-\frac{1}{\eta_0^2}Y_n\left(\Psi_1^{(2)}\right)\right)+$$
$$\frac{\Lambda^{(2)}}{\eta_0}\left(-\zeta_1^{(2)}Y_n^I\left(\Psi_1^{(2)}\right)+\frac{1}{\eta_0}Y_n\left(\Psi_1^{(2)}\right)\right),$$

$$\beta_{51}\left(\zeta_1^{(2)},\Psi_1^{(2)}\right)=M^{(2)}\left[\begin{array}{l}\left(-\left(\zeta_1^{(2)}\right)^2 J_1^{II}\left(\Psi_1^{(2)}\right)\right)+\\ \left(-\dfrac{1}{\eta_0^2}J_1\left(\Psi_1^{(2)}\right)+\dfrac{\zeta_1^{(2)}}{\eta_0}J_1^I\left(\Psi_1^{(2)}\right)\right)\end{array}\right],$$

$$\beta_{51}\left(\zeta_1^{(2)},\Psi_1^{(2)}\right)=M^{(2)}\left[\begin{array}{l}\left(-\left(\zeta_1^{(2)}\right)^2 J_1^{II}\left(\Psi_1^{(2)}\right)\right)+\\ \left(-\dfrac{1}{\eta_0^2}J_1\left(\Psi_1^{(2)}\right)+\dfrac{\zeta_1^{(2)}}{\eta_0}J_1^I\left(\Psi_1^{(2)}\right)\right)\end{array}\right],$$

$$\beta_{52}\left(\zeta_1^{(2)},\Psi_1^{(2)}\right)=M^{(2)}\left[\begin{array}{l}\left(-\left(\zeta_1^{(2)}\right)^2 Y_1^{II}\left(\Psi_1^{(2)}\right)\right)+\\ \left(-\dfrac{1}{\eta_0^2}Y_1\left(\Psi_1^{(2)}\right)+\dfrac{\zeta_1^{(2)}}{\eta_0}Y_1^I\left(\Psi_1^{(2)}\right)\right)\end{array}\right],$$

$$\beta_{61}\left(\zeta_1^{(2)},\Psi_1^{(2)}\right)=\frac{M^{(2)}}{\eta_0}J_1\left(\Psi_1^{(2)}\right)$$

$$\beta_{62}\left(\zeta_1^{(2)},\Psi_1^{(2)}\right)=\frac{M^{(2)}}{\eta_0}Y_1\left(\Psi_1^{(2)}\right)$$

$$\beta_{71}\left(\zeta_1^{(2)},\Psi_1^{(2)}\right)=\frac{1}{\eta_0}J_1\left(\Psi_1^{(2)}\right)$$

$$\beta_{72}\left(\zeta_1^{(2)},X_2^{(2)}\right)=\frac{1}{\eta_0}Y_1\left(\Psi_1^{(2)}\right)$$

$$\beta_{81}\left(\zeta_1^{(2)},\Psi_1^{(2)}\right)=-\zeta_1^{(2)}J_1^I\left(\Psi_1^{(2)}\right)$$

$$\beta_{82}\left(\zeta_1^{(2)},\Psi_1^{(2)}\right)=-\zeta_1^{(2)}Y_1^I\left(\Psi_1^{(2)}\right)$$

$$\beta_{13}\left(\zeta_2^{(2)},X_{2h^{(2)}}^{(2)}\right)=\left(\Lambda^{(2)}+2M^{(2)}\right)\left(\left(\zeta_2^{(2)}\right)^2 J_1^{II}\left(X_{2h^{(2)}}^{(2)}\right)\right)+$$
$$\frac{\Lambda^{(2)}}{\eta_1}\left(\frac{1}{\eta_1}J_1\left(X_{2h^{(2)}}^{(2)}\right)+\zeta_2^{(2)}J_1^I\left(X_{2h^{(2)}}^{(2)}\right)\right)+$$
$$\Lambda^{(2)}\left(-\left(\Lambda^{(2)}+2M^{(2)}\right)\left(\beta_2^{(2)}/\left(\Lambda^{(2)}+M^{(2)}\right)J_1\left(X_{2h^{(2)}}^{(2)}\right)\right)\right),$$

$$\beta_{14}\left(\zeta_2^{(2)},X_{2h^{(2)}}^{(2)}\right)=\left(\Lambda^{(2)}+2M^{(2)}\right)\left(\left(\zeta_2^{(2)}\right)^2 Y_1^{II}\left(X_{2h^{(2)}}^{(2)}\right)\right)+$$
$$\frac{\Lambda^{(2)}}{\eta_1}\left(\frac{1}{\eta_1}Y_1\left(X_{2h^{(2)}}^{(2)}\right)+\zeta_2^{(2)}Y_1^I\left(X_{2h^{(2)}}^{(2)}\right)\right)+$$
$$\Lambda^{(2)}\left(-\left(\Lambda^{(2)}+2M^{(2)}\right)\left(\beta_2^{(2)}/\left(\Lambda^{(2)}+M^{(2)}\right)Y_1\left(X_{2h^{(2)}}^{(2)}\right)\right)\right),$$

$$\beta_{23}\left(\zeta_2^{(2)},X_{2h^{(2)}}^{(2)}\right)=M^{(2)}\left[\begin{array}{l}\left(\dfrac{1}{\eta_1^2}J_1\left(X_{2h^{(2)}}^{(2)}\right)-\dfrac{\zeta_2^{(2)}}{\eta_1}J_1^I\left(X_{2h^{(2)}}^{(2)}\right)\right)+\\ \left(-\dfrac{\zeta_2^{(2)}}{\eta_1}J_1^I\left(X_{2h^{(2)}}^{(2)}\right)+\dfrac{1}{\eta_1^2}J_1\left(X_{2h^{(2)}}^{(2)}\right)\right)\end{array}\right],$$

$$\beta_{24}\left(\zeta_2^{(2)},X_{2h^{(2)}}^{(2)}\right)=M^{(2)}\left[\begin{array}{l}\left(\dfrac{1}{\eta_1^2}Y_1\left(X_{2h^{(2)}}^{(2)}\right)-\dfrac{\zeta_2^{(2)}}{\eta_1}Y_1^I\left(X_{2h^{(2)}}^{(2)}\right)\right)+\\ \left(-\dfrac{\zeta_2^{(2)}}{\eta_1}Y_1^I\left(X_{2h^{(2)}}^{(2)}\right)+\dfrac{1}{\eta_1^2}Y_1\left(X_{2h^{(2)}}^{(2)}\right)\right)\end{array}\right],$$

$$\beta_{33}\left(\zeta_2^{(2)},X_{2h^{(2)}}^{(2)}\right)=M^{(2)}\left[\begin{array}{l}\zeta_2^{(2)}J_1^I\left(X_{2h^{(2)}}^{(2)}\right)+\\ \dfrac{\Lambda^{(2)}+2M^{(2)}}{\Lambda^{(2)}+M^{(2)}}\zeta_2^{(2)}\beta_2^{(2)}J_1^I\left(X_{2h^{(2)}}^{(2)}\right)\end{array}\right],$$

