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Size-dependent plastic buckling behavior of micro-beam structures by using conventional mechanism-based strain gradient plasticity

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Abstract. Since the actuators with small- scale structures may be exposed to external reciprocal actions lead to create undesirable loads causing instability, the buckling behaviors of them are interested to make reliable or accurate actions. Therefore, the purpose of this paper is to analyze plastic buckling behavior of the micro beam structures by adopting a Conventional Mechanism-based Strain Gradient plasticity (CMSG) theory. The effect of length scale on critical force is considered for three types of boundary conditions, i.e. the simply supported, cantilever and clamped - simply supported micro beams. For each case, the stability equations of the buckling are calculated to obtain related critical forces. The constitutive equation involves work hardening phenomenon through defining an index of multiple plastic hardening exponent. In addition, the Euler-Bernoulli hypothesis is used for kinematic of deflection. Corresponding to each length scale and index of the plastic work hardening, the critical forces are determined to compare them together.

Keywords: conventional mechanism-based strain gradient theory; length scale; plastic buckling; micro beam

1. Introduction

Novel Microelectromechanical systems (MEMS) and related micro-scale actuators are technology of microscopic devices, particularly utilized with moving parts of mechatronics, composed of arbitrary beam, plate and shell structures. However, these structures may be exposed by small interactions of internal and external devices, but at the micro- scale, these reciprocal actions may lead to create undesirable loads to make instability. Therefore, their buckling or instable behaviors are interested to make reliable or accurate actions. Although, there are so many works representing to elastic behaviors of cantilever actuators in the micro and nanoscales for various loading conditions and geometries. However, there are little efforts available for the permanent plastic behavior of the MEMS structures and actuators. The observations of experiments have shown the features of a case strongly size-dependent mechanical behavior at the small scales (Liu *et al*. 2017). It is known that prediction of material behavior in the micro-scale deformation to be failed if the classic continuum mechanic principles are used alone (Aifantis 2009). Examples of these experiments include the wire torsion (Fleck *et al*. 1994) thin-film bulge (Xiang and Vlassak 2006) and (Cordill *et al*. 2004), micro bend (Stölken and Evans 1998), and the indentation test (Stelmashenko *et al*. 1993) and (Cordill *et al*. 2004). So, the classic plasticity relations should be developed to can explain the behavior of metal in micro and sub-micron scale and include the parameter of size effect. So far a particular physical concept that describes the length scale parameter and acceptable for the researcher has not offered. Indeed, the only thing that we can say with certainty that's the length scale in the form a multiple of high-order strain gradient come in equations and were trying to help justify behavior is dependent on the size of the material. However, efforts to find a physical explanation for the length scale of done. Initial efforts have tried to the inner length scale be linked micro-structure (Nix and Gao 1998). The non-local continuum is provided for the gradient formulations as the higher order gradients imply to the coefficients representing dependency of the parameters on the material dimensions. Particularly, several gradient plasticity (SGP) theories have been introduced to related problems of the small and large deformations. In this regard, comprehensive literature reviewing the SGP variety with discussions can be found in (Liu *et al*. 2017), (Darvishvand and Zajkani 2019a) and (Darvishvand and Zajkani 2019b) divided into two categories:

Firstly, the low-order theories were introduced by Acharya *et al*. (Acharya and Beaudoin 2000). Then, Chen and Wang (Chen and Wang 2000) proposed a new hardening law for the strain gradient plasticity. Huang and *et al*. (Huang *et al*. 2004) presented the conventional mechanism-based strain gradient (CMSG) plasticity theory. These theories have the conventional theory of plasticity with a plastic strain gradient in the form of quasi their time or tangential module. In other words, a consistent term can conjugate with the effects of the strain gradient plasticity in a yield function as such way that the classical equations remain with conventional boundary conditions. Based on

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the C-W low-order strain gradient theory, a new strain gradient theory including damage effect was successfully proposed recently (Ban *et al*. 2017), which could explain the coupling effect of size and damage induced by deformation on the mechanical behavior of thin-wire torsion, thin-beam bending (Ban *et al*. 2017), and the tensile and compressive strength of micro-particle reinforced composites (Ban *et al*. 2019).

The second type is the high-order theories that they have additional boundary conditions with high order stress. Firstly, a model based on an inner length scale was presented by (Aifantis 1987). Then, Fleck and Hutchinson reported alternate model according to considerations bases on three inner length scales (Fleck and Hutchinson 1997). Moreover, by using a Taylor dislocation model, Gao and *et al*. (Gao *et al*. 1999; Huang *et al*. 2000) proposed an efficient mechanism-based strain gradient (MSG) plasticity theory. This theory is in two frameworks; first, at the micro scale, the Taylor formula of the flow strength, suitably modified to account for both SSD (statistically stored dislocation) and GND (geometrically necessary dislocation) and this framework haven't high order stress (that explained in part 2). At the second case for the mesoscale, it is sufficiently large to connect the notion of geometrically necessary dislocation to the gradient of strain field and to institute the constitutive framework based on the Taylor, dislocation model.

Results obtained from MSG model well agree with micro-indentation experiments of bulk copper (Elhaney *et al*. 1998), indentation tests of aluminum thin film on a glass substrate (Shi *et al*. 2008), micro-torsion (Fleck *et al*. 1994), micro-bending experiments (Stölken and Evans 1998) and on metal matrix composites (Xue *et al*. 2002b). Also, it has been successfully applied to study a few Important problems at the micron and submicron scales as the micro-electro-mechanical systems (Xue *et al*. 2002a), fracture (Jiang *et al*. 2001; Lü *et al*. 2000), and plastic flow localization (Shi *et al*. 2000).

