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Multi-objective topology and geometry optimization of statically determinate beams

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Abstract. The paper concerns topology and geometry optimization of statically determinate beams with arbitrary number of supports. The optimization problem is treated as a bi-criteria one, with the objectives of minimizing the absolute maximum bending moment and the maximum deflection for a uniform gravity load. The problem is formulated and solved using the Pareto optimality concept and the lexicographic ordering of the objectives. The non-dominated sorting genetic algorithm NSGA-II and the local search method are used for the optimization in the Pareto sense, whereas the genetic algorithm and the exhaustive search method for the lexicographic optimization. Trade-offs between objectives are examined and sets of Pareto-optimal solutions are provided for different topologies. Lexicographically optimal beams are found assuming that the maximum moment is a more important criterion. Exact formulas for locations and values of the maximum deflection are given for all lexicographically optimal beams of any topology and any number of supports. Topologies with lexicographically optimal geometries are classified into equivalence classes, and specific features of these classes are discussed. A qualitative principle of the division of topologies equivalent in terms of the maximum moment into topologies better and worse in terms of the maximum deflection is found.

Keywords: Pareto optimality; lexicographic ordering; beams; topology and geometry optimization; bending moment; deflection

1. Introduction

The structural optimization usually involves many requirements which should be met simultaneously to achieve a satisfactory design. Therefore, the majority of structural optimization problems are multi-criteria. The optimization of the different criteria can be performed simultaneously or hierarchically. Simultaneous multi-criteria optimization leads to Pareto-optimal solutions based on the principle of non-dominance (Parmee 2001). The Pareto-optimality concept is the most widely accepted in multi-objective optimization. The Pareto method takes all objectives into consideration concurrently during the optimization process and no objective is considered more or less important than any other. Lexicographic multi-criteria optimization utilizes ranking among objectives, which are considered in a hierarchical manner according to their importance (Coello Coello et al. 2002, Ehrgott 2005). A set of solutions which fulfill the first most important criterion is identified first. Then, the solutions which meet the criterion second in importance are selected from the set found in the first step and so on. This a priori approach requires preference information before the optimization process and is used when objectives are easily ranked. Both variants of multi-criteria optimization are considered in the paper.

The main purpose of a structure is to have sufficient strength to carry imposed loads without failure. The goal of maximizing strength is often equivalent to minimizing the maximum stress and, especially in the case of beams, to minimizing the absolute maximum bending moment. The second purpose of the structure is to have adequate stiffness to restrict deformations. Multi-criteria beam optimization for a criterion related to the bending stress and the deflection occurs among others in Ostwald and Rodak (2013). Optimization of beams to minimize the maximum moment is presented in Wang (2006) and Kozikowska (2011, 2014, 2016). Optimization of beams for decreasing the maximum deflection is discussed in Wang (2004), Jang et al. (2009), Loureiro et al. (2014). Minimization of the maximum moment and the maximum deflection of beams is considered in Imam and Al-Shihri (1996) and Dems and Turant (1997). These articles, however, do not concern multi-criteria optimization of beam topology and geometry. Ostwald and Rodak present results of bi-criteria Pareto optimization of the cross-section for the simply supported beam. Al-Shihri determines optimal locations of supports for the symmetric overhanging beam in single-criterion optimization problems. Dems and Turant consider general formulas of sensitivity analysis and scalar optimization problems for beams and frames with rigid and elastic supports and hinges. They present solutions for a single-span and a two-span beam. They take into account only such topographical changes that consist in moving one support away from the end of the beam towards the center. The previous articles by the author concern topology and geometry optimization of statically determinate beams with the absolute maximum moment as the scalar objective for

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different variants of fixed and most unfavorably distributed load. In these earlier papers, the author does not deal with either optimization for displacement or multi-criteria optimization. The current article by the author focuses on bi-criteria optimization of topology and geometry of these beams with the absolute maximum moment and the maximum deflection as the objectives. Results of all these articles show that beam optimization problems with topological and geometrical design variables, such as locations of supports and hinges, can greatly improve the structural behavior and the quality of structural design. Therefore, topology and geometry bi-criteria optimization of beams with the absolute maximum moment and the maximum deflection as objectives is performed in the paper. It is assumed that beams are subjected to a uniformly distributed transverse load (gravitational load for horizontal beams). For such a load, the maximum deflection of a beam is always downward (considered as positive), and absolute value does not have to be determined. Multi-span beams are statically determinate if all their reaction forces may be calculated using three global equilibrium conditions and moment-balance conditions of hinges. Beams studied in the article carry only transverse loads and do not undergo horizontal displacements and forces. Therefore, their statical determinacy is preserved without introducing roller supports.

Statically determinate multi-span beams are used in real structures due to their many advantages. They are more easily manufactured, transported, and erected. Temperature changes, support settlements, and fabrication errors cause no stresses in them. A proper arrangement of their supports and hinges leads to lower stresses or higher stiffness compared to statically indeterminate beams with the same number of the same supports, with a smaller number of hinges, or without hinges. It is because the optimization of a statically determinate beam in its geometric space with a larger number of dimensions (number of supports and hinges) allows to find solutions with smaller values of maximum moment or deflection.

The deflection of a beam under distributed load is highly nonlinear with respect to positions of supports and hinges (Biro and Cveticanin 2016). Furthermore, the point of the maximum deflection is not known in advance and occurs at different locations for different values of topological and geometrical variables. Changes of locations of supports and hinges can cause abrupt changes in the derivatives of an objective function associated with displacements. Analytical optimization methods based on gradient descent are not suitable for the bi-criteria beam optimization, while genetic algorithms are well adapted to deal with it. Genetic algorithms do not explicitly assume any properties of the objective functions, improves no single solution but a set of solutions and can effectively consider conflicting requirements. Because of this, the genetic algorithm is used in the paper for lexicographic optimization and the non-dominated sorting genetic algorithm NSGA-II (Coello Coello et al. 2002, Branke et al. 2008, Kaveh et al. 2013) for optimization in the Pareto sense.

Evolutionary optimization techniques to solve multi-objective problems began to appear in the literature in the eighties of the twentieth century. Since the late nineties, there has been a considerable increase in the number of applications of multi-objective evolutionary algorithms (Coello Coello et al. 2002, Branke et al. 2008, Munk et al. 2015). Today, evolutionary algorithms, including genetic algorithms, are considered as some of the most valuable and auspicious methods for solving varied multi-criteria optimization problems. But, the genetic algorithm requires a relatively long time to obtain a good description of the Pareto set, so a method of local improvement is applied for this purpose. Though a great number of publications are devoted to methods of multi-criteria structural optimization, there are not many studies in which the specific features of optimal solutions of such optimization problems are discussed. Therefore, the article concerns the multi-criteria optimization of beams with particular emphasis on the characteristics of optimal solutions.

The beams are modeled in the paper according to the classical Euler-Bernoulli beam theory. They are subjected to lateral loads only. Their cross-sections and deflections are small compared with their lengths. Therefore, deflections due to shear are assumed to be not significant, and deflections due to bending are only considered. Horizontal deflections are negligible, and slopes of deflection curves remain small. All beams have the same cross-section and are made of the same linear elastic material, so the flexural rigidity of beams EI is constant. Cross-sections are symmetrical about the planes of bending, and an unsymmetrical bending does not take place. Equations for slopes and deflections are determined by integrating the differential equations for bending of beams. Calculations of deflections by the direct integration method were previously hindered during the optimization due to weak hardware capabilities of computers. Magnitudes and locations of the maximum deflection usually were computed in an approximate way. Today's computers are a lot faster, and maximum deflections are calculated in the paper accurately (within the assumptions applicable to the differential equation of the elastic curve). Deflections in points of zero slope, at free ends of beams, and in hinges are checked to find the maximum value. All deflection curves presented in figures are magnified with respect to the lengths of beams to improve the readability of drawings. All the simulations, whose results are presented in the paper, are performed using a software written by the author.

2. Problem domain: Beam topology and geometry

The construction of all statically determinate beam topologies starts with the topology with supports at all ends of bars (beam ends and hinges). Then supports can be shifted from the ends of bars. The topology $\mathbf{t}$ of an n-support statically determinate beam is represented by n-element vector

$$\mathbf{t} = [t_1, \ldots, t_n] \tag{1}$$

where the topological code t_i describes a shift of support i away from the end of the beam for $i = 1$ and $i = n$ or from a hinge for $i = 2, \ldots, n-1$ (see Fig. 1). A shift to the

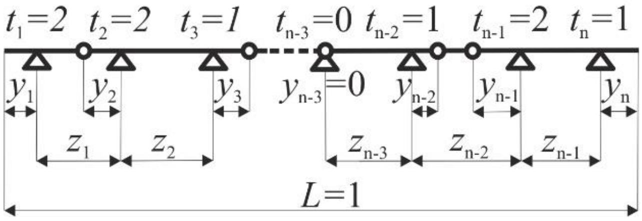


Fig. 1 Beam topology: topological codes t_i. Beam geometry: cantilever lengths y_i, span lengths z_i

left is encoded as 1, a shift to the right as 2, and no shift as 0. The size $|\mathbf{T}^n|$ of the set $\mathbf{T}^n$ of all beam topologies with n supports is equal to

$$\left|\mathbf{T}^n\right| = 4 \cdot 3^{n-2} \tag{2}$$

The geometry **g** of a beam with a fixed topology **t** is described by two sets of geometric parameters: $n-1$ variables z_i and n variables y_i (see Fig. 1). The parameters z_i include dimensionless lengths of spans between neighboring supports

$$0 < z_i < 1, \quad i \in \{1, 2, \ldots, n-1\} \tag{3}$$

The parameters y_i represents dimensionless lengths of cantilevers (external for $i = 1$ and $i = n$, internal for $i = 2, \ldots, n-1$)

$$\begin{aligned} y_i = 0 \quad \text{if } t_i = 0, \quad i \in \{1, 2, \ldots, n\} \\ 0 < y_i < 1 \quad \text{if } t_i \neq 0, \quad i \in \{1, 2, \ldots, n\} \end{aligned} \tag{4}$$

All beams have the same length, normalized to unity

$$L = y_1 + z_1 + z_2 + \ldots + z_{n-1} + y_n = 1 \tag{5}$$

A more detailed description of the topological and geometrical parameters is presented in Rychter and Kozikowska (2009).