$$\beta_{34}\left(\zeta_2^{(2)},X_{2h^{(2)}}^{(2)}\right)=M^{(2)}\left[\begin{array}{l}\zeta_2^{(2)}Y_1^I\left(X_{2h^{(2)}}^{(2)}\right)+\\ \dfrac{\Lambda^{(2)}+2M^{(2)}}{\Lambda^{(2)}+M^{(2)}}\zeta_2^{(2)}\beta_2^{(2)}Y_1^I\left(X_{2h^{(2)}}^{(2)}\right)\end{array}\right],$$

$$\begin{gathered}\beta_{43}\left(\zeta_2^{(2)},X_2^{(2)}\right)=\left(\Lambda^{(2)}+2M^{(2)}\right)\left(\left(\zeta_2^{(2)}\right)^2 J_1^{II}\left(X_2^{(2)}\right)\right)+\\ \frac{\Lambda^{(2)}}{\eta_0}\left(\frac{1}{\eta_0}J_1\left(X_2^{(2)}\right)+\zeta_2^{(2)}J_1^{I}\left(X_2^{(2)}\right)\right)+\\ \Lambda^{(2)}\left(-\left(\Lambda^{(2)}+2M^{(2)}\right)\left(\beta_2^{(2)}/\left(\Lambda^{(2)}+M^{(2)}\right)J_1\left(X_2^{(2)}\right)\right)\right),\end{gathered}$$

$$\begin{gathered}\beta_{44}\left(\zeta_2^{(2)},X_2^{(2)}\right)=\left(\Lambda^{(2)}+2M^{(2)}\right)\left(\left(\zeta_2^{(2)}\right)^2 Y_1^{II}\left(X_2^{(2)}\right)\right)+\\ \frac{\Lambda^{(2)}}{\eta_0}\left(\frac{1}{\eta_0}Y_1\left(X_2^{(2)}\right)+\zeta_2^{(2)}Y_1^{I}\left(X_2^{(2)}\right)\right)+\\ \Lambda^{(2)}\left(-\left(\Lambda^{(2)}+2M^{(2)}\right)\left(\beta_2^{(2)}/\left(\Lambda^{(2)}+M^{(2)}\right)Y_1\left(X_2^{(2)}\right)\right)\right),\end{gathered}$$

$$\beta_{53}\left(\zeta_2^{(2)},X_2^{(2)}\right)=M^{(2)}\left[\begin{gathered}\left(\frac{1}{\eta_0{}^2}J_1\left(X_2^{(2)}\right)-\frac{\zeta_2^{(2)}}{\eta_0}J_1^{I}\left(X_2^{(2)}\right)\right)+\\ \left(-\frac{\zeta_2^{(2)}}{\eta_0}J_1^{I}\left(X_2^{(2)}\right)+\frac{1}{\eta_0{}^2}J_1\left(X_2^{(2)}\right)\right)\end{gathered}\right],$$

$$\beta_{54}\left(\zeta_2^{(2)},X_2^{(2)}\right)=M^{(2)}\left[\begin{gathered}\left(\frac{1}{\eta_0{}^2}Y_1\left(X_2^{(2)}\right)-\frac{\zeta_2^{(2)}}{\eta_0}Y_1^{I}\left(X_2^{(2)}\right)\right)+\\ \left(-\frac{\zeta_2^{(2)}}{\eta_0}Y_1^{I}\left(X_2^{(2)}\right)+\frac{1}{\eta_0{}^2}Y_1\left(X_2^{(2)}\right)\right)\end{gathered}\right],$$

$$\beta_{63}\left(\zeta_2^{(2)},X_2^{(2)}\right)=M^{(2)}\left[\begin{gathered}\zeta_2^{(2)}J_1^{I}\left(X_2^{(2)}\right)+\\ \frac{\Lambda^{(2)}+2M^{(2)}}{\Lambda^{(2)}+M^{(2)}}\zeta_2^{(2)}\beta_2^{(2)}J_1^{I}\left(X_2^{(2)}\right)\end{gathered}\right],$$

$$\beta_{64}\left(\zeta_2^{(2)},X_2^{(2)}\right)=M^{(2)}\left[\begin{gathered}\zeta_2^{(2)}Y_1^{I}\left(X_2^{(2)}\right)+\\ \frac{\Lambda^{(2)}+2M^{(2)}}{\Lambda^{(2)}+M^{(2)}}\zeta_2^{(2)}\beta_2^{(2)}Y_1^{I}\left(X_2^{(2)}\right)\end{gathered}\right],$$

$$\beta_{73}\left(\zeta_2^{(2)},X_2^{(2)}\right)=\zeta_2^{(2)}J_1^{I}\left(X_2^{(2)}\right)$$

$$\beta_{74}\left(\zeta_2^{(2)},X_2^{(2)}\right)=\zeta_2^{(2)}Y_1^{I}\left(X_2^{(2)}\right)$$

$$\beta_{83}\left(\zeta_2^{(2)},X_2^{(2)}\right)=-\frac{1}{\eta_0}J_1\left(X_2^{(2)}\right)$$

$$\beta_{84}\left(\zeta_2^{(2)},X_2^{(2)}\right)=-\frac{1}{\eta_0}Y_1\left(X_2^{(2)}\right)$$

$$\beta_{93}\left(\zeta_2^{(2)},X_2^{(2)}\right)=\frac{\Lambda^{(2)}+2M^{(2)}}{\Lambda^{(2)}+M^{(2)}}\beta_2^{(2)}J_1\left(X_2^{(2)}\right)$$

$$\beta_{94}\left(\zeta_2^{(2)},X_2^{(2)}\right)=\frac{\Lambda^{(2)}+2M^{(2)}}{\Lambda^{(2)}+M^{(2)}}\beta_2^{(2)}Y_1\left(X_2^{(2)}\right)$$

$$\begin{gathered}\beta_{15}\left(\zeta_3^{(2)},X_{3h^{(2)}}^{(2)}\right)=\left(\Lambda^{(2)}+2M^{(2)}\right)\left(\left(\zeta_3^{(2)}\right)^2 J_1^{II}\left(X_{3h^{(2)}}^{(2)}\right)\right)+\\ \frac{\Lambda^{(2)}}{\eta_1}\left(\frac{1}{\eta_1}J_1\left(X_{3h^{(2)}}^{(2)}\right)+\zeta_3^{(2)}J_1^{I}\left(X_{3h^{(2)}}^{(2)}\right)\right)+\\ \Lambda^{(2)}\left(-\left(\Lambda^{(2)}+2M^{(2)}\right)\left(\beta_3^{(2)}/\left(\Lambda^{(2)}+M^{(2)}\right)J_1\left(X_{3h^{(2)}}^{(2)}\right)\right)\right),\end{gathered}$$

