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Elastic bending analysis of irregular-shaped plates

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Abstract. An approximate method for analyzing the bending problems of irregular-shaped plates is proposed. In this paper irregular-shaped plates are such plates as plate with opening, circular plate, semi-circular plate, elliptic plate, triangular plate, skew plate, rhombic plate, trapezoidal plate or the other polygonal plates which are not uniform rectangular plates. It is shown that these irregular-shaped plates can be considered finally as a kind of rectangular plates with non-uniform thickness. An opening in a plate can be considered as an extremely thin part of the plate, and a non-rectangular plate can be translated into a circumscribed rectangular plate whose additional parts are extremely thin or thick according to the boundary conditions of the original plate. Therefore any irregular-shaped plate can be replaced by the equivalent rectangular plate with non-uniform thickness. For various types of irregular-shaped plates the convergency and accuracy of numerical solution by proposed method are investigated.

Key words: irregular-shaped plate; bending problem; equivalent rectangular plate; discrete solution.

1. Introduction

The irregular-shaped plates in this paper are such plates as plate with opening, circular plate, semi-circular plate, elliptic plate, triangular plate, skew plate, rhombic plate, trapezoidal plate or the other polygonal plates which are not uniform rectangular plates. But these irregular-shaped plates can be considered finally as a kind of rectangular plates with non-uniform thickness.

An opening in a plate can be considered as an extremely thin part of the plate, and a non-rectangular plate can be translated into a circumscribed rectangular plate whose additional parts are extremely thin or thick according to the boundary conditions of the original plate. Namely, the additional part connected with a fixed edge is considered as extremely thick, and the additional part connected with a free edge is considered as extremely thin. Therefore any irregular-shaped plate can be replaced by the equivalent rectangular plate with non-uniform thickness.

Timoshenko and Krieger (1959) presented the analytical solutions for fixed circular plate, semi-circular plate, elliptic plate or simple-free skew plate. Fletcher (1959) analyzed isosceles right triangular plates with a fixed or simply supported diagonal edge and the other two simply supported edges. Conway (1962) analyzed triangular plates by point matching. Iwahara (1980) investigated the bending problem of square plate with central square opening hole by applying the mapping method. Ohta *et al.* (1962) analyzed fixed rhombic plates on the basis of the energy method, Saito *et al.* (1958) analyzed elliptic plates subjected to a concentrated load at an arbitrary

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point.

In this paper an approximate method is proposed for analyzing the bending problem of various types of irregular-shaped plates by applying the discrete general solution obtained by Sakiyama and Matsuda (1983) for the rectangular plate with non-uniform thickness, and the convergency and accuracy of numerical solution by proposed method are investigated. The discrete general solution of a differential equation is obtained by applying the numerical integration method, and theoretically it can be considered as a discrete-type expression of an analytical solution of differential equation. Therefore the numerical solutions by proposed method always approach to the values corresponding with the analytical solution by increasing the number of discrete points, and moreover they can be directly obtained without using the assumed displacement function in FEM or the fundamental solution in BEM.

2. Fundamental differential equation of plate with variable thickness and point supports

The fundamental differential equations of plate with variable thickness and point supports at each discrete points (x_c, y_d) as shown in Fig. 1 are given by following equations.

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+\bar{q}+\sum_{c=0}^{m}\sum_{d=0}^{n}\bar{P}_{1cd}\,\delta(x-x_c)\,\delta(y-y_d)=0 \tag{1-a}$$

$$\frac{\partial M_y}{\partial y}+\frac{\partial M_{xy}}{\partial x}-Q_y+\bar{m}_x+\sum_{c=0}^{m}\sum_{d=0}^{n}\bar{P}_{2cd}\,\delta(x-x_c)\,\delta(y-y_d)=0 \tag{1-b}$$

$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x-\bar{m}_y-\sum_{c=0}^{m}\sum_{d=0}^{n}\bar{P}_{3cd}\,\delta(x-x_c)\,\delta(y-y_d)=0 \tag{1-c}$$

$$\frac{\partial \theta_x}{\partial x}+\nu\frac{\partial \theta_y}{\partial y}=\frac{M_x}{D} \tag{1-d}$$

$$\frac{\partial \theta_y}{\partial y}+\nu\frac{\partial \theta_x}{\partial x}=\frac{M_y}{D} \tag{1-e}$$

$$\frac{\partial \theta_x}{\partial y}+\frac{\partial \theta_y}{\partial x}=\frac{2}{(1-\nu)}\frac{M_{xy}}{D} \tag{1-f}$$

$$\frac{\partial w}{\partial y}+\theta_x=\frac{Q_x}{Gt_s} \tag{1-g}$$

$$\frac{\partial w}{\partial y}+\theta_y=\frac{Q_y}{Gt_s} \tag{1-h}$$

where Q_x, Q_y the shearing forces, M_{xy} the twisting moment, M_x, M_y the bending moments, θ_x, θ_y the slopes, w the deflection, $D=Eh^3/12(1-\nu^2)$: the bending rigidity, E, G: modulus, shear modulus of elasticity, ν: Poisson's ratio, $h=h(x, y)$: the thickness of plate, $t_s=h/1.2$, $\bar{q}=\bar{q}(x, y)$: distributed load, $\bar{m}_x=\bar{m}_x(x, y)$, $\bar{m}_y=\bar{m}_y(x, y)$: distributed moment loads around x-, y-axes, $\bar{P}_{1cd}$: vertical reaction of point support at (x_c, y_d), $\bar{P}_{2cd}$, $\bar{P}_{3cd}$: moment reactions around x-, y-axes of point support at (x_c, y_d), $\delta(x-x_c)$, $\delta(y-y_d)$: Dirac's delta functions.

