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The two-scale analysis method for bodies with small periodic configurations

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Abstract. The mechanical behaviours of the structure made from composite materials or the structure with periodic configurations depend not only on the macroscopic conditions of structure, but also on the detailed configurations. The Two-Scale Analysis (TSA) method for these structures, which couples the macroscopic characteristics of structure with its detailed configurations, is presented for 2 or 3 dimensional case in this paper. And the finite element algorithms based on TSA are developed, and some results of numerical experiments are given. They show that TSA with its finite element algorithms is more effective.

Key words: two-scale analysis method; composite material; structure with periodic configuration.

1. Introduction

Composite materials have been widely used in high technology engineering as well as ordinary industrial products because they have many elegant qualities, such as high strength, high stiffness, high temperature resistance, corrosion resistance and fatigue resistance. Most of the composite materials have periodic configurations. For example, in composite materials reinforced with metallic fibres, the reinforce, the matrix and their interfaces are regularly distributed, and they have very strong coordinate effects. Moreover, structures with periodic configurations have many advantages, for example, under bearing certain loadings the amount of materials can be reduced.

In order to understand the coordinate effects so as to improve the design and manufacture of

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new composite materials and structures with periodic configurations, many experts have been working in this field from macroscopic, detailed or microscopic points of view. They have achieved useful results in experiment, theory and applications. R. Hill, Z. Hashin. J. Reddy *et al.* and J.L. Lions, O.A. Oleinik *et al.* have proposed important theories in effective constants of material, multi-laminated plate model and homogenization methods. But so far there are no unified model and effective numerical method to analyze in detail and precisely the problems arisen from these fields. This might be due to the size ε of the basic cell is very small and the composition of the reinforce, the matrix and their interfaces is very complicated. This paper contributes to develop a kind of two-scale variable model and the corresponding FEM algorithms to analyze the problems in detail and more efficiently.

The mechanical behaviours of these structures and the mechanism of failure depend not only on the macroscopic conditions, the geometry of the structure, the effective constants of materials, the loadings and the constraints, but also on the detailed configuration, especially the states of strains and stresses within a basic cell since they vary sharply in locale. Therefore using the analysis from either macroscopic or microscopic scale, one cannot obtain an accurate estimation. We consider that it is a reasonable strategy to construct a method for coupling the macroscopic behaviours of the structure with its detailed configuration.

The Two-Scale Analysis method (TSA) that couples both characteristics of global behaviour of the structure and its detailed configurations is studied for problems in two and three dimensional cases. The mathematical model of the problem is given in section 2. The asymptotic expansion of the solution and its approximation are briefly discussed in section 3. The finite element algorithms based on TSA and some results of numerical experiments are shown presented in section 4.

2. Mathematical model

In structural engineering and in the design and manufacture of new industrial products, the analysis problems for the structures with the following features will be encountered:

- The whole structure or the main part of the structure has periodicity, that is, it is composed of same basic configurations, for example, the bricked wall shown in Fig. 1(a).
- The structure is made from composite materials with the same cells shown in Fig. 1(b).
- Rock structures with probable models.

In two or three dimensional cases, the analysis of these structures leads to the following elasto-static problems with material constants changing sharply and periodically. For simplicity, following conventional notations are introduced: Assume that a group of loads and constraints are imposed on the structure Ω

$$\begin{array}{lll} \text{body force:} & f(x) = (f_1(x), \cdots, f_n(x))^T, & x \in \Omega \\ \text{surface forces:} & p(x) = (p_1(x), \cdots, p_n(x))^T, & x \in S_\sigma \\ \text{given displacement:} & \bar{u}(x) = (\bar{u}_1(x), \cdots, \bar{u}_n(x))^T, & x \in S_u \end{array} \tag{1}$$

where $n=2$ or 3, $x=(x_1, \cdots, x_n)^T$ denotes the coordinates of structure, and the surface $\partial\Omega$ is divided into two parts S_σ and S_u, $\partial\Omega = S_u \cup S_\sigma$ and $S_u \cap S_\sigma = \phi$. The displacement is denoted by the vector valued function $u(x) = (u_1(x), \cdots, u_n(x))^T$.

Based on solid mechanics the problem can be formulated as following elasto-static problem

$$\frac{\partial}{\partial x_j}\left(a_{ijhk}^{\varepsilon}(x)\frac{1}{2}\left(\frac{\partial u_h}{\partial x_k}+\frac{\partial u_k}{\partial x_h}\right)\right)=f_i(x),\quad i=1,\ldots,n,\ x\in\Omega \tag{2}$$

$$\nu_j\sigma_{ji}(u)=p_i(x),\ x\in S_\sigma,\quad i=1,\ldots,n \tag{3}$$

$$u(x)=\bar{u}\ ,\ x\in S_u \tag{4}$$

where $\{a_{ijhk}^{\varepsilon}(x)\}$ is the tensor of material parameters which are periodic functions with period ε_i in the direction e_i, e_i is a base vector. In this paper, suppose that $\varepsilon_1=\cdots=\varepsilon_n=\varepsilon$. Then

$$a_{ijhk}^{\varepsilon}(x+\varepsilon e_m)=a_{ijhk}^{\varepsilon}(x) \tag{5}$$

$\nu=(\nu_1,\cdots,\nu_n)^T$ denotes the normal direction on Γ_σ. Usually material constants have the following properties:

Property 2.1 Material constants $\{a_{ijhk}^{\varepsilon}(x)\}$ are bounded, measurable and symmetric, i.e.,

$$a_{ijhk}^{\varepsilon}=a_{ijkh}^{\varepsilon}=a_{jihk}^{\varepsilon} \tag{6}$$

and they satisfy the $E(\mu_1,\mu_2)$ condition

$$\mu_1\eta_{ih}\eta_{ih}\le a_{ijhk}^{\varepsilon}\eta_{ih}\eta_{jk}\le\mu_2\eta_{ih}\eta_{ih} \tag{7}$$

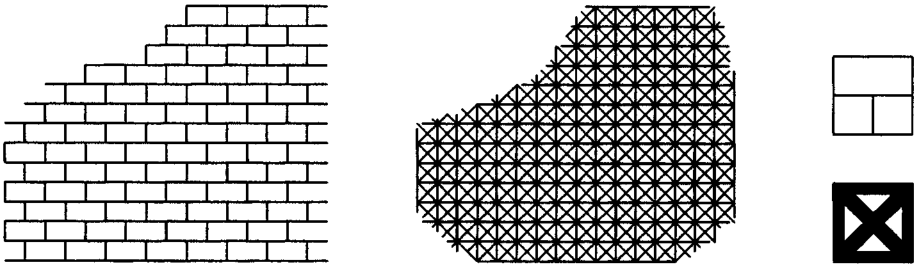
where $\{\eta_{ih}\}$ is a symmetric matrix, and μ_1,μ_2 are constants with $0<\mu_1<\mu_2$.

