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# Concerning the tensor-based flexural formulation: Theory

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**Abstract.** Since the days of yore, plate's flexural analysis and formulation were dependent on the assumed coordinate system. In uncovering the coordinates-independent flexural interpretation, in this study, the plate bending analysis has been interpreted in terms of the tensor's components of curvatures and bending moments, in accordance with the continuum mechanics. The paper herein presents the theoretical formulations and conceptual perspectives of the Hydrostatic Method of Analysis (HM) that combines the continuum mechanics with the elasticity theory; the graphical statics and analysis; the theory of thin isotropic and orthotropic plates.

**Keywords:** hydrostatic formulation; plates flexural; spatial curvature; tensors; coordinates independent formulation

## 1. Introduction

The curvature and its associated moment-concept play a central role in physics, cosmology, structural and gravitational analysis. Historically, driven by musical phenomena, the theory of plates and shells had begun with Leonhard Euler's works on acoustics and curvature (Euler L 1766), a mathematical answer was offered on how bells and drums produce sounds. Based on Euler's work, the internal stresses and moment-curvature relations were offered through introducing the proportionality constants and uncovering their dimensions. Moreover, Bernoulli's attempted the formulation of the mathematical expression for plates deflection. Though this attempt resulted in a deficient equation lacking the twisting term with its mixed operator and these findings facilitated the introduction of the Germain-Lagrange biharmonic equation.

In 1811, in response to a contest held by Paris Academy of Sciences concerning Chladni's experiments, Sophie Germain submitted a mathematical explanation, which has been corrected by Joseph Lagrange to form the equation of vibration of thin plates, known as Germain-Lagrange plates biharmonic equation or the governing differential equation for deflection of thin plates (Love A 1888, Sihame *et al.* 2018, Adim *et al.* 2018, Behravan *et al.* 2017, Tran *et al.* 2017, Murat 2014, Szilard 2004). In 1829, based on Germain work (Gray 1978), Poisson introduced the governing plate's differential equation in polar coordinates (Poisson 1829). Between 1828 and 1830, Cauchy (1900) and Poisson (1829) successfully formulated through the theory of elasticity's equations the plate bending problem. Implementing the powers of the distance measured from a middle surface into a series expansion, retaining only the first terms of significance. Later, the tensors formulations were introduced to the structural theory with Cauchy (1900), Quinn and Stubblefield (2012), Capaldi (2012), and Lame (1866). In the second half of the nineteenth century, Kirchhoff refined the theory of thin plates with his widely accepted hypotheses (Ventsel and Krauthammer 2001, Kirchhoff 1850, Kurrer 2008). Deriving the differential equation of plates deflection through the principle of virtual displacements, generalizing the constant of rigidity and introducing a consistent formulation of the boundary conditions (Ventsel and Krauthammer 2001, Kirchhoff 1850, Kurrer 2008, Kirchhoff 1897).

Regretfully, the integration of higher curvature conceptions and formulations into the structural theory had been left-behind, following the adoption of Euler's formulation in buckling analysis, and Germain-Lagrange in plate's biharmonics. On the bases of whom-concepts and conventions, ensuing contribution immerged tackling various structural problems. However, their formulations coordinates-dependence has resulted in many formulations for the same geometrical problem in various coordinate systems, and yet with an enormous amount of simplifying assumptions. In this regard, Germain, Lagrange and Poisson conception in plate's formulation has failed to offer such criterion. Thus, has failed to reveal the formulation of the natural law of flexural biharmonics, that possess the "general (or diffeomorphism) covariance" (Norton 1993, Einstein and Minkowski 1920). Consequently, a wide gap has formed, between the structural theory, and the curvature theory of surfaces and manifolds, as the later had succeeded into astrophysics through tensors formulations with Riemann contributions.

Mohr circle; named after the German civil engineer Christian Otto Mohr (Oct 1835-Oct 1918) is a graphical representation of Cauchy static stress tensor's transformation (Ugural 2010, Parry 2005, Ugural and Fenster 2008). It falls in the field of graphical statics and analysis introduced by Parry (2005) and expanded by Kurrer (2008), Mohr (1906), Maxwell and Föppl (1900). In Mohr circle any symmetric 2-dimensional

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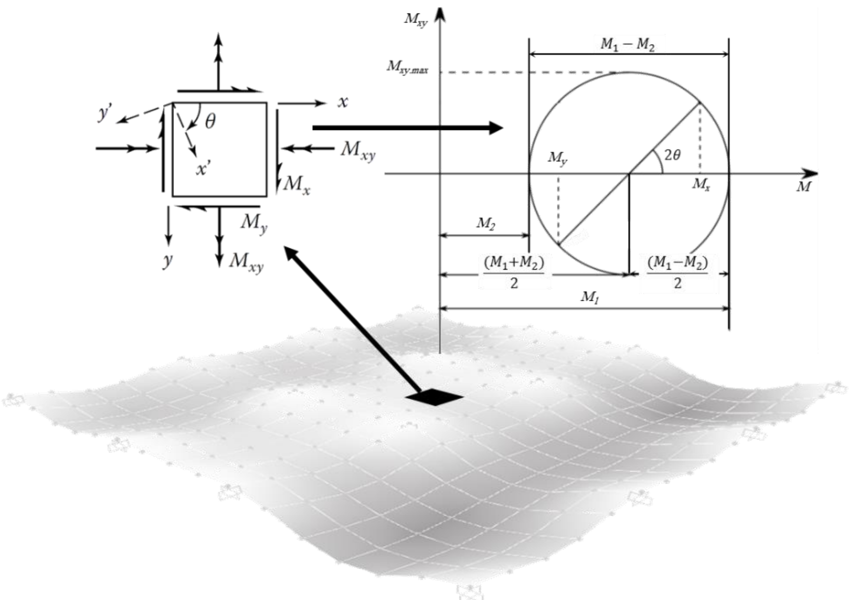


Fig. 1 Mohr circle concept illustration for an infinitesimal plate element

second-order tensor can be represented using the same principals, this includes stress, strain, moment and curvature. Of particular interest in this context (in dealing with the problem of flexural analysis) is the moment and curvature circles, since, the stress and strain have no single value at a particular section. Moreover, different materials across the thickness alter the stress distribution according to their respective stiffness. The moment and curvature, therefore, have the advantage of having a single Mohr circle representation at each section of the plate. In this study, moment representation has been considered for this reason, in addition to its importance in plates design and practice.