3. Problem formulation of multi-objective beam optimization

The vector of objective functions for topology and geometry optimization of n-support statically determinate beams is

$$\begin{aligned} \mathbf{f}(\mathbf{t}, \mathbf{g}) &= \left(f_1(\mathbf{t}, \mathbf{g}), f_2(\mathbf{t}, \mathbf{g}) \right) \\ &= \left(\max_{x \in (0,1)} |M(\mathbf{t}, \mathbf{g}, x)|, \max_{x \in (0,1)} d(\mathbf{t}, \mathbf{g}, x) \right) \end{aligned} \tag{6}$$

where **t** is the beam topology described by the topological codes t_i, **g** is the beam geometry described by the parameters z_i, y_i, M**(t, g,** x**)** and d**(t, g,** x**)** are the bending moment and the deflection of the beam at the point of a coordinate x.

The topology and geometry optimization problem is to minimize the function **f(t, g)** over all spaces G of all admissible beam geometries **g** for all n-support topologies **t** from the set $\mathbf{T}^n$

$$\underset{\mathbf{t} \in \mathbf{T}^n : \mathbf{g} \in G}{\text{minimize}} \; \mathbf{f}(\mathbf{t}, \mathbf{g}) \tag{7}$$

The optimization problem expressed by equation (7) is solved in two stages. The first stage is to optimize the geometry **g** of each beam with a fixed topology **t** for all topologies from the set $\mathbf{T}^n$. The second stage is to find optimal topologies among all the topologies belonging to the set $\mathbf{T}^n$ on the basis of optimal values of the absolute maximum moment and the maximum deflection found for each topology in the first step. The two optimization stages are dependent on whether the optimization of the objectives is performed simultaneously or sequentially. Both objectives f_1 and f_2 are equally important and considered at the same time in the case of Pareto optimization, whereas the first objective f_1 is more important and considered first in the case of lexicographic optimization. The simultaneous optimization of topological and geometrical variables is not carried out because it does not guarantee that all optimal topologies will be found. The final solutions of optimization depend on starting points. A bad selection of initial topologies in the simultaneous optimization causes that only some parts of the full geometric-topological search space are explored, and only some optimal topological layouts are reached. Since all statically determinate beam topologies are known, the exhaustive search of these topologies with optimal geometries is carried out in the paper, and all global and local optima in the full geometric-topological search space are found.

4. Pareto multi-objective optimization

4.1 Problem formulation of Pareto geometry optimization for a fixed topology

The Pareto optimization problem is to minimize the function **f(g)** over the space G of all feasible beam geometries for a fixed topology **t** with no preference in the order in which the objective functions are optimized. The optimization problem can be formulated as follows

$$\begin{aligned} \underset{\mathbf{g} \in G}{\text{Pareto-minimize}} \; \mathbf{f}(\mathbf{g}) &= \left(f_1(\mathbf{g}), f_2(\mathbf{g}) \right) \\ &= \left(\max_{x \in (0,1)} |M(\mathbf{g}, x)|, \max_{x \in (0,1)} d(\mathbf{g}, x) \right) \end{aligned} \tag{8}$$

$$\text{Subject to} \begin{cases} 0 < z_i < 1, \quad i \in \{1, 2, \ldots, n-1\} \\ 0 < y_i < 1 \quad \text{for } t_j \neq 0, \quad j \in \{1, 2, \ldots, n\} \\ y_1 + z_1 + z_2 + \ldots + z_{n-1} + y_n = 1 \end{cases} \tag{9}$$

where components z_i and y_j of the decision (variable) vectors of beam geometry **g** belong to a nonempty feasible region $G \subset R^{n-1+c_E+c_H}$, objective vectors **f(g)** belong to a feasible objective region $F \subset R^2$, $M(\mathbf{g}, x)$ and $d(\mathbf{g}, x)$ are the bending moment and the deflection of the beam at the point of a coordinate x. The feasible objective region is the set of values of the objective functions f_1 and f_2 from equation (8) which may be attained for all admissible values of the geometry vector $\mathbf{g} \in G$ described by equation (9).

A beam geometry $\mathbf{g}^{PO} \in G$ is a Pareto-optimal or non-dominated solution for the given function $\mathbf{f(g)}$ if there exists no beam $\mathbf{g} \in G$ such that $f_1(\mathbf{g}) \le f_1(\mathbf{g}^{PO})$ and $f_2(\mathbf{g}) \le f_2(\mathbf{g}^{PO})$, and at least one of the inequalities is strict (Parmee 2001). A beam geometry $\mathbf{g}^{\mathrm{WPO}} \in G$ is a weakly Pareto-optimal solution if there exists no beam $\mathbf{g} \in G$ such that $f_1(\mathbf{g}) \le f_1(\mathbf{g}^{WPO})$ and $f_2(\mathbf{g}) \le f_2(\mathbf{g}^{WPO})$. The set of all Pareto-optimal decision vectors $\mathbf{g}^{PO}$ is called the Pareto-optimal set and is denoted by $P(G)$ in the decision space. Appropriately, an objective vector is Pareto-optimal if the corresponding decision vector is Pareto-optimal and the set of all Pareto-optimal objective vectors is called Pareto front or Pareto frontier and is denoted by $P(F)$ in the objective space. All Pareto front points are weak Pareto front points, but not vice versa. The terminology about multi-objective optimization problems has been taken from Branke *et al.* (2008).

The geometric variables z_i and y_j are constrained only by minimum distances between supports and hinges. These constraints affect the sizes of feasible regions G, but do not affect optimal solutions, which lie away from the boundaries of these feasible regions.

4.2 Method of Pareto geometry optimization for a fixed topology

Pareto optimization of the beam geometry for a fixed topology is performed by a hybrid optimization method: the non-dominated sorting genetic algorithm NSGA-II and a local search technique. In the binary tournament selection procedure of NSGA-II, a new population is created based on front numbers and crowding distances of individuals in the old population. All individuals which are non-dominated by any other have front number 1. Individuals with front number 2 are dominated only by individuals with front number 1, and so on. The crowding distance value of a particular solution is the average distance of its two neighboring solutions. Each binary tournament is won by the individual with a lower front number or by the individual with a higher crowding distance if both individuals have the same front numbers. Nevertheless, the genetic algorithm is not a good choice to find accurate Pareto front due to the low rate of convergence. Therefore, the Pareto local search method based on mutation is used in the paper to find the best possible approximations of Pareto-optimal solutions in a vicinity of points found by the genetic algorithm. Individuals with front number 1 in the last population of NSGA-II algorithm create an initial actual Pareto set for the Pareto local search procedure. New candidates for members of the Pareto set are formed from members of an actual Pareto set by small random changes in some of their geometrical variables. The procedure is accomplished by comparing objective function values of candidates and members, including the non-dominated candidates to the Pareto set, and removing dominated members from the set. The size of the Pareto set varies. The pseudo code of the algorithm is given in Fig. 2.

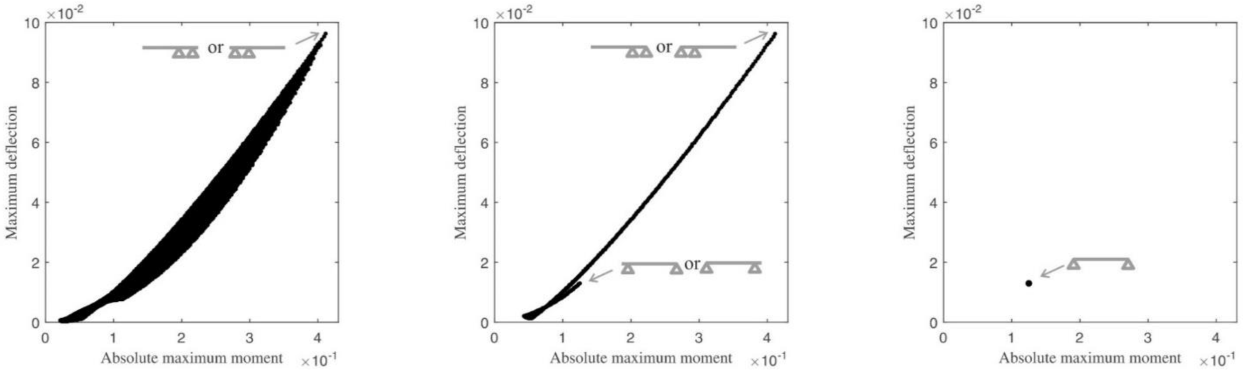
```
FUNCTION Pareto_local_search_method()
INPUT: initial Pareto set: s_PS individuals with front number 1 in last population of NSGA-II
OUTPUT: final Pareto set
WHILE time limit is exceeded OR minimal size of actual Pareto set is exceeded
   FOR i starts at 1, i<= s_PS, increment i DO
      CREATE candidate to Pareto set by mutation of geometrical variables of i member
                in actual Pareto set
   END FOR
   CALL give_Pareto_set()
END WHILE
END FUNCTION

FUNCTION give_Pareto_set()
INPUT: actual Pareto set (empty), checked set (s_CS individuals: all member of actual Pareto set
                                                   and all candidates to the set)
OUTPUT: new actual Pareto set with s_PS members
SORT the checked set into ascending order, first by the first objective, then by the second
FOR i starts at s_CS, i >=1, decrement i DO
   WHILE actual Pareto is not empty AND second objective of last individual in actual Pareto set
                                     >= second objective of i individual in checked set
      DELETE last individual in actual Pareto set
   END WHILE
   ADD i individual of checked set at the end of actual Pareto set
END FOR
END FUNCTION

s_PS size of actual Pareto set
s_CS size of checked set
```