$$\begin{gathered}\beta_{16}\left(\zeta_3^{(2)},X_{3h^{(2)}}^{(2)}\right)=\left(\Lambda^{(2)}+2M^{(2)}\right)\left(\left(\zeta_3^{(2)}\right)^2 Y_1^{II}\left(X_{3h^{(2)}}^{(2)}\right)\right)+\\ \frac{\Lambda^{(2)}}{\eta_1}\left(\frac{1}{\eta_1}Y_1\left(X_{3h^{(2)}}^{(2)}\right)+\zeta_3^{(2)}Y_1^{I}\left(X_{3h^{(2)}}^{(2)}\right)\right)+\\ \Lambda^{(2)}\left(-\left(\Lambda^{(2)}+2M^{(2)}\right)\left(\beta_3^{(2)}/\left(\Lambda^{(2)}+M^{(2)}\right)Y_1\left(X_{3h^{(2)}}^{(2)}\right)\right)\right),\end{gathered}$$

$$\beta_{25}\left(\zeta_3^{(2)},X_{3h^{(2)}}^{(2)}\right)=M^{(2)}\left[\begin{gathered}\left(\frac{1}{\eta_1{}^2}J_1\left(X_{3h^{(2)}}^{(2)}\right)-\frac{\zeta_3^{(2)}}{\eta_1}J_1^{I}\left(X_{3h^{(2)}}^{(2)}\right)\right)+\\ \left(-\frac{\zeta_3^{(2)}}{\eta_1}J_1^{I}\left(X_{3h^{(2)}}^{(2)}\right)+\frac{1}{\eta_1{}^2}J_1\left(X_{3h^{(2)}}^{(2)}\right)\right)\end{gathered}\right],$$

$$\beta_{26}\left(\zeta_3^{(2)},X_{3h^{(2)}}^{(2)}\right)=M^{(2)}\left[\begin{gathered}\left(\frac{1}{\eta_1{}^2}Y_1\left(X_{3h^{(2)}}^{(2)}\right)-\frac{\zeta_3^{(2)}}{\eta_1}Y_1^{I}\left(X_{3h^{(2)}}^{(2)}\right)\right)+\\ \left(-\frac{\zeta_3^{(2)}}{\eta_1}Y_1^{I}\left(X_{3h^{(2)}}^{(2)}\right)+\frac{1}{\eta_1{}^2}Y_1\left(X_{3h^{(2)}}^{(2)}\right)\right)\end{gathered}\right],$$

$$\beta_{35}\left(\zeta_3^{(2)},X_{3h^{(2)}}^{(2)}\right)=M^{(2)}\left[\begin{gathered}\zeta_3^{(2)}J_1^{I}\left(X_{3h^{(2)}}^{(2)}\right)+\\ \frac{\Lambda^{(2)}+2M^{(2)}}{\Lambda^{(2)}+M^{(2)}}\zeta_3^{(2)}\beta_3^{(2)}J_1^{I}\left(X_{3h^{(2)}}^{(2)}\right)\end{gathered}\right],$$

$$\beta_{36}\left(\zeta_3^{(2)},X_{3h^{(2)}}^{(2)}\right)=M^{(2)}\left[\begin{gathered}\zeta_3^{(2)}Y_1^{I}\left(X_{3h^{(2)}}^{(2)}\right)+\\ \frac{\Lambda^{(2)}+2M^{(2)}}{\Lambda^{(2)}+M^{(2)}}\zeta_3^{(2)}\beta_3^{(2)}Y_1^{I}\left(X_{3h^{(2)}}^{(2)}\right)\end{gathered}\right],$$

$$\beta_{36}\left(\zeta_3^{(2)},X_{3h^{(2)}}^{(2)}\right)=M^{(2)}\left[\begin{gathered}\zeta_3^{(2)}Y_1^{I}\left(X_{3h^{(2)}}^{(2)}\right)+\\ \frac{\Lambda^{(2)}+2M^{(2)}}{\Lambda^{(2)}+M^{(2)}}\zeta_3^{(2)}\beta_3^{(2)}Y_1^{I}\left(X_{3h^{(2)}}^{(2)}\right)\end{gathered}\right],$$

$$\begin{gathered}\beta_{45}\left(\zeta_3^{(2)},X_3^{(2)}\right)=\left(\Lambda^{(2)}+2M^{(2)}\right)\left(\left(\zeta_3^{(2)}\right)^2 J_1^{II}\left(X_3^{(2)}\right)\right)+\\ \frac{\Lambda^{(2)}}{\eta_0}\left(\frac{1}{\eta_0}J_1\left(X_3^{(2)}\right)+\zeta_3^{(2)}J_1^{I}\left(X_3^{(2)}\right)\right)+\\ \Lambda^{(2)}\left(-\left(\Lambda^{(2)}+2M^{(2)}\right)\left(\beta_3^{(2)}/\left(\Lambda^{(2)}+M^{(2)}\right)J_1\left(X_3^{(2)}\right)\right)\right),\end{gathered}$$

$$\beta_{46}\left(\zeta_3^{(2)},X_3^{(2)}\right)=\left(\Lambda^{(2)}+2M^{(2)}\right)\left(\left(\zeta_3^{(2)}\right)^2 Y_1^{II}\left(X_3^{(2)}\right)\right)+$$
$$\frac{\Lambda^{(2)}}{\eta_0}\left(\frac{1}{\eta_0}Y_1\left(X_3^{(2)}\right)+\zeta_3^{(2)}Y_1^{I}\left(X_3^{(2)}\right)\right)+$$
$$\Lambda^{(2)}\left(-\left(\Lambda^{(2)}+2M^{(2)}\right)\left(\beta_3^{(2)}/\left(\Lambda^{(2)}+M^{(2)}\right)Y_1\left(X_3^{(2)}\right)\right)\right),$$

$$\beta_{55}\left(\zeta_3^{(2)},X_3^{(2)}\right)=M^{(2)}\left[\begin{array}{l}\left(\frac{1}{\eta_0^{\,2}}J_1\left(X_3^{(2)}\right)-\frac{\zeta_3^{(2)}}{\eta_0}J_1^{I}\left(X_3^{(2)}\right)\right)+\\ \left(-\frac{\zeta_3^{(2)}}{\eta_0}J_1^{I}\left(X_3^{(2)}\right)+\frac{1}{\eta_0^{\,2}}J_1\left(X_3^{(2)}\right)\right)\end{array}\right],$$

$$\beta_{56}\left(\zeta_3^{(2)},X_3^{(2)}\right)=M^{(2)}\left[\begin{array}{l}\left(\frac{1}{\eta_0^{\,2}}Y_1\left(X_3^{(2)}\right)-\frac{\zeta_3^{(2)}}{\eta_0}Y_1^{I}\left(X_3^{(2)}\right)\right)+\\ \left(-\frac{\zeta_3^{(2)}}{\eta_0}Y_1^{I}\left(X_3^{(2)}\right)+\frac{1}{\eta_0^{\,2}}Y_1\left(X_3^{(2)}\right)\right)\end{array}\right],$$