By introducing the following non-dimensional expressions,

$$[X_1, X_2]=\frac{a^2}{D_0(1-\nu^2)}[Q_y, Q_x],\ [X_3, X_4, X_5]=\frac{a}{D_0(1-\nu^2)}[M_{xy}, M_y, M_x]$$

$$[X_6, X_7, X_8] = [\theta_y, \theta_x, w/a],$$

the differential Eqs. (1-a)~(1-h) can be rewritten as follows.

$$\sum_{e=1}^{8}[F_{1te}\frac{\partial X_e}{\partial \zeta}+F_{2te}\frac{\partial X_e}{\partial \eta}+F_{3te}X_e]+\sum_{f=1}^{3}[p_f+\sum_{c=0}^{m}\sum_{d=0}^{n}P_{fcd}\delta(\eta-\eta_c)\delta(\zeta-\zeta_d)]\delta_{ft}=0 \tag{2}$$

where t=1~8, $[p_1, p_2, p_3]=[\bar{q}a, \bar{m}_x, -\bar{m}_y]\mu a^2/D_0(1-\nu^2)$, $\mu=b/a$

$\eta=x/a$, $\zeta=y/b$, $D_0=Eh_0^3/12(1-\nu^2)$: standard bending rigidity

h_0: standard thickness of plate, a,b: breadth, length of rectangular plate

$[P_{1cd}, P_{2cd}, P_{3cd}]=[\bar{P}_{1cd}a, \bar{P}_{2cd}, -\bar{P}_{3cd}]/D_0(1-\nu^2)$, δ_{ft}: Kronecker's delta

$F_{1te}, F_{2te}, F_{3te}$: Appendix I

3. Discrete solution of fundamental differential equation

With a rectangular plate divided vertically into m equal-length parts and horizontally n equal-length parts as shown in Fig. 2, the plate can be considered as a group of discrete points which are the intersections of the $(m+1)$-vertical and $(n+1)$-horizontal dividing lines. In this paper, the rectangular area, $0\leq\eta\leq\eta_i$, $0\leq\zeta\leq\zeta_j$, corresponding to the arbitrary intersection (i, j) as shown in Fig. 2 is denoted as the area $[i, j]$ the intersection (i, j) denoted by ◎ is called the main point of the area $[i, j]$, the intersections denoted by ○ are called the inner dependent points of the area, and the intersections denoted by ● are called the boundary dependent points of the area.

By integrating the Eq. (2) over the area $[i, j]$, the following integral equation is obtained.

$$\sum_{e=1}^{8}\left\{\begin{array}{l} F_{1te}\int_0^{\eta_i}[X_e(\eta,\zeta_j)-X_e(\eta,0]d\eta+F_{2te}\int_0^{\zeta_j}[X_e(\eta_i,\zeta)-X_e(0,\zeta)d\zeta \\ +F_{3te}\int_0^{\eta_i}\int_0^{\zeta_j}X_e(\eta,\zeta)d\eta d\zeta \end{array}\right\}$$

$$+\sum_{f=1}^{3}\left\{\int_0^{\eta_i}\int_0^{\zeta_j}p_f(\eta,\zeta)d\eta d\zeta+\sum_{c=0}^{m}\sum_{d=0}^{n}P_{fcd}u(\eta_i-\eta_c)u(\zeta_j-\zeta_d)\right\}\delta_{ft}=0 \tag{3}$$

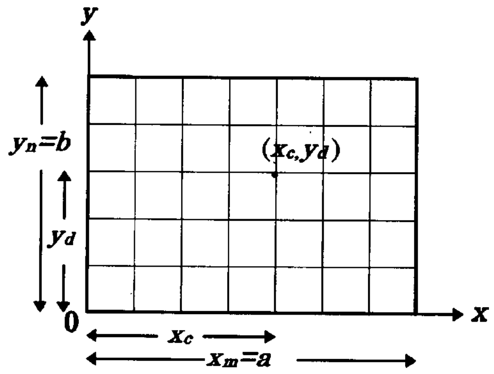


Fig. 1 Position of point support

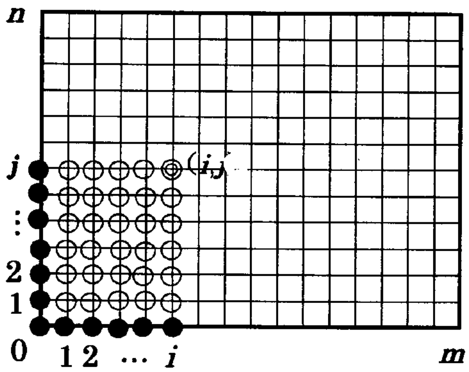


Fig. 2 Discrete points on plate

where $u(\eta_i-\eta_c)=0(i<c)$, $0.5(i=c)$, $1(i>c)$, $u(\zeta_j-\zeta_d)=0(j<d)$, $0.5(j=d)$, $1(j>d)$.
$t=1\sim8$, $i=1\sim m$, $j=1\sim n$.

Next, by applying the numerical integration method the simultaneous equation for the unknown quantities $X_{eij}=X_e(\eta_i, \zeta_j)$ at the main point (i, j) of the area $[i, j]$ is obtained as follows.

$$\sum_{e=1}^{8}\left\{F_{1te}\sum_{k=0}^{i}\beta_{ik}(X_{ekj}-X_{ek0})+F_{2te}\sum_{l=0}^{j}\beta_{jl}(X_{eil}-X_{e0l})+F_{3te}\sum_{k=0}^{i}\sum_{l=0}^{j}\beta_{ik}\beta_{jl}X_{ekl}\right\}$$
$$+\sum_{f=1}^{3}\left\{\sum_{k=0}^{i}\sum_{l=0}^{j}\beta_{ik}\beta_{jl}p_{fkl}+\sum_{c=0}^{m}\sum_{d=0}^{n}P_{fcd}u_{ic}u_{jd}\right\}\delta_{ft}=0 \tag{4}$$

where $\beta_{ik}=\alpha_{ik}/\mathrm{m}$, $\beta_{jl}=\alpha_{jl}/\mathrm{n}$, $t=1\sim8$, $i=1\sim m$, $j=1\sim n$,

$$u_{ic}=\begin{cases}0 & (i<c)\\ 0.5 & (i=c),\\ 1 & (i>c)\end{cases}\quad u_{jd}=\begin{cases}0 & (j<d)\\ 0.5 & (j=d),\\ 1 & (j>d)\end{cases}\quad \alpha_{ik}=\begin{cases}0.5(k=0,i)\\ 1\ \ (k\neq0,i)\end{cases},\ \alpha_{jl}=\begin{cases}0.5(l=0,j)\\ 1\ \ (l\neq0,j)\end{cases},$$