Fig. 2 Pseudo code for Pareto local search method

4.3 Results of Pareto geometry optimization for a fixed topology

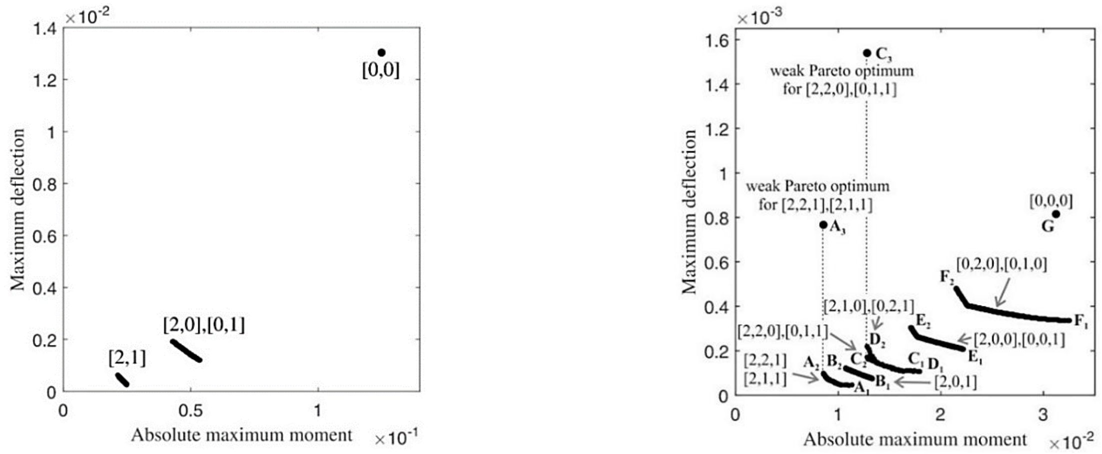
Graphical representations of feasible objective regions F and Pareto frontiers $P(F)$ in the two-dimensional objective space allow to detect conflicts between criteria and toestimate the range and the shape of Pareto fronts. Feasible objective regions and Pareto frontiers presented in the paper are performed for the beam length L, the intensity of uniformly distributed load q and the flexural rigidity EI all equal to unity. The feasible objective regions for all two-support beam topologies are shown in Figs. 3(a)-(c). The feasible objective region is one point for the simply supported beam with the topology [0,0] and only one geometry in Fig. 3(c), is a curve for topologies with different geometries obtained by changing only one parameter (for example for the symmetrical topologies [2,0] and [0,1] in Fig. 3(b)), and is an area for topologies with different geometries obtained by changing more than one parameter (for example for the topology [2,1] in Fig. 3(a)).

Pareto-optimal frontiers of two- and three-support topologies are presented in Fig. 4. We can notice that the optimized objective functions are conflicting with each other with respect to beam geometrical variables except for the topologies [0,0] and [0,0,0] without any cantilevers. The geometry of the beam with the topology [0,0] is fixed and is not subject to optimization. Thus, its Pareto front coincides with the feasible objective region and is one point in the two-dimensional objective space. For topologies with only zero topological code elements, there is no conflict between the objective functions for the number of supports $n > 2$. Geometry optimization which tends to equalize span lengths of such beams improves both the absolute maximum moment and the maximum deflection, and the Pareto front is also one point (the point G for the topology [0,0,0] in Fig. 4(b)). In the case of all other topologies with at least one non-zero topological code element, it does not exist a single solution which simultaneously optimizes both objectives, and there is always a set of Pareto-optimal solutions. Topologies with two or more the same neighboring topological code elements 2 or 1 (for example with topologies [2,2…], […2,2,2…], […2,2,1,1…],



(a) Topology [2,1] (b) Topologies [2,0], [0,1] (c) Topology [0,0]

Fig. 3 Feasible objective regions *F* for all two-support beam topologies



(a) Two-support beam topologies (b) Three-support beam topologies

Fig. 4 Pareto frontiers *P(F)*

[…1,1…]) have not only continuous Pareto fronts but also single weakly Pareto-optimal solutions (the points A_3 and C_3 in Fig. 4(b)). Geometries of the weakly Pareto-optimal solutions, represented by the points A_3 and C_3, are shown in Fig. 5.

Lower bounds of Pareto-optimal sets can be obtained by minimizing the absolute maximum moment and the maximum deflection individually. Points representing results of the single-objective optimization procedures lie at the extreme ends of the Pareto-optimal fronts in the objective space. The points for three-support topologies are denoted as A_1, A_2, … G in Fig. 4(b). Beams corresponding to these points are shown in Fig. 5. Values of moments, slopes and deflections of the beams are drawn on all subfigures proportionally. The beams which contain a pair 2,2 or 1,1 in their topological codes and are optimal for the absolute maximum moment have two different values of the maximum deflection for the hinge in different zero moment points. The maximum deflections of the beams are significantly smaller for the shorter internal cantilever than for the longer one. The shorter internal cantilever is created by placing the hinge in zero moment point near the support corresponding to this hinge (beams A_2 and C_2 in Fig. 5) while the longer internal cantilever – in zero moment point away from the corresponding support (beams A_3 and C_3 in Fig. 5).

4.4 Results of Pareto topology optimization

Results of multi-criteria topology optimization in Pareto sense for the absolute maximum moment and the maximum deflection are shown in Figs. 4 and 6. Fig. 4 shows us that both objective functions are consistent in terms of optimal topologies. Topologies with the maximum number of external cantilevers $c_E = 2$ and internal cantilevers $c_H = n\text{-}2$ are optimal for both the absolute maximum moment and the maximum deflection for any number of supports. However, topology optimization of beams should not only rely on finding the best solution among all possible topologies. Design conditions can limit the topological search space to a certain set of topologies. For example, it may be necessary to place supports at the ends of a beam or to place both ends of neighboring bars on a support. Therefore, it is advisable to compare Pareto fronts of all topologies in the space of both criteria.

Different topologies with the same numbers of external and internal cantilevers (with the same optimal value of absolute maximum moment) have the same or very much the same Pareto frontiers if all the topologies have at least one pair of neighboring elements 2,1 or have no pair 2,1 in their topological codes. Pareto fronts of topologies with the same numbers of external and internal cantilevers, but topologically different in terms of the presence of the pair

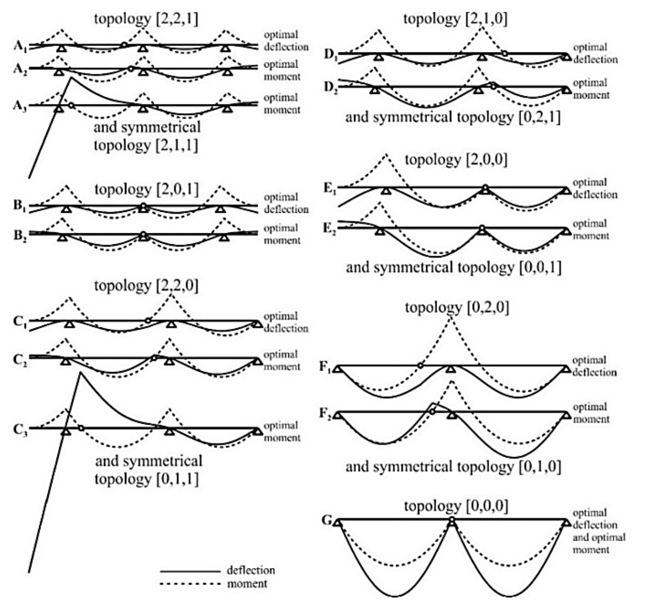


Fig. 5 Beams corresponding to extreme ends of Pareto fronts

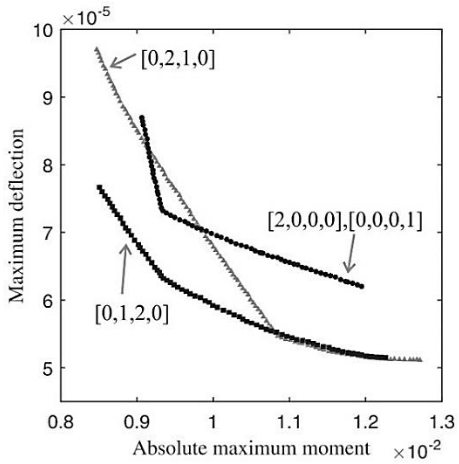


Fig. 6 Pareto frontiers for a few four-support beam topologies

2,1, have Pareto fronts which are partially overlapping in the parts corresponding to smaller values of the maximum deflections and markedly different in parts with smaller values of the maximum moments (see Pareto fronts of the topologies [2,1,0], [0,2,1], [2,2,0] and [0,1,1] in Fig. 4(b) and the topologies [0,2,1,0] and [0,1,2,0] in Fig. 6). Values of the maximum deflection are larger for beams with the part 2,1 in their topological codes.