$$\beta_{65}\left(\zeta_3^{(2)},X_3^{(2)}\right)=M^{(2)}\left[\begin{array}{l}\zeta_3^{(2)}J_1^{I}\left(X_3^{(2)}\right)+\\ \frac{\Lambda^{(2)}+2M^{(2)}}{\Lambda^{(2)}+M^{(2)}}\zeta_3^{(2)}\beta_3^{(2)}J_1^{I}\left(X_3^{(2)}\right)\end{array}\right],$$

$$\beta_{66}\left(\zeta_3^{(2)},X_3^{(2)}\right)=M^{(2)}\left[\begin{array}{l}\zeta_3^{(2)}Y_1^{I}\left(X_3^{(2)}\right)+\\ \frac{\Lambda^{(2)}+2M^{(2)}}{\Lambda^{(2)}+M^{(2)}}\zeta_3^{(2)}\beta_3^{(2)}Y_1^{I}\left(X_3^{(2)}\right)\end{array}\right],$$

$$\beta_{75}\left(\zeta_3^{(2)},X_3^{(2)}\right)=\zeta_3^{(2)}J_1^{I}\left(X_3^{(2)}\right)$$

$$\beta_{76}\left(\zeta_3^{(2)},X_3^{(2)}\right)=\zeta_3^{(2)}Y_1^{I}\left(X_3^{(2)}\right)$$

$$\beta_{85}\left(\zeta_3^{(2)},X_3^{(2)}\right)=-\frac{1}{\eta_0}J_1\left(X_3^{(2)}\right)$$

$$\beta_{86}\left(\zeta_3^{(2)},X_3^{(2)}\right)=-\frac{1}{\eta_0}Y_1\left(X_3^{(2)}\right)$$

$$\beta_{95}\left(\zeta_3^{(2)},X_3^{(2)}\right)=\frac{\Lambda^{(2)}+2M^{(2)}}{\Lambda^{(2)}+M^{(2)}}\beta_3^{(2)}J_1\left(X_3^{(2)}\right)$$

$$\beta_{96}\left(\zeta_3^{(2)},X_3^{(2)}\right)=\frac{\Lambda^{(2)}+2M^{(2)}}{\Lambda^{(2)}+M^{(2)}}\beta_3^{(2)}Y_1\left(X_3^{(2)}\right)$$

$$\beta_{47}\left(\zeta_1^{(1)},\Psi_1^{(1)}\right)=-\frac{\mu_0^{(2)}}{\mu_0^{(1)}}\left[\begin{array}{l}\left(\Lambda^{(1)}+2M^{(1)}\right)\left(\frac{\zeta_1^{(1)}}{\eta_0}J_1^{I}\left(\Psi_1^{(1)}\right)-\frac{1}{\eta_0^{\,2}}J_1\left(\Psi_1^{(1)}\right)\right)+\\ \frac{\Lambda^{(2)}}{\eta_0}\left(-\zeta_1^{(1)}J_1^{I}\left(\Psi_1^{(1)}\right)+\frac{1}{\eta_0}J_1\left(\Psi_1^{(1)}\right)\right)\end{array}\right],$$

$$\beta_{48}\left(\zeta_1^{(1)},\Psi_1^{(1)}\right)=-\frac{\mu_0^{(2)}}{\mu_0^{(1)}}\left[\begin{array}{l}\left(\Lambda^{(1)}+2M^{(1)}\right)\left(\frac{\zeta_1^{(1)}}{\eta_0}Y_1^{I}\left(\Psi_1^{(1)}\right)-\frac{1}{\eta_0^{\,2}}Y_1\left(\Psi_1^{(1)}\right)\right)+\\ \frac{\Lambda^{(1)}}{\eta_0}\left(-\zeta_1^{(1)}Y_1^{I}\left(\Psi_1^{(1)}\right)+\frac{1}{\eta_0}Y_1\left(\Psi_1^{(1)}\right)\right)\end{array}\right],$$

$$\beta_{57}\left(\zeta_1^{(1)},\Psi_1^{(1)}\right)=-\frac{\mu_0^{(2)}}{\mu_0^{(1)}}M^{(1)}\left[\begin{array}{l}\left(-\left(\zeta_1^{(1)}\right)^2 J_1^{II}\left(\Psi_1^{(1)}\right)\right)\\ +\left(-\frac{1}{\eta_0^{\,2}}J_1\left(\Psi_1^{(1)}\right)+\frac{\zeta_1^{(1)}}{\eta_0}J_1^{I}\left(\Psi_1^{(1)}\right)\right)\end{array}\right],$$

$$\beta_{58}\left(\zeta_1^{(1)},\Psi_1^{(1)}\right)=-\frac{\mu_0^{(2)}}{\mu_0^{(1)}}M^{(1)}\left[\begin{array}{l}\left(-\left(\zeta_1^{(1)}\right)^2 Y_1^{II}\left(\Psi_1^{(1)}\right)\right)+\\ \left(-\frac{1}{\eta_0^{\,2}}Y_1\left(\Psi_1^{(1)}\right)+\frac{\zeta_1^{(1)}}{\eta_0}Y_1^{I}\left(\Psi_1^{(1)}\right)\right)\end{array}\right],$$

$$\beta_{67}\left(\zeta_1^{(1)},\Psi_1^{(1)}\right)=-\frac{\mu_0^{(2)}}{\mu_0^{(1)}}\frac{M^{(1)}}{\eta_0}J_1\left(\Psi_1^{(1)}\right)$$

$$\beta_{68}\left(\zeta_1^{(1)},\Psi_1^{(1)}\right)=-\frac{\mu_0^{(2)}}{\mu_0^{(1)}}\frac{M^{(1)}}{\eta_0}Y_1\left(\Psi_1^{(1)}\right)$$

$$\beta_{77}\left(\zeta_1^{(1)},\Psi_1^{(1)}\right)=-\frac{\mu_0^{(2)}}{\mu_0^{(1)}}\frac{1}{\eta_0}J_1\left(\Psi_1^{(1)}\right)$$

$$\beta_{78}\left(\zeta_1^{(1)},X_2^{(1)}\right)=-\frac{\mu_0^{(2)}}{\mu_0^{(1)}}\frac{1}{\eta_0}Y_1\left(\Psi_1^{(1)}\right)$$

$$\beta_{87}\left(\zeta_1^{(1)},\Psi_1^{(1)}\right)=\zeta_1^{(1)}J_1^{I}\left(\Psi_1^{(1)}\right)$$

$$\beta_{88}\left(\zeta_1^{(1)},\Psi_1^{(1)}\right)=\zeta_1^{(1)}Y_1^{I}\left(\Psi_1^{(1)}\right)$$