Mohr circle has the advantage of representing both normal and twisting moments with simple representation. Thus, through implementing the circle, the three moments problem are reduced into two circle parameters problem; namely, a centre and a radius (also referred to as, Mean and Deviatoric Moments in continuum mechanics). **Fig. 1** illustrates Mohr circle representation for bending moment of infinitesimal plate elements, in which the radius and the centre expressions are depicted. The moment transformation equations that establishes the bases of Mohr circle representation are as follows (Ugural 2010, Parry 2005, Ugural and Fenster 2008, Lisle and Robinson 1995).

$$M_{x'} = \frac{\left(M_x + M_y\right)}{2} + \frac{\left(M_x - M_y\right)}{2} \cdot cos(2\theta) + M_{xy} \cdot sin(2\theta) \tag{1}$$

$$M_{y'} = \frac{\left(M_x + M_y\right)}{2} - \frac{\left(M_x - M_y\right)}{2} \cdot cos(2\theta) - M_{xy} \cdot sin(2\theta) \tag{2}$$

$$M_{x'y'} = -\frac{\left(M_x - M_y\right)}{2} \cdot sin(2\theta) - M_{xy} \cdot cos(2\theta) \tag{3}$$

where (x, y) coordinates are to be transformed into (x`, y`) coordinates orientation. These equations provide the transformation for bending moment in x and y; (i.e. around y and x, respectively) and the twisting moment on the plate element for a coordinate system rotation with an angle θ. For every 2D-static tensor (whether it is a stress, moment, strain or a curvature tensor), there need be a special coordinate's orientation, which expels the shear, twisting moment, shear strain and twisting curvature from the tensor matrix, resulting in pure normal stresses actions. This is referred to as the principal moment's plane. This plane can be found through equating Eq. 3 to zero, which result in the following plane orientation.

$$\tan\left(2\theta_p\right) = \frac{2M_{xy}}{\left(M_x - M_y\right)} \tag{4}$$

Finally, introducing the principal plane angle into Eq. 1 and 2 would result in the following expression for the first and second principal moments; (the highest and the lowest bending moments acting on the respective element of concern).

$$M_{1,2} = \frac{\left(M_x + M_y\right)}{2} \pm \sqrt{\left(\frac{\left(M_x - M_y\right)}{2}\right)^2 + \left(M_{xy}\right)^2} \tag{5}$$

In the preceding equation, the first term locates the centroid of the Mohr circle, and the square root term symbolizes the circle radius. One of the most interesting properties of the Mohr circle representation is in its correlation to various tensor properties. For instance, the centroid of Mohr circle is the mean moment for the particular-tensor that results in pure bending action and thus, has no contribution to twisting distortion. This centre moment is also twice the first moment invariant, which equal the summation of the normal moments in the tensor and is constant at a particular-element in the plate regardless of the orientation in concern. The radius, on the other hand, characterizes the shear or twisting action and

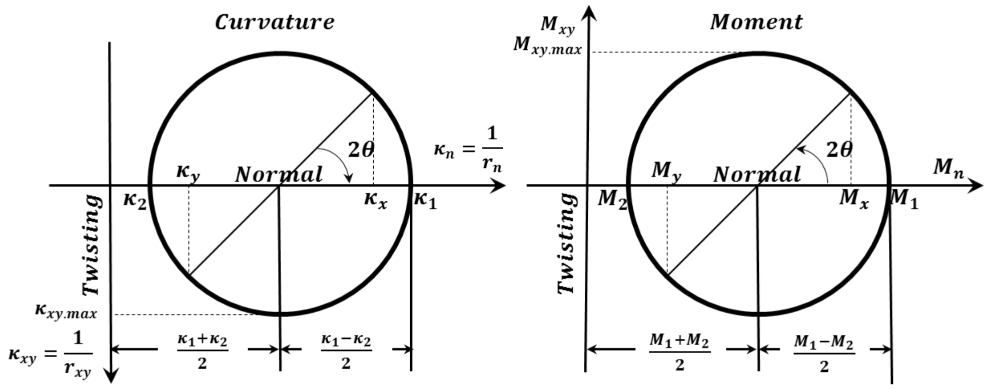


Fig. 2 Mohr circle representation and sign convention for curvature and moment

the deviatoric margin of the moment tensor that contribute to the shearing or twisting distortion. This radius is equal also to the maximum twisting moment $M_{xy.max}$ at the particular-element under consideration.

$$M_{xy.max} = \sqrt{\left(\frac{(M_x - M_y)}{2}\right)^2 + (M_{xy})^2} \tag{6}$$

The equations of moment transformation with respect to the principal plane orientation can be obtained from equation Eq. 1 and 2, resulting in the following.

$$M_{x'} = \frac{(M_1 + M_2)}{2} + \frac{(M_1 - M_2)}{2} \cdot \cos(2\theta) \tag{7}$$

$$M_{y'} = \frac{(M_1 + M_2)}{2} - \frac{(M_1 - M_2)}{2} \cdot \cos(2\theta) \tag{8}$$

In a similar manner, the curvature can be used in bending problems to yield almost the same meaning as that of the moment with a sign convention similar-to that of the strain Mohr circle. due to the variation of adopted sign conventions for bending moment and curvature; while a positive curvature results from the increase in the rate of change of the slope-angle as consideration moves with respect to the distance along the curve (concaved downward), a positive bending moment is the moment resulting in an opposite deformation (concaved upward deformed shape) (Ugural 2010, Ugural and Fenster 2008). **Fig. 2** depicts the Mohr circles for curvature and moment based on the aforementioned-equations.

## 2. Hydrostatic moment phenomenon

This section is devoted to the Phenomenon and concept Establishment in relation to continuum mechanics tensors formulation. According to the principals of continuum mechanics, any generic static stress tensor can be decomposed into a hydrostatic part, and a deviatoric tensor, thus,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma_{iso}} + \boldsymbol{S} \tag{9}$$

where $\boldsymbol{\sigma}$ is a general stress tensor, $\boldsymbol{\sigma_{iso}}$ is the isotropic hydrostatic stress tensor and $\boldsymbol{S}$ is the deviatoric, shear-producing stress tensor. In this research, this concept has been brought to the moment tensors, resulting in the following conscious representation.

$$\boldsymbol{M} = \boldsymbol{M_{iso}} + \boldsymbol{S_M} \tag{10}$$

Introducing this equation in matrix representation yields the following; noting that twisting moment vanishes for the isotropic hydrostatic moment state.