Immeasurable of the micro beam applications in electro-mechanical microsystems encourage researchers to analyze the behavior of that. Micro-beam bending in the elastic and plastic are highly regarded research (Challamel and Wang 2008; Chen and Feng 2011; Park and Gao 2006; Patel *et al*. 2017; Wang *et al*. 2003). Idiart (Idiart *et al*. 2009) using Fleck and Willis gradient plasticity theory (Fleck and Willis 2009) analyzed the size effects in the curvature of thin beams. Shi *et al*. (Shi *et al*. 2008) studied the behavior of elastoplastic micro beam using couple stress plasticity. Mao (Mao *et al*. 2013) examined the strain gradient effects and stress-strain curves on the micro-scale micro-beam bending. Arsenlis (Arsenlis 1999) analyzed the elastoplastic micro-beams made of smart materials using a mechanism-based strain gradient plasticity theory. Feng (Chen and Feng 2011) did a bending analysis of the micro cantilevers based on the Chen-Wang strain gradient plasticity theory. Moreover, Lubarda (Lubarda 2017) presented analysis of pure bending of rigid-plastic beams in strain gradient plasticity.

In two past decades, there are so many works around of the elastic behavior of the micro and the Nano beams founded for various loading, geometries, and conditions. But, from author's knowledge, there is a little effort available in the plastic area. In this regard, we can refer to the references inside (Liu *et al*. 2017). Therefore, the main purpose of the paper is to analyze the plastic buckling behavior of the micro beam in three boundary conditions; simply supported, cantilever and clamp-simply supported micro beams. For each case, we calculate the eigenvalue, critical force and get government equation of buckling for the micro beams by illustrating the first three modes of the buckled micro beams. In addition, the effect of length scale on a significant force is determined by using a CMSG plasticity model in conjunction with the multiple plastic work hardening exponents. This model can be categorized as a novel structural work dedicated as the micro devices studies when their plastic buckling behavior is especially important during their implementation in the microsystems applications.

2. Conventional Mechanism-based strain gradient (CMSG) plasticity

The CMSG plasticity theory is based on the Taylor model of dislocation hardening. Taylor explained that the Peach-Koehler force caused by the interaction of a pair of dislocations is proportional to:

$$\tau \propto \frac{\mu b}{L_0} \tag{1}$$

where μ is the shear modulus, b is the magnitude of the Burgers vector, and L_0 is the average of the distance between dislocations. The critical resolved shear stress for the moving dislocation to overcome this stress field is defined as

$$\tau = \frac{\alpha\mu b}{L_0} = \alpha\mu b\sqrt{\rho} \tag{2}$$

That α is an empirical coefficient, which takes values between 0.3 and 0.5, and ρ is the dislocation density.

Nix and Gao (Nix and Gao 1998) generalized the Taylor's model for the dislocation hardening criterion by showing the relation between the dislocation density and shear stress as:

$$\tau = \alpha\mu b\sqrt{\rho_T} = \alpha\mu b\sqrt{\rho_S + \rho_G} \tag{3}$$

where ρ_T is the total dislocation density, ρ_S is the density of statistically stored dislocations (SSD), ρ_G is the density of geometrically necessary dislocations (GND), and ρ_G can be related to the effective plastic strain gradient

$$\rho_G = \bar{r}\frac{\eta}{b} \tag{4}$$

where $\bar{r}$ is a Nye-factor and is around 1.90 for the face-centered - cubic (FCC) poly-crystals (Arsenlis 1999). The tensile flow stress flow σ_{flow} is related to the shear flow stress τ by

$$\sigma_{flow} = M\tau = M\alpha\mu b\sqrt{\rho_s + \bar{r}\frac{\eta}{b}} \tag{5}$$

M is Taylor factor and M = 3.06 for the FCC metals. In the absence of the strain gradient term, the equation (5) degenerates to the classical plasticity law

$$\sigma_{flow} = M\alpha\mu b\sqrt{\rho_s} = \sigma_Y f(\bar{\varepsilon}_p) \tag{6}$$

where σ_Y is an initial yield stress, f is a non-dimensional function of plastic strain $\bar{\varepsilon}_p$, and $\sigma_Y f(\bar{\varepsilon}_p)$ represents the stress-plastic strain curve in an uniaxial tension. The combination of relations (5) and (6) gives the flow stress accounting for the non-uniform plastic deformation associated with the geometrically necessary dislocations as

$$\begin{aligned}\sigma_{flow} &= \sqrt{[\sigma_Y f(\bar{\varepsilon}_p)]^2 + M^2\bar{r}\alpha^2\mu^2 b\eta^p} \\ &= \sigma_Y\sqrt{f^2(\bar{\varepsilon}_p) + l\eta}\end{aligned} \tag{7}$$

which

$$l = M^2\bar{r}\alpha^2\left(\frac{\mu}{\sigma_Y}\right)^2 b = 18\alpha^2\left(\frac{\mu}{\sigma_Y}\right)^2 b \tag{8}$$

where l is an intrinsic material length in the strain gradient plasticity theories. From the uniaxial power law stress-strain relationship,

$$f(\bar{\varepsilon}_p) = \bar{\varepsilon}_p{}^N \tag{9}$$

where N is a plastic work hardening exponent. The flow stress is then obtained from Taylor's relationship as

$$\sigma_{flow} = \sigma_Y\sqrt{\bar{\varepsilon}_p{}^{2N} + l\eta} \tag{10}$$

That $\bar{\varepsilon}_p$ is the effective strain and defined as

$$\bar{\varepsilon}_p = \sqrt{\frac{2}{3}\varepsilon'_{ij}\varepsilon'_{ij}} \tag{11}$$

and ε'_{ij} is the deviatoric strain. The components of strain gradient corresponding to displacement field are given by

$$\eta_{ijk} = u_{k,ij} \tag{12}$$

The effective plastic strain gradient η in relation (10) is:

$$\eta = \sqrt{\frac{1}{4}\eta'_{ij}\eta'_{ij}} \tag{13}$$

where η'_{ij} is the deviatoric strain gradient and defined as:

$$\eta'_{ijk} = \eta_{ijk} - \frac{1}{4}(\delta_{ik}\eta_{jpp} - \delta_{jk}\eta_{ipp}) \tag{14}$$