$$\beta_{107}\left(\zeta_1^{(1)},\Psi_{2h^{(1)}}^{(1)}\right)=\frac{\mu_0^{(2)}}{\mu_0^{(1)}}\left[\begin{array}{l}\left(\Lambda^{(1)}+2M^{(1)}\right)\left(\begin{array}{l}\frac{\zeta_1^{(1)}}{\eta_2}J_1^{I}\left(\Psi_{2h^{(1)}}^{(1)}\right)\\ -\frac{1}{\eta_2^{\,2}}J_1\left(\Psi_{2h^{(1)}}^{(1)}\right)\end{array}\right)+\\ \frac{\Lambda^{(2)}}{\eta_2}\left(-\zeta_1^{(1)}J_1^{I}\left(\Psi_{2h^{(1)}}^{(1)}\right)+\frac{1}{\eta_2}J_1\left(\Psi_{2h^{(1)}}^{(1)}\right)\right)\end{array}\right],$$

$$\beta_{108}\left(\zeta_1^{(1)},\Psi_{2h^{(1)}}^{(1)}\right)=\frac{\mu_0^{(2)}}{\mu_0^{(1)}}\left[\begin{array}{l}\left(\Lambda^{(1)}+2M^{(1)}\right)\left(\begin{array}{l}\frac{\zeta_1^{(1)}}{\eta_2}Y_1^{I}\left(\Psi_{2h^{(1)}}^{(1)}\right)\\ -\frac{1}{\eta_2^{\,2}}Y_1\left(\Psi_{2h^{(1)}}^{(1)}\right)\end{array}\right)+\\ \frac{\Lambda^{(2)}}{\eta_2}\left(-\zeta_1^{(1)}Y_1^{I}\left(\Psi_{2h^{(1)}}^{(1)}\right)+\frac{1}{\eta_2}Y_1\left(\Psi_{2h^{(1)}}^{(1)}\right)\right)\end{array}\right],$$

$$\beta_{117}\left(\zeta_1^{(1)},\Psi_{2h^{(1)}}^{(1)}\right)=\frac{\mu_0^{(2)}}{\mu_0^{(1)}}M^{(1)}\left[\begin{array}{l}\left(-\left(\zeta_1^{(1)}\right)^2 J_1^{II}\left(\Psi_{2h^{(1)}}^{(1)}\right)\right)+\\ \left(-\frac{1}{\eta_2{}^2}J_1\left(\Psi_{2h^{(1)}}^{(1)}\right)+\frac{\zeta_1^{(1)}}{\eta_2}J_1^I\left(\Psi_{2h^{(1)}}^{(1)}\right)\right)\end{array}\right],$$

$$\beta_{118}\left(\zeta_1^{(1)},\Psi_{2h^{(1)}}^{(1)}\right)=\frac{\mu_0^{(2)}}{\mu_0^{(1)}}M^{(1)}\left[\begin{array}{l}\left(-\left(\zeta_1^{(1)}\right)^2 Y_1^{II}\left(\Psi_{2h^{(1)}}^{(1)}\right)\right)+\\ \left(-\frac{1}{\eta_2{}^2}Y_1\left(\Psi_{2h^{(1)}}^{(1)}\right)+\frac{\zeta_1^{(1)}}{\eta_2}Y_1^I\left(\Psi_{2h^{(1)}}^{(1)}\right)\right)\end{array}\right],$$

$$\beta_{127}\left(\zeta_1^{(1)},\Psi_{2h^{(1)}}^{(1)}\right)=\frac{\mu_0^{(2)}}{\mu_0^{(1)}}\frac{M^{(1)}}{\eta_2}J_1\left(\Psi_{2h^{(1)}}^{(1)}\right)$$

$$\beta_{128}\left(\zeta_1^{(1)},\Psi_{2h^{(1)}}^{(1)}\right)=\frac{\mu_0^{(2)}}{\mu_0^{(1)}}\frac{M^{(1)}}{\eta_2}Y_1\left(\Psi_{2h^{(1)}}^{(1)}\right)$$

$$\beta_{49}\left(\zeta_2^{(1)},X_2^{(1)}\right)=-\frac{\mu_0^{(2)}}{\mu_0^{(1)}}\left[\begin{array}{l}\left(\Lambda^{(1)}+2M^{(1)}\right)\left(\left(\zeta_2^{(1)}\right)^2 J_1^{II}\left(X_2^{(1)}\right)\right)+\\ \frac{\Lambda^{(1)}}{\eta_0}\left(\frac{1}{\eta_0}J_1\left(X_2^{(1)}\right)+\zeta_2^{(1)}J_1^I\left(X_2^{(1)}\right)\right)+\\ \Lambda^{(1)}\left(\begin{array}{l}-\left(\Lambda^{(1)}+2M^{(1)}\right)\times\\ \left(\beta_2^{(1)}/\left(\Lambda^{(1)}+M^{(1)}\right)J_1\left(X_2^{(1)}\right)\right)\end{array}\right)\end{array}\right],$$

$$\beta_{59}\left(\zeta_2^{(1)},X_2^{(1)}\right)=-\frac{\mu_0^{(2)}}{\mu_0^{(1)}}M^{(1)}\left[\begin{array}{l}\left(\frac{1}{\eta_0{}^2}J_1\left(X_2^{(1)}\right)-\frac{\zeta_2^{(1)}}{\eta_0}J_1^I\left(X_2^{(1)}\right)\right)+\\ \left(-\frac{\zeta_2^{(1)}}{\eta_0}J_1^I\left(X_2^{(1)}\right)+\frac{1}{\eta_0{}^2}J_1\left(X_2^{(1)}\right)\right)\end{array}\right],$$

$$\beta_{510}\left(\zeta_2^{(1)},X_2^{(1)}\right)=-\frac{\mu_0^{(2)}}{\mu_0^{(1)}}M^{(1)}\left[\begin{array}{l}\left(\frac{1}{\eta_0{}^2}Y_1\left(X_2^{(1)}\right)-\frac{\zeta_2^{(1)}}{\eta_0}Y_1^I\left(X_2^{(1)}\right)\right)+\\ \left(-\frac{\zeta_2^{(1)}}{\eta_0}Y_1^I\left(X_2^{(1)}\right)+\frac{1}{\eta_0{}^2}Y_1\left(X_2^{(1)}\right)\right)\end{array}\right],$$

$$\beta_{69}\left(\zeta_2^{(1)},X_2^{(1)}\right)=-\frac{\mu_0^{(2)}}{\mu_0^{(1)}}M^{(1)}\left[\begin{array}{l}\zeta_2^{(1)}J_1^I\left(X_2^{(1)}\right)+\\ \frac{\Lambda^{(1)}+2M^{(1)}}{\Lambda^{(1)}+M^{(1)}}\zeta_2^{(1)}\beta_2^{(1)}J_1^I\left(X_2^{(1)}\right)\end{array}\right],$$