For a number of supports $n \leq 3$, Pareto fronts of all topologies with the same number of supports, but with a different number of external or internal cantilevers, do not have common points, and there is no conflict between the objective functions in these cases. A topology that is better in terms of the absolute maximum moment is also always better in terms of the maximum deflection. The Pareto front of a worse topology is dominated by the Pareto front of a better topology. For a number of supports $n > 3$, the Pareto fronts of topologies with the same number of supports and with a little difference in the number of cantilevers can intersect (see Pareto fronts of the topologies [2,0,0,0] [0,0,0,1] and [0,2,1,0] in Fig. 6). If the intersection of Pareto fronts occurs for two topologies, none of these topologies is better. Some Pareto points of one topology are dominated by some Pareto points of the other topology and vice versa.

5. Lexicographic multi-objective beam optimization

The simultaneous multi-objective optimization approach produces all possible Pareto-optimal solutions and then leaves the user to choose from them by expressing preferences among the conflicting objectives. However, the set of Pareto-optimal solutions is usually too large, so that one needs certain additional rules to reduce it. Therefore, there are developed methodologies for finding preferred solutions from the Pareto set, instead of the complete set. One of the possible approaches is to utilize the lexicographic order of objectives.

Under a hierarchical lexicographic approach, both criteria must be ranked in order of importance. Beams deflect and stress when a transverse load is applied to them, and main longitudinal normal stresses are caused by the bending moment. Exceeding the permissible stresses causes the failure of the beam while exceeding the allowable deflection rather results in aesthetic and functional problems. Thus, there is a natural ranking among the objectives, and the bi-criteria optimization of statically determinate beams can be considered as lexicographic with the absolute maximum bending moment ranked as a more important criterion than the maximum deflection.

5.1 Problem formulation of lexicographic geometry optimization for a fixed topology

The optimization problem is to minimize the function **f(g)** over the space G of all feasible beam geometries for a fixed topology **t** with the preference in the order in which the objective functions are optimized. The optimization problem can be formulated as follows

$$\underset{\mathbf{g}\in G}{\text{Lex-minimize}}\ \mathbf{f}(\mathbf{g}) = (f_1(\mathbf{g}), f_2(\mathbf{g})) = \left(\max_{x\in(0,1)} |M(\mathbf{g},x)|, \max_{x\in(0,1)} d(\mathbf{g},x)\right) \tag{10}$$

$$\text{Subject to} \begin{cases} 0 < z_i < 1, \quad i \in \{1,2,\ldots,n-1\} \\ 0 < y_i < 1 \quad \text{for}\, t_j \neq 0, \quad j \in \{1,2,\ldots,n\} \\ y_1 + z_1 + z_2 + \ldots + z_{n-1} + y_n = 1 \end{cases} \tag{11}$$

where components z_i and y_j of the decision (variable) vectors of beam geometry **g** belong to a nonempty feasible region $G \subset R^{n-1+c_E+c_H}$, objective vectors **f(g)** belong to a feasible objective region $F \subset R^2$, $M(\mathbf{g}, x)$ and $d(\mathbf{g}, x)$ are the bending moment and the deflection of the beam at the point of a coordinate x.

A beam geometry $\mathbf{g}^{LO} \in G$ is a Lexicographic optimum (lex-optimum) if

$$\forall \mathbf{g} \in G: \quad \mathbf{f}(\mathbf{g}^{LO}) \prec \mathbf{f}(\mathbf{g}) \tag{12}$$

where $\prec$ is the lexicographic ordering indicating that $\mathbf{f}(\mathbf{g}^{LO}) = \mathbf{f}(\mathbf{g})$ or the first nonzero component of $\mathbf{f}(\mathbf{g}^{LO}) - \mathbf{f}(\mathbf{g})$ is negative (Ben-Tal 1979). The geometry $\mathbf{g}^{LO}$ lexicographically minimizes the vector function $\mathbf{f}(\mathbf{g})$ if $\mathbf{g}^{LO}$ minimizes f_1 and $\mathbf{g}^{LO}$ minimizes f_2 under the constraining condition that $f_1(\mathbf{g}) = f_1(\mathbf{g}^{LO})$.

Lexicographic beam optimization is carried out in two steps. The absolute maximum moment is minimized first. If the problem has a unique solution, it is the solution $\mathbf{g}^{LO}$ of the whole bi-objective optimization problem. Otherwise, $\mathbf{g}^{LO}$ is a solution of minimization task for the maximum deflection, subject to achieving the minimum with respect to the absolute maximum moment. The value of the absolute maximum moment M of the geometry $\mathbf{g}^{LO}$ is optimal both in scalar optimization for the absolute maximum moment and in lexicographic bi-objective optimization expressed by equations (10)-(11). The value of the maximum deflection d of the geometry $\mathbf{g}^{LO}$ is only optimal in lexicographic bi-objective optimization. Therefore, M is described as the optimal value while d as the lexicographically optimal or lex-optimal value, both corresponding to the geometry $\mathbf{g}^{LO}$.

5.2 Results of single-objective geometry optimization for absolute maximum moment for a fixed topology

The first stage of lexicographic bi-objective geometry optimization expressed by equations (10) and (11) is single-objective geometry optimization for the absolute maximum bending moment. The results of the optimization problem under a uniform load are given in Kozikowska (2011). The exact formulas for the locations of supports and hinges of beams optimal for the absolute maximum moment are found for all topologies. Lengths of beam segments with the optimal moment diagram at the bottom l^{n,c_E,c_H}, lengths of external cantilevers l_E^{n,c_E,c_H}, lengths of shorter internal cantilevers l_E^{n,c_E,c_H} and lengths of longer internal cantilevers $l_H^{n,c_E,c_H} + l^{n,c_E,c_H}$ can be expressed for a beam optimal for the absolute maximum moment with a fixed topology using the following formulas

$$l^{n,c_E,c_H} = \frac{L}{D^{n,c_E,c_H}} \quad l_E^{n,c_E,c_H} = \frac{L}{2D^{n,c_E,c_H}} \qquad l_H^{n,c_E,c_H} = \frac{\left(\sqrt{2}-1\right)L}{2D^{n,c_E,c_H}} \tag{13}$$

where $D^{n,c_E,c_H} = n + \frac{\sqrt{2}}{2}c_E + (\sqrt{2}-1)c_H - 1$, L is the length of the beam, the numbers of external cantilevers c_E and internal cantilevers c_H depend on the beam topology (Kozikowska, 2011). The value of the absolute maximum bending moment M^{n,c_E,c_H} of a beam optimal for the absolute maximum moment can be calculated from the formula

$$M^{n,c_E,c_H} = \frac{q\left(l^{n,c_E,c_H}\right)^2}{8} \tag{14}$$

The optimal moment diagram has the same extreme moment values equal to M^{n,c_E,c_H} at supports which were moved away from the ends of bars and in spans. The moment diagram of a beam with at least one pair of neighboring topological code elements 2,2 or 1,1 corresponds to many optimal geometries with the same locations of supports and different locations of hinges. Each pair 2,2 or 1,1 in a beam topological code corresponds to a span with one hinge and two zero moment points inside. Each such a span allows two alternative locations of the hinge in zero moment points. The total number of different optimal geometries g_H for a fixed topology equals the number of different locations of such single hinges in n_H spans

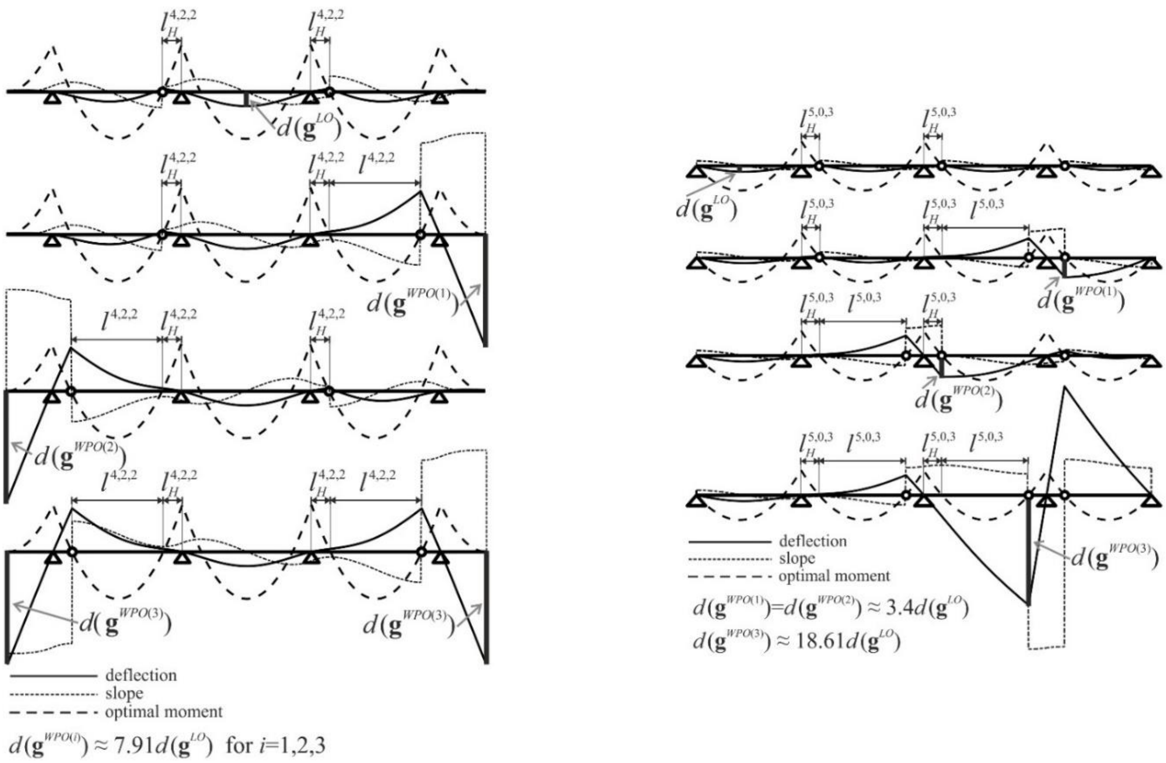
$$g_H = 2^{n_H} \tag{15}$$

where n_H is the number of pairs 2,2 or 1,1 in the topological code $\mathbf{t}$ where 2,2 indicates one pair, 2,2,2 – two pairs and so on. A single hinge which is placed in zero moment point near the support associated with this hinge creates shorter internal cantilever of the length l_H^{n,c_E,c_H} while placed in zero moment point distant from the associated support creates longer internal cantilever of the length $l_H^{n,c_E,c_H} + l^{n,c_E,c_H}$ (see Figs. 5 and 7).