Substitution of relation (10) into flow rule, a constitutive equation of the deformation theory of the CMSG plasticity is obtained as

$$\sigma'_{ij} = \frac{2\varepsilon_{ij}}{3\bar{\varepsilon}_p}\sigma_{flow} = \frac{2\varepsilon_{ij}}{3\varepsilon}\sigma_Y\sqrt{\bar{\varepsilon}_p{}^{2N} + l\eta} \tag{15}$$

Also, the effective flow stress is

$$\bar{\sigma}_p = \sqrt{\frac{2}{3}\sigma'_{ij}\sigma'_{ij}} \tag{16}$$

3. Plastic buckling of micro-beam

Firstly, displacement fields of the beam based on the Euler-Bernoulli kinematics are given as

$$u_1 = -zw'(x), \quad u_2 = 0, \quad u_3 = -w(x) \tag{17}$$

The components of strain corresponding to this displacement field are written by

$$\begin{gathered}\varepsilon_{xx} = -z\frac{d^2w(x)}{dx^2} \\ \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{xy} = \varepsilon_{yz} = \varepsilon_{zx} = 0\end{gathered} \tag{18}$$

On the other hand, the deviatoric strains can be defined as

$$\begin{gathered}\varepsilon'_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{mm}\delta_{ij} \\ \varepsilon'_{xx} = -2\varepsilon'_{yy} = -2\varepsilon'_{zz} = -\frac{2}{3}z\frac{d^2w(x)}{dx^2}\end{gathered} \tag{19}$$

By substituting relation (19) into (11), the effective strain can be obtained as

$$\bar{\varepsilon}_p = \frac{2}{3}z\frac{d^2w(x)}{dx^2} \tag{20}$$

Moreover, the non-vanishing strain gradients in the Cartesian reference frame are given by

$$\begin{gathered}\eta_{111} = -z\frac{d^3w}{dx^3} \\ \eta_{131} = \eta_{311} = -\eta_{113} = -z\frac{d^2w}{dx^2}\end{gathered} \tag{21}$$

By substitution of relation (21) in Eq. (14) and then rearranging the formula (13), we will be able to drive the effective strain gradient η given by

$$\eta = \frac{1}{2}\sqrt{\frac{z^2}{2}\left(\frac{d^3w}{dx^3}\right)^2 + 3\left(\frac{d^2w}{dx^2}\right)^2} \tag{22}$$

The non-vanishing stress in formula (15) is given by

$$\sigma'_x = \frac{2\varepsilon_x}{3\bar{\varepsilon}_p}\sigma_Y\sqrt{\bar{\varepsilon}_p{}^{2N} + l\eta} \tag{23}$$

By substitution relations (18), (20) and (22) into terms of relation (23), the σ'_x is obtained as follows

$$\sigma'_x = -\sigma_Y\sqrt{\left(\frac{2}{3}z\frac{d^2w}{dx^2}\right)^{2N} + \frac{l}{2}\sqrt{\frac{z^2}{2}\left(\frac{d^3w}{dx^3}\right)^2 + 3\left(\frac{d^2w}{dx^2}\right)^2}} \tag{24}$$

By multiplying (24) into ε'_x, we get the following expression

$$\sigma'_x\varepsilon'_x = \frac{2\sigma_Y z}{3}\frac{d^2w}{dx^2}\sqrt{\left(\frac{2}{3}z\frac{d^2w}{dx^2}\right)^{2N} + \frac{l}{2}\sqrt{\frac{z^2}{2}\left(\frac{d^3w}{dx^3}\right)^2 + 3\left(\frac{d^2w}{dx^2}\right)^2}} \quad (25)$$

Consequently, the strain energy is defined as

$$U = \frac{1}{2}\iiint_v \sigma'_{ij}\varepsilon'_{ij}\,dv = \frac{b}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_0^L \sigma'_x\varepsilon'_x\,dxdz \quad (26)$$

By substitution the relations (25) into (26), we can get

$$U = \int_{-\frac{h}{2}}^{\frac{h}{2}}\int_0^L \frac{b\sigma_Y z}{3}\frac{d^2w}{dx^2}\sqrt{\left(\frac{2}{3}z\frac{d^2w}{dx^2}\right)^{2N} + \frac{l}{2}\sqrt{\frac{z^2}{2}\left(\frac{d^3w}{dx^3}\right)^2 + 3\left(\frac{d^2w}{dx^2}\right)^2}}\,dxdz \quad (27)$$