In this paper we discuss the TSA method for the structure composed of the entire basic configurations without holes and the boundary condition of displacements is given. Let εQ_c denote a basic configuration that is a ε-square with prescribed composition of materials shown in Fig. 1(c), Q is a 1-square, and Q_c is shrunk to εQ_c in ε, then Ω can be denoted as

$$\Omega=\bigcup_{Z\in T}\varepsilon(Q+Z) \tag{8}$$

where Z is an n-dimensional integer vector, and T is the set of Z, such that $\varepsilon(Q+Z)\in\Omega$. From Eq. (5) $\{a_{ijhk}^{\varepsilon}(x)\}$ can be expressed as

$$a_{ijhk}^{\varepsilon}(x)=a_{ijhk}\left(\frac{x}{\varepsilon}\right)=a_{ijhk}(\xi),\quad \xi=\frac{x}{\varepsilon} \tag{9}$$



(a) (b) (c) Basic configuration

Fig. 1 Composite materials and structures with periodicity

$\{a_{ijhk}(\xi)\}$ are 1-periodic functions.

In the following discussion it is also supposed that $a_{ijhk}(x/\varepsilon) \in C^1(\Omega)$ and $f(x) \in C^\infty(\Omega)$.

3. Two-scale asymptotic analysis

Consider the following elasto-static problem

$$\frac{\partial}{\partial x_j}\left(a_{ijhk}(\xi)\frac{1}{2}\left(\frac{\partial u_h}{\partial x_k}+\frac{\partial u_k}{\partial x_h}\right)\right)=f_i(x),\ x\in\Omega,\ i=1,\cdots,n \tag{10}$$

$$u=\bar{u}\quad x\in\partial\Omega$$

Since the deformations of the structure with periodic configurations depend on both its global behaviours and basic configurations, the displacements of the structure can be expressed as the vector valued function $u(x)=u(x, \xi)$ of the two scale variables x and ξ. Assume that $u(x)$ can be expanded into a series in the following form:

$$u(x)=\sum_{l=0}^{\infty}\varepsilon^l W_l(x,\xi) \tag{11}$$

and that $W_l(x, \xi)$ for $l=0, 1, 2, \cdots$ can be expressed in separated variables as follows

$$W_l=\sum_{\alpha}N_\alpha(\xi)v_\alpha(x) \tag{12}$$

where $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_l)$, for $j=1, \cdots, l$, α_j can take any integer satisfying $1\le\alpha_j\le n$.

$$N_\alpha(\xi)=\begin{bmatrix}N_{\alpha 11}(\xi),\cdots,N_{\alpha 1n}(\xi)\\ \vdots\ ,\cdots,\ \vdots\\ N_{\alpha n1}(\xi),\cdots,N_{\alpha nn}(\xi)\end{bmatrix},\ v_\alpha(x)=\begin{bmatrix}v_{\alpha 1}(x)\\ \vdots\\ v_{\alpha n}(x)\end{bmatrix} \tag{13}$$

every $N_{\alpha ij}(\xi)\ (i, j=1,\cdots,n$ and $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_l)$, $l=0, 1, 2, \cdots)$ is a 1-periodic function defined on R^n, and $v_\alpha(x)$ is a vector valued function defined on Ω.

$$u_i(x)=\sum_{l=0}^{\infty}\varepsilon^l\sum_{\alpha}N_{\alpha im}(\xi)v_{\alpha m}(x),\ i=1,\cdots,n \tag{14}$$

Every $N_{\alpha ij}(\xi)$ and $v_{\alpha m}(x)$ will be determined below. Respecting

$$\frac{\partial u_i}{\partial x_j}=\frac{\partial u_i}{\partial x_j}+\frac{1}{\varepsilon}\frac{\partial u_i}{\partial \xi_j} \tag{15}$$

one obtains that

$$\frac{\partial}{\partial x_j}\left(a_{ijhk}(\xi)\frac{1}{2}\left(\frac{\partial u_h}{\partial x_k}+\frac{\partial u_k}{\partial x_h}\right)\right)$$

$$
\begin{aligned}
&=\sum_{l=0}^{\infty}\varepsilon^{l}\sum_{\alpha}a_{ijhk}(\xi)\,\frac{1}{2}\left[N_{\alpha hm}(\xi)\frac{\partial^{2}v_{\alpha m}}{\partial x_{k}\partial x_{j}}+N_{\alpha km}(\xi)\frac{\partial^{2}v_{\alpha m}}{\partial x_{h}\partial x_{j}}\right]\\
&+\sum_{l=0}^{\infty}\varepsilon^{l-1}\sum_{\alpha}a_{ijhk}(\xi)\,\frac{1}{2}\left[\frac{\partial N_{\alpha hm}(\xi)}{\partial\xi_{k}}+\frac{\partial N_{\alpha km}(\xi)}{\partial\xi_{h}}\right]\frac{\partial v_{\alpha m}}{\partial x_{j}}\\
&+\sum_{l=0}^{\infty}\varepsilon^{l-1}\sum_{\alpha}\frac{1}{2}\left[\frac{\partial}{\partial\xi_{j}}\left(a_{ijhk}(\xi)N_{\alpha hm}(\xi)\right)\frac{\partial v_{\alpha m}}{\partial x_{k}}+\frac{\partial}{\partial\xi_{j}}\left(a_{ijhk}(\xi)N_{\alpha km}(\xi)\right)\frac{\partial v_{\alpha m}}{\partial x_{h}}\right]\\
&+\sum_{l=0}^{\infty}\varepsilon^{l-2}\sum_{\alpha}\frac{\partial}{\partial\xi_{j}}\,\frac{1}{2}\left[\alpha_{ijhk}(\xi)\left(\frac{\partial N_{\alpha hm}(\xi)}{\partial\xi_{k}}+\frac{\partial N_{\alpha km}(\xi)}{\partial\xi_{h}}\right)\right]v_{\alpha m}(x)\\
&=f_{i}(x)\qquad i=1,\cdots,n,\quad x\in\Omega,\ \xi\in\Omega/\varepsilon
\end{aligned}
\tag{16}
$$