5.3 Results of geometry optimization for maximum deflection in the set of beams optimal for absolute maximum moment for a fixed topology

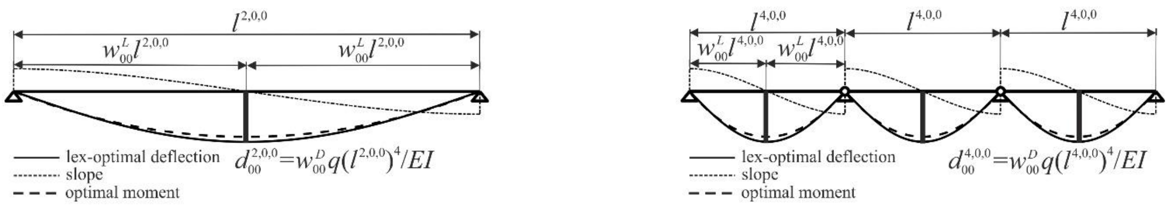
The second stage of the lexicographic bi-objective geometry optimization, expressed by equations (10) and (11), is to find geometries with the smallest maximum deflection from all the geometries optimal for the absolute maximum moment. Beams without single hinges in spans with two zero moment points (without any pair 2,2 or 1,1 in topological codes) are unique solutions of the single-objective optimization for the absolute maximum moment. They are also lex-optimal solutions of the bi-objective optimization for the absolute maximum moment and the maximum deflection, and the second stage of the lexicographic bi-objective optimization is then not needed. Otherwise, the single-objective optimization for the absolute maximum moment does not have a unique solution, and the second stage is accomplished by the exhaustive search of all different geometries g_H with the same optimal moment diagram and different locations of single hinges in spans with two zero values of this moment diagram. The results of this exhaustive search are shown for the topologies [2,2,1,1] and [0,1,1,1,0] in Fig. 7. Values of moments, slopes and deflections are drawn on both subfigures proportionally. The smallest maximum deflection, marked as $d(\mathbf{g}^{LO})$, have beam geometries with all shorter internal cantilevers of the length l_H^{n,c_E,c_H}, expressed by equation (13). These beam geometries are also Pareto optimal. Beam geometries with at least one longer internal cantilever of the length $l_H^{n,c_E,c_H} + l^{n,c_E,c_H}$ have much higher values of the maximum deflection, marked as $d(\mathbf{g}^{\mathrm{WPO}})$. These beam geometries are not lex-optimal; they are only weakly Pareto optimal.

Exact locations and values of the maximum deflection



(a) Topology [2,2,1,1] (b) Topology [0,1,1,1,0]

Fig. 7 All geometries optimal for absolute maximum moment



(a) Topology [0,0], $c_E = 0$, $c_H = 0$ (b) Topology [0,0,0,0], $c_E = 0$, $c_H = 0$

Fig. 8 Lexicographically optimal beam geometries with maximum deflection d_{00}

of all lexicographically optimal beams are found in zero slope points. The maximum deflection of a beam with any number of supports and any topology corresponds to one of three types. The types depend on the beam topology and are marked with subscripts 00, 21, and 20/01. A statically determinate multi-span beam consists of primary and secondary beams, and at least one primary beam is required. The maximum deflection d_{00} occurs when all primary beams are simply supported, the maximum deflection d_{21} – when at least one primary beam has both ends extending beyond supports, and the maximum deflection $d_{20/01}$ – when at least one primary beam has an overhang on one side and no primary beam has overhangs on both sides. The maximum deflections can be expressed as the product of one of three constant coefficients, designated w_{00}^D, w_{21}^D, $w_{21/01}^D$, and the expression $q(l^{n,c_E,c_H})^4$ / EI. Similarly, the horizontal distance from the point of the maximum deflection to the nearest support is the product of one of three constant coefficients, designated as w_{00}^L, w_{21}^L, $w_{20/01}^L$, and the length l^{n,c_E,c_H}.

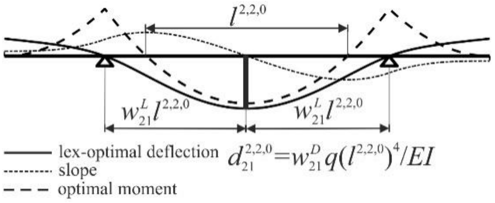
If the topological code of a beam consists of only zero elements (the beam does not have any cantilevers) then the maximum deflection of the beam of any number of supports can be calculated as for the simply supported beam with the topology [0,0] (see Fig. 8). Values of moment, slope and deflection have been scaled so that all maximum values are the same in Figs. 6-8. The maximum deflection is in the middle of each segment of the length l^{n,c_E,c_H} and the horizontal distance from the point of the maximum deflection to the nearest support can be expressed by the equation

$$l_{A00}^{n,0,0} = w_{00}^{L}\, l^{n,0,0} \tag{16}$$

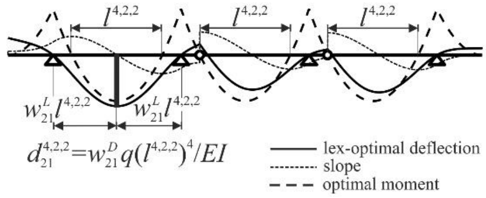
where the constant coefficient w_{00}^L= 1/2 and $l^{n,0,0}$= L/(n−1) in accordance with equation (13) for $c_E = 0$ and $c_H = 0$. The value of the maximum deflection is given by

$$d_{00}^{n,0,0} = w_{00}^{D} \frac{q\left(l^{n,0,0}\right)^4}{EI} \tag{17}$$

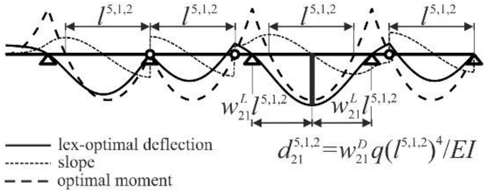
where $w_{00}^D = 5/348$, $\boldsymbol{q}$ is an intensity of evenly distributed load and EI is a constant flexural rigidity of the



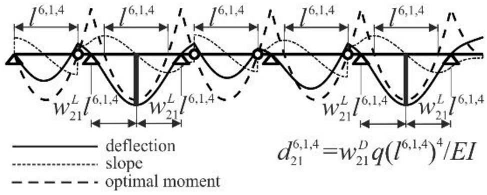
(a) Topology [2,1], $c_E = 2$, $c_H = 0$



(b) Topology [2,1,1,1], $c_E = 2$, $c_H = 2$

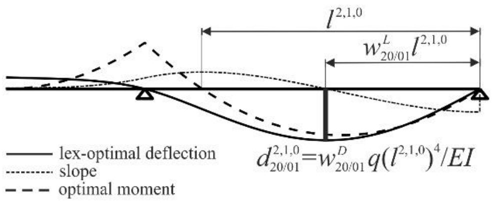


(c) Topology [2,0,2,1,0], $c_E = 1$, $c_H = 2$

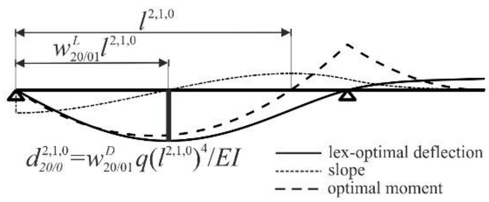


(d) Topology [0,2,1,2,2,1], $c_E = 1$, $c_H = 4$

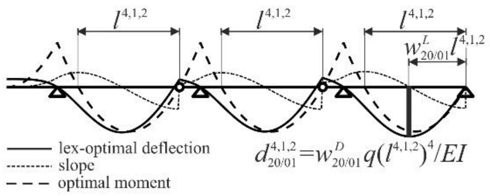
Fig. 9 Lexicographically optimal beam geometries with maximum deflection d_{21}



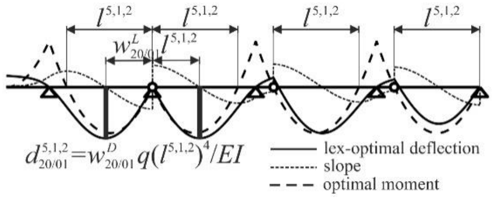
(a) Topology [2,0], $c_E = 1$, $c_H = 0$



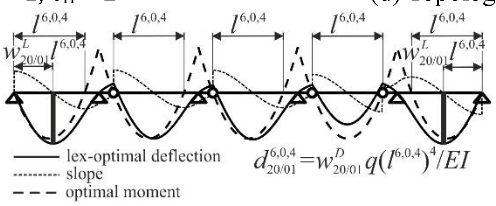
(b) Topology [0,1], $c_E = 1$, $c_H = 0$



(c) Topology [2,2,2,0], $c_E = 1$, $c_H = 2$



(d) Topology [2,0,1,1,0], $c_E = 1$, $c_H = 2$



(e) Topology [0,1,1,1,2,0], $c_E = 1$, $c_H = 4$

Fig. 10 Lexicographically optimal beam geometries with maximum deflection $d_{20/01}$

beam. Values of moment, slope and deflection have been scaled so that all maximum values are the same in Figs. 8-10.