If a twofold Taylor expansion is used, the radical term in Eq. (27) can be expressed in the following form

$$\begin{aligned}
&\sqrt{\left(\frac{2}{3}z\frac{d^2w}{dx^2}\right)^{2N} + \frac{l}{2}\sqrt{\frac{z^2}{2}\left(\frac{d^3w}{dx^3}\right)^2 + 3\left(\frac{d^2w}{dx^2}\right)^2}} \\
&= \frac{1}{(2z^2+12)^{\frac{3}{2}}\left(2^{2N+2}\,9^{-N}\,z^{2N} + 9l\sqrt{2z^2+12}\right)^{3/2}} \\
&\left\{\left(\frac{d^3w}{dx^3}\right)\left[l^2z^4\sqrt{2z^2+12}\right.\right. \\
&+15l^2z^2\sqrt{2z^2+12} + \frac{l(N+1)2^{2N+2}z^{2N+4}}{9^N} \\
&\left. + \frac{6l(N+1)2^{2N+2}z^{2N+2}}{9^N}\right] \\
&+\left(\frac{d^2w}{dx^2}\right)\left[\frac{N2^{4N+4}z^{4N+2}}{81^N}\sqrt{2z^2+12}\right. \\
&+\frac{6N2^{4N+4}z^{4N+1}}{81^N}\sqrt{2z^2+12} \\
&+15l^2z^3\sqrt{2z^2+12} \\
&36l^2z\sqrt{2z^2+12} + \frac{3\,l\,N2^{2N+2}z^{2N+4}}{9^N} \\
&+\frac{6l(5N+2)2^{2N+2}z^{2N+2}}{9^N} \\
&\left.+\frac{36l(2N+1)2^{2N+2}z^{2N}}{9^N}\right] \\
&+\left(\frac{d^3w}{dx^3}\frac{d^2w}{dx^2}\right)\left[-9l^2z^2\sqrt{2z^2+12} - \frac{l\,N2^{2N+2}z^{2N+4}}{9^N}\right. \\
&\left.-\frac{6l(N+1)2^{2N+2}z^{2N+2}}{9^N}\right] \\
&+\left[\frac{(1-N)2^{4N+4}z^{4N+2}}{81^N}\sqrt{2z^2+12}\right. \\
&+\frac{6(1-N)2^{4N+4}z^{4N}}{81^N}\sqrt{2z^2+12} \\
&+l^2z^4\sqrt{2z^2+12} \\
&+3l^2z^2\sqrt{2z^2+12} + 36l^2z\sqrt{2z^2+12} \\
&+\frac{3l(1-N)2^{2N+2}z^{2N+4}}{9^N} \\
&\left.\left.+\frac{30l(1-N)2^{2N+2}z^{2N+2}}{9^N} + \frac{36l(3-2N)2^{2N+2}z^{2N}}{9^N}\right]\right\}
\end{aligned} \quad (28)$$

Using Eq. (28) in relation (27), we can draw the following equation

$$\begin{aligned}
U = \frac{1}{3}b\sigma_Y\int_0^L\left(\frac{d^2w}{dx^2}\right)&\left\{\alpha\left(\frac{d^3w}{dx^3}\right) + \beta\left(\frac{d^2w}{dx^2}\right)\right. \\
&\left.+\eta\left(\frac{d^3w}{dx^3}\right)\left(\frac{d^2w}{dx^2}\right) + \psi\right\}dx
\end{aligned} \quad (29)$$

where

$$\begin{aligned}
\alpha = l^2I_5 + 15l^2I_3 &+ \frac{l(N+1)2^{2N+2}}{9^N}K_5 \\
&+\frac{6l(N+1)2^{2N+2}}{9^N}K_3
\end{aligned} \quad (30)$$

$$\begin{aligned}
\beta = \frac{2^{4N+4}}{81^N}NJ_3 + 6\frac{2^{4N+4}}{81^N}NJ_1 &+ 15l^2I_3 + 36l^2I_1 \\
&+3\frac{lN2^{2N+2}}{9^N}K_5 \\
&+\frac{6l(5N+2)2^{2N+2}}{9^N}K_3 \\
&+\frac{36l(2N+1)2^{2N+2}}{9^N}K_1
\end{aligned} \quad (31)$$

$$\eta = -9l^2I_3 - \frac{lN2^{2N+2}}{9^N}K_5 - \frac{6l(N+1)2^{2N+2}}{9^N}K_3 \quad (32)$$

$$\begin{aligned}
\psi = \frac{(1-N)2^{4N+4}}{81^N}J_3 &+ \frac{6(1-N)2^{4N+4}}{81^N}J_1 + l^2I_5 \\
&+3l^2I_3 + 36l^2I_1 \\
&+\frac{3l(1-N)2^{2N+2}}{9^N}K_5 \\
&+\frac{30l(1-N)2^{2N+2}}{9^N}K_3 \\
&+\frac{36l(3-2N)2^{2N+2}}{9^N}K_1
\end{aligned} \quad (33)$$

In the above equations, integrals of I_m, J_m and K_m $(m = 1,3,5)$ can be obtained as below

$$\begin{aligned}
I_m &= \int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{(2z^2+12)^{-1}\,z^m\,dz}{\left(2^{2N+2}\,9^{-N}\,z^{2N} + 9l\sqrt{2z^2+12}\right)^{3/2}} \\
J_m &= \int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{(2z^2+12)^{-1}\,z^{4N+m}dz}{\left(2^{2N+2}\,9^{-N}\,z^{2N} + 9l\sqrt{2z^2+12}\right)^{3/2}} \\
K_m &= \int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{(2z^2+12)^{-3/2}\,z^{2N+m}dz}{\left(2^{2N+2}\,9^{-N}\,z^{2N} + 9l\sqrt{2z^2+12}\right)^{3/2}}
\end{aligned} \quad (34)$$