$$\boldsymbol{M} = \begin{bmatrix} M_x & M_{xy} \\ M_{xy} & M_y \end{bmatrix} = \begin{bmatrix} M_{iso} & 0 \\ 0 & M_{iso} \end{bmatrix} + \begin{bmatrix} M_x - M_{iso} & M_{xy} \\ M_{xy} & M_y - M_{iso} \end{bmatrix} \tag{11}$$

$$\boldsymbol{M} = \begin{bmatrix} \frac{M_x + M_y}{2} & 0 \\ 0 & \frac{M_x + M_y}{2} \end{bmatrix} + \begin{bmatrix} \frac{M_x - M_y}{2} & M_{xy} \\ M_{xy} & \frac{M_x - M_y}{2} \end{bmatrix} \tag{12}$$

One of the phenomena of vital importance in this context henceforth is related to the static moment tensors in the hydrostatic moment state. In which, a state of an isotropic couple that mimics the isotropic confining stress given by the weight of water above a specific point, acting on an infinitesimal submerged particle, in accordance with the continuum mechanics definition (Capaldi 2012, Quinn and Stubblefield 2012). Subsequently, this situation when introduced to the moment tensors can be summarized in the following conditions.

$$M_1 = M_2 = M_{x`} = M_{y`} = M_{iso} \;\; ; \;\; M_{xy} = 0 \tag{13}$$

This when introduced to the continuum mechanics representation for stress tensors decomposition results in the omission of the $\boldsymbol{S}$ term in Eq. 9, and when introduced to the moment tensor equivalent representation results in the following.

$$\boldsymbol{M} = \begin{bmatrix} M_{iso} & 0 \\ 0 & M_{iso} \end{bmatrix} \tag{14}$$

This novel conception of the hydrostatic moment tensor state can be interpreted in natural, mathematical and physical meaning as the case of stress tensors in continuum mechanics. That's, while a hydrostatic stress tensor is described as the state of a submerged particle, a hydrostatic

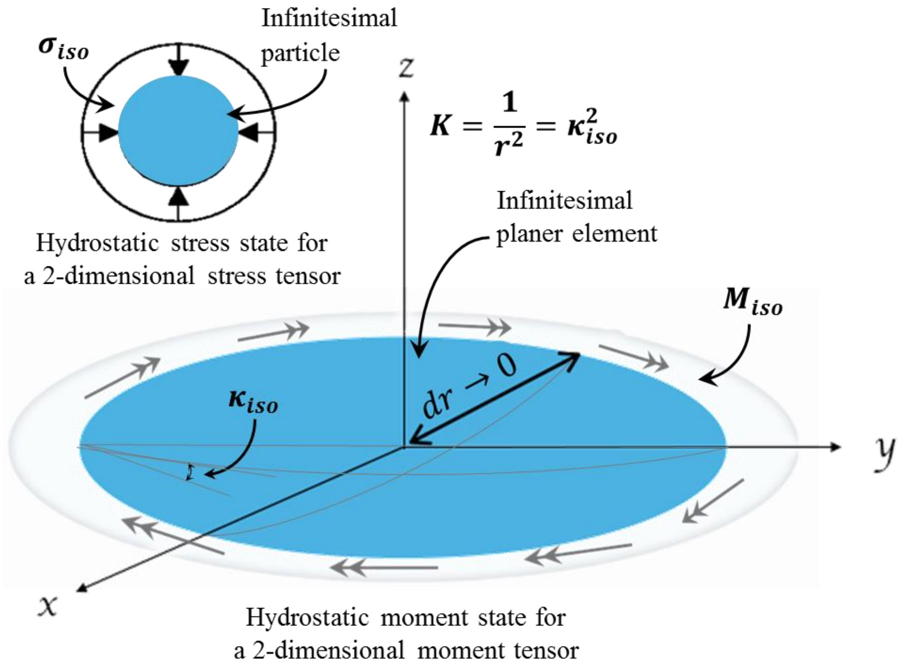


Fig. 3 Hydrostatic Moment and curvature representation in contrast to the hydrostatic stress representation

moment tensor state can be described as the state of bending of an infinitesimal continuum axisymmetric circular element, exhibiting a domain, loading and boundaries conditions axisymmetry. This definition results in pure isotropic normal curvature. The surface's Gaussian curvature for such state is equal to the second and third tensor invariants and to the hydrostatic curvature squared, which in turn is half the first tensor invariant (Harutyunyan 2017). These conditions explicitly suggest the development of a spherical surface (a part of a sphere). The aforementioned-conditions are summarized in terms of the following expression; Eq. 15.

$$\kappa_{iso} = \frac{I_1}{2} = \sqrt{K} \quad ; \quad K = I_2 = I_3 = \kappa_{iso}^2 \quad ; \quad I_1 = 2\kappa_{iso} \quad (15)$$

where:

$\kappa_{iso}$: Mean or hydrostatic or isotropic curvature tensor based on equivalent terminologies to that of continuum mechanics in defining Mean or hydrostatic or isotropic strain tensor.

$I_1 = \kappa_1 + \kappa_2$; First curvature tensor invariant at an infinitesimal plate element, as per Cauchy definition its constant for a particular-tensor regardless of the coordinates orientation.

$I_2 = \kappa_1 * \kappa_2$; Second curvature tensor invariant at an infinitesimal plate element, as per Cauchy definition its constant for a particular-tensor regardless of the coordinates orientation (note that the twisting terms have vanished due to the definition of the principal-plane orientation).

$I_3 = \kappa_1 * \kappa_2$; Third curvature tensor invariant at an infinitesimal plate element. Likewise, as for other invariants, its constant for a particular-tensor regardless of the coordinates system orientation (note that the twisting terms have vanished due to the definition of the principal-plane orientation).

$K = \kappa_1 * \kappa_2$; Gaussian curvature of a surface, which characterizes the curvature of various surfaces (Backus 1966). Based on Gauss curvature, the only surface to have an identical curvature at all points is a spherical surface as per Minding and Liebmann's theorems (Kühnel 2015).

A graphical representation of the hydrostatic moment tensor state, as described in an analogues relation to the continuum mechanics representation of the hydrostatic stress concept, is shown in **Fig. 3** for an infinitesimal element.

## 3. HM conceptual formulation

This section is devoted to the Major Hypotheses Statement and the establishment of the boundary value problem dimensions and hypotheses and basic notations and relations between the parameters to be used in the following section. Through the meditative consideration of the flexural problem, on structural and natural aspects, with a profound review of previous assumptions, hypotheses, equations and most importantly, conceptions, the visualization of a four-dimensional boundary value problem (4D-BVP) was suggested. In implementing this 4D-BVP system visualization, the following flexural behaviour hypotheses holds between the axes of various system variables.