If there is at least one pair of neighboring elements 2 and 1 in the topological code of a beam (where 2 appears before 1 and not vice versa) then the maximum deflection of the beam with the lexicographically optimal geometry occurs in the center of each two-support beam with the topology [2,1] overhanging supports (see Fig. 9). The horizontal distance from the point of the maximum deflection to the nearest support is

$$l_{A21}^{n,c_E,c_H} = w_{21}^{L} l^{n,c_E,c_H} \tag{18}$$

where the constant coefficient $w_{21}^{L}=\sqrt{2}/2$. The maximum deflection can be expressed as follows

$$d_{21}^{n,c_E,c_H} = w_{21}^{D}\frac{q\left(l^{n,c_E,c_H}\right)^4}{EI} \tag{19}$$

where w_{21}^{D}= 1/48.

If there is at least one non-zero element and no pair 2,1 in the topological code of a beam (there is at least one primary beam overhanging one support and there is no primary beam overhanging both supports) then the maximum deflection of the beam with the lex-optimal geometry is near the center of each two-support beam with the topology [2,0] or [0,1] (see Fig. 10). The horizontal distance from the point of the maximum deflection to the nearest support (right support for the topology [2,0] and left support for the topology [0,1]) is equal to

$$l_{A20/01}^{n,c_E,c_H} = w_{20/01}^{L}\, l^{n,c_E,c_H} \tag{20}$$

where the constant coefficient

$$w_{20/01}^{L} = \left(\frac{1}{2} + \sin\left(\frac{1}{3}\sin^{-1}\left(\frac{3}{8}\left(\sqrt{2}-1\right)\right)\right)\right) \approx 0.551964$$

is the trigonometric root of cubic slope function. The maximum deflection in this zero slope point is equal to

$$d_{20/01}^{n,c_E,c_H} = w_{20/01}^{D} \frac{q\left(l^{n,c_E,c_H}\right)^4}{EI} \tag{21}$$

Where

$$w_{20/01}^{D} = \frac{\left(w_{20/01}^{L}\right)^4}{24} - \frac{\left(w_{20/01}^{L}\right)^3}{12} + \frac{\left(3\sqrt{2}+5\right) w_{20/01}^{L}}{192} \approx 0.016425$$

5.4 Equivalence relation of beam topologies with lex-optimal geometries

$\mathbf{T}^n$ is the set of n-support beam topologies. Any two different topologies $\mathbf{t}_i$ and $\mathbf{t}_j$ of the set $\mathbf{T}^n$ are equivalent with respect to the relation R^{Md} if optimal values of the absolute maximum moments M_i and M_j, and lex-optimal values of the maximum deflections d_i and d_j of these topologies are equal. It has been shown in Kozikowska (2011) that M_i and M_j are equal if the numbers of external cantilevers c_E and internal cantilevers c_H of these topologies are the same. Thus, the equivalence relation R^{Md} can be expressed as

$$\begin{aligned} &\mathbf{t}_i \equiv_{R^{Md}} \mathbf{t}_j \\ &\text{if } M_i^{n,c_E,c_H} = M_j^{n,c_E,c_H} \ \wedge \ d_i^{n,c_E,c_H} = d_j^{n,c_E,c_H} \end{aligned} \tag{22}$$

where M_i^{n,c_E,c_H}, M_j^{n,c_E,c_H}, d_i^{n,c_E,c_H}, d_j^{n,c_E,c_H} are optimal values of the absolute maximum moment and lex-optimal values of the maximum deflection for the topology $\boldsymbol{t}_i$ and $\boldsymbol{t}_j$ with c_E external cantilevers and c_H internal cantilevers. The equivalent condition (22) can be expressed on the basis of the properties of two different n-support beam topologies $\boldsymbol{t}_i$ and $\boldsymbol{t}_j$ as

$$\begin{aligned} &\mathbf{t}_i \equiv_{R^{Md}} \mathbf{t}_j \quad \text{if } \underbrace{\left(c_{E,i} = c_{E,j} \wedge c_{H,i} = c_{H,j}\right)}_{M_i^{n,c_E,c_H} = M_j^{n,c_E,c_H}} \\ &\wedge \left(\underbrace{\begin{matrix}\text{there is at least one pair} \\ \text{2,1 in both topologies}\end{matrix}}_{d_i^{n,c_E,c_H} = d_j^{n,c_E,c_H} = d_{21}} \vee \underbrace{\begin{matrix}\text{there is no pair 2,1} \\ \text{in both topologies}\end{matrix}}_{d_i^{n,c_E,c_H} = d_j^{n,c_E,c_H} = d_{20/01}} \right) \end{aligned} \tag{23}$$

Based on the relation R^{Md}, the set $\mathbf{T}^n$ can be divided into disjoint equivalence classes of beam topologies called topological classes $\mathbf{T}_i^{nMd}$.

5.5 Results of lexicographic topology optimization: topological classes $\mathbf{T}_i^{nMd}$

The sequence $\{\mathbf{T}_i^{nMd}\}$ is the set of n-support topological classes $\mathbf{T}_i^{nMd}$ which are sorted according to the lexicographical order of the values M_i^{nMd} and d_i^{nMd}. The class $\mathbf{T}_i^{nMd}$ precedes the class $\mathbf{T}_j^{nMd}$ in the sequence $\{\mathbf{T}_i^{nMd}\}$ if

$$\left(M_i^{nMd} < M_j^{nMd}\right) \vee \left(M_i^{nMd} = M_j^{nMd} \wedge d_i^{nMd} < d_j^{nMd}\right) \tag{24}$$

where M_i^{nMd}, d_i^{nMd}, M_j^{nMd} and d_j^{nMd} are optimal values of the absolute maximum moment and lex-optimal values of the maximum deflection in the topological class $\mathbf{T}_i^{nMd}$ and $\mathbf{T}_j^{nMd}$, respectively.

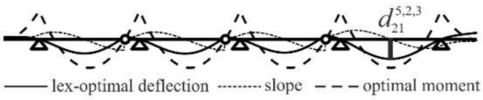
Beam topologies from a topological class $\mathbf{T}_i^{n}$ (Kozikowska 2011) can have the same value or two different values of the maximum deflection. The class $\mathbf{T}_i^{n}$ with equal values of the maximum deflection for all topologies remains unchanged as a class $\mathbf{T}_i^{nMd}$. Such a class can have values of the maximum deflection d_{21}, $d_{20/01}$, or d_{00}. The class $\mathbf{T}_i^{n}$ with two different values of the maximum deflection is divided into two distinct classes: $\mathbf{T}_j^{nMd}$ with the smaller value $d_{20/01}$ and $\mathbf{T}_{j+1}^{nMd}$ with the larger value d_{21}.

The first topological class $\mathbf{T}_1^{nMd}$ with maximum possible number of cantilevers $c_E = 2$ and $c_H = n-2$ has at least one pair 2,1 in all topological codes of beams for any number of supports. Values of the maximum deflection are the same for all beams and can be calculated from the formula (19). The first two-support class $\mathbf{T}_1^{2Md}$ is presented in Fig. 9a, the first three-support class $\mathbf{T}_1^{3Md}$ – in Fig. 14(a), and the first five-support class $\mathbf{T}_1^{5Md}$ – in Fig. 11. Moment diagrams, slope diagrams, and deflection curves are plotted in Figs 11-13 maintaining proportions.

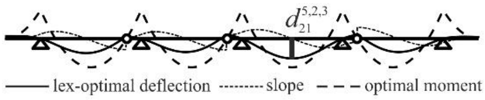
If all beams in a topological class $\mathbf{T}_i^{n}$ do not have any pair 2,1 and have at least one non-zero element in their topological codes (at least one primary beam has one overhang and no primary beam has two overhangs) then the class $\mathbf{T}_i^{n}$ is equal to a topological class $\mathbf{T}_j^{nMd}$ with maximum deflections $d_{20/01}$ expressed by equation (21). There are three such classes $\mathbf{T}_j^{nMd}$ for a fixed $n > 2$, corresponding to different values of the parameters c_E and c_H: the class $\mathbf{T}_{4n-10}^{nMd}$ for $n \in \langle 3,4 \rangle$ and $\mathbf{T}_{6n-20}^{nMd}$ for $n \geq 5$ with $c_E = 2$ and $c_H = 0$ (for example the class $\mathbf{T}_2^{3Md}$ in Fig. 14(b)), the class $\mathbf{T}_{6n-13}^{nMd}$ with $c_E = 1$ and $c_H = 0$ (for example the class $\mathbf{T}_5^{3Md}$ in Fig. 14(e)) and the penultimate class $\mathbf{T}_{6n-12}^{nMd}$ with $c_E = 0$ and $c_H = 1$ (for example the class $\mathbf{T}_6^{3Md}$ in Fig. 14(f)).

There is only one topology without any cantilevers ($c_E = 0$ and $c_H = 0$) in the last n-support topological class with the maximum deflection d_{00} expressed by equation (17). Such a class is the same as the last class $\mathbf{T}_{3(n-1)}^{n}$. The class $\mathbf{T}_3^{2Md}$ is given in Fig. 8(a), $\mathbf{T}_7^{3Md}$ – in Fig. 14(g), and $\mathbf{T}_{13}^{4Md}$ – in Fig. 8(b).