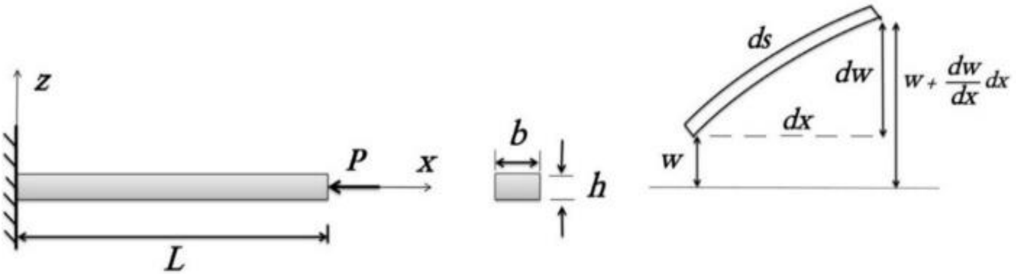

Fig. 1 Schematics of buckling in a micro-beam structure

For the work, we need to calculate the displacement. Therefore,

$$ds - dx \approx \sqrt{dx^2 + dw^2} - dx = dx\left(\left(1 + \left(\frac{dw}{dx}\right)^2\right)^{1/2} - 1\right) - dx \tag{35}$$

Considering $Y = (dw/dx)^2$ with the Maclaurin expansion of $(1+Y)^{1/2}$, relation (35) get as:

$$ds - dx \approx \frac{1}{2}\left(\frac{dw}{dx}\right)^2 dx \tag{36}$$

And the work in buckling is defined as

$$W_P = \frac{1}{2}\int_0^L P\left(\frac{dw}{dx}\right)^2 dx \tag{37}$$

By using the principle of minimum of potential energy and taking the first variation respect to the displacement,

$$\delta\pi = \delta U - \delta W_P \tag{38}$$

By substitution of the relations (29) and (37) into (38), the governing equation is given as

$$2\beta'\left(\frac{d^4w}{dx^4}\right) + P\left(\frac{d^2w}{dx^2}\right) = 0, \quad \beta' = \frac{1}{3}b\sigma_Y\beta \tag{39}$$

On the other hands, second variation of the potential energy for the buckling analysis results the same equation. So, the solution of equation (39) is

$$w(x) = c_1 + c_2x + c_3\sin(Ax) + c_4\cos(Ax) \tag{40}$$

where

$$A = \frac{\sqrt{6}}{2}\sqrt{\frac{P}{b\sigma_Y\beta}} \tag{41}$$

Also, the boundary conditions can be drawn

$$\left[-2\beta'\left(\frac{d^3w}{dx^3}\right) - P\left(\frac{dw}{dx}\right)\right]\delta w\Big]\Big]_0^L = 0$$

$$\left[2\beta\left(\frac{d^2w}{dx^2}\right) + \psi\right]\delta\frac{dw}{dx}\Big]\Big]_0^L = 0 \tag{42}$$

$$\left[\alpha\left(\frac{d^2w}{dx^2}\right) + \eta\left(\frac{d^2w}{dx^2}\right)^2\right]\delta\frac{d^2w}{dx^2}\Big]\Big]_0^L = 0$$

As explained in the section of the introduction, there are additional boundary conditions. Therefore, the essential boundary conditions should be chosen for the exact physical description of the beam. We investigate the problem in three boundary conditions, i.e., the simply supported, cantilever and clamp-simply supported micro beam. For each case, we calculate the eigenvalue and critical force by obtaining the governing equation of buckling micro beam and evaluating the appropriate buckled modes.

3.1 Simply-supported micro beam

We know the following boundary conditions are suitable for the simply- supported beam

$$\delta w\Big]\Big]_0^L = 0 \tag{43}$$

$$\delta\frac{d^2w}{dx^2}\Big]\Big]_0^L = 0 \tag{44}$$

By using the boundary conditions $w(0) = 0$ and $w''(0) = 0$, the equation (40) converts to equation (45) as

$$w(x) = c_2x + c_3\sin(Ax) \tag{45}$$

In addition, by using the boundary condition $w''(L) = 0$, the eigenvalue yields

$$P = 2\beta'\left(\frac{n\pi}{L}\right)^2 = \frac{2}{3}b\sigma_Y\beta\left(\frac{n\pi}{L}\right)^2 \tag{46}$$

when n denote the buckling mode.

Considering strain hardening caused by the size and gradient- dependent flow stress in equation (10), we may arrange the eigenvalue by the following dimensionless relation:

$$\Omega = \frac{P}{\sigma_{flow}hb} = \frac{2\beta}{3h\sqrt{\bar{\varepsilon}_p^{\,2N} + l\eta}}\left(\frac{n\pi}{L}\right)^2 \tag{47}$$

which $\Omega \geq 1$ indicates at least possible mode number for the plastic yield exceed in in axial compressive load:

$$n_{cr} \geq \sqrt{\frac{3hL^2\sqrt{\bar{\varepsilon}_p^{\,2N} + l\eta}}{2\beta\pi^2}} \tag{48}$$

If relation (46) is rearranged by the initial yield stress, the smallest value of P is an actual critical force for the plastic buckling obtained at the critical mode number corresponding to Eq. (48):

$$\frac{P}{\sigma_Y hb} = \frac{2\beta\pi^2}{3hL^2} {n_{cr}}^2 = p_{cr} \tag{49}$$

Since Ω is always less than $P/\sigma_Y hb$, thus depending to different length scales, strain gradient and hardening, n_{cr} can be achieved in different values $n_{cr} \geq 1$. Although, in particular case of rigid-perfectly plastic without concerning hardening and size dependency, the first mode $n = 1$ cannot represent the desired buckled mode. But, to illustrate conventional buckling values, a benchmark dimensionless parameter is used ordinary at the prescribed first mode $n = 1$, in order to compare magnitudes of calculated results as

$$\tilde{p}_{cr} = \frac{2\beta\pi^2}{3hL^2} \tag{50}$$

The eigenvalue and critical force are related to β. Thus, corresponding to relation (31), they are dependent to length scale, plastic work hardening exponent, thickness, and height of the micro beam. By using boundary condition $w(L) = 0$, the equation of buckling of simply support micro beam is