$$\beta_{610}\left(\zeta_2^{(1)},X_2^{(1)}\right)=-\frac{\mu_0^{(2)}}{\mu_0^{(1)}}M^{(1)}\left[\begin{array}{l}\zeta_2^{(1)}Y_1^I\left(X_2^{(1)}\right)+\\ \frac{\Lambda^{(1)}+2M^{(1)}}{\Lambda^{(1)}+M^{(1)}}\zeta_2^{(1)}\beta_2^{(1)}Y_1^I\left(X_2^{(1)}\right)\end{array}\right],$$

$$\beta_{79}\left(\zeta_2^{(1)},X_2^{(1)}\right)=-\zeta_2^{(1)}J_1^I\left(X_2^{(1)}\right)$$

$$\beta_{710}\left(\zeta_2^{(1)},X_2^{(1)}\right)=-\zeta_2^{(1)}Y_1^I\left(X_2^{(1)}\right)$$

$$\beta_{89}\left(\zeta_2^{(1)},X_2^{(1)}\right)=\frac{1}{\eta_0}J_1\left(X_2^{(1)}\right)$$

$$\beta_{810}\left(\zeta_2^{(1)},X_2^{(1)}\right)=\frac{1}{\eta_0}Y_1\left(X_2^{(1)}\right)$$

$$\beta_{99}\left(\zeta_2^{(1)},X_2^{(1)}\right)=\frac{\Lambda^{(1)}+2M^{(1)}}{\Lambda^{(1)}+M^{(1)}}\beta_2^{(1)}J_1\left(X_2^{(1)}\right)$$

$$\beta_{910}\left(\zeta_2^{(1)},X_2^{(1)}\right)=\frac{\Lambda^{(1)}+2M^{(1)}}{\Lambda^{(1)}+M^{(1)}}\beta_2^{(1)}Y_1\left(X_2^{(1)}\right)$$

$$\beta_{119}\left(\zeta_2^{(1)},X_{2h^{(2)}}^{(1)}\right)=\frac{\mu_0^{(2)}}{\mu_0^{(1)}}M^{(1)}\left[\begin{array}{l}\left(\frac{1}{\eta_2{}^2}J_1\left(X_{2h^{(1)}}^{(1)}\right)-\frac{\zeta_2^{(1)}}{\eta_2}J_1^I\left(X_{2h^{(1)}}^{(1)}\right)\right)+\\ \left(-\frac{\zeta_2^{(1)}}{\eta_2}J_1^I\left(X_{2h^{(1)}}^{(1)}\right)+\frac{1}{\eta_2{}^2}J_1\left(X_{2h^{(1)}}^{(1)}\right)\right)\end{array}\right],$$

$$\beta_{711}\left(\zeta_3^{(1)},X_3^{(1)}\right)=-\zeta_3^{(1)}J_1^I\left(X_3^{(1)}\right)$$

$$\beta_{712}\left(\zeta_3^{(1)},X_3^{(1)}\right)=-\zeta_3^{(1)}Y_1^I\left(X_3^{(1)}\right)$$

$$\beta_{109}\left(\zeta_2^{(1)},X_{2h^{(1)}}^{(1)}\right)=\frac{\mu_0^{(2)}}{\mu_0^{(1)}}\left[\begin{array}{l}\left(\Lambda^{(1)}+2M^{(1)}\right)\left(\left(\zeta_2^{(1)}\right)^2 J_1^{II}\left(X_{2h^{(1)}}^{(1)}\right)\right)+\\ \frac{\Lambda^{(1)}}{\eta_2}\left(\frac{1}{\eta_2}J_1\left(X_{2h^{(1)}}^{(1)}\right)+\zeta_2^{(1)}J_1^I\left(X_{2h^{(1)}}^{(1)}\right)\right)+\\ \Lambda^{(1)}\left(\begin{array}{l}-\left(\Lambda^{(1)}+2M^{(1)}\right)\times\\ \left(\beta_2^{(1)}/\left(\Lambda^{(1)}+M^{(1)}\right)J_1\left(X_{2h^{(1)}}^{(1)}\right)\right)\end{array}\right)\end{array}\right],$$

$$\beta_{1010}\left(\zeta_2^{(1)},X_{2h^{(1)}}^{(1)}\right)=\frac{\mu_0^{(2)}}{\mu_0^{(1)}}\left[\begin{array}{l}\left(\Lambda^{(1)}+2M^{(1)}\right)\left(\left(\zeta_2^{(1)}\right)^2 Y_1^{II}\left(X_{2h^{(1)}}^{(1)}\right)\right)+\\ \frac{\Lambda^{(1)}}{\eta_2}\left(\frac{1}{\eta_2}Y_1\left(X_{2h^{(1)}}^{(1)}\right)+\zeta_2^{(1)}Y_1^I\left(X_{2h^{(1)}}^{(1)}\right)\right)+\\ \Lambda^{(1)}\left(\begin{array}{l}-\left(\Lambda^{(1)}+2M^{(1)}\right)\times\\ \left(\beta_2^{(1)}/\left(\Lambda^{(1)}+M^{(1)}\right)Y_1\left(X_{2h^{(1)}}^{(1)}\right)\right)\end{array}\right)\end{array}\right],$$

$$\beta_{1110}\left(\zeta_2^{(1)},X_{2h^{(2)}}^{(1)}\right)=\frac{\mu_0^{(2)}}{\mu_0^{(1)}}M^{(1)}\begin{bmatrix}\left(\frac{1}{\eta_2{}^2}Y_1\left(X_{2h^{(1)}}^{(1)}\right)-\frac{\zeta_2^{(1)}}{\eta_2}Y_1^I\left(X_{2h^{(1)}}^{(1)}\right)\right)+\\ \left(-\frac{\zeta_2^{(1)}}{\eta_2}Y_1^I\left(X_{2h^{(1)}}^{(1)}\right)+\frac{1}{\eta_2{}^2}Y_1\left(X_{2h^{(1)}}^{(1)}\right)\right)\end{bmatrix},$$

$$\beta_{129}\left(\zeta_2^{(1)},X_{2h^{(1)}}^{(1)}\right)=\frac{\mu_0^{(2)}}{\mu_0^{(1)}}M^{(1)}\begin{bmatrix}\zeta_2^{(1)}J_1^I\left(X_{2h^{(1)}}^{(1)}\right)+\\ \frac{\Lambda^{(1)}+2M^{(1)}}{\Lambda^{(1)}+M^{(1)}}\zeta_2^{(1)}\beta_2^{(1)}J_1^I\left(X_{2h^{(1)}}^{(1)}\right)\end{bmatrix},$$

$$\beta_{1210}\left(\zeta_2^{(1)},X_{2h^{(1)}}^{(1)}\right)=\frac{\mu_0^{(2)}}{\mu_0^{(1)}}M^{(1)}\begin{bmatrix}\zeta_2^{(1)}Y_1^I\left(X_{2h^{(1)}}^{(1)}\right)+\\ \frac{\Lambda^{(1)}+2M^{(1)}}{\Lambda^{(1)}+M^{(1)}}\zeta_2^{(1)}\beta_2^{(1)}Y_1^I\left(X_{2h^{(1)}}^{(1)}\right)\end{bmatrix},$$