Suppose that the expression (16) is valid for arbitrary ε and basic configuration and load $f(x)$. Compare the coefficients of ε^{l} on both sides of (16), and one obtains a series of equations with respect to $N_{\alpha m}(\xi)$ and $v_{\alpha m}(x)$. First consider the coefficients of ε^{-2}, and one obtains that

$$
\frac{\partial}{\partial\xi_{j}}\left[a_{ijhk}(\xi)\frac{1}{2}\left(\frac{\partial N_{0hm}(\xi)}{\partial\xi_{k}}+\frac{\partial N_{0km}(\xi)}{\partial\xi_{h}}\right)\right]v_{0m}(x)=0,x\in\Omega,\xi\in\Omega/\varepsilon\qquad i=1,\cdots,n
\tag{17}
$$

Since $v_0(x)=(v_{01}(x),\ \cdots,\ v_{0n}(x))^T$ arises from the macroscopic behaviour of the structure, geometry, loads and constraints, it cannot indentify to zero. Thus, from $N_{0im}(\xi)$s periodicity, it follows that for m=1, $\cdots$, n

$$
\frac{\partial}{\partial\xi_{j}}\left[a_{ijhk}(\xi)\frac{1}{2}\left(\frac{\partial N_{0hm}(\xi)}{\partial\xi_{k}}+\frac{\partial N_{0km}(\xi)}{\partial\xi_{h}}\right)\right]=0,\quad \xi\in Q
\tag{18}
$$

$N_{0m}(\xi)=\left(N_{01m}(\xi),\cdots,N_{0nm}(\xi)\right)^{T}$ is a 1-periodic vector valued function.

It can be proved that problem (18) for $N_{0m}(\xi)$ has a constant solution. Thus $N_0(\xi)$ can be chosen as

$$
N_0(\xi)=\begin{bmatrix}1&\cdots&0\\ \vdots&\cdots&\vdots\\ 0&\cdots&1\end{bmatrix}
\tag{19}
$$

Now compare the coefficients of ε^{-1} on both sides of Eq. (16) one obtains that

$$
\sum_{\alpha_1=1}^{n}\frac{\partial}{\partial\xi_{j}}\left[a_{ijhk}(\xi)\,\frac{1}{2}\left(\frac{\partial N_{\alpha_1 hm}(\xi)}{\partial\xi_{k}}+\frac{\partial N_{\alpha_1 km}(\xi)}{\partial\xi_{h}}\right)\right]v_{\alpha_1 m}
$$

$$+\frac{1}{2}\left[\frac{\partial}{\partial\xi_j}(a_{ijhk}(\xi)N_{0hm}(\xi))\frac{\partial v_{0m}}{\partial x_k}+\frac{\partial}{\partial\xi_j}(a_{ijhk}(\xi)N_{0km}(\xi))\frac{\partial v_{0m}}{\partial x_h}\right]$$
$$=0,\ x\in\Omega,\ \xi\in\Omega/\varepsilon,\ \text{and}\ i,\alpha_1=1,\cdots,n$$

Further

$$v_{\alpha_1 m}=\frac{\partial v_{0m}}{\partial x_{\alpha_1}},\ \text{and}\ \frac{\partial v_{0m}}{\partial x_{\alpha_1}}\neq 0,\quad \forall x\in\Omega,\alpha_1,\ m=1,\cdots,n$$

Respecting the symmetry of $\{a_{ijhk}(\xi)\}$ and $N_{\alpha_1 km}{}'(\xi)$s 1-periodicity then one can obtain the Lame equations satisfied by vector-valued function $N_{\alpha_1 m}(\xi)=(N_{\alpha_1 1m}(\xi),\cdots,N_{\alpha_1 nm}(\xi))^T$

$$\frac{\partial}{\partial\xi_j}\left[a_{ijhk}(\xi)\frac{1}{2}\left(\frac{\partial N_{\alpha_1 hm}(\xi)}{\partial\xi_k}+\frac{\partial N_{\alpha_1 km}(\xi)}{\partial\xi_h}\right)\right]=-\frac{\partial a_{ijm\alpha_1}}{\partial\xi_j},\ \xi\in Q,i=1,\cdots,n \tag{20}$$

Attach the following boundary condition on ∂Q

$$N_{\alpha_1 m}(\xi)=0,\qquad \forall\,\xi\in\partial Q \tag{21}$$

From Korn's inequality and Lax-Milgram Lemma, (20) and (21) have a unique solution for the pair (α_1, m).

Next, compare the coefficients of ε^0 on both sides of Eq. (16), and one obtains that

$$\frac{\partial}{\partial\xi_j}\left[a_{ijhk}(\xi)\frac{1}{2}\left(\frac{\partial N_{\alpha_1\alpha_2 hm}}{\partial\xi_k}+\frac{\partial N_{\alpha_1\alpha_2 km}}{\partial\xi_h}\right)\right]v_{\alpha_1\alpha_2 m}$$
$$+\frac{1}{2}\frac{\partial}{\partial\xi_j}(a_{ijhk}(\xi)N_{\alpha_1 hm}(\xi))\frac{\partial v_{\alpha_1 m}}{\partial x_k}+\frac{1}{2}\frac{\partial}{\partial\xi_j}(a_{ijhk}(\xi)N_{\alpha_1 km}(\xi))\frac{\partial v_{\alpha_1 m}}{\partial x_h}$$
$$+a_{ijhk}(\xi)\frac{1}{2}\left(\frac{\partial N_{\alpha_1 hm}}{\partial\xi_k}+\frac{\partial N_{\alpha_1 km}}{\partial\xi_h}\right)\frac{\partial v_{\alpha_1 m}}{\partial x_j}$$
$$+a_{ijhk}(\xi)\frac{1}{2}\left(N_{0hm}(\xi)\frac{\partial^2 v_{0m}}{\partial x_k\partial x_j}+N_{0km}(\xi)\frac{\partial^2 v_{0m}}{\partial x_h\partial x_j}\right)$$
$$=f_i(x),\ x\in\Omega,\ \xi\in\Omega/\varepsilon\ \text{and}\ i,m=1,\cdots,n,\alpha_1,\alpha_2=1,\cdots,n \tag{22}$$