$$w(x) = c\left(sin(Ax) - \frac{sin(AL)}{L}x\right) \tag{51}$$

3.2 Cantilever micro beam

The following boundary conditions are suitable for the cantilever beam

$$\left[\left(\frac{d^3w}{dx^3}\right) + A^2(\frac{dw}{dx})\right]\delta w\rrbracket_0^L = 0$$

$$\delta\frac{dw}{dx}\rrbracket_0 = 0 \tag{52}$$

$$\delta\frac{d^2w}{dx^2}\rrbracket_L = 0$$

By using boundary conditions $w(0) = 0$ and $w'(0) = 0$, the equation (40) may convert to equation (53) as

$$w(x) = c_1(cos(Ax) - 1) + c_3(sin(Ax) - Ax) \tag{53}$$

The boundary condition $(d^3w/dx^3) + A^2(dw/dx)$ leads to give $c_3 = 0$ and use of boundary condition $w''(L) = 0$ gives

$$P = 2\beta'\left(\frac{(2n+1)\pi}{2L}\right)^2 \tag{54}$$

As the same of previous section in equations (47) and (48), we have

$$n_{cr} \geq \frac{1}{2}\left(\sqrt{\frac{6hL^2\sqrt{{\bar{\varepsilon}_p}^{2N} + l\eta}}{\beta\pi^2}} - 1\right) \tag{55}$$

Also, the dimensionless prescribed critical force for the first mode corresponding to $n = 1$ is

$$\tilde{p}_{cr} = \frac{3\beta\pi^2}{2hL^2} \tag{56}$$

By using boundary condition $w(L) = 0$, the equation of buckling of simply supported micro beam is

$$w(x) = c(cos(Ax) - 1) \tag{57}$$

3.3 Clamped-simply supported micro beam

Boundary conditions (58) are suitable for the beam with one end fixed and the another end is the simply supported

$$\delta w\rrbracket_0^L = 0$$

$$\delta\frac{dw}{dx}\rrbracket_0 = 0 \tag{58}$$

$$\delta\frac{d^2w}{dx^2}\rrbracket_L = 0$$

By using the boundary conditions $w(0) = 0$ and $w'(0) = 0$, the equation (41) converts into equation (59)

$$w(x) = c_1(cos(Ax) - 1) + c_3(sin(Ax) - Ax) \tag{59}$$

Also, by using boundary condition $w(L) = 0$ and $w''(L) = 0$, we may have

$$c_1(cos(AL) - 1) + c_3(sin(AL) - AL) = 0 \tag{60}$$

$$c_1A^2 cos(AL) + c_3A^2 sin(AL) = 0 \tag{61}$$

Equations (59) and (61) denote a system of two homogeneous algebraic equations which c_1 and c_3 are unknowns. For a nontrivial solution of c_1 and c_3 we set determinant of the coefficients of c_1 and c_3 in Eqs. (59) and (61) be zero to obtain

$$tan(AL) = -AL \tag{62}$$

Substituting the relation (42) into Eq. (62) gives

$$P = 2\beta'\left(\frac{(2n-1)\pi}{L}\right)^2 \tag{63}$$

Similarly,

$$n_{cr} \geq \frac{1}{2}\left(\sqrt{\frac{6hL^2\sqrt{{\bar{\varepsilon}_p}^{2N} + l\eta}}{\beta\pi^2}} + 1\right) \tag{64}$$

And prescribed critical force for the first mode corresponding to $n = 1$ is

$$\tilde{p}_{cr} = \frac{2\beta\pi^2}{3hL^2} \tag{65}$$

Equation (60) lead to

$$c_3 = -c_1\frac{(cos(AL) - 1)}{(sin(AL) - AL)} \tag{66}$$

Finally, by substitution of the relation (66) in equation (58), we have

$$w(x) = c\left[-\frac{(cos(AL)-1)}{(sin(AL)-AL)}(sin(Ax)-Ax) + (cos(Ax)-1)\right] \tag{67}$$

4. Results and discussion

Since there is no experimental or analytical study on plastic buckling of micro beam, we should verify these results with (Chen and Feng 2011) which have been used for the Chen-Wang theory in bending of cantilever micro beam in similar way with (Darvishvand and Zajkani 2019a). Therefore, we considered a concentrated force (F) at the end of the beam that the work in this case is

$$W_F = -Fw(x = L) \tag{68}$$

By substitution of relations (29) and (67) into (37), governing equation is given for bending as

$$2\beta'\left(\frac{d^4w}{dx^4}\right) = 0 \tag{69}$$

The response of the equation (69) is

$$w(x) = \frac{c_1}{6}x^3 + \frac{c_2}{2}x^2 + c_3x + c_4 \tag{70}$$

By using the boundary conditions (52) with particular details for cantilever micro beam,

$$w(0) = 0, \quad w'(0) = 0, \quad \left(\frac{d^3w}{dx^3}\right) + A^2\left(\frac{dw}{dx}\right) = 0 \tag{71}$$

So, the constants are getting to obtain response of the equation (71) for bending deflection as below

$$w(x) = \left(\frac{F}{2\beta'}\right)x^3 - \left(\frac{FL}{4\beta'}\right)x^2 \tag{72}$$

To verify our model, we choose ratio of length to thickness; $L/h = 40$, the ratio of wide to thickness; $b/h = 6$, the ratio of length scale to thickness $l/h = 0.2$, $N = 0$, and $F/\sigma_Y hb = 0.0012$.

In Figure 2, the normalized deflections of the beam W/h obtained by two theories of the CMSG and Chen-Wang (CW) are illustrated.