The solution X_{pij} of the simultaneous Eq. (4) is obtained as follows.

$$X_{pij}=\sum_{e=1}^{8}\left\{\begin{array}{l}\sum_{k=0}^{i}\beta_{ik}A_{pe}[X_{ek0}-X_{ekj}(1-\delta_{ki})]+\sum_{l=0}^{j}\beta_{jl}B_{pe}[X_{e0l}-X_{eil}(1-\delta_{lj})]\\ +\sum_{k=0}^{i}\sum_{l=0}^{j}\beta_{ik}\beta_{jl}C_{pekl}X_{ekl}(1-\delta_{ki}\delta_{lj})\end{array}\right\}$$
$$-\sum_{f=1}^{3}\gamma_{pf}\left\{\sum_{k=0}^{i}\sum_{l=0}^{j}\beta_{ik}\beta_{jl}p_{fkl}+\sum_{c=0}^{m}\sum_{d=0}^{n}P_{fcd}u_{ic}u_{jd}\right\} \tag{5}$$

where $p=1, 2, \cdots, 8$, $i=1, 2, \cdots, m$, $j=1, 2, \cdots, n$, A_{pe}, B_{pe}, C_{pelk}, γ_{pf}: Appendix II

In the Eq. (5), the quantity X_{pij} at the main point (i,j) of the area $[i, j]$ is related to the quantities X_{ek0} and X_{e0l} at the boundary dependent points of the area and the quantities X_{ekj}, X_{eil} and X_{ekl} at the inner dependent points of the area. With the spreading of the area $[i, j]$ according to the regular order as [1, 1], [1, 2], $\cdots$, [1, n], [2, 1], [2, 2], $\cdots$, [2, n], $\cdots$, [m, 1], [m, 2], $\cdots$, [m, n], a main point of smaller area becomes one of the inner dependent points of the following larger areas. Whenever the quantity X_{pij} at the main point (i, j) is obtained by using the Eq. (5) in above mentioned order, the quantities X_{ekj}, X_{eil} and X_{ekl} at the inner dependent points of the following larger areas can be eliminated by substituting the obtained results into the corresponding terms of the right side of Eq. (5). By repeating this process, the equation X_{pij} at the main point is related to only the quantities X_{rk0}, $(r=1, 3, 4, 6, 7, 8)$ and X_{s0l}, $(s=2, 3, 5, 6, 7, 8)$ which are six independent quantities at the each boundary dependent points along the horizontal axis and the vertical axis in Fig. 2 respectively. The result is as follows.

$$X_{pij}=\sum_{k=0}^{i}\left\{\begin{array}{l}a_{1pijk1}(Q_y)_{k0}+a_{1pijk2}(M_{xy})_{k0}+a_{1pijk3}(M_y)_{k0}\\+a_{1pijk4}(\theta_y)_{k0}+a_{1pijk5}(\theta_x)_{k0}+a_{1pijk6}(w)_{k0}\end{array}\right\}$$

$$+\sum_{l=0}^{j}\left\{\begin{array}{l}a_{2pijl1}(Q_x)_{0l}+a_{2pijl2}(M_{xy})_{0l}+a_{2pijl3}(M_x)_{0l}\\+a_{2pijl4}(\theta_y)_{0l}+a_{2pijl5}(\theta_x)_{0l}+a_{2pijl6}(w)_{0l}\end{array}\right\}$$

$$+\sum_{f=1}^{3}\left\{q_{fpij}+\sum_{c=0}^{m}\sum_{d=0}^{n}\bar{q}_{fpijcd}P_{fcd}\right\}\quad (i=1\sim m,\ j=1\sim n) \tag{6}$$

where $(Q_y)=X_1$, $(Q_x)=X_2$, $(M_{xy})=X_3$, $(M_y)=X_4$, $(M_x)=X_5$
$(\theta_y)=X_6$, $(\theta_x)=X_7$, $(w)=X_8$, a_{hpijuv}, q_{fpij}, $\bar{q}_{fpijcd}$: Appendix III

The Eq. (6) gives the discrete solution of the fundamental differential Eq. (2) of plate bending problem.

4. Integral constant and boundary condition of rectangular plate

The integral constants $(Q_y)_{k0}$, $(M_{xy})_{k0}$, $\cdots$, $(w)_{k0}$, $(Q_x)_{0l}$, $(M_{xy})_{0l}$, $\cdots$, $(w)_{0l}$ being involved in the discrete solution (6) are to be evaluated by the boundary conditions of a rectangular plate. The combinations of the integral constants and the boundary conditions for some cases are shown in Fig. 3~Fig. 6, in which the integral constants and the boundary conditions at the four corners are shown in the boxes. The integral constants and the boundary conditions along the four edges are given at the each equally-spaced discrete points. In this paper simply supported, fixed and free edges are denoted by solid line ——, thick solid line —— and dotted line ········.

5. Equivalent rectangular plate of irregular-shaped plate

Such irregular-shaped plates as plate with opening, circular plate, semi-circular plate, triangular