Further let

$$v_{\alpha_1\alpha_2 m}(x)=\frac{\partial v_{\alpha_1 m}}{\partial x_{\alpha_2}} \tag{23}$$

Assume that $f(x)$ is an integrable function on domain $G\in R^n$ and let ~ denote the homogenization operator defined by:

$$\tilde{f}=\frac{1}{|G|}\int_G f(x)\,dv \tag{24}$$

It is noticed that Eq. (22) is an equation of two - scale variables (x, ξ), but on the right hand side there exists only one variable x. Since all functions of variable ξ in Eq. (22) are 1-periodic and $N_{m\alpha_1} = N_{\alpha_1 m}$, one can impose a homogenization operator for the variable ξ on both sides of Eq. (22) and obtain

$$\hat{a}_{ijhk} \frac{\partial}{\partial x_j} \frac{1}{2} \left(\frac{\partial v_{0h}}{\partial x_k} + \frac{\partial v_{0k}}{\partial x_h} \right) = f_i(x), \qquad x \in \Omega,\ i = 1, \cdots, n \tag{25}$$

where

$$\hat{a}_{ijhk} = \int_Q [a_{ijhk}(\xi) + a_{ijpq}(\xi)\varepsilon_{pq}(N_{hk})]\, d\xi,\ \ \varepsilon_{pq}(N_{hk}) = \frac{1}{2} \left(\frac{\partial N_{hpk}}{\partial \xi_q} + \frac{\partial N_{hqk}}{\partial \xi_p} \right) \tag{26}$$

In general $\{\hat{a}_{ijhk}\}$ is called the homogenization constants of materials, or macroscopic parameters. Let

$$v_0(x) = \bar{u}\ (x), \quad x \in \partial\Omega \tag{27}$$

then Eq. (25) and Eq. (27) compose a Dirichlet problem with constant coefficients. It can be proved that the homogenization constants $\{\hat{a}_{ijhk}\}$ satisfy the $E(\mu_1, \mu_2)$ conditions (Aboudi 1991). Then Eq. (25) and Eq. (27) have one unique solution denoted by $u_0(x)$, and it can be proved that $u_0(x)$ is sufficiently smooth in arbitrary internal sub-domain Ω' of Ω.

Further substituting $u_0(x)$ for $v_0(x)$ in Eq. (22), and making use of Eqs. (25) and (6) one obtains

$$\frac{\partial}{\partial \xi_j} \frac{1}{2} \left[a_{ijhk}(\xi) \left(\frac{\partial N_{\alpha_1\alpha_2 hm}}{\partial \xi_k} + \frac{\partial N_{\alpha_1\alpha_2 km}}{\partial \xi_h} \right) \right] \tag{28}$$

$$= \hat{a}_{i\alpha_2 m\alpha_1} - a_{i\alpha_2 m\alpha_1}(\xi) - a_{i\alpha_2 hk}(\xi) \frac{\partial N_{\alpha_1 hm}}{\partial \xi_k} - \frac{\partial}{\partial \xi_j} (a_{ijh\alpha_2}(\xi) N_{\alpha_1 hm}(\xi))$$
$$\xi \in Q,\ i,\ m = 1, \cdots, n \text{ and } \alpha_1,\ \alpha_2 = 1, \cdots, n$$

Attach

$$N_{\alpha_1\alpha_2 m}(\xi) = 0, \quad \xi \in \partial Q \tag{29}$$

then Eq. (28) and (29) determine $N_{\alpha_1\alpha_2 m}(\xi)$ for $\alpha_1, \alpha_2 = 1, \cdots, n$.

Similarly, letting the coefficients of ε^l, $l = 1, 2, \cdots$ on both sides of Eq. (16),

$$v_{\alpha m}(x) = \frac{\partial v_{\alpha_1 \cdots \alpha_{l-1} m}}{\partial x_{\alpha_l}} \tag{30}$$

from which, one obtains the following problems that determine $N_{\alpha_1 \cdots \alpha_l m}(\xi)$ $(1 \le \alpha_i \le n$, $m = 1, \cdots, n,\ l = 3, 4, \cdots)$

$$\frac{\partial}{\partial \xi_j} \left[a_{ijhk}(\xi) \frac{1}{2} \left(\frac{\partial N_{\alpha_1 \cdots \alpha_l hm}}{\partial \xi_k} + \frac{\partial N_{\alpha_1 \cdots \alpha_l km}}{\partial \xi_h} \right) \right]$$

$$= -\left[a_{i\alpha_l m\alpha_{l-1}}(\xi)N_{\alpha_1\cdots\alpha_{l-2}hm}(\xi) + \frac{\partial}{\partial\xi_j}\left(a_{ijh\alpha_l}(\xi)N_{\alpha_1\cdots\alpha_{l-1}hm}(\xi)\right) + a_{i\alpha_l hk}(\xi)\frac{\partial N_{\alpha_1\cdots\alpha_{l-1}hm}}{\partial\xi_k}\right],\ \xi\in Q,\quad i,\ m=1,\cdots,n \tag{31}$$

$$N_{\alpha_1\cdots\alpha_l m}(\xi)=0,\qquad \xi\in\partial Q \tag{32}$$

To sum up, one obtains the following theorem.