$$\beta_{411}\left(\zeta_3^{(1)},X_3^{(1)}\right)=-\frac{\mu_0^{(2)}}{\mu_0^{(1)}}\begin{bmatrix}\left(\Lambda^{(1)}+2M^{(1)}\right)\left(\left(\zeta_3^{(1)}\right)^2J_1^{II}\left(X_3^{(1)}\right)\right)+\\ \frac{\Lambda^{(1)}}{\eta_0}\left(\frac{1}{\eta_0}J_1\left(X_3^{(1)}\right)+\zeta_3^{(1)}J_1^I\left(X_3^{(1)}\right)\right)+\\ \Lambda^{(1)}\begin{pmatrix}-\left(\Lambda^{(1)}+2M^{(1)}\right)\times\\ \left(\beta_3^{(1)}/\left(\Lambda^{(1)}+M^{(1)}\right)J_1\left(X_3^{(1)}\right)\right)\end{pmatrix}\end{bmatrix},$$

$$\beta_{412}\left(\zeta_3^{(1)},X_3^{(1)}\right)=-\frac{\mu_0^{(2)}}{\mu_0^{(1)}}\begin{bmatrix}\left(\Lambda^{(1)}+2M^{(1)}\right)\left(\left(\zeta_3^{(1)}\right)^2Y_1^{II}\left(X_3^{(1)}\right)\right)+\\ \frac{\Lambda^{(1)}}{\eta_0}\left(\frac{1}{\eta_0}Y_1\left(X_3^{(1)}\right)+\zeta_3^{(1)}Y_1^I\left(X_3^{(1)}\right)\right)+\\ \Lambda^{(1)}\begin{pmatrix}-\left(\Lambda^{(1)}+2M^{(1)}\right)\times\\ \left(\beta_3^{(1)}/\left(\Lambda^{(1)}+M^{(1)}\right)Y_1\left(X_3^{(1)}\right)\right)\end{pmatrix}\end{bmatrix},$$

$$\beta_{511}\left(\zeta_3^{(1)},X_3^{(1)}\right)=-\frac{\mu_0^{(2)}}{\mu_0^{(1)}}M^{(1)}\begin{bmatrix}\left(\frac{1}{\eta_0{}^2}J_1\left(X_3^{(1)}\right)-\frac{\zeta_3^{(1)}}{\eta_0}J_1^I\left(X_3^{(1)}\right)\right)+\\ \left(-\frac{\zeta_3^{(1)}}{\eta_0}J_1^I\left(X_3^{(1)}\right)+\frac{1}{\eta_0{}^2}J_1\left(X_3^{(1)}\right)\right)\end{bmatrix},$$

$$\beta_{512}\left(\zeta_3^{(1)},X_3^{(1)}\right)=-\frac{\mu_0^{(2)}}{\mu_0^{(1)}}M^{(1)}\begin{bmatrix}\left(\frac{1}{\eta_0{}^2}Y_1\left(X_3^{(1)}\right)-\frac{\zeta_3^{(1)}}{\eta_0}Y_1^I\left(X_3^{(1)}\right)\right)+\\ \left(-\frac{\zeta_3^{(1)}}{\eta_0}Y_1^I\left(X_3^{(1)}\right)+\frac{1}{\eta_0{}^2}Y_1\left(X_3^{(1)}\right)\right)\end{bmatrix},$$

$$\beta_{6\,11}\left(\zeta_3^{(1)},X_3^{(1)}\right)=-\frac{\mu_0^{(2)}}{\mu_0^{(1)}}M^{(1)}\begin{bmatrix}\zeta_3^{(1)}J_1^I\left(X_3^{(1)}\right)+\\ \frac{\Lambda^{(1)}+2M^{(1)}}{\Lambda^{(1)}+M^{(1)}}\zeta_3^{(1)}\beta_3^{(1)}J_1^I\left(X_3^{(1)}\right)\end{bmatrix},$$

$$\beta_{6\,12}\left(\zeta_3^{(1)},X_3^{(1)}\right)=-\frac{\mu_0^{(2)}}{\mu_0^{(1)}}\begin{bmatrix}M^{(1)}\zeta_3^{(1)}Y_1^I\left(X_3^{(1)}\right)+\\ M^{(1)}\frac{\Lambda^{(1)}+2M^{(1)}}{\Lambda^{(1)}+M^{(1)}}\zeta_3^{(1)}\beta_3^{(1)}Y_1^I\left(X_3^{(1)}\right)\end{bmatrix},$$

$$\beta_{811}\left(\zeta_3^{(1)},X_3^{(1)}\right)=\frac{1}{\eta_0}J_1\left(X_3^{(1)}\right)$$

$$\beta_{812}\left(\zeta_3^{(1)},X_3^{(1)}\right)=\frac{1}{\eta_0}Y_1\left(X_3^{(1)}\right)$$

$$\beta_{1011}\left(\zeta_3^{(1)},X_{3h^{(1)}}^{(1)}\right)=\frac{\mu_0^{(2)}}{\mu_0^{(1)}}\begin{bmatrix}\left(\Lambda^{(1)}+2M^{(1)}\right)\left(\left(\zeta_3^{(1)}\right)^2J_1^{II}\left(X_{3h^{(1)}}^{(1)}\right)\right)+\\ \frac{\Lambda^{(1)}}{\eta_2}\left(\frac{1}{\eta_2}J_1\left(X_{3h^{(1)}}^{(1)}\right)+\zeta_3^{(1)}J_1^I\left(X_{3h^{(1)}}^{(1)}\right)\right)+\\ \Lambda^{(1)}\begin{pmatrix}-\left(\Lambda^{(1)}+2M^{(1)}\right)\times\\ \left(\beta_3^{(1)}/\left(\Lambda^{(1)}+M^{(1)}\right)J_1\left(X_{3h^{(1)}}^{(1)}\right)\right)\end{pmatrix}\end{bmatrix},$$

$$\beta_{911}\left(\zeta_3^{(1)},X_3^{(1)}\right)=-\frac{\Lambda^{(1)}+2M^{(1)}}{\Lambda^{(1)}+M^{(1)}}\beta_3^{(1)}J_1\left(X_3^{(1)}\right)$$

