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An exact finite element for a beam on a two-parameter elastic foundation: a revisit

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Abstract. An analytical solution for the shape functions of a beam segment supported on a generalized two-parameter elastic foundation is derived. The solution is general, and is not restricted to a particular range of magnitudes of the foundation parameters. The exact shape functions can be utilized to derive exact analytic expressions for the coefficients of the element stiffness matrix, work equivalent nodal forces for arbitrary transverse loads and coefficients of the consistent mass and geometrical stiffness matrices. As illustration, each distinct coefficient of the element stiffness matrix is compared with its conventional counterpart for a beam segment supported by no foundation at all for the entire range of foundation parameters.

Key words: two-parameter elastic foundation; Winkler model; Pasternak model; element stiffness matrix; work equivalent nodal force; consistent mass matrix; consistent geometrical stiffness matrix.

1. Introduction

The need to perform a rational stress analysis of transversely loaded beams supported upon or wrapped by an elastic medium emerges in many diverse applications, e.g., buried pipelines, piles, foundation elements, reinforcing filaments in composite materials, and even endodontics, to name a few (Kerr 1964, Scott 1981, Eisenberger and Bielak 1989, Tekkaya and Aydln 1991). The theory is traced to Winkler's original work on stresses in railway tracks where the idea of representing the soil underneath tracks by a series of closely packed, but otherwise unconnected, springs led to a solution for the internal forces, Hetenyi (1946). This type of a foundation is characterized by a single parameter called the Winkler foundation parameter. Subsequent refinements of this theory (e.g., by Pasternak 1954, Kerr 1964) have accounted for the possible interaction between these springs, and have enabled a wealth of diverse applications to be made (Zhaohua and Cook 1983, Williams and Kennedy 1987, Razaqpur and Shah 1991). Stiffness and mass matrices for beam segments embedded in these more general types of foundations have found use also in vibration and stability problems. Examination of the differential equation (which will be developed later) for the most general case will show that a beam segment not supported by any foundation is only a special case of a beam segment supported by a single-parameter Winkler foundation which in turn is a subclass of the more general two-parameter case which is known as the Pasternak

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foundation (Kerr 1964). This suggests that element matrices for the simpler cases should be derived from the Pasternak solution by examining successively two limiting situations. This way, stress, stability and vibration analyses for arbitrary structural configurations consisting of one-dimensional members supported by foundations which possess any combination of these characteristics can be carried out routinely with the use of only one set of matrix expressions. In contrast, much of the previous work in this area appears to be fragmented either because specific foundation parameters have been selected for specific applications or because not all frame element matrices have been derived within the same development. A unified derivation of all linear element matrices for all combinations of foundation parameters is desirable. This is the motivation for the present reexamination.

In this paper we derive first the exact shape functions for a beam segment supported by a two-parameter elastic foundation. One of our objectives is to demonstrate graphically how these shapes deviate from the cubic (Hermitian) polynomial expressions for particular combinations of the foundation parameters. This derivation is not limited in any way to a particular set of foundation parameters, and we will show that the shape functions blend into one another smoothly across the different ranges of the differential equation solution for the complete spectrum of these parameters. Equipped with exact one-dimensional shape functions, it becomes possible to derive exact expressions for coefficients of the element stiffness matrix, or the consistent geometric and mass matrices for beam segments as well as work-equivalent joint forces for a variety of concentrated or distributed loads. Coefficients of the element stiffness matrix are again compared graphically with their counterparts for the simple frame element, demonstrating that they can be "cloned" from the general solution which is presented. Most of the expressions for element matrices are lengthy and cumbersome but this is outweighed by virtue of the fact that once they are made part of a structural analysis software, they enable exact analyses to be performed with far fewer elements. Accounting for the appropriate force boundary condition at the free end of a finite beam on an infinitely long foundation via the artifice of providing a virtual extension of the beam beyond its physical end is also discussed.

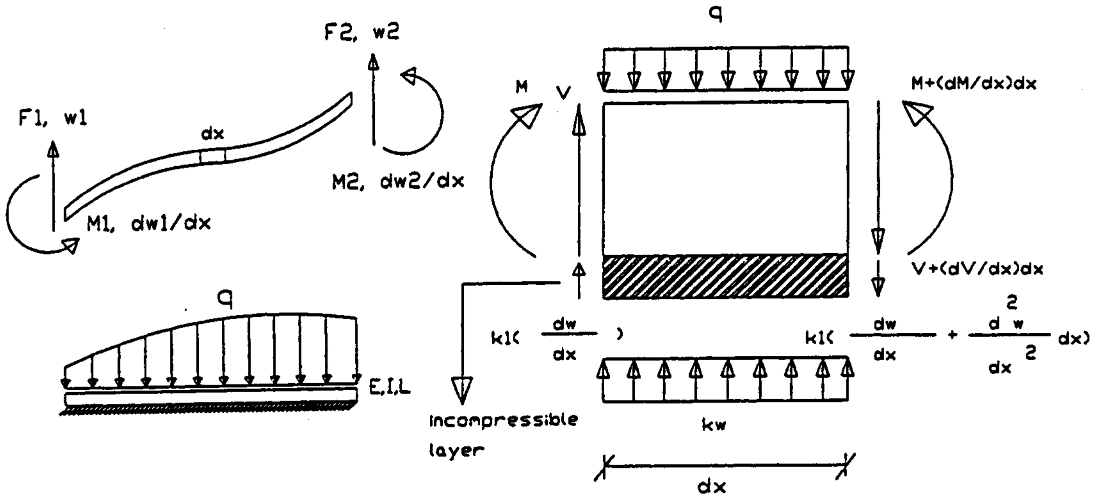

Fig. 1 Beam segment on a general foundation

2. The two-parameter foundation

A differential part of a beam supported by a two-parameter elastic foundation which terminates at the ends of the beam is shown in Fig. 1. The first of these parameters is representative of the foundation's resistance to transverse translations, and is called the Winkler parameter k, with dimensions F/L^2. The second parameter for which the dimensions are F/rotation is representative of the foundation's resistance to shear deformations, and is denoted by k_1, the Pasternak parameter. In the Winkler formulation each translational spring can deflect independently of springs immediately adjacent to it, but in the Pasternak theory an interaction between these springs is admitted. Other interpretations of this parameter are possible (Kerr 1964). We note that, for the problem at hand both of these parameters must be positive constants.