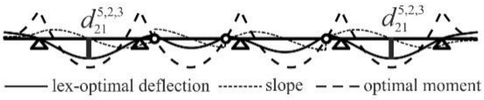
All four beam topologies in the topological class $\mathbf{T}_{11}^{5Md}$ from Fig. 12, with $c_E = 0$ and $c_H = 3$ do not include a pair 2,1 and have the maximum deflection $d_{20/01}$, expressed



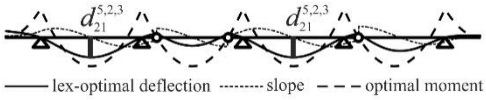
(a) Topology [2,2,2,2,1] and symmetrical



(b) Topology [2,2,2,1,1] and symmetrical

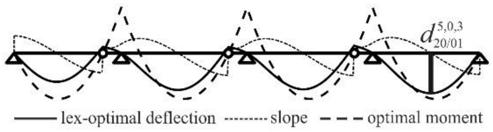


(c) Topology [2,1,2,2,1] and symmetrical

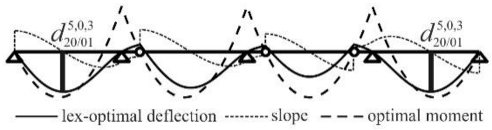


(d) Topology [2,1,2,1,1] and symmetrical

Fig. 11 The class $\mathbf{T}_1^{5Md}$ with maximum deflection d_{21}

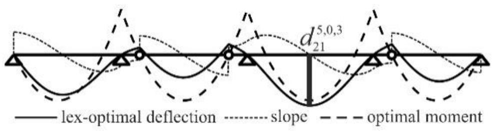


(a) Topology [0,2,2,2,0] and symmetrical

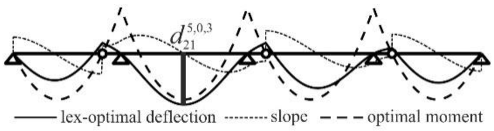


(b) Topology [0,1,1,2,0] and symmetrical

Fig. 12 The class $\mathbf{T}_{11}^{5Md}$ with maximum deflection $d_{20/01}$



(a) Topology [0,1,2,1,0] and symmetrical



(b) Topology [0,2,1,1,0] and symmetrical

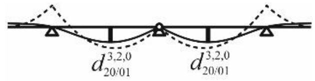
Fig. 13 The class $\mathbf{T}_{12}^{5Md}$ with maximum deflection d_{21}

by equation (21). Topologies in the class $\mathbf{T}_{12}^{5Md}$, presented in Fig. 13, have the same number of supports and cantilevers as the topologies from Fig.12. However, they form a separate class because their topological codes include a pair 2,1, and their maximum deflection d_{21} is expressed by equation (19). The maximum deflection $d_{20/01}$ is smaller than the maximum deflection d_{21} for beams with the same number of supports and cantilevers (the same length l^{n,c_E,c_H}). The class $\mathbf{T}_{11}^{5Md}$ precedes the class $\mathbf{T}_{12}^{5Md}$ because classes with equal values of the absolute maximum moment are ordered by increasing values of the maximum deflection. The class $\mathbf{T}_7^5$ is the sum of the classes $\mathbf{T}_{11}^{5Md}$ (Fig. 12) and $\mathbf{T}_{12}^{5Md}$ (Fig. 13).

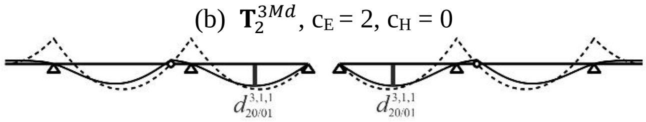
The whole set of three-support topological classes $\mathbf{T}_3^{nMd}$ under a uniform load with all optimal moment diagrams and lex-optimal deflection curves is presented in Fig. 14. Values of moments and deflections are drawn proportionally in all subfigures. The total number of classes $\mathbf{T}_i^n$ equivalent in terms of the optimal moment only is equal to 3(*n*-1) according to equation (5.9) in Kozikowska (2011). The number of the classes $\mathbf{T}_i^n$ with only one formula for the maximum deflection is equal to 5 for $n \geq 3$ (the first class with the deflection d_{21}, the last class with the deflection d_{00}, and three classes with the deflection $d_{20/01}$). Each of the other $3(n-1) - 5 = 3n-8$ classes $\mathbf{T}_i^n$ is divided into two classes $\mathbf{T}_i^{nMd}$ equivalent in terms of the optimal moment and the lex-optimal deflection: of smaller deflection $d_{20/01}$ and of larger deflection d_{21}. The total number of *n*-support topological classes $\mathbf{T}_i^{nMd}$ is equal to

$$p^{nMd} = \begin{cases} 3 & \text{for } n = 2 \\ 5 + 2(3n-8) = 6n-11 & \text{for } n \geq 3 \end{cases} \tag{25}$$

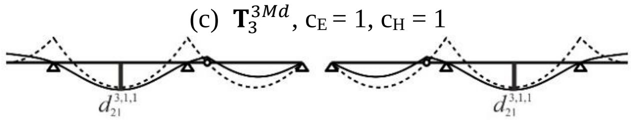
(a) $\mathbf{T}_1^{3Md}$, $c_E = 2$, $c_H = 1$



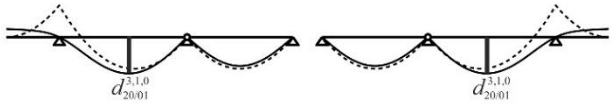
(b) $\mathbf{T}_2^{3Md}$, $c_E = 2$, $c_H = 0$



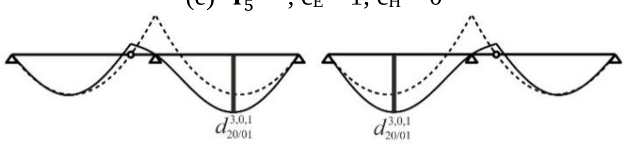
(c) $\mathbf{T}_3^{3Md}$, $c_E = 1$, $c_H = 1$



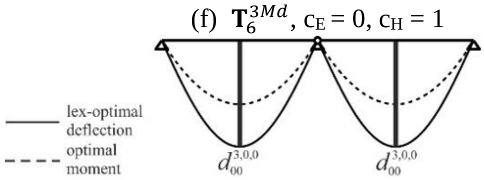
(d) $\mathbf{T}_4^{3Md}$, $c_E = 1$, $c_H = 1$



(e) $\mathbf{T}_5^{3Md}$, $c_E = 1$, $c_H = 0$

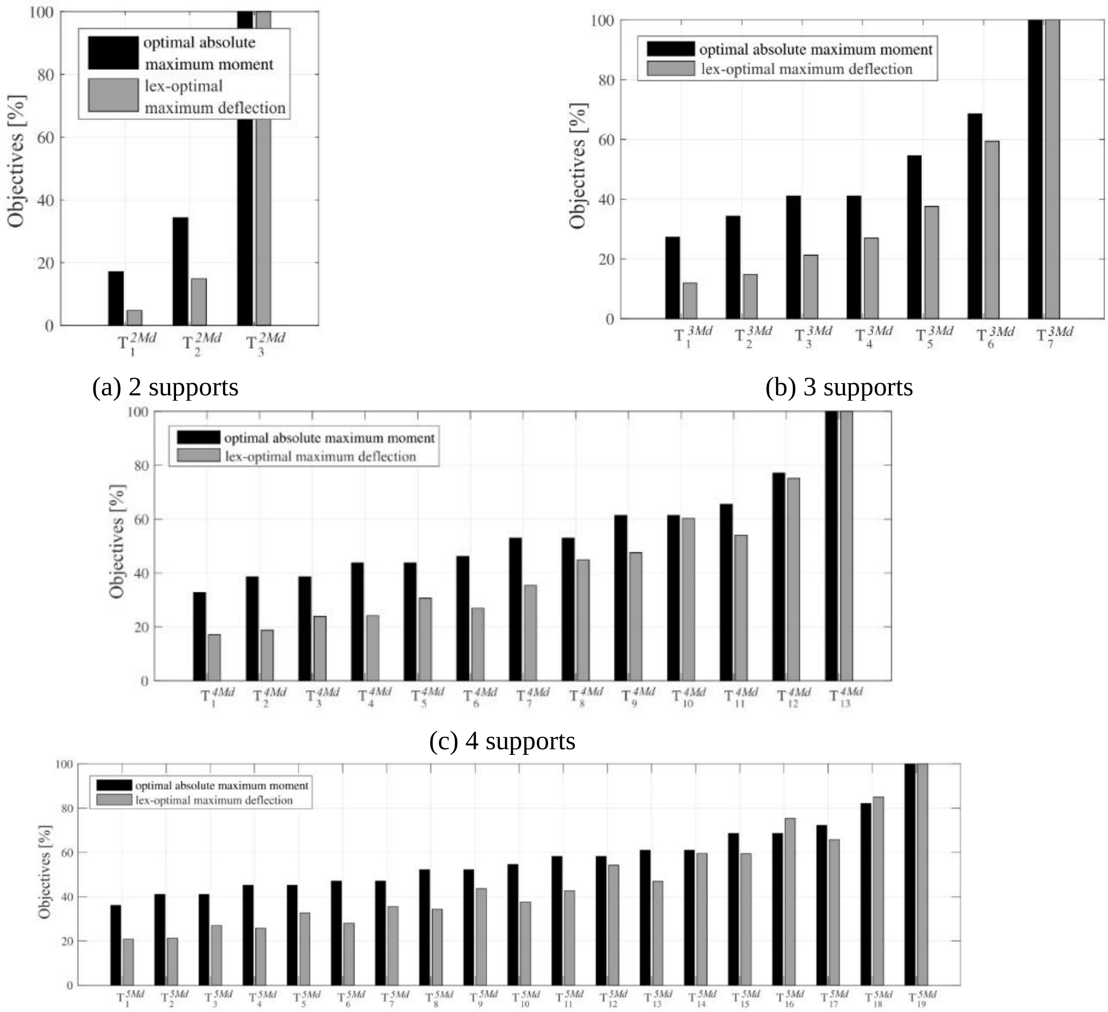


(f) $\mathbf{T}_6^{3Md}$, $c_E = 0$, $c_H = 1$



(g) $\mathbf{T}_7^{3Md}$, $c_E = 0$, $c_H = 0$

Fig. 14 All three-support classes



(a) 2 supports (b) 3 supports

(c) 4 supports

(d) 5 supports

Fig. 15 Values of the objectives in topological classes $\mathbf{T}_i^{nMd}$ for different numbers of supports

The division of beam topologies into classes $\mathbf{T}_i^{nMd}$ is very useful in the design of statically determinate beams with a constant cross-section, made of the same material. Proper selection of topologies allows us to obtain the maximum deflection over 20% lower for beams with the same optimal absolute maximum moment ($d_{20/01}^{n,c_E,c_H}/d_{21}^{n,c_E,c_H}$ ≈ 0.788).