$$\beta_{1012}\left(\zeta_3^{(1)},X_{3h^{(1)}}^{(1)}\right)=\frac{\mu_0^{(2)}}{\mu_0^{(1)}}\begin{bmatrix}\left(\Lambda^{(1)}+2M^{(1)}\right)\left(\left(\zeta_3^{(1)}\right)^2Y_1^{II}\left(X_{3h^{(1)}}^{(1)}\right)\right)+\\ \frac{\Lambda^{(1)}}{\eta_2}\left(\frac{1}{\eta_2}Y_1\left(X_{3h^{(1)}}^{(1)}\right)+\zeta_3^{(1)}Y_1^I\left(X_{3h^{(1)}}^{(1)}\right)\right)+\\ \Lambda^{(1)}\begin{pmatrix}-\left(\Lambda^{(1)}+2M^{(1)}\right)\times\\ \left(\beta_3^{(1)}/\left(\Lambda^{(1)}+M^{(1)}\right)Y_1\left(X_{3h^{(1)}}^{(1)}\right)\right)\end{pmatrix}\end{bmatrix},$$

$$\beta_{912}\left(\zeta_3^{(1)},X_3^{(1)}\right)=-\frac{\Lambda^{(1)}+2M^{(1)}}{\Lambda^{(1)}+M^{(1)}}\beta_3^{(1)}Y_1\left(X_3^{(1)}\right)$$

$$\beta_{1111}\left(\zeta_3^{(1)},X_{3h^{(1)}}^{(1)}\right)=\frac{\mu_0^{(2)}}{\mu_0^{(1)}}M^{(1)}\begin{bmatrix}\left(\frac{1}{\eta_2{}^2}J_1\left(X_{3h^{(1)}}^{(1)}\right)-\frac{\zeta_3^{(1)}}{\eta_2}J_1^I\left(X_{3h^{(1)}}^{(1)}\right)\right)+\\ \left(-\frac{\zeta_3^{(1)}}{\eta_2}J_1^I\left(X_{3h^{(1)}}^{(1)}\right)+\frac{1}{\eta_2{}^2}J_1\left(X_{3h^{(1)}}^{(1)}\right)\right)\end{bmatrix},$$

$$\beta_{1112}\left(\zeta_3^{(1)},X_{3h^{(1)}}^{(1)}\right)=\frac{\mu_0^{(2)}}{\mu_0^{(1)}}M^{(1)}\begin{bmatrix}\left(\frac{1}{\eta_2{}^2}Y_1\left(X_{3h^{(1)}}^{(1)}\right)-\frac{\zeta_3^{(1)}}{\eta_2}Y_1^I\left(X_{3h^{(1)}}^{(1)}\right)\right)+\\ \left(-\frac{\zeta_3^{(1)}}{\eta_2}Y_1^I\left(X_{3h^{(1)}}^{(1)}\right)+\frac{1}{\eta_2{}^2}Y_1\left(X_{3h^{(1)}}^{(1)}\right)\right)\end{bmatrix},$$

$$\beta_{1211}\left(\zeta_3^{(1)}, X_{3h^{(1)}}^{(1)}\right)=\frac{\mu_0^{(2)}}{\mu_0^{(1)}}M^{(1)}\begin{bmatrix}\zeta_3^{(1)}J_1^I\left(X_{3h^{(1)}}^{(1)}\right)+\\ \frac{\Lambda^{(1)}+2M^{(1)}}{\Lambda^{(1)}+M^{(1)}}\zeta_3^{(1)}\beta_3^{(1)}J_1^I\left(X_{3h^{(1)}}^{(1)}\right)\end{bmatrix},$$

$$\beta_{1212}\left(\zeta_3^{(1)}, X_{3h^{(1)}}^{(1)}\right)=\frac{\mu_0^{(2)}}{\mu_0^{(1)}}M^{(1)}\begin{bmatrix}\zeta_3^{(1)}Y_1^I\left(X_{3h^{(1)}}^{(1)}\right)+\\ \frac{\Lambda^{(1)}+2M^{(1)}}{\Lambda^{(1)}+M^{(1)}}\zeta_3^{(1)}\beta_3^{(1)}Y_1^I\left(X_{3h^{(1)}}^{(1)}\right)\end{bmatrix},$$

$$\beta_{91}=\beta_{92}=\beta_{97}=\beta_{98}=0$$

$$\beta_{1n}=0,\beta_{2n}=0,\beta_{3n}=0,\beta_{10m}=0,\beta_{11m}=0,\beta_{12m}=0 \quad ,$$

n=7,8..12 and m=1,2..6

B(1)

Where

$$J_1^I(\mathrm{x})=\frac{dJ_1}{dx},\ J_1^{II}(\mathrm{x})=\frac{d^2J_1}{dx^2},\ Y_1^I(\mathrm{x})=\frac{dY_1}{dx},\ Y_1^{II}(\mathrm{x})=\frac{d^2Y_1}{dx^2}$$

$$\Psi_1^{(1)}=kR\zeta_1^{(1)},\ \Psi_1^{(2)}=kR\zeta_1^{(2)},\ X_2^{(1)}=kR\zeta_2^{(1)},$$

$$X_2^{(2)}=kR\zeta_2^{(2)},\ X_3^{(1)}=kR\zeta_3^{(1)},\ X_3^{(2)}=kR\zeta_3^{(2)}$$

$$\Psi_{2h^{(2)}}^{(2)}=kR\left(1-\frac{h^{(2)}}{R}\right)\zeta_1^{(2)}, X_{2h^{(2)}}^{(2)}=kR\left(1-\frac{h^{(2)}}{R}\right)\zeta_2^{(2)}$$

$$X_{3h^{(2)}}^{(2)}=kR\left(1-\frac{h^{(2)}}{R}\right)\zeta_3^{(2)},\ \Psi_{2h^{(1)}}^{(1)}=kR\left(1+\frac{h^{(1)}}{R}\right)\zeta_1^{(1)}$$

$$X_{2h^{(1)}}^{(1)}=kR\left(1+\frac{h^{(1)}}{R}\right)\zeta_2^{(1)}, X_{3h^{(1)}}^{(1)}=kR\left(1+\frac{h^{(1)}}{R}\right)\zeta_3^{(1)}$$

$$\eta_0=kR,\ \eta_1=kR\left(1-h^{(2)}\right),\eta_2=kR\left(1+h^{(1)}\right)$$

$$X_{3h^{(2)}}^{(2)}=kR\left(1-\frac{h^{(2)}}{R}\right)\zeta_3^{(2)},\ X_{3h^{(1)}}^{(1)}=kR\left(1+\frac{h^{(1)}}{R}\right)\zeta_3^{(1)}$$

$$\eta_0=kR,\ \eta_1=kR\left(1-h^{(2)}\right),\eta_2=kR\left(1+h^{(1)}\right),$$

$$\beta_2^{(m)}=-\left(\zeta_2^{(m)}\right)^2-\frac{M^{(m)}}{\Lambda^{(m)}+2M^{(m)}}\left(1-\frac{\omega/k}{M^{(m)}}\right),$$

$$\beta_3^{(m)}=-\left(\zeta_3^{(m)}\right)^2-\frac{M^{(m)}}{\Lambda^{(m)}+2M^{(m)}}\left(1-\frac{\omega/k}{M^{(m)}}\right),m=1,2$$

(B2)