Theorem 3.1 Problem (10) has a formal solution as follows

$$u(x)=u_0(x)+\sum_{l=1}^{\infty}\varepsilon^l\sum_{\alpha}N_{\alpha_1\cdots\alpha_l}(\xi)\frac{\partial^l u_0}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_l}} \tag{33}$$

where $N_{\alpha_1\cdots\alpha_l}(\xi)=(N_{\alpha 1}(\xi),\cdots,N_{\alpha n}(\xi))$, every $N_{\alpha_1\cdots\alpha_l m}(\xi)(m=1,\cdots,n)$ is a 1-periodic vector valued function defined on $\boldsymbol{R}$ and can be solved on 1-square Q, and $u_0(x)$ is the solution of homogenized problem defined on Ω. They are the solutions of following problems respectively:

1. For l=1, $N_{\alpha_1 m}(\xi)\,(\alpha_1, m=1,\cdots,n)$ is the solution of the problem defined by Eqs. (20) and (21) on 1-square Q.
2. $u_0(x)$ is the solution of the homogenization problem of Eqs. (25) and (27) on Ω; it and homogenization constants are defined by Eq. (26).
3. For $l=2$, $N_{\alpha_1\alpha_2 m}(\xi)\,(\alpha_1, \alpha_2, m=1,\cdots,n)$ is the solution of the problem defined by Eqs. (28) and (29) on 1-square Q.
4. For $l=3,\ 4,\ \cdots$, $N_{\alpha_1\alpha_2\cdots\alpha_l m}(\xi)\,(\alpha_1, \alpha_2,\cdots,\alpha_l, m=1,\cdots,n)$ is the solution of the problem defined by Eqs. (31) and (32) on 1-square Q.

It is easy to see that the right side of Eq. (20) only depends on the composition of $\{a_{ijhk}(\xi)\}$ on a basic configuration, and the right side of Eq. (28) depends on the distribution of $\{a_{ijhk}(\xi)\}$ and $N_{\alpha_1 m}(\xi)$, and for $l=3,\ 4,\ \cdots$, the right side of Eq. (31) depends on the distribution of $\{a_{ijhk}(\xi)\}$ and $N_{\alpha_1\alpha_2\cdots\alpha_{l-2}m}(\xi)$, $N_{\alpha_1\alpha_2\cdots\alpha_{l-1}m}(\xi)$, and they can be computed in a recursive formula.

For practical computation, let

$$u^{(M)}(x)=u_0(x)+\sum_{l=1}^{M}\varepsilon^l\sum_{\alpha}N_{\alpha_1\cdots\alpha_l}(\xi)\frac{\partial^l u_0}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_l}} \tag{34}$$

where $M=2,\ 3,\ \cdots$.

One can prove the following theorem, see (Cui, Shih, Shin and Wang)

Theorem 3.2 Let $u(x)$ be the true solution of problem (10), and suppose that Ω is a convex domain with piecewise smooth boundaries and $f(x)$ satisfies the conditions such that $u_0(x)\in C^{M+2}(\Omega)$, then

$$\| u^{(M)}(x)-u(x)\|_{H_0^1(\Omega)}\le (A\,\varepsilon)^{M-1}C \tag{35}$$

where A and C are constants independent of ε.

4. FE algorithms based on TSA

4.1. Virtual work equations on TSA

Using the integration in part, one can prove that the previous problems in Eqs. (25), (26) and

(27) for homogenization solution $u_0(x)$ on Ω, the problem Eqs. (20) and (21) for periodic solution $N_{\alpha_1 m}(\xi)$, the problem Eqs. (28) and (29) for $N_{\alpha_1\alpha_2 m}(\xi)$, and the problem Eqs. (31) and (32) for $N_{\alpha_1\cdots\alpha_l m}(\xi)$ are equivalent to following virtual work equations, respectively

$$\int_Q \left(\frac{\partial v_i}{\partial \xi_j}+\frac{\partial v_j}{\partial \xi_i}\right) a_{ijhk}(\xi)\left(\frac{\partial N_{\alpha_1 hm}}{\partial \xi_k}+\frac{\partial N_{\alpha_1 km}}{\partial \xi_h}\right) d\xi = -4\int_Q a_{ijm\alpha_1}\frac{\partial v_i}{\partial \xi_j}\, d\xi$$

$$i,\ j=1,\ \cdots,n,\quad \forall v\in H_0^1(Q) \tag{36}$$

$$\int_\Omega \left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)\hat{a}_{ijhk}\left(\frac{\partial u_{0h}}{\partial x_k}+\frac{\partial u_{0k}}{\partial x_h}\right) dx = -4\int_\Omega f_i(x)\,v_i\,dx,\quad \forall v\in H_0^1(\Omega), \tag{37}$$

$$\int_Q \left(\frac{\partial v_i}{\partial \xi_j}+\frac{\partial v_j}{\partial \xi_i}\right) a_{ijhk}(\xi)\left(\frac{\partial N_{\alpha_1\alpha_2 hm}}{\partial \xi_k}+\frac{\partial N_{\alpha_1\alpha_2 km}}{\partial \xi_h}\right) d\xi$$

$$=-4\int_Q\left[\left(\hat{a}_{i\alpha_2 m\alpha_1}-a_{i\alpha_2 m\alpha_1}(\xi)-a_{i\alpha_2 hk}(\xi)\frac{\partial N_{\alpha_1 hm}}{\partial \xi_k}\right)v_i + a_{ijh\alpha_2}N_{\alpha_1 hm}(\xi)\frac{\partial v_i}{\partial \xi_j}\right]d\xi$$

$$\alpha_1,\ \alpha_2,\ m=1,\ \cdots,\ n,\quad \forall v\in H_0^1(Q) \tag{38}$$

$$\int_Q \left(\frac{\partial v_i}{\partial \xi_j}+\frac{\partial v_j}{\partial \xi_i}\right) a_{ijhk}(\xi)\left(\frac{\partial N_{\alpha_1\cdots\alpha_l hm}}{\partial \xi_k}+\frac{\partial N_{\alpha_1\cdots\alpha_l km}}{\partial \xi_h}\right) d\xi$$

$$=4\int_Q\left[\left(a_{i\alpha_l m\alpha_{l-1}}(\xi)N_{\alpha_1\cdots\alpha_{l-2}hm}(\xi)+a_{i\alpha_l hk}(\xi)\frac{\partial N_{\alpha_1\cdots\alpha_{l-1}hm}}{\partial \xi_k}\right)v_i \right. \tag{39}$$

$$\left. -\,a_{ijh\alpha_l}N_{\alpha_1\cdots\alpha_{l-1}hm}(\xi)\frac{\partial v_i}{\partial \xi_j}\right]d\xi,\ \alpha_1,\ \cdots,\ \alpha_l,\ m=1,\ n,\quad \forall v\in H_0^1(Q)$$