In Table 1, the normalized critical forces $\tilde{p}_{cr}$ are abbreviated for the beam with the simply- supported boundary condition. In these results, the normalized parameters are: $L/h = 40$, $L/b = 10$ and $b/h = 4$. These results are obtained for the first mode $n = 1$ and listed versus several length scales and the plastic work hardening exponent.

By increasing the length scale and plastic work hardening exponent; the critical force increase. Corresponding to these boundary condition and the geometrical normalized parameters, the buckled micro beam are plotted for first three modes; $n = 1$, 2 and 3 in Figure 3.

In Table 2, normalized prescribed critical forces $\tilde{p}_{cr}$ are abbreviated for the cantilever boundary condition. In these results, the normalized parameters are: $L/h = 40$, $L/b = 10$ and $b/h = 4$. These results are obtained for the first mode $n = 1$ and listed versus several length scales and the plastic work hardening exponent. By increasing length scale

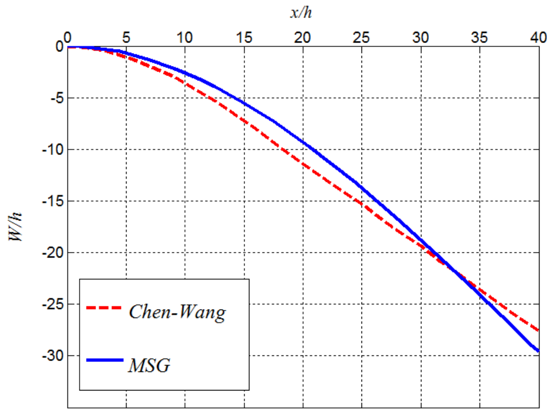

Fig. 2 Normalized deflection in two theories CMSG ((Darvishvand and Zajkani 2019a)) and CW theory (Chen and Feng 2011)

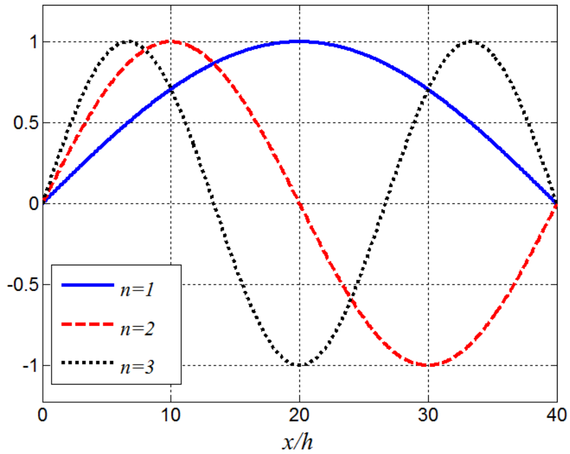

Fig. 3 The buckled simply- supported micro beam for first three mode $n = 1$, 2 and 3

Table 1 The variety of the normalized (prescribed) critical forces in the simply - supported micro-beam; $\tilde{p}_{cr}$ is related to the first mode with the normalized parameters $L/h = 40$, $L/b = 10$ and $b/h = 4$

	l/h				
N	0.2	0.4	0.6	0.8	1
0.1	0.223	0.2231	0.2232	0.2233	0.2234
0.2	0.1293	0.1294	0.1295	0.1296	0.1297
0.3	0.0565	0.0567	0.0569	0.0571	0.0573
0.4	0.0224	0.0230	0.0236	0.0241	0.0246
0.5	0.0095	0.01	0.012	0.013	0.014

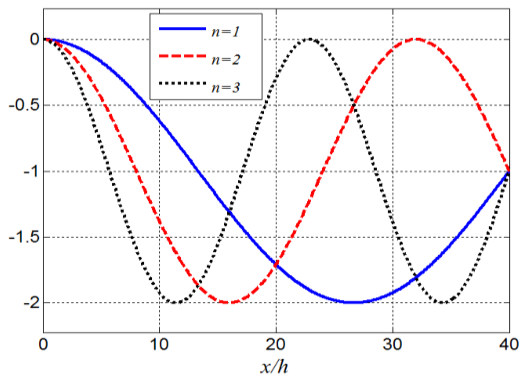

Fig. 4 The buckled cantilever micro beam for first three mode n=1, 2 and 3

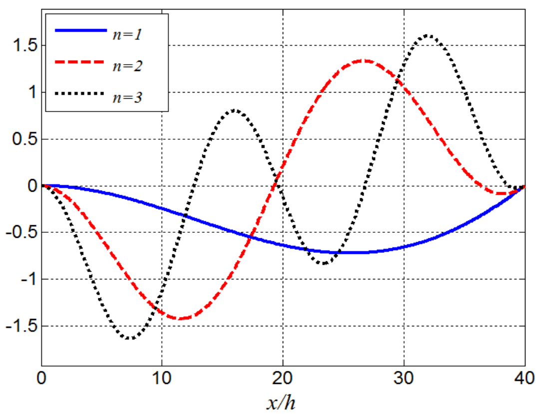

Fig. 5 The buckled beam with one end fixed and other end be simply supported for first three mode n=1, 2 and 3

and plastic work hardening exponent; the critical force increase. Corresponding to these boundary condition and the geometrical normalized parameters, the buckled micro beam are plotted for first three modes; $n = 1$, 2 and 3 in Figure 4.

In Table 3, the normalized (prescribed) critical forces $\tilde{p}_{cr}$ are abbreviated for the beam with one end fixed which another end is simply - supported condition. In these results, the normalized parameters are: $L/h = 40$, $L/b = 10$ and $b/h = 4$. These results are obtained for the first mode $n = 1$ and listed versus several length scales and the plastic work hardening exponent. By increasing length scale and plastic work hardening exponent; the critical force increase. Corresponding to these boundary condition and the geometrical normalized parameters, the buckled micro beam are plotted for first three modes; $n = 1$, 2 and 3 in Figure 5.