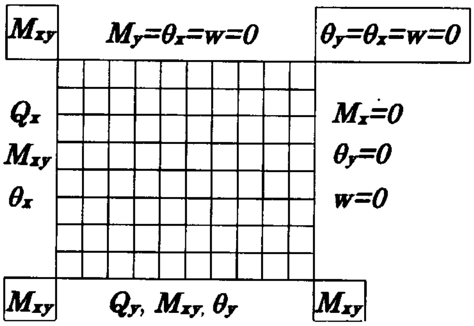


Fig. 3 Simply supported plate

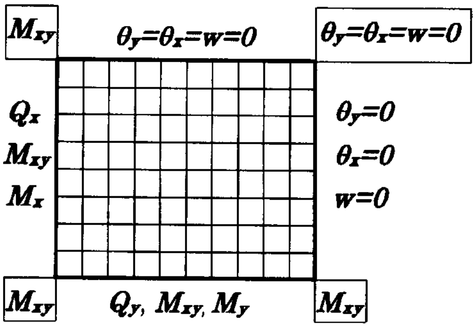


Fig. 4 Fixed plate

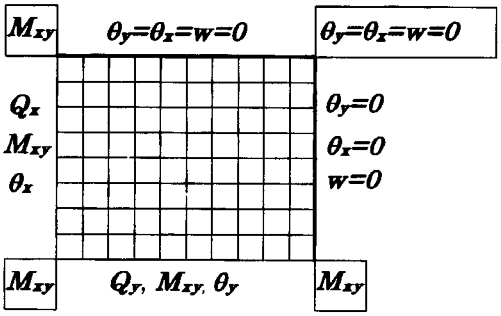


Fig. 5 Plate with simple edges and fixed edges

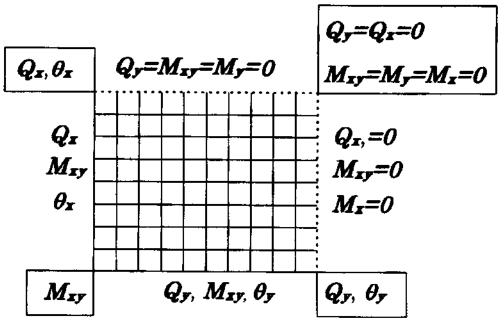


Fig. 6 Plate with simple edges and free edges

plate, skew plate, rhombic plate, trapezoidal plate or the other polygonal plate are quite different from uniform rectangular plates, but they can be considered as a kind of rectangular plate by translating them into the equivalent rectangular plate with non-uniform thickness.

An opening in an original irregular-shaped plate can be considered as an extremely thin part of the equivalent rectangular plate, and a non-rectangular plate can be translated into the circumscribed equivalent rectangular plate whose additional parts are extremely thin or thick according to the boundary condition of the original plate.

The thickness of the actual part of original irregular-shaped plates is expressed by h_0, and the thickness of additional parts of each equivalent rectangular plates is expressed by h in this paper. And the thickness at a point on the border line between the actual part and the additional part of the equivalent rectangular plate is taken as $(h_0+h)/2$.

A typical translation from an original irregular-shaped plate to its equivalent rectangular plate is shown in Fig. 7, in which the inclined simply-supported edge of original plate is translated into many point supports along the edge. The values of three reactions P_{1cd}, P_{2cd}, P_{3cd} at each point support of the equivalent plate are determined by following three conditions, $M_t=0$, $\theta_n=0$, $w=0$.

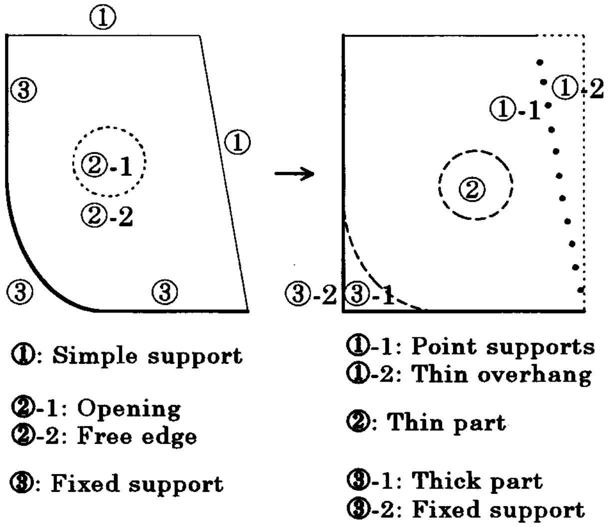


Fig. 7 Irregular-shaped plate and its equivalent rectangular plate

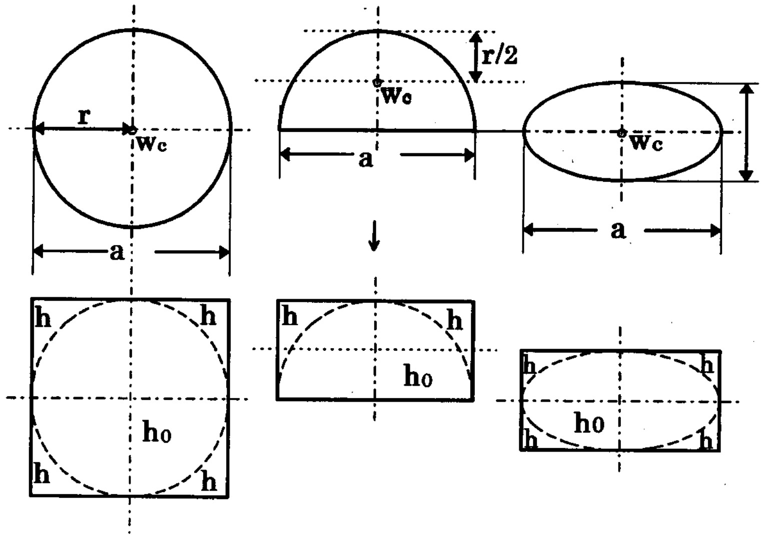


Fig. 8 Fixed circular plate, semi-circular plate and elliptic plate

The first condition means that the bending moment around the tangential axis of the line of point supports is zero at each point support. The second condition means that the slope around the normal axis of the line of point supports is zero, and the third condition means zero deflection at point support.

6. Numerical works

The convergency and accuracy of numerical solutions have been investigated for some irregular-shaped plates with uniform load q_0 or a concentrated load P such as circular plate, semi-circular plate, triangular plates, rectangular plates with opening, skew plate, rhombic plate and trapezoidal plate.

The numerical solutions for these irregular-shaped plates have been obtained by using Richardson's extrapolation formula for two case of combinations of divisional numbers m and n.

6.1. Circular plate, semi-circular plate and elliptic plate

Numerical solutions for deflections of a fixed circular plate, a semi-circular plate and elliptic plate shown in Fig. 8 are given in Table 1 for two cases of external load of uniform load and concentrated load. The numerical solutions were obtained by using Richardson's extrapolation formula for the two case of divisional numbers $m(=n)$ of 8 and 12 for the one fourth part or a half part of each equivalent rectangular plate. Table 1 involves the theoretical values by Timoshenko and Krieger (1959) or Saito *et al.* (1958), and it shows the good convergency and satisfiable accuracy of the numerical solutions by present method.