Further it is easy to see (Cui, Shih, Shin and Wang) that one can obtain the FE solutions $u_0^{\bar{h}}(x)$ for $u_0(x)$, $N_{\alpha_1 m}^{h_0}(\xi)$, $N_{\alpha_1\alpha_2 m}^{h_0}(\xi)$ and $N_{\alpha_1\cdots\alpha_l m}^{h_0}(\xi)$ for $N_{\alpha_1 m}(\xi), N_{\alpha_1\alpha_2 m}(\xi), \cdots, N_{\alpha_1\cdots\alpha_l m}(\xi)$, by solving the following FE virtual work equation, respectively

$$\int_Q \left(\frac{\partial v_i}{\partial \xi_j}+\frac{\partial v_j}{\partial \xi_i}\right) a_{ijhk}(\xi)\left(\frac{\partial N^{h_0}_{\alpha_1 hm}}{\partial \xi_k}+\frac{\partial N^{h_0}_{\alpha_1 km}}{\partial \xi_h}\right) d\xi = -4\int_Q a_{ijm\alpha_1}\frac{\partial v_i}{\partial \xi_j}\, d\xi$$

$$i,\ j=1,\ \cdots,\ n,\qquad \forall v\in S^{h_0}(Q) \tag{40}$$

$$\int_\Omega \left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)\hat{a}^{h_0}_{ijhk}\left(\frac{\partial u^{\bar{h}}_{0h}}{\partial x_{xk}}+\frac{\partial u^{\bar{h}}_{0k}}{\partial_{xh}}\right) dx = -4\int_\Omega f_i(x)v_i\,dx,\quad \forall v\in S^{\bar{h}}(\Omega), \tag{41}$$

$$\int_Q \left(\frac{\partial v_i}{\partial \xi_j}+\frac{\partial v_j}{\partial \xi_i}\right) a_{ijhk}(\xi)\left(\frac{\partial N^{h_0}_{\alpha_1\alpha_2 hm}}{\partial \xi_k}+\frac{\partial N^{h_0}_{\alpha_1\alpha_2 km}}{\partial \xi_h}\right) d\xi$$

$$=-4\int_Q \left[\left(\hat{a}^{h_0}_{i\alpha_2 m\alpha_1}-a_{i\alpha_2 m\alpha_1}(\xi)-a_{i\alpha_2 hk}(\xi)\,\varepsilon_{hk}(N^{h_0}_{\alpha_1 m})\right) v_i + a_{ijh\alpha_2}N^{h_0}_{\alpha_1 hm}(\xi)\frac{\partial v_i}{\partial \xi_j}\right] d\xi$$

$$\alpha_1,\ \alpha_2,\ m=1,\cdots,n,\quad \forall v\in S^{h_0}(Q) \tag{42}$$

$$\int_Q \left(\frac{\partial v_i}{\partial \xi_j}+\frac{\partial v_j}{\partial \xi_i}\right) a_{ijhk}(\xi)\left(\frac{\partial N^{h_0}_{\alpha_1\cdots\alpha_l hm}}{\partial \xi_k}+\frac{\partial N^{h_0}_{\alpha_1\cdots\alpha_l km}}{\partial \xi_h}\right) d\xi$$

$$=4\int_Q \left[\left(a_{i\alpha_l m\alpha_{l-1}}(\xi)N^{h_0}_{\alpha_1\cdots\alpha_{l-2}hm}(\xi)+a_{i\alpha_l hk}(\xi)\,\varepsilon_{hk}(N^{h_0}_{\alpha_1\cdots\alpha_{l-1}m})\right) v_i\right.$$

$$\left. - a_{ijh\alpha_l}N^{h_0}_{\alpha_1\cdots\alpha_{l-1}hm}(\xi)\frac{\partial v_i}{\partial \xi_j}\right] d\xi,\ \alpha_1,\cdots,\alpha_l, m=1,\cdots,n,\ \forall v\in S^{h_0}(Q) \tag{43}$$

where

$$\hat{a}^{h_0}_{ijhk}=\int_Q \left[a_{ijhk}(\xi)+a_{ijpq}(\xi)\,\varepsilon_{pq}\left(N^{h_0}_{hk}\right)\right] d\xi,\quad \varepsilon_{pq}\left(N^{h_0}_{hk}\right)=\frac{\partial N^{h_0}_{hpk}}{\partial \xi_q}+\frac{\partial N^{h_0}_{hqk}}{\partial \xi_p} \tag{44}$$

4.2. The FE algorithm procedure for TSA:

(1) Set up the mechanical and mathematical model

- Form and verify the geometry of the structure, the material properties of the components, the loading conditions and constraints.
- Form and verify the composition of the basic configurations for every component with periodicity, matrix, reinforce and their interfaces.

(2) Set up the FE model

- Partition the structure into a set of finite elements according to the composition of the structure. Let h denote the maximum diameter of finite elements in Ω.
- Partition the 1-square domain into a set of finite elements (triangles or quadrangles) according to the composition of every basic configuration. Let h_0 denote the maximum diameter of finite elements for 1-square Q.

(3) Compute $N^{h_0}_{\alpha_1 m}(\xi)(\alpha_1, m=1,\cdots,n)$ on a 1-square domain according to the material properties of the basic configurations by the FE program, and then compute $\varepsilon_{pq}(N^{h_0}_{\alpha_1 m})$ in formula Eq. (44). Since the left sides of all equations satisfied by $N_{\alpha_1\cdots\alpha_l m}(\xi)(\alpha_j, m=1,\cdots,n,\ j=1,\cdots,l,$ and $l=2,3,\cdots)$ are the same as those by $N_{\alpha_1 m}(\alpha_1, m=1,\cdots,n)$, it is necessary to compute the stiffness matrix A_N of the FE equations for $N_{\alpha_1\cdots\alpha_l m}(\xi)$, decompose A_N into $A_N=LDL^T$ only one time, and save L and D.

(4) Compute the homogenization constants by formula Eq. (44), where $N^{h_0}_{hk}(\xi)$ are obtained by

step 3.

(5) Compute $u_0^{\bar{h}}(x)$ on the whole structure by FE program based on the homogenization constants obtained by step 4. Then evaluate

$$\varepsilon_{ij}(u_0^{\bar{h}}) = \frac{\partial u_{0i}^{\bar{h}}}{\partial x_j} + \frac{\partial x_{0j}^{\bar{h}}}{\partial x_i} \tag{45}$$

(6) Compute the following approximations of the higher-order derivatives

$$\frac{\partial^l u_{0m}^{\bar{h}}}{\partial x_{\alpha_1} \cdots \partial x_{\alpha_l}}, \ (l = 1, \cdots, M, \ 1 \le \alpha_i \le n, i = 1, \cdots l)$$

using the data processing technique of the average of relative elements (Cui, Shih, Shin and Wang).