- The 4D-BVP representation is independent of the time axis, and thus, vibrations and dynamic responses are omitted, and when considered a general solution for dynamic flexural analysis is attainable under the 5D-IVP representation.
- Various axes (variables) are interdependent, with complex interconnected relations, which is governed by continuum mechanics and material constitutive relations and modelling.
- The boundary condition variable is dependent on the

material-variable under consideration along with the geometrical (aspect ratio) variable. That's, the material condition (elasticity, plasticity, fractures and damage, etc.) will affect the boundary conditions. Likewise, the geometrical (aspect ratio) variable will affect the dependency of the boundary conditions on the material variable.

- The geometrical variable of the system in flexure is dependent on both material and boundary conditions. Where a change in the material condition, would alter the boundaries. Likewise, a change in the boundaries due to material and loading conditions would affect the geometrical domain.
- The material condition is dependent on the boundaries and the geometrical variables. Both of which would govern the extent of the response, and thus, the material conditions.
- The response (moment, curvature) is dependent on the three-dimensional variables. The alteration of any variable would alter the system response.
- In plates bending, for every boundary conditions there exist an aspect ratio that renders the maximum response a hydrostatic point moment tensor, having a zero radius (a zero deviatoric moment tensor). At such point, the principal moment's curves intersect in the 4D-BVP system. This Hydrostatic moment tensor state is to be developed based on the theory of elasticity and continuum mechanics as per Cauchy and Lamé (1866). The concepts and notations introduced here are to be incorporated in the development of the Hydrostatic Method formulation in the following sections for various structural and natural problems.

This marks the end of hypotheses introduction; these hypotheses are introduced to redefine the moment boundary value problem, solidifying its differential mathematical origin and the dependency of the system on various dimensions of the BVP. The following sub-sections introduce the basic notations and relations for moment and curvature hydrostatic analysis.

### *3.1 HM-moment formulation*

Pursuing the introduction of afore assumptions and hypotheses in relation to the flexural problem, in this section, notations and main parameters relations are established. Firstly, Mohr circle representation of the static moment tensor relies on various parameters. However, two parameters are of the major importance and can establish the full representation; these are the centre and radius, which happens to have the most crucial role in the continuum mechanics tensor representation as they coincide with the mean and deviatoric tensors, respectively. The central moment representing the portion of the tensor responsible for pure normal Gaussian curvature is expressed as follows.

$$HM_c = M_{iso} = \frac{(M_1 + M_2)}{2} \tag{16}$$

where:

$HM_c$: **H**ydrostatic **M**oment **C**enter ≡ $M_{iso}$; Mean or hydrostatic or isotropic moment tensor based on equivalent terminologies to that of continuum mechanics in defining Mean or hydrostatic or isotropic stress tensor.

$M_1$: First principal moment at an infinitesimal plate element. and

$M_2$: Second principal moment at an infinitesimal plate element.

On the other hand, the radius of the Mohr circle representation symbolizes the deviatoric portion of the moment tensor and is denoted as $HM_r$ henceforth, and it characterizes the tendency of the specific 4D-BVP problem of concern to reveal a twisting action and thus, to hinder the development of uniform bending resistance (Two-way action in case of rectangular slabs). This radius is given by the expression.

$$HM_r = S_M = \frac{(M_1 - M_2)}{2} \tag{17}$$

In adopting these definitions to the flexural problem, the graphical statics and analysis with Mohr moment circle representation and its underneath continuum mechanics tensor and constitutive relations, Eq. 5 results in the following expression for the principal first and second moment.

$$M_1 = HM_c + HM_r \tag{18}$$

$$M_2 = HM_c - HM_r \tag{19}$$

Similarly, the bending moment for any plane orientation in the plate flexural analysis can be obtained from the following equations, resulting from modifying Eq. 1 and 2 through the introduction of abovementioned designation.

$$M_{x'} = HM_c + HM_r \cdot \cos(2\theta) \tag{20}$$

$$M_{y'} = HM_c - HM_r \cdot \cos(2\theta) \tag{21}$$

In the same manner, the twisting moment can be found through the transformation equation of the radius (deviatoric) moment in any coordinates through the following expression for plane making an angle θ with the principal shortest dimension.

$$M_{xy} = HM_r \cdot \sin(2\theta) \tag{22}$$

Finally, the maximum twisting moment is equivalent to the radius in this representation, which can be written in the following manner.

$$M_{xy.max} = HM_r \tag{23}$$

So, through this representation, the plate bending analysis can yield all variables of interest through only two parameters that have a profound meaning both in graphical statics and analysis, in theory of elasticity and the broader field of continuum mechanics. Introducing these designations into moment tensors matrix representation yields the following; noting that twisting moment vanishes for the isotropic hydrostatic moment state.

$$\begin{aligned} M = \begin{bmatrix} M_x & M_{xy} \\ M_{xy} & M_y \end{bmatrix} = \begin{bmatrix} HM_c & 0 \\ 0 & HM_c \end{bmatrix} \\ + \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \cdot HM_r \end{aligned} \tag{24}$$

Or through symbolizing *sin(2θ)* with *S* and *cos(2θ)* with *C* in Eq. 24, results in the following concise form,

$$M = \begin{bmatrix} M_x & M_{xy} \\ M_{xy} & M_y \end{bmatrix} = \begin{bmatrix} HM_c & 0 \\ 0 & HM_c \end{bmatrix} + \begin{bmatrix} C & S \\ S & -C \end{bmatrix} \cdot HM_r \quad (25)$$

For the hydrostatic moment state, the moment tensor is characterized by the following two conditions.

$$\begin{cases} HM_c = M_1 = M_2 \\ HM_r = 0 \end{cases} \quad (26)$$

Introducing these conditions into Eq. 25 representation for moment tensors decomposition results in the omission of the $HM_r$ term, resulting in the following.

$$M = \begin{bmatrix} HM_c & 0 \\ 0 & HM_c \end{bmatrix} = HM_c \quad (27)$$

### *3.2 HM curvature formulation*

This sub-section is to emphasize the applicability of the same concept to the curvature, which is a direct result of the direct correlation between moment and curvature. however, this curvature notation is not adopted and not ensued. The sections to follow will adopt the moment notation, solely, in the mathematical derivations, and the curvature will serve in the conceptual and natural relation through Gaussian curvature and geometry. In contrast to moment representation, the curvature representation is of less significance in design and practice due to the correlation of material strength and resistance with moment capacity rather than the curvature capacity. However, for the purpose of completeness, the curvature representation is to be given here. The designation of curvature parameters can be presented adopting the same analogy used for moment representation as follows,

$$HM_{\kappa c} = \frac{(\kappa_1 + \kappa_2)}{2} \quad (28)$$

$$HM_{\kappa r} = \frac{(\kappa_1 - \kappa_2)}{2} \quad (29)$$

where:

$HM_{\kappa c}$: **H**ydrostatic **M**ethod **C**urvature **C**enter ≡ $\kappa_{iso}$; Mean or hydrostatic or isotropic curvature tensor based on equivalent terminologies to that of continuum mechanics in defining Mean or hydrostatic or isotropic strain tensor.