6.2. Isosceles right triangular plates

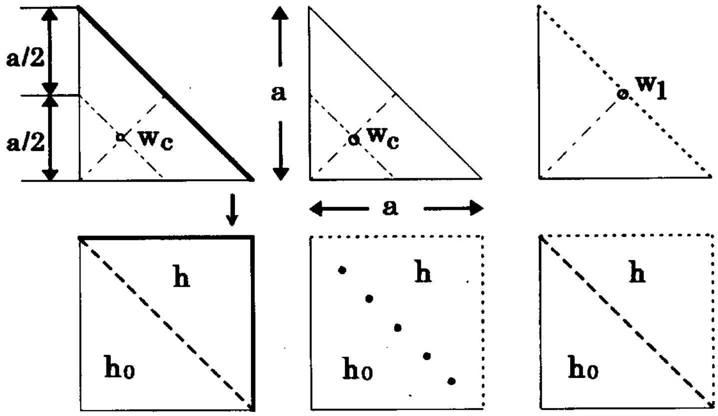
Numerical solutions for deflections of the right triangular plates with three type of boundary conditions shown in Fig. 9 are given in Table 2. The numerical solutions were obtained by using

Table 1 Fixed circular plate, semi-circular plate and elliptic plate (ν=0.3)

h/h_0	Cirucular plate $w_cD_0/q_0a^4\times10^4$				Semi-circular plate $w_cD_0/q_0a^4\times10^4$				Elliptic plate (b/a=0.5) $w_cD_0/q_0a^4\times10^4$			
	m		Nu.	Ref.	m		Nu.	Ref	m		Nu.	Ref.
	8	12	Solu.	[1]	8	12	Solu.	[1]	8	12	Solu.	[1]
4	7.96	8.73	9.35		1.06	1.16	1.24		1.16	1.24	1.30	
6	7.54	8.58	9.42		1.05	1.15	1.23		1.15	1.23	1.29	
8	7.32	8.54	9.52		1.04	1.14	1.22		1.15	1.23	1.29	
10	7.20	8.52	9.59	9.77	1.04	1.14	1.22	1.26*	1.14	1.23	1.29	1.32
12	7.12	8.52	9.63		1.04	1.14	1.22		1.14	1.22	1.29	
20	6.98	8.51	9.73		1.04	1.14	1.22		1.14	1.23	1.29	
h/h_0	Circular plate $w_cD_0/Pa^2\times10^3$				Semi-circular plate $w_cD_0/Pa^2\times10^4$				Elliptic plate (b/a=0.5) $w_cD_0/Pa^2\times10^3$			
	m		Nu.	Ref.	m		Nu.	Ref.	m		Nu.	Ref.
	8	12	Solu.	[1]	8	12	Solu.	[1]	8	12	Solu.	[2]
4	4.15	4.52	4.82		6.44	7.43	8.23		1.45	1.59	1.70	
6	4.04	4.49	4.84		6.40	7.41	8.21		1.45	1.59	1.70	
8	3.98	4.47	4.87		6.39	7.40	8.21		1.44	1.58	1.70	
10	3.95	4.47	4.89	4.97	6.38	7.39	8.20	-	1.44	1.58	1.70	1.71
12	3.93	4.47	4.90		6.38	7.39	8.20		1.44	1.58	1.70	
20	3.89	4.47	4.93		6.37	7.39	8.20		1.44	1.58	1.70	

1.26*; deflection at neighboring point, which is maximum deflection of the plate
Ref. [1]: Timoshenko and Krieger (1959), Ref.[2]: Saito Shimazaki and Kimura (1958)

the two case of divisional numbers m (=n) of 8 and 12 for the whole part of the equivalent plate. Table 2 involves the theoretical values by Fletcher (1959), and it also shows the good convergency and satisfiable accuracy of the numerical solutions by present method.



Fixed diagonal Simply supported diagonal Free diagonal

Fig. 9 Isosceles right triangular plates

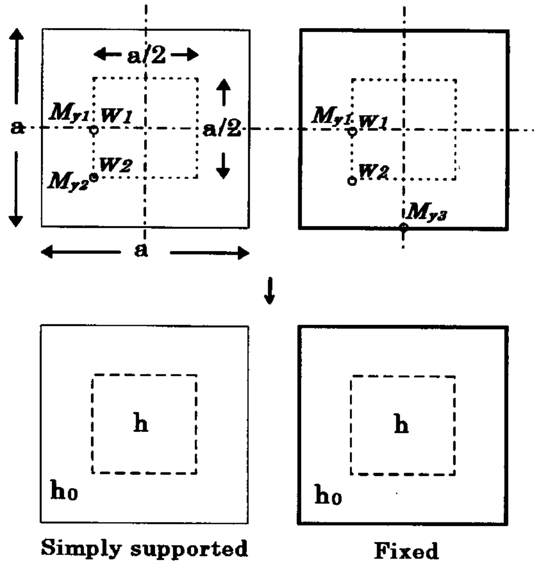


Fig. 10 Square plate with square opening

Table 2 Isosceles right triangular plates with fixed, simply supported or free diagonal edge and the other simply supported two edges (ν=0.3)

h/h_0	Fixed diagonal $w_cD_0/q_0a^4\times10^4$				Simply supported diagonal $w_cD_0/q_0a^4\times10^4$					Free diagonal $w_lD_0/q_0a^4\times10^2$		
	m		Nu.	Ref.	h/h$_0$		m	Nu.	Ref.	m		Nu.
	8	12	Solu.	[3]	8	12		Solu.	[3]	8	12	Solu.
4	3.74	3.97	4.16		1/4	4.78	5.42	5.93		2.61	2.04	1.95
6	3.18	3.54	3.82		1/6	4.80	5.42	5.92		2.20	2.07	1.97
8	3.02	3.41	3.72	3.69	1/8	4.81	5.42	5.91	6.07	2.23	2.08	1.97
10	2.95	3.35	3.66		1/10	4.81	5.42	5.91		2.25	2.09	1.96
12	2.91	3.31	3.62		1/12	4.81	5.42	5.91		2.27	2.09	1.95