(7) Compute $N_{\alpha_1 \cdots \alpha_l m}^{h0}(\xi)\,(l = 1, \cdots, M, 1 \le \alpha_1 \le n, i = 1, \cdots l)$ recurrently from the former results $N_{\alpha_1 \cdots \alpha_{l-1} m}^{h0}(\xi)$ and $N_{\alpha_1 \cdots \alpha_{l-2} m}^{h0}(\xi)$, and then compute

$$\varepsilon_{hk}(N_{\alpha_1 \cdots \alpha_l m}^{h0}) = \frac{\partial N_{\alpha_1 \cdots \alpha_l hm}^{h0}}{\partial \xi_k} + \frac{\partial N_{\alpha_1 \cdots \alpha_l km}^{h0}}{\partial \xi_h} \tag{46}$$

(8) Compute the final results by following formulas

$$u_i^{h(M)}(x) = u_{0i}^{\bar{h}}(x) + \sum_{l=1}^{M} \varepsilon^l \sum_{\alpha} N_{a_1 \cdots \alpha_l im}^{h0}\left(\frac{x}{\varepsilon}\right) \frac{\partial^l u_{0m}^{\bar{h}}}{\partial x_{\alpha_1} \cdots \partial x_{\alpha_l}}, \ i = 1, \cdots, n \tag{47}$$

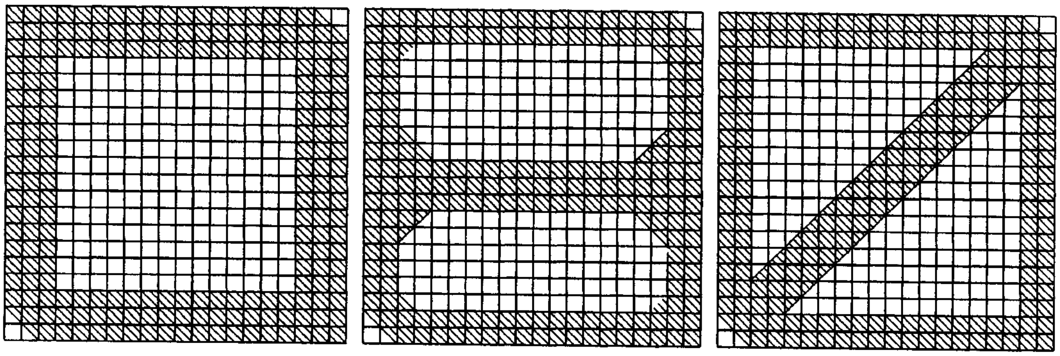
$$\begin{aligned}
\varepsilon_{hk}^{(M)}(x) = {} & \varepsilon_{hk}(u_0^{\bar{h}}) + \sum_{l=1}^{M} \varepsilon^{l-1} \varepsilon_{hk}\left(N_{a_1 \cdots \alpha_l m}^{h0}\left(\frac{x}{\varepsilon}\right)\right) \frac{\partial^l u_{0m}^{\bar{h}}}{\partial x_{\alpha_1} \cdots \partial x_{\alpha_l}} \\
& + \sum_{l=1}^{M} \varepsilon^l \frac{1}{2}\left[N_{a_1 \cdots \alpha_l hm}^{h0}\left(\frac{x}{\varepsilon}\right) \frac{\partial^{l+1} u_{0m}^{\bar{h}}}{\partial x_{\alpha_1} \cdots \partial x_{\alpha_l} \partial x_k}\right. \\
& \left. + N_{a_1 \cdots \alpha_l km}^{h0}\left(\frac{x}{\varepsilon}\right) \frac{\partial^{l+1} u_{m0}^{\bar{h}}}{\partial x_{\alpha_1} \cdots \partial x_{\alpha_l} \partial x_h}\right]
\end{aligned} \tag{48}$$

$$\sigma_{ij}^{(M)}(x) = a_{ijhk}\left(\frac{x}{\varepsilon}\right) \varepsilon_{hk}^{(M)}(x) \tag{49}$$

4.3. Numerical results

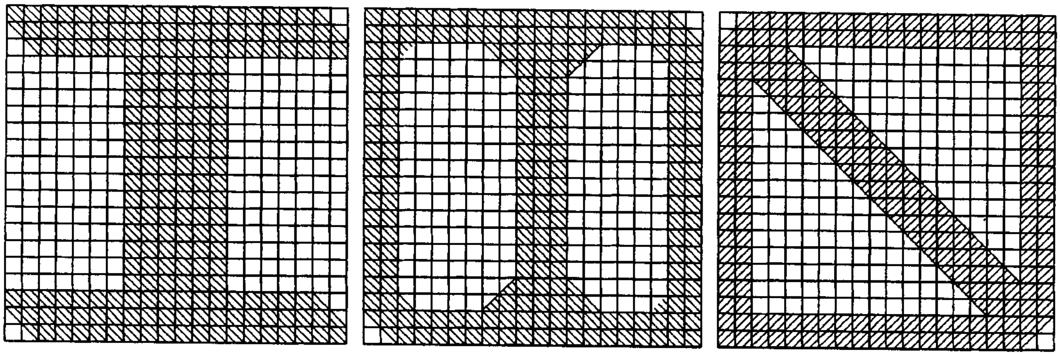
We have coded the computing program of previous algorithms using COREP, which is a FE software developed by the authors, and some numerical experiments are evaluated to verify the effectiveness of the FE algorithms based on TSA.