$\kappa_1$: First principal curvature at an infinitesimal plate element.

$\kappa_2$: Second principal curvature at an infinitesimal plate element.

Based on this definition, the following equations can be implemented in identifying the principal curvatures upon the estimation of the circle parameters from the solution of the governing equations of plate bending.

$$\kappa_1 = HM_{\kappa c} + HM_{\kappa r} \quad (30)$$

$$\kappa_2 = HM_{\kappa c} - HM_{\kappa r} \quad (31)$$

For orientations other than that of the principal plane, one can apply the following equations for the curvatures in coordinates of *x* and *y*, respectively.

$$\kappa_{x'} = HM_{\kappa c} + HM_{\kappa r} \cdot \cos(2\theta) \quad (32)$$

$$\kappa_{y'} = HM_{\kappa c} - HM_{\kappa r} \cdot \cos(2\theta) \quad (33)$$

Finally, the twisting and maximum twisting curvatures can be found, respectively, through the following expressions; both of which depends solely on the radius parameter.

$$\kappa_{xy} = HM_{\kappa r} \cdot \sin(2\theta) \quad (34)$$

$$\kappa_{xy.max} = HM_{\kappa r} \quad (35)$$

Likewise, when the hydrostatic case appears, these curvature expressions results in the omission of the radius terms $HM_{\kappa r}$, and thus, in similar expressions to those of the hydrostatic moment state.

**Fig. 4** illustrates the terminology and some cases encountered in a pure two-way system. including the extreme hydrostatic point with a point Mohr circle and a zero radius. The pure beam action at an edge has negligible rotational stiffness in which the radius and centre parameters are equal. The corner support or corner column case with the negative mean moment. Finally, the general case for an arbitrary element is presented. Including Hydrostatic extreme circle of pure two-way action, beam circle of pure one-way action, support circle of the negative mean circle. Henceforth, the moment terminology will be solely adopted in the formulation and discussion of flexural analysis and particularly plates bending problem.

The concept now has been established, the hypotheses, notation and essential relations had been briefly introduced. Yet, till this point, no reference to any of the advantages and attainments has been made. This is left to the following sections. Through which, the accomplishments that have revised the work of Euler (1766), Love (1888), Poisson (1829) and Kirchhoff (1897) were uncovered. All of whom hadn't been able to achieve such understanding of the bending problem. Their misconception resulted in the coordinate-dependent formulations, where every plate geometry (and thus coordinate system) requires a distinctive formulation. The introduction of such misconception to the plates theory resulted in the persistent adoption of the strip method in treating slabs like beam strips. This misconception, however, is justified, since tensor formulations and the concept of hydrostatic and deviatoric tensors was introduced almost fifty years later by Cauchy and Lame (1866). Moreover, it was almost a century later when C. Otto Mohr identified the flexural rigidity (Kurrer 2008, Mohr 1906), which was a constant of proportionality when first introduced by J.II Bernoulli. Thus, it is now left for the following sections to show a glance of the evolutional that has been attained. Regarding the plate theory, a novel and unique coordinates-independent partial differential equation was developed. Such development has allowed for deeper understanding of the natural, physical and structural curvature problems.

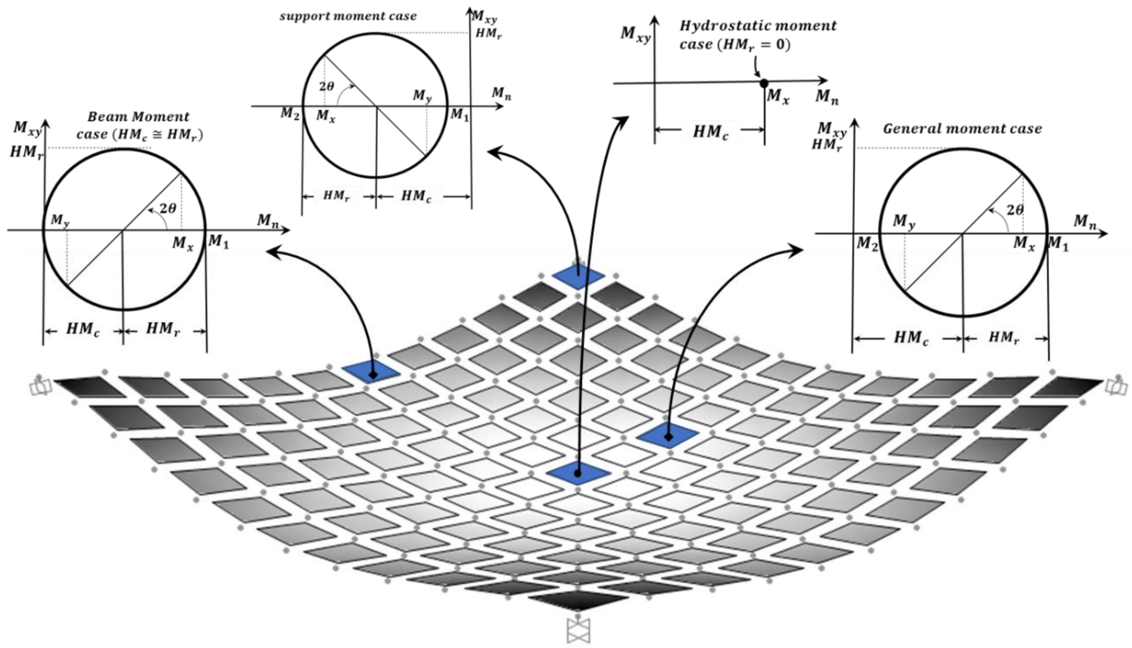


Fig. 4 Representation of HM designation for elements of interests having special cases

## 4. HM-Isotropic thin plates formulation

The assumption of thin isotropic plates, based on the work of Germain (1823, 1826) and Navier (1823, 1826, 1828), in the form presented by Kirchhoff (1850, 1897) is appropriate for steel and plain concrete plates, having almost identical material and geometrical (section) properties in all direction. The behaviour of which is to satisfy the following partial differential equation of deformation (Bramble and Payne 1962).