Ref.[3]: Fletcher (1959)

Table 3 Simply supported square plate with square opening (ν=0.3)

h/h_0	$w_1D_0/q_0a^4\times10^3$			$w_2D_0/q_0a^4\times10^3$			$M_{y1}/q_0a^2\times10^2$			$M_{y2}/q_0a^2\times10^2$		
	Nu.	Ref.[4]		Nu.	Ref.[4]		Nu.	Ref.[4]		Nu.	Ref.[4]	
	Solu.	Map.	FEM	Solu.	Map.	FEM	Solu.	Map.	FEM	Solu.	Map.	FEM
1/4	3.21			2.29			2.22			2.75		
1/6	3.27			2.33			2.23			2.81		
1/8	3.29	3.14	3.23	2.34	2.28	2.34	2.24	2.23	2.22	2.82	-	-
1/10	3.29			2.34			2.24			2.83		
1/12	3.29			2.34			2.24			2.83		

Ref.[4]: Iwahara (1980), Map.: Mapping function method

Table 4 Fixed square plate with square opening (ν=0)

h/h_0	$w_1D_0/q_0a^4\times10^4$			$w_2D_0/q_0a^4\times10^4$			$M_{y1}/q_0a^2\times10^3$			$-M_{y3}/q_0a^2\times10^2$		
	Nu. Solu.	Ref.[4] Map.	Ref.[4] FEM	Nu. Solu.	Ref.[4] Map.	Ref.[4] FEM	Nu. Solu.	Ref.[4] Map.	Ref.[4] FEM	Nu. Solu.	Ref.[4] Map.	Ref.[4] FEM
1/4	3.68			2.55			3.01			2.73		
1/6	3.70			2.56			3.00			2.74		
1/8	3.70	3.56	3.80	2.56	2.50	2.67	3.00	3.60	2.7	2.74	2.76	2.75
1/10	3.70			2.56			3.00			2.74		
1/12	3.70			2.56			3.00			2.74		

Ref.[4]: Iwahara (1980), Map.: Mapping function method

6.3. Square plate with square opening

Numerical solutions for deflections and bending moments of rectangular plates with square opening at the center and two type of boundary conditions shown in Fig. 10 are given in Table 3 and 4. The numerical solutions were obtained by using the two case of divisional numbers m (=n) of 8 and 12 for the one fourth part of the equivalent plate. Tables 3 and 4 involve the theoretical values by Iwahara (1980), and they show the good convergency and satisfiable accuracy of the numerical solutions for deflections and bending moments by present method.

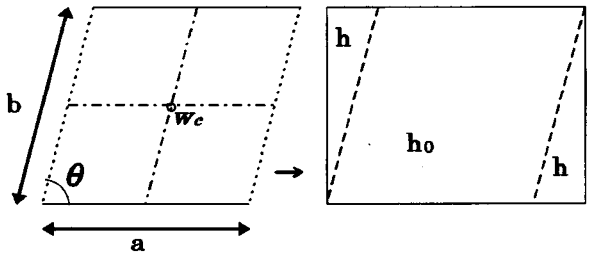


Fig. 11 Simple-free skew plate

Table 5 Simple-free skew plates (ν=0.3)

h/h_0	$\theta=75°$ $w_cD_0/q_0a^4\times10^3$				$\theta=60°$ $w_cD_0/q_0a^4\times10^3$				$\theta=45°$ $w_cD_0/q_0a^4\times10^3$			
	m 12	m 14	Nu. Solu.	Ref. [1]	m 12	m 14	Nu. Solu.	Ref. [1]	m 12	m 14	Nu. Solu.	Ref. [1]
4	6.96	7.08	7.42		6.56	6.61	6.77		3.53	3.49	3.38	
6	7.18	7.40	8.00		6.93	7.02	7.27		3.73	3.69	3.56	
8	7.27	7.51	8.18	-	7.08	7.23	7.66	7.91	3.76	3.75	3.73	3.93
10	7.30	7.56	8.29		7.14	7.25	7.57		3.80	3.78	3.73	
12	732	7.58	8.32		7.17	7.29	7.65		3.82	3.80	3.74	

Ref.[1]: Timoshenko and Krieger (1959)

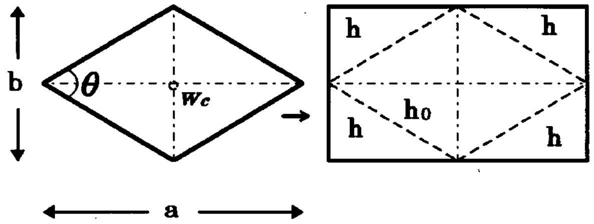


Fig. 12 Fixed rhombic plate

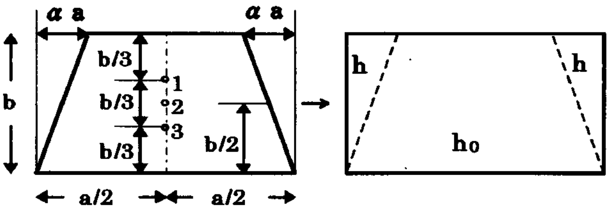


Fig. 13 Fixed trapezoidal plate

6.4. Skew plate

Numerical solutions for deflections of skew plates with two simply and freely supported opposite sides shown in Fig. 11 are given in Table 5. The numerical solutions for three case of aspect angles were obtained by using the two case of divisional numbers m (=n) of 12 and 14 for the whole part of the equivalent plate. Table 5 involves the theoretical values by Timoshenko and Krieger (1959), and it also shows the satisfiable convergency and accuracy of the numerical solutions by present method.

6.5. Rhombic plate

Numerical solutions for deflections of fixed rhombic plates with three case of aspect angles shown in Fig. 12 are given in Table 6. The numerical solutions were obtained by using the two case of divisional numbers m (=n) of 8 and 12 for the one fourth part of the equivalent plate.