In order to test the FE computation of homogenization constants, making use of two kinds of isotropic materials, we designed three kinds of basic configurations for periodic structures shown in Fig. 2. The basic material constants corresponding to the shaded and white areas respectively



$$\begin{pmatrix} 1143604 & 175017 & 0 \\ 175017 & 1143604 & 0 \\ 0 & 0 & 286746 \end{pmatrix}, \begin{pmatrix} 1334341 & 188121 & 0 \\ 188121 & 937491 & 0 \\ 0 & 0 & 245477 \end{pmatrix}, \begin{pmatrix} 1029547 & 266973 & 85244 \\ 266973 & 1029547 & 85244 \\ 85244 & 85244 & 343238 \end{pmatrix}$$

(a) (b) (c)



$$\begin{pmatrix} 1134231 & 182435 & 0 \\ 182435 & 1134231 & 0 \\ 0 & 0 & 277228 \end{pmatrix}, \begin{pmatrix} 932821 & 171846 & 0 \\ 171846 & 1300548 & 0 \\ 0 & 0 & 240848 \end{pmatrix}, \begin{pmatrix} 1029378 & 266772 & -85104 \\ 266772 & 1029378 & -85104 \\ -85104 & -85104 & 343146 \end{pmatrix}$$

(d) (e) (f)

Fig. 2 The basic periodic configuration and their homogenization constants

are following:

$$\begin{pmatrix} 3125000 & 625000 & 0 \\ 625000 & 3125000 & 0 \\ 0 & 0 & 1250000 \end{pmatrix} \text{ and } \begin{pmatrix} 178571 & 71428 & 0 \\ 71428 & 178571 & 0 \\ 0 & 0 & 53571 \end{pmatrix},$$

In the basic configurations, though the areas of two materials are equal, the corresponding homogenization constants are very different, not only in magnitude but also in the number of independent constants shown in Figs. 2(a)-(f).

In order to demonstrate the accuracy and efficiency of FE algorithms based on TSA, we computed the displacements and stresses of the periodic structure in Fig. 3. For $u_0^h(x)$ the whole structure is partitioned into 38×94 rectangles, and for $N_{\alpha m}^{h_0}(\xi)$ 1-square Q is divided into 40×40

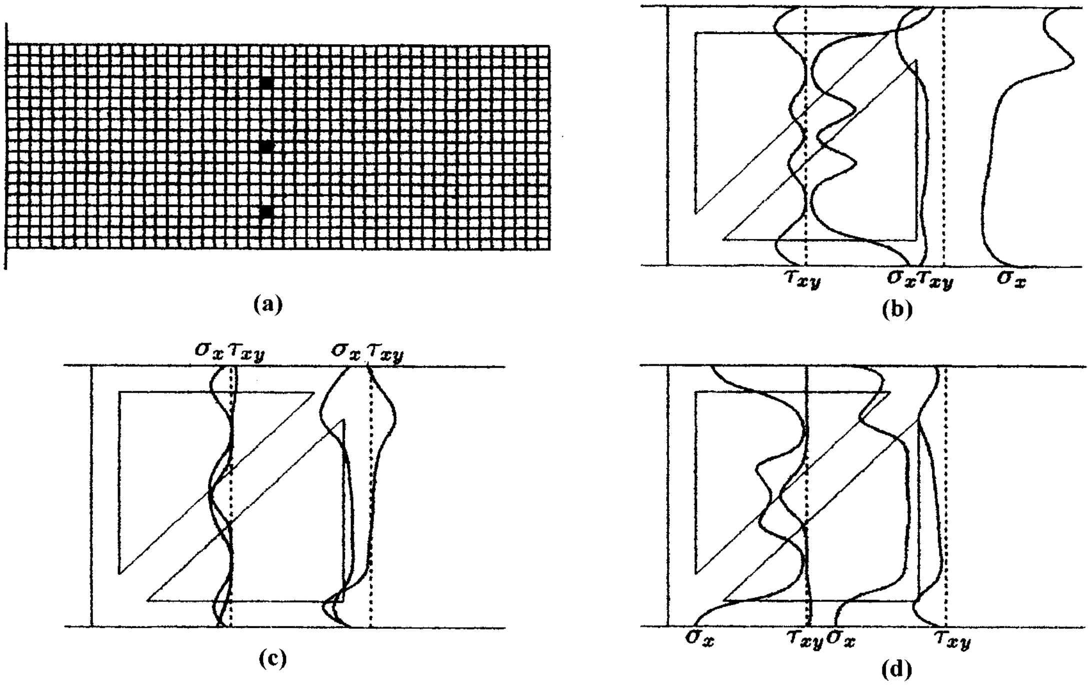


Fig. 3 Structure with periodic configuration and some stress results

rectangles. The results of detailed stress distributions inside every basic configuration are obtained. Owing to the limitation of space here the stress results on the three black squares in Fig. 3(a) with the basic configuration in Fig. 2(c) are shown in Fig. 3(b)-(d). For conventional FE method, much more refined rectangular meshes might be needed to achieve the same accuracy. Therefore the FE method based on TSA is very efficient for problems raised from composite materials and structures with a small period.

5. Conclusions

The mechanical behaviours of the structures made from composite materials or the structures with periodic configurations depend not only on the macroscopic conditions, such as the geometry of the structure, the effective constants of materials, the loadings and the constraints, but also on the detailed configurations. So the elasto-static problems for these structures cannot be reasonably and precisely analysed by using either macroscopic scale analysis or detailed cell scale analysis. The two-scale variable model and its FE algorithms form an effective method for analysing these kinds of structures.

The two-scale variable expressions of displacements simplify the complexity of solving the problem. The approximation of displacements $u^{\varepsilon}(\xi)$ can be evaluated by solving one homogenization problem on Ω and several simple problems with the same basic configurations on 1-squire Q, respectively.

From the previous discussion, all left sides of PDE equations satisfied by $N_{\alpha_1 m}(\xi)$, $N_{\alpha_1\alpha_2 m}(\xi)$, $N_{\alpha_1\cdots\alpha l m}(\xi)$ $(m, \alpha_1, \cdots \alpha_n = 1, \cdots, n, l = 1, 2, \cdots, M)$ are the same, and the right sides can be recursively computed one by one. So it is easy to obtain their approximations using finite element method and its software. The finite element meshes for evaluating $N^{h0}_{\alpha_1 m}(\xi)$, $N_{\alpha_1\alpha_2 m}(\xi)$, $N_{\alpha_1\cdots\alpha_l m}(\xi)$ $(m, \alpha_1, \cdots \alpha_n = 1, \cdots, n, l = 1, 2, \cdots, M)$ are chosen the same, and then the global stiffness matrix is computed, decomposed and saved only one time. Therefore the computing amount of FE algorithms based on TSA is smaller than the classical macroscopic scale FE algorithms used to obtain the results with the same precision.

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