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p(x,y)}{D} \tag{36}$$

However, this equation is based on the introduction of the following normal and twisting moment-curvature relations Eq. 37, 38 and 39 into the statics-based equilibrium equation of moments Eq. 40. The following equations describe the moment-curvature relations and moment-displacement relations in Cartesian coordinates for normal moments.

$$M_x = -D \cdot (\kappa_x + \nu \cdot \kappa_y) = -D \cdot \left(\frac{\partial^2 w}{\partial x^2} + \nu \cdot \frac{\partial^2 w}{\partial y^2}\right) \tag{37}$$

$$M_y = -D \cdot (\kappa_y + \nu \cdot \kappa_x) = -D \cdot \left(\frac{\partial^2 w}{\partial y^2} + \nu \cdot \frac{\partial^2 w}{\partial x^2}\right) \tag{38}$$

On the other hand, the equation for twisting moment in planer elements is presented as follows,

$$M_{xy} = -D \cdot (1-\nu)\kappa_{xy} = -D \cdot (1-\nu)\frac{\partial^2 w}{\partial x \partial y} \tag{39}$$

That's, the flexural behavior of plates is described in terms of static equilibrium, and thus, suggests a three-dimensional boundary value problem 3D_BVP rather than a four-dimensional one 4D_BVP. This had led others to ignore the dependence of moment on material properties, which based on the introduced hypotheses in this study, has a direct effect on both the domain and the boundary conditions of the problem, resulting therefore in the four-dimensional problem. The moment equilibrium relation is as follows,

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -p \ (40) \tag{40}$$

Now, in introducing the concept of the hydrostatic method of analysis to the thin plates formulation, consider the principal plane of the extreme point in a plate. Based on the terminology developed in the previous section, this results in Eq. 37 and 38 for principal normal moments being written in the following form; noting that for the principal plane the twisting moment is omitted.

$$M_1 = HM_c + HM_r = -D \cdot \left(\frac{\partial^2 w}{\partial x^2} + \nu \cdot \frac{\partial^2 w}{\partial y^2}\right) \tag{41}$$

$$M_2 = HM_c - HM_r = -D \cdot \left(\frac{\partial^2 w}{\partial y^2} + \nu \cdot \frac{\partial^2 w}{\partial x^2}\right) \tag{42}$$

These two equations, representing the assumption of thin isotropic plates, when solved simultaneously for the unknown Mohr circle parameters $HM_c$ and $HM_r$, would result in the following expression for the hydrostatic method centre or mean moment parameters,

$$HM_c = -D \cdot \frac{1+\nu}{2} \cdot \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \tag{43}$$

Which can be written in the following concise form;

$$HM_c = -D \cdot \frac{1+\nu}{2} \cdot \nabla^2 w \tag{44}$$

Noting that $\nabla^2$ represents the Laplace operator also known as the Laplacian (symbolized with Nabla); where $\nabla^2 = \Delta = (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$. Substituting the resulting center or mean moment expression in the equations would result in finding the expression of the radius or deviatoric moment parameter for thin isotropic plates as given by,

$$HM_r = -D \cdot \frac{1-\nu}{2} \cdot \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}\right) \tag{45}$$

Recalling that $D$ refer to the isotropic flexural rigidity of plates; which is given with the following expression (Timoshenko 1953).

$$D = \frac{E \cdot t^3}{12 \cdot (1-\nu^2)} \tag{46}$$

This expression, when introduced in the earlier parameters equations, results in the following two expressions for the centre or hydrostatic or mean circle parameter $HM_c$ and the deviatoric or radius circle parameter $HM_r$ as follows,

$$HM_c = -\frac{1}{24} \cdot \frac{E \cdot t^3}{(1-\nu)} \cdot \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \blacksquare \tag{47}$$

$$HM_r = -\frac{1}{24} \cdot \frac{E \cdot t^3}{(1+\nu)} \cdot \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}\right) \blacksquare \tag{48}$$

where:

$HM_c$: **H**ydrostatic **M**ethod **C**enter ≡ $M_{iso}$; Mohr circle centre or Mean or hydrostatic or isotropic moment tensor based on equivalent terminologies to those of continuum mechanics in defining Mean or hydrostatic or isotropic stress tensor; in units of $[M \cdot L^2 \cdot T^{-2}/L]$ or moment per unit distance.

$HM_r$: **H**ydrostatic **M**ethod **R**adius ≡ $M_S$; Mohr circle radius or deviatoric moment tensor based on equivalent terminologies to that of continuum mechanics in defining deviatoric stress tensor; in units of moment per unit distance or $[M \cdot L^2 \cdot T^{-2}/L]$.

$E$: Modulus of elasticity (Young's Modulus) for isotropic material; in units of pressure or $[M\,L^{-1}\,T^{-2}]$.

ν: Poisson's ratio of lateral strain to normal strain; dimensionless.

$w$: Vertical normal displacement in plate element; in length $[L]$ dimension.

$t$: total uniform thickness of a plate element; in length $[L]$ dimension.

In a reflection of aforementioned-interpretation, the beneficial aspects of applying this concept are revealed through examining the introduced expressions for thin isotropic plates; it can be noted that the material dimension has been isolated successfully from the geometry-and-boundary-dependent differentials with respect to the Cartesian coordinate system. This allows for the reduction of the parameters to be found from three in the case of Cartesian coordinates; namely, the two normal moments and one twisting moment, into two parameters, namely, the hydrostatic and deviatoric moments represented in Mohr as centre and radius parameters. Recalling that if the twisting moment is of concern, it can be found merely through applying Eq. 22 and 23, where

$$M_{xy.max} = HM_r \tag{49}$$

Thus, in adopting this conception, the plate's differential equations gain a rather spacious physical parametric meaning than the narrow coordinates-dependent definition. The problem of HM-Isotropic thin plates is governed by the following partial differential equation resulting from the consideration of the equilibrium of an infinitesimal plate element with respect to the introduced concept,

$$\nabla^2[HM_c] + \left(\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)[HM_{rn}] + 2\frac{\partial^2}{\partial x \partial y}[HM_{rt}]\right) = -p(50) \tag{50}$$

where

$HM_{rn}$: The normal component of the moment deviatoric tensor, or the projection of the moment-radius on the normal moment axis in Mohr circle representation; $HM_{rn} = HM_r \cdot \cos(2\theta)$.

$HM_{rt}$: The twisting component of the moment deviatoric tensor, or the projection of the moment-radius on the twisting moment axis in Mohr circle representation; $HM_{rt} = HM_r \cdot \sin(2\theta)$.