Table 6 Fixed rhombic plates (ν=0.3)

h/h_0	θ=45° $w_cD_0/q_0a^4\times10^5$				θ=60° $w_cD_0/q_0a^4\times10^5$				θ=75° $w_cD_0/q_0a^4\times10^4$			
	m		Nu. Solu.	Ref. [5]	m		Nu. Solu.	Ref. [5]	m		Nu. Solu.	Ref. [5]
	8	12			8	12			8	12		
4	3.05	3.25	3.40		8.21	8.69	9.08		1.74	1.83	1.91	
6	2.89	3.12	3.30		7.62	8.18	8.64		1.58	1.70	1.79	
8	2.83	3.07	3.26	3.24	7.41	8.02	8.50	8.54	1.53	1.65	1.75	1.77
10	2.80	3.05	3.25		7.31	7.94	8.45		1.51	1.63	1.74	
12	2.78	3.04	3.25		7.25	7.90	8.42		1.49	1.62	1.73	

Ref.[5]: Ota, Hamada, Sagijima, Nishimura and Masui (1962)

Table 7 Fixed triangular, trapezoidal and rectangular plates (ν=0.3)

	α=0.5; regular triangular plate											
h/h_0	$w_1D_0/q_0a^4\times10^4$				$w_2D_0/q_0a^4\times10^4$				$w_3D_0/q_0a^4\times10^4$			
	m		Nu.	Ref.	m		Nu.	Ref.	m		Nu.	Ref.
	8	12	Solu.	[6]	8	12	Solu.	[6]	8	12	Solu.	[6]
4	.428	.508	.571		1.15	1.33	1.47		1.62	1.77	1.90	
6	.244	.331	.401		.861	1.07	1.24		1.37	1.56	1.71	
8	.189	.279	.352	-	.766	.991	1.17	-	1.28	1.49	1.66	1.67
10	.166	.258	.332		.726	.957	1.14		1.25	1.46	1.64	
12	.155	.248	.322		.705	.940	1.13		1.23	1.45	1.63	

	α=0.25; symmetric trapezoidal plate											
h/h_0	$w_1D_0/q_0a^4\times10^4$				$w_2D_0/q_0a^4\times10^4$				$w_3D_0/q_0a^4\times10^4$			
	m		Nu.	Ref.	m		Nu.	Ref.	m		Nu.	Ref.
	8	12	Solu.		8	12	Solu.		8	12	Solu.	
4	3.14	3.44	3.68		4.17	4.59	4.92		3.77	4.09	4.35	
6	2.88	3.24	3.52		3.84	4.36	4.77		3.52	3.93	4.27	
8	2.80	3.17	3.47	-	3.74	4.29	4.72	-	3.44	3.88	4.24	-
10	2.77	3.14	3.44		3.70	4.26	4.70		3.40	3.86	4.23	
12	2.76	3.13	3.43		3.67	4.24	4.69		3.38	3.85	4.23	

	α=0; rectangular plate											
	$w_1D_0/q_0a^4\times10^4$				$w_2D_0/q_0a^4\times10^4$				$w_3D_0/q_0a^4\times10^4$			
	m		Nu.	Ref.	m		Nu.	Ref.	m		Nu.	Ref.
	8	12	Solu.	[7]	8	12	Solu.	[7]	8	12	Solu.	[7]
	7.39	7.45	7.49	7.47	9.39	9.25	9.29	9.26	7.39	7.45	7.49	7.47

Ref.[6]: Conway (1962), Ref.[7]: Sakiyama and Matsuda (1983)

Table 6 involves the theoretical values by Ota *et al.* (1962), and it also shows the good convergency and satisfiable accuracy of the numerical solutions by present method.

6.6. Trapezoidal plate and regular triangular plate

Numerical solutions for deflections of a fixed trapezoidal plates and a fixed regular triangular plates as one of the special case of trapezoidal plate shown in Fig. 13 are given in Table 7 with a fixed rectangular plate as the other special case of trapezoidal plate. The numerical solutions are obtained by using the two case of divisional numbers m (=n) of 8 and 12 for the half part of the equivalent plate. Table 7 involves the theoretical values by Conway (1962) or Sakiyama and Matsuda (1983), and it shows the good convergency and satisfiable accuracy of the numerical solutions by present method.

7. Conclusions

Under the concept that the irregular-shaped plates such as plate with opening, circular plate, semi-circular plate, elliptic plate, triangular plate, skew plate, rhombic plate, trapezoidal plate or the other polygonal plates can be considered finally as a kind of rectangular plates with nonuniform thickness, an approximate method was proposed for analyzing the bending problem of various types of irregular-shaped plates by using the discrete general solution for the equivalent rectangular plate with non-uniform thickness.

As a result of numerical works, it was shown that the numerical solutions by proposed method had the good convergency and satisfiable accuracy for various types of irregular-shaped plates.

In this paper the results of numerical works were shown mainly for the case of uniform load, because the main purpose of this paper was to investigate the propriety of the equivalent rectangular plate. Naturally numerical solutions for the other cases of external loads such as concentrated loads or non-uniform loads can be obtained similarly by proposed method.

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Appendix I

$$F_{111}=F_{123}=F_{134}=F_{146}=F_{167}=F_{178}=F_{188}=1,\quad F_{212}=F_{225}=F_{233}=F_{257}=F_{266}=\mu$$
$$F_{156}=\nu,\quad F_{247}=\nu\mu,\quad F_{322}=F_{331}=-\mu,\quad F_{344}=F_{355}=-I,\quad F_{363}=-J,\quad F_{372}=-\kappa$$
$$F_{377}=1,\quad F_{381}=-\mu\kappa,\quad F_{386}=\mu,\quad \text{other } F_{1te}, F_{2te}, F_{3te}=0$$
$$I=\mu(1-\nu^2)(h_0/h)^3,\quad J=2\mu(1+\nu)(h_0/h)^3,\quad \kappa=(1/10)(E/G)(h_0/a)^2(h_0/h)$$