The plot in Fig. 15 shows values of the absolute maximum moment and the maximum deflection in all classes $\mathbf{T}_i^{nMd}$ with lexicographically optimal geometries, under a uniform load, for a number of supports $n \in$ <2,5>. The values of the absolute maximum moment and the maximum deflection of the last class are assumed to be all equal to 100% in each subfigure with a different number of supports. Optimal values of the absolute maximum moment in consecutive classes $\mathbf{T}_i^{nMd}$ for a fixed n form a non-decreasing sequence $\{M_i^{nMd}\}$ (strictly increasing or constant). For this sequence the following relationships hold

$$\forall i \in \langle 1, 6n-12 \rangle \quad M_i^{nMd} \le M_{i+1}^{nMd} \wedge M_i^{nMd} < M_{i+2}^{nMd} \tag{26}$$

where M_i^{nMd} is the minimal value of the absolute maximum moment in topological class $\mathbf{T}_i^{nMd}$.

Lexicographically optimal values of the maximum deflection in consecutive classes $\mathbf{T}_i^{nMd}$ for a fixed n form a non-monotonic sequence $\{d_i^{nMd}\}$. For this sequence the following relationships hold

$$\begin{gathered}\forall i \in \langle 1, 6n-12 \rangle \wedge (i \ne 6n-21 \text{ for } n > 4) \\ \left(d_i^{nMd} < d_{i+1}^{nMd}\right) \vee \left(d_i^{nMd} > d_{i+1}^{nMd} \wedge d_i^{nMd} < d_{i+2}^{nMd}\right)\end{gathered} \tag{27}$$

where d_i^{nMd} is the lex-optimal value of the maximum deflection in a topological class $\mathbf{T}_i^{nMd}$. The lex-optimal value of the maximum deflection $d_{6n-21}^{nMd} = d_{21}^{n,1,2}$ in the class $\mathbf{T}_{6n}^{nMd}$ with c_E = 1 and c_H = 2, for n >4, meets the conditions

$$\begin{gathered} d_{6n-21}^{nMd} > d_{6n-20}^{nMd} \wedge d_{6n-21}^{nMd} \\ > d_{6n-19}^{nMd} \wedge d_{6n-21}^{nMd} < d_{6n-18}^{nMd} \end{gathered} \tag{28}$$

Let us consider the sequence $\{r_n^d\}_{n=2}^{\infty}$ whose members are ratios of lexicographically optimal values of the maximum deflection, d_{6n-11}^{nMd} for n ≥ 3 (d_3^{nMd} for n = 2) and d_1^{nMd} of the extreme classes $\mathbf{T}_{6n-11}^{nMd}$ for n ≥ 3 ($\mathbf{T}_3^{nMd}$ for n = 2) and $\mathbf{T}_1^{nMd}$, respectively

$$r_n^d = \frac{d_{6n-11}^{nMd}}{d_1^{nMd}} = \frac{5}{8}\left(\frac{\sqrt{2}n-\sqrt{2}+1}{n-1}\right)^4 \qquad (29)$$

The sequence $\{r_n^d\}$ starts with the value $\frac{5(1+\sqrt{2})^4}{8} \approx 21.23$ for n = 2, then it decreases monotonically and converges to the limit 5/2. This means that the values of the maximum deflection of extreme classes become closer to each other with a growing number of supports, but the deflection of the worst class $\mathbf{T}_{6n-11}^{nMd}$ is at least two and a half times greater than the maximum deflection of the best class $\mathbf{T}_1^{nMd}$. Properties of the sequence $\{r_n^M\}^{\infty}{}_{n=2}$, whose members are ratios of optimal values of the absolute maximum moments of the extreme classes, are discussed in Kozikowska (2011).

6. Conclusions

The paper has dealt with the bi-objective optimization of statically determinate beams of any number of supports and any topology. The absolute maximum bending moment and the maximum deflection have been chosen as optimization criteria, and topological and geometrical parameters have been used as design variables. Pareto and lexicographic optimization variants have been considered and studied.

The feasible objective regions have been presented in the objective space. The non-dominated sorting genetic algorithm coupled with the local search technique has been used to obtain Pareto-optimal solutions of the optimization problem. Conflicts between both criteria have been detected, and representative Pareto-optimal fronts and solutions have been examined for many topologies.

The use of lexicographic approach is particularly justified in the bi-objective optimization of statically determinate beams because we rather do not have difficulties in putting the objective functions into an absolute order of importance. Furthermore, the single-objective optimization for the absolute maximum moment has led to alternative optima of different values of the maximum deflection for beams with pairs of 1,1 or 2,2 in their topological codes. As a result of the lexicographic optimization, beams optimal for the maximum deflection have been found among these alternatives, and their geometric characteristics have been determined. An important contribution of the paper has been to find the exact analytical expressions for the locations and values of the maximum deflection for all lexicographically optimal beams of any topology and any number of supports.

Beam topologies with lexicographically optimal geometries have been split up into topological classes with the same values of both objectives. The characteristic features of these classes have been presented. It has been detected that beams with different topologies but with the same number of cantilevers and the same optimal value of the absolute maximum moment can differ in values of the maximum deflection. The maximum deflection of lex-optimal beams with at least one pair 2,1 in their topological codes is almost 27% larger than the deflection of beams without this pair.

Locations of beam supports in real structures are usually determined based on architectural aesthetic requirements and structural design limitations. A lot of support layouts of optimal statically determinate beams are symmetrical, regular, and meet aesthetic standards. The beams have minimum values of the absolute maximum moment and also handle structural constraints. The right selection of topologies can significantly reduce the maximum deflection of the beams. The existence of many optimal beams provides huge design opportunities because it offers a variety of satisfactory solutions. Therefore, the results of the paper can be used in many practical design tasks.

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Notations

c_E, c_H	number of external and internal cantilevers
$d(\ldots,x)$	deflection at the point of coordinate x
EI	constant flexural rigidity of beam
$\mathbf{f}$	bi-objective optimization function with components f_1, f_2
F	feasible objective region in objective space
$\mathbf{g}$	beam geometry described by parameters z_i, y_i
$\mathbf{g}^{PO}$, $\mathbf{g}^{WPO}$	Pareto-optimal and weakly Pareto-optimal beam geometry
g_H	number of different geometries optimal for absolute maximum moment
G	feasible region in decision space
l	length of beam segment with optimal moment diagram at the bottom
l_E, l_H, L	length of external and internal cantilever of optimal beam and length of beam
$M(\ldots, x)$	bending moment at the point of coordinate x
M, M_i	optimal value of absolute maximum moment (for topology $\mathbf{t}_i$)
n	number of supports
n_H	number of pairs 2,2 or 1,1 in beam topology
p	number of topological classes
P	Pareto front
q	intensity of uniformly distributed load
S_{CS}	size of checked set
S_{PS}	size of actual Pareto set
$\mathbf{t}$, $\mathbf{t}_i$	beam topology (topological code of beam)
t_i	topological code of support i, $i = 1, 2, \ldots, n$
$\mathbf{T}^n$	set of all topologies with n supports
$\mathbf{T}_i^n$	class of beam topologies with geometries optimal for absolute maximum moment
x	axial coordinate
y_i	dimensionless length of cantilever, $i = 1, 2, \ldots, n$
z_i	dimensionless length of span, $i = 1, 2, \ldots, n-1$
$(.)^{n,c_E,c_H}$	quantity for n-support optimal beam with c_E external and c_H internal cantilevers

For lex-optimal beam geometries

d	lex-optimal value of maximum deflection
d_i, d_i^{nMd}	lex-optimal value of maximum deflection (for topology $\mathbf{t}_i$ and class $\mathbf{T}_i^{nMd}$)
$\mathbf{g}^{LO}$	lex-optimal beam geometry
l_A	horizontal distance from point of maximum deflection to the nearest support
M_i^{nMd}	optimal value of absolute maximum moment in class $\mathbf{T}_i^{nMd}$
R^{Md}	equivalence relation of beam topologies with lex-optimal geometries
$\{r_n^{d,M}\}$	sequence of ratios of d or M in extreme classes $\mathbf{T}_i^{nMd}$
$\mathbf{T}_i^{nMd}$	class of beam topologies with lex-optimal geometries
w^D, w^L	coefficient for calculating value and position of maximum deflection
$(.)_{00}$	quantity for topology with only zero elements
$(.)_{20/01}$	quantity for topology with at least one non-zero element and without any pair 2,1
$(.)_{21}$	quantity for topology with at least one pair 2,1