Recalling that the angle of (2θ) in previous expressions corresponds to element rotation with an angle (θ) and through polar-cartesian conversion of the angle; based on the fact, that,

$$\mathrm{x} = \mathrm{r} \cdot \cos(\theta) \quad ; \quad y = r \cdot \sin(\theta) \quad ; \quad r = \sqrt{x^2+y^2} \quad (51) \tag{51}$$

Finally, implementing the deviatoric partial differential term in the derived partial differential equation of Eq. 50 concludes in the following (Eq. 52) simple form, governing differential equation for moment tensors in thin isotropic plates, shortened as HM-Isotropic PDE.

$$\nabla^2[HM_c] + \Lambda([HM_r]) = -p \blacksquare \tag{52}$$

Through a delicate examination of this equation, the apparent feature to be noticed is the full separation of tensor parameters. That's, the centre, mean or hydrostatic term has been isolated completely from the radius or deviatoric. This has been achieved in addition to the reduction of the number of terms that represents the problem (for e.g., from three in case of Eq. 40 to be described with only two parameters having more concise physical meaning).

Moreover, through meticulous perceptiveness, it can be observed that through introducing this partial differential equation, the plate problem has been split in two. This feature would better be illustrated with consideration to the various cases of **Fig. 4**.

Firstly, through equating the deviatoric or radius term in the HM-Isotropic PDE to zero the resulting partial differential equation would have the form

$$\nabla^2[HM_c] = -p \blacksquare \tag{53}$$

This equation governs all hydrostatic spherically curved elements to be formed in plates under flexural pure two-way actions. This includes all hydrostatic extreme and non-

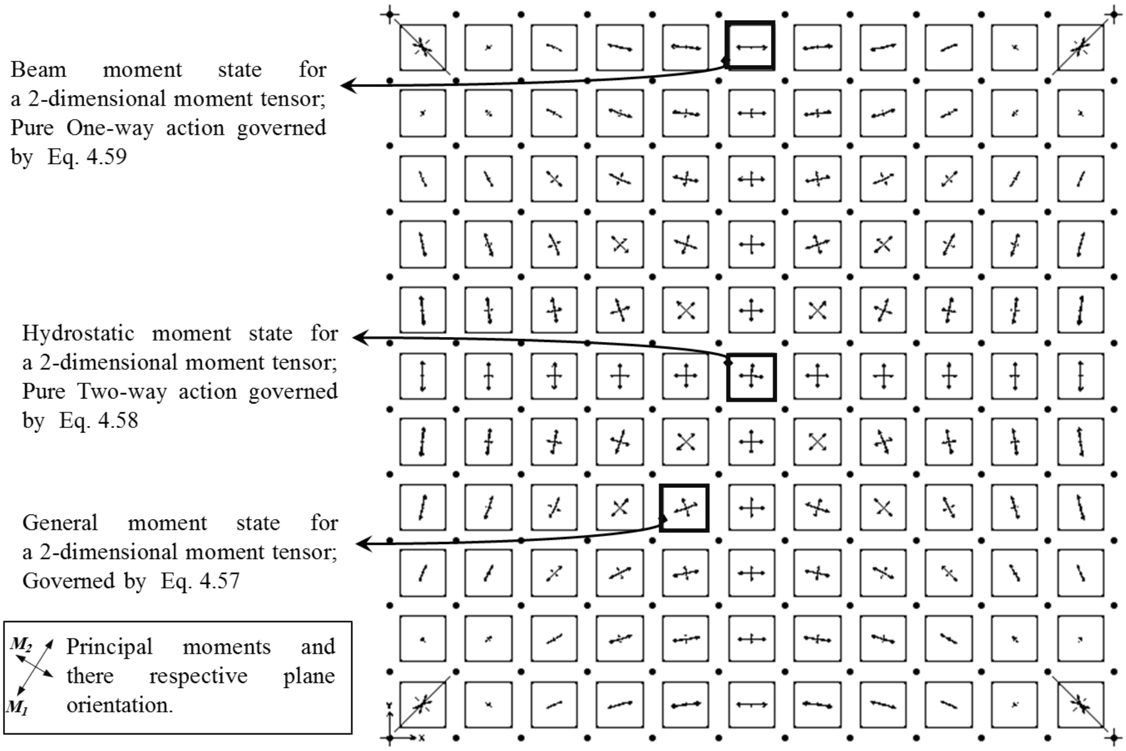


Fig. 5 Graphical illustration of the domain of applicability of various sub-terms of the introduced HM-Isotropic partial differential equation

extreme points to present in plates subjected to out-of-plane loading.

Secondly, in equating the proportions of the mean or hydrostatic moment to the deviatoric moment in the HM-Isotropic PDE, exposing the case of pure One-way moment resistance which is governed by the following partial differential equation.

$$2\cdot\nabla^2[HM_c] = -p \ \blacksquare \tag{54}$$

Tertiary, in the case of the mean or hydrostatic moment tensor being equal to zero, resulting in a plane orientation with pure twisting action. This case involves the situation of continuous plates, where an element with negative moment presenting at a certain direction, undergoes an equal positive moment on the perpendicular direction, the magnitude of each of which is equal to the deviatoric radius moment, and subsequently, identical to the twisting moment. Elements posing this state are parts of a surface having a negative Gaussian curvature (Huang and Lin 1998), and are governed by the following partial differential equation.

$$\Lambda([HM_r]) = -p \tag{55}$$

These cases of elements tensor-states are presented in **Fig. 5** along with various other states in **Fig. 4** for visual illustration purposes of the different application domain of the aforementioned-partial differential equations.

Finally, this section is concluded with the mathematical proof of the *HM-Isotropic partial differential equation*; Eq. 52. This can be done through simple substitution of the Eq. 47 and 48 into Eq. 50 along with the expressions of $HM_{rn}$ and $HM_{rt}$. This results in the following,

$$\begin{gathered}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)\left[-\frac{1}{24}\cdot\frac{E\cdot t^3}{(1-\nu)}\cdot\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)\right]\dots\\ +\left(\begin{array}{c}\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)\left[-\frac{1}{24}\cdot\frac{E\cdot t^3}{(1+\nu)}\left(\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w}{\partial y^2}\right)\cdot\cos(2\theta)\right]\dots\\ +2\frac{\partial^2}{\partial x\partial y}\left[-\frac{1}{24}\cdot\frac{E\cdot t^3}{(1+\nu)}\left(\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w}{\partial y^2}\right)\cdot\sin(2\theta)\right]\end{array}\right)\\ =-p\end{gathered} \tag{56}$$