It is notable that according to experimental research and theory in strain gradient elasticity by (Lam *et al.* 2003), the length scale are suggested as $l = 15$ µm and $l = 8.14$ µm.

Moreover, (Shrotriya *et al.* 2003) calibrated the intrinsic plasticity length scale parameter for LIGA nickel foils by means of experimental study to present $l = 6.5$ µm and $l = 4.8$ µm. We obtained prescribed critical buckling loads for various length scales and hardening indexes. These prescribed values in Table 1 are lower than unite, while the most contents of Table 2 and 3 are larger than unit

Table 2 The variations of the normalized (prescribed) critical forces in the cantilever micro beam; $\tilde{p}_{cr}$ related to the first mode of the CMSG theory with the normalized parameters $L/h = 40$, $L/b = 10$ and $b/h = 4$

			l/h		
N	0.2	0.4	0.6	0.8	1
0.1	20.11	20.201	20.32	20.45	20.54
0.2	11.65	11.75	11.85	11.91	12.10
0.3	5.15	5.25	5.34	5.46	5.54
0.4	2.24	2.18	2.27	2.38	2.47
0.5	0.94	1.52	1.14	1.27	1.38

Table 3 The variations of the normalized critical forces in the clamped - simply micro-beam; $\tilde{p}_{cr}$; related to the first mode of the CMSG theory with the normalized parameters $L/h = 40$, $L/b = 10$ and $b/h = 4$

			l/h		
N	0.2	0.4	0.6	0.8	1
0.1	8.97	9.12	9.15	9.28	9.37
0.2	5.27	5.38	5.47	5.57	5.64
0.3	2.31	2.81	2.94	38	3.17
0.4	1.11	1.16	1.29	1.37	1.48
0.5	0.41	0.52	0.65	0.76	0.87

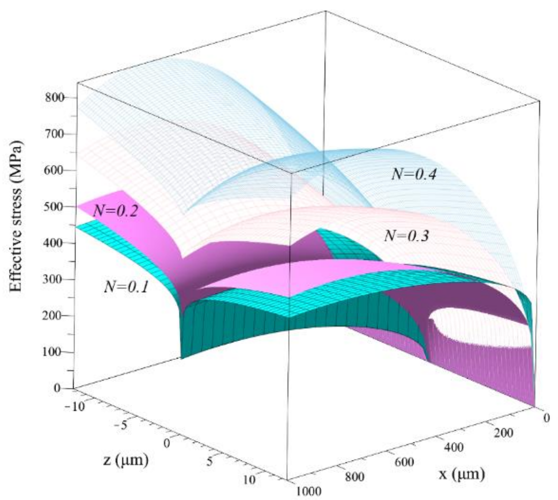

Fig. 6 The effective plastic stress in buckled simply-supported micro beam for first mode n=1

($\tilde{p}_{cr} > 1$). On the other hands, to make the sufficient way for comparing results and interpreting them with the actual size and gradient-dependent plastic buckling amounts for the work hardening material, the effective plastic stresses are illustrated by contours Figures 6 and 7.

These illustrations are calculated for the prescribed values in the first mode $n = 1$ for the simply- supported and cantilever micro beams, respectively. For these results, we take some data e.g. $h = 25$ µm, $\sigma_Y = 400$ MPa from experimental reports by (Shrotriya *et al.* 2003). In addition, other normalized parameters are selected as $L/h = 40$,

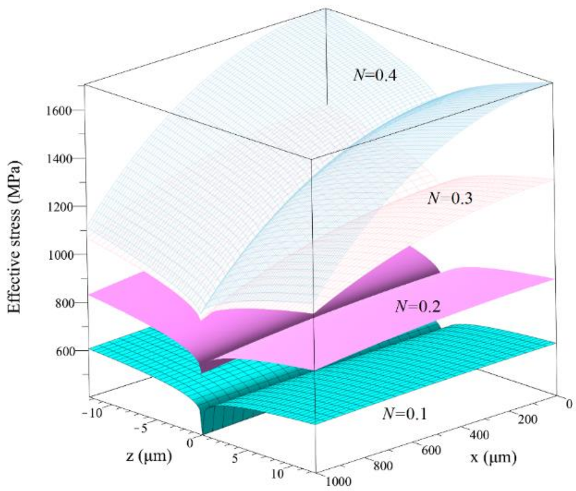

Fig. 7 The effective plastic stress in the buckled cantilever micro beam for first mode n=1

$L/b = 10$ and $b/h = 4$ as the same of Tables 1-3. To be more convenient with contour plots of Figures 6 and 7, we select the intrinsic length scale $l/h = 0.6$ or $l = 15$ μm for the strain hardening exponents of $N = 0.1$, 0.2 and 0.3.

It is completely clear that results of Figure 7 are upper than Figure 6 due to corresponding larger loads in Table 1 and 2, which can generate larger effective stresses. Increasing hardening exponent cause growing up the effective flow stresses.

5. Conclusions

In this paper, a plastic buckling of the micro-beam was studied for three boundary conditions such as the simply-supported, cantilever, and clamped - simply supported by using the conventional mechanism-based strain gradient plasticity theory. The effects of length scale and plastic work hardening on the buckling Eigenvalues were considered as well as the critical forces. For each case, we got the governing equations of the buckling for the micro beam obtaining three buckled modes. By increasing the length scale and plastic work hardening exponent; the critical force increase. We implied that this study might be regarded as a novel first work dedicated especially for the buckling of a micro beam structure based on a non-classical continuum mechanics. It was drawn mentioning that the classical continuum theories can't predict the plastic structural behaviour of the material at the microscale.

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