Appendix II

$$A_{p1}=\gamma_{p1},\quad A_{p2}=0,\quad A_{p3}=\gamma_{p3}=\gamma_{p2},\quad A_{p4}=\gamma_{p3},\quad A_{p5}=0,\quad A_{p6}=\gamma_{p4}+\nu\gamma_{p5},\quad A_{p7}=\gamma_{p6}$$
$$A_{p1}=\gamma_{p7},\quad B_{p1}=0, B_{p2}=\mu\gamma_{p1},\quad B_{p3}=\mu\gamma_{p3},\quad B_{p4}=0,\quad B_{p5}=\mu\gamma_{p2},\quad B_{p6}=\mu\gamma_{p6}$$
$$B_{p7}=\mu(\nu\gamma_{p4}+\gamma_{p5}),\quad B_{p8}=\gamma_{p8},\quad C_{p1kl}=\mu(\gamma_{p3}+\kappa_{kl}\gamma_{p7}),\quad C_{p2kl}=\mu\gamma_{p2}+\kappa_{kl}\gamma_{p8}$$
$$C_{p3kl}=J_{kl}\gamma_{p6},\quad C_{p4kl}=I_{kl}\gamma_{p4},\quad C_{p5kl}=I_{kl}\gamma_{p5},\quad C_{p6kl}=-\mu\gamma_{p7},\quad C_{p7kl}=-\gamma_{p8}$$

$C_{p8kl}=0,\ [\gamma_{pk}]=[\bar{\gamma}_{pk}]^{-1},\ \bar{\gamma}_{11}=\beta_{ii},\ \bar{\gamma}_{12}=\mu\beta_{jj},\ \bar{\gamma}_{22}=-\mu\beta_{ij},\ \bar{\gamma}_{23}=\beta_{ii},\ \bar{\gamma}_{25}=\mu\beta_{jj}$
$\bar{\gamma}_{31}=-\mu\beta_{ij},\ \bar{\gamma}_{33}=\mu\beta_{jj},\ \bar{\gamma}_{34}=\beta_{ii},\ \bar{\gamma}_{44}=-I_{ij}\ \beta_{ij},\ \bar{\gamma}_{46}=\beta_{ii},\ \bar{\gamma}_{47}=\mu\nu\beta_{jj},\ \bar{\gamma}_{55}=-I_{ij}\ \beta_{ij}$
$\bar{\gamma}_{56}=\nu\beta_{ii},\ \bar{\gamma}_{57}=\mu\beta_{jj},\ \bar{\gamma}_{63}=-J_{ij}\beta_{ij},\ \bar{\gamma}_{66}=\mu\beta_{jj},\bar{\gamma}_{67}=\beta_{ii},\ \bar{\gamma}_{71}=-\mu\kappa_{ij}\beta_{ij},\ \bar{\gamma}_{76}=\mu\beta_{ij}$
$\bar{\gamma}_{78}=\beta_{ii},\ \bar{\gamma}_{82}=-\kappa_{ij}\beta_{ij},\ \bar{\gamma}_{87}=\beta_{ij},\ \bar{\gamma}_{88}=\beta_{jj},$ other $\bar{\gamma}_{pk}=0,\ \beta_{ij}=\beta_{ii}\beta_{jj}$

Appendix III

$a_{11i0il}=a_{13i0i2}=a_{14i0i3}=a_{16i0i4}=a_{17i0i5}=a_{18i0i6}=1,\ a_{15i0i3}=\nu$
$a_{220jj1}=a_{230jj2}=a_{250jj3}=a_{260jj4}=a_{270jj5}=a_{280jj6}=1,\ a_{240jj3}=\nu,\ a_{230002}=0$

$$a_{hpiju\nu}=\sum_{e=1}^{8}\left\{\begin{array}{l}\sum_{k=0}^{i}\beta_{ik}A_{pe}[a_{hek0u\nu}-a_{hekju\nu}(1-\delta_{ki})]+\sum_{l=0}^{j}\beta_{jl}B_{pe}[a_{heolu\nu}-a_{heilu\nu}(1-\delta_{lj})]\\ +\sum_{k=0}^{i}\sum_{l=0}^{j}\beta_{ik}\beta_{jl}C_{pekl}a_{heklu\nu}(1-\delta_{ki}\delta_{lj})\end{array}\right\}$$

where $h=1,2,\ p=1,2,\cdots,8,\ i=1,2,\cdots,m,\quad j=1,2,\cdots,n,\ \nu=1,2,\cdots,6$
$u=0,1,\cdots,i\ (h=1),\ 0,1,\cdots,j\ (h=2)$

$$q_{fpij}=\sum_{e=1}^{8}\left\{\begin{array}{l}\sum_{k=0}^{i}\beta_{ik}A_{pe}[q_{fek0}-q_{fekj}(1-\delta_{ki})]+\sum_{l=0}^{j}\beta_{jl}B_{pe}[q_{fe0l}-q_{feil}(1-\delta_{lj})]\\ +\sum_{k=0}^{i}\sum_{l=0}^{j}\beta_{ik}\beta_{jl}C_{pekl}a_{fekl}(1-\delta_{ki}\delta_{lj})\end{array}\right\}$$
$$-\gamma_{pf}\sum_{k=0}^{i}\sum_{l=0}^{j}\beta_{ik}\beta_{jl}p_{fkl}$$

where $f=1,2,3$

$$\bar{q}_{fpijcd}=\sum_{e=1}^{8}\left\{\begin{array}{l}\sum_{k=0}^{i}\beta_{ik}A_{pe}[\bar{q}_{fek0cd}-\bar{q}_{fekjcd}(1-\delta_{ki})]+\sum_{l=0}^{j}\beta_{jl}B_{pe}[\bar{q}_{fe0lcd}-\bar{q}_{feilcd}(1-\delta_{lj})]\\ +\sum_{k=0}^{i}\sum_{l=0}^{j}\beta_{ik}\beta_{jl}C_{pekl}\bar{q}_{feklcd}(1-\delta_{ki}\delta_{lj})\end{array}\right\}$$
$$-\gamma_{pf}\sum_{k=0}^{i}\sum_{l=0}^{j}u_{ik}u_{jl}\bar{u}_{fkl}$$

where

$$\bar{u}_{fkl}=\begin{cases}0\text{: not existing point support}\\ 1\text{: existing point support}\end{cases}$$