Extracting the flexural rigidity term from Eq. 56, yields,

$$\begin{gathered}-\frac{E\cdot t^3}{12(1-\nu^2)}\\ \cdot\left(\frac{(1+\nu)}{2}\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)\dots\right.\\ \left.+\frac{(1-\nu)}{2}\left(\begin{array}{c}\left[\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)\left[\left(\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w}{\partial y^2}\right)\cdot\cos(2\theta)\right]\right]\dots\\ +2\frac{\partial^2}{\partial x\partial y}\left[\left(\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w}{\partial y^2}\right)\cdot\sin(2\theta)\right]\end{array}\right)\right)=-p\end{gathered} \tag{57}$$

Now, considering the Mohr circle representation of curvature; **Fig. 2**, which can be reintroduced in the form of differentials relations as per **Fig. 6**. Thus, serving as a differentiation alternative in finding trigonometric expressions in term of differentials. The sinusoidal term based on which is shown to be as follows,

$$\left(\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w}{\partial y^2}\right)\cdot\sin(2\theta) = 2\cdot\frac{\partial^2 w}{\partial x\partial y} \tag{58}$$

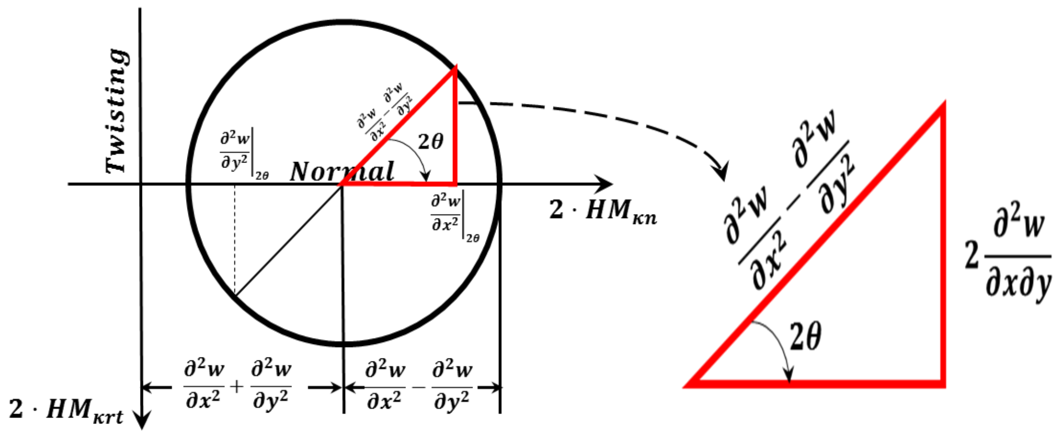


Fig. 6 The differential form of Mohr circle representation of curvature

Thus, substituting the sine trigonometric expression of Eq. 58 from **Fig. 6** into Eq. 57, yields the following,

$$-D \cdot \left( \frac{(1+\nu)}{2}(\nabla^4 w) + \frac{(1-\nu)}{2} \cdot \left( \begin{gathered} \left[ \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) \left[ \left( \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} \right) \cdot \cos(2\theta) \right] \right] \ldots \\ +2\frac{\partial^2}{\partial x \partial y}\left[ \left( \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} \right) \cdot \frac{2 \cdot \frac{\partial^2 w}{\partial x \partial y}}{\left( \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} \right)} \right] \end{gathered} \right) \right) \ldots = -p \tag{59}$$

From which, and since the angle used in the definition is the principal plane angle for which tan(2θ) =sin (2θ) and for which cos(2θ) =1. Eq. 59 simplifies into the following,

$$-D \cdot \left( \frac{(1+\nu)}{2}(\nabla^4 w) + \frac{(1-\nu)}{2} \cdot \left( \begin{gathered} \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) \left[ \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} \right] \ldots \\ +4\frac{\partial^4 w}{\partial x^2 \partial y^2} \end{gathered} \right) \right) = -p \tag{60}$$

Or

$$-D \cdot \left( \frac{(1+\nu)}{2}(\nabla^4 w) + \frac{(1-\nu)}{2} \cdot \left( \begin{gathered} \frac{\partial^4 w}{\partial x^4} - 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \ldots \\ +4\frac{\partial^2 w}{\partial x^2 \partial y^2} \end{gathered} \right) \right) = -p \tag{61}$$

Finally, upon rearrangement of common partial differential terms in Eq. 61 and simplification, the Germain-Lagrange governing partial differential equation for thin plates appears as given by Eq. 36, indicating the end of the proof and the correctness of the derived *HM-Iso partial differential equation for thin plates; Eq. 52*.

$$-D \cdot (\nabla^4 w) = -p \blacksquare \tag{62}$$

## 5. Conclusions

Based on the established conception and formulations, the following conclusions can be drawn:

1. Any 2-dimensional moment (curvature) tensor can be resolved into two tensors. These are; a hydrostatic and a deviatoric tensors. This can be represented as follows,

$$\boldsymbol{\kappa} = \boldsymbol{HM_{\kappa c}} + \boldsymbol{HM_{\kappa r}} = \boldsymbol{\kappa_{iso}} + \boldsymbol{S_\kappa} \tag{63}$$

$$\boldsymbol{M} = \boldsymbol{HM_c} + \boldsymbol{HM_r} = \boldsymbol{M_{iso}} + \boldsymbol{S_M} \tag{64}$$

2. A hydrostatic curvature tensor describes the development of a spherical surface with a positive constant Gaussian curvature.

3. The principal moments at any point in a plate can be described in relation to the hydrostatic and deviatoric moments as follows,

$$\boldsymbol{M_{1,2}} = \boldsymbol{HM_c} \pm \boldsymbol{HM_r} \tag{65}$$

Whereas, the moment at plane orientations other than the principal orientation can be represented as follows,

$$\boldsymbol{M_{x',y'}} = \boldsymbol{HM_c} \pm \boldsymbol{HM_r} \cdot \boldsymbol{\cos(2\theta)} \tag{66}$$

4. The twisting moment at any point in a plate under flexural deformations can be represented through the following expression,

$$\boldsymbol{M_{xy}} = \boldsymbol{HM_r} \cdot \boldsymbol{\sin(2\theta)} \tag{67}$$

While the maximum twisting moment can be represented by the following expression,

$$\boldsymbol{M_{xy.max}} = \boldsymbol{HM_r} \tag{68}$$

5. The redundant (coordinates-dependent) flexural variables were eliminated through the implementation of the coordinates-independent tensor formulations.

6. A full separation of the flexural tensor parameters was achieved. That's, the centre, mean or hydrostatic term has been isolated completely from the radius or deviatoric.

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