The governing differential equation of the beam segment in Fig. 1 for transverse displacement w is derived from equilibrium:

$$EI\frac{d^4w}{dx^4} - k_1\frac{d^2w}{dx^2} + kw = q(x) \tag{1}$$

Defining $A=k_1/EI$ and $B=k/EI$ the complementary solution of the fourth order Eq. (1) can be computed as any of the expressions in Eqs. (2a)-(2c) depending on which of the respective conditions $A<2\sqrt{B}$, $A=2\sqrt{B}$, or $A>2\sqrt{B}$ holds:

$$w(x)=c_1\cos\beta x\cosh\alpha x + c_2\cos\beta x\sinh\alpha x + c_3\sin\beta x\cosh\alpha x + c_4\sin\beta x\sinh\alpha x \tag{2a}$$

$$w(x)=c_1e^{\sqrt[4]{B}x} + c_2xe^{\sqrt[4]{B}x} + c_3e^{-\sqrt[4]{B}x} + c_4xe^{-\sqrt[4]{B}x} \tag{2b}$$

$$w(x)=c_1\cosh\beta x\cosh\alpha x + c_2\sinh\beta x\cosh\alpha x + c_3\cosh\beta x\sinh\alpha x + c_4\sinh\beta x\sinh\alpha x \tag{2c}$$

In Eqs. (2) the auxiliary variables are defined by

$$\alpha=\sqrt{\lambda^2+\delta} \tag{3}$$

$$\beta=\sqrt{\lambda^2-\delta}\ \text{ in Eq. (2a)} \qquad \beta=\sqrt{\delta-\lambda^2}\ \text{ in Eq. (2c)} \tag{4}$$

$$\lambda=\sqrt[4]{\frac{k}{4EI}}\quad\text{and}\quad\delta=\frac{k_1}{4EI} \tag{5}$$

When k_1 approaches zero, all three expressions for $w(x)$ in Eqs. (2) reduce to:

$$w(x)=c_1\sin\lambda x\sinh\lambda x + c_2\sin\lambda x\cosh\lambda x + c_3\cos\lambda x\sinh\lambda x + c_4\cos\lambda x\cosh\lambda x \tag{6}$$

We note that Eq. (6) is the solution of the governing differential equation for the Winkler foundation:

$$EI\frac{d^4w}{dw^4} + kw = q(x) \tag{7}$$

When k approaches zero, Eq. (1) becomes identical to the governing differential equation for a beam-column subjected, for the negative sign in Eq. (1), to a tensile force of k_1. We will not consider this case here. When both foundation constants are zero, then Eq. (1) reverts to the beam equation the complementary solution of which is a cubic polynomial.

Possibly because they were inspired by geotechnical engineering considerations, most solutions to date for beams supported by generalized Pasternak type foundations have considered the "practical" case in Eq. (2a) for which $A<2\sqrt{B}$ (Scott 1981, Zhaohua and Cook 1983, Williams and

Kennedy 1987, Chiwanga and Valsangkar 1988), although on physical grounds the justification for this restriction is debatable because for particular combinations of the parameters this condition may not be fulfilled. This is noted also by Razaqpur and Shah (1991).

We note that Eqs. (2) may be written conveniently in matrix form

$$w = \boldsymbol{B}^T \boldsymbol{C} \tag{8}$$

where $\boldsymbol{C}$ is the column vector containing the four constants of integration c_1-c_4 and $\boldsymbol{B}$ is the column vector which contains any one of the three functions in Eqs. (2) arranged appropriately.

At this stage we define four boundary conditions which are the two transverse displacements w_1 and w_2 and the two rotations dw_1/dx and dw_2/dx at each end of the segment in Fig. 1. Inserting the four generalized displacements into a column matrix $\boldsymbol{W}=\{w_1,\ dw_1/dx,\ w_2,\ dw_2/dx\}^T$ for the proper values of A and B, we obtain

$$\boldsymbol{W} = \boldsymbol{HC} \tag{9}$$

When $\boldsymbol{C}$ is solved symbolically from Eq. (9) and substituted into Eq. (8) we obtain

$$\begin{aligned} w &= \boldsymbol{B}^T \boldsymbol{H}^{-1} \boldsymbol{W} \\ &= \boldsymbol{NW} \end{aligned} \tag{10}$$

The row vector $\boldsymbol{N}$ includes four shape functions each corresponding to a unit value of the generalized displacements. Taking the inverse of matrix $\boldsymbol{H}$ and premultiplying it by matrix $\boldsymbol{B}^T$ is an arduous procedure. Therefore, the computer software Mathematica (Wolfram 1988) was used to perform all symbolic calculations throughout this study. We obtain shape functions for each set of solutions stated in Eq. (2) for the different ranges of the Winkler and the Pasternak parameters. To give a brief idea, we present the first shape function N_1 that represents unit vertical displacement at left end while all remaining displacements are set equal to zero.

For $A < 2\sqrt{B}$:

$$\begin{aligned} N_1 = [&\beta^2 \cos\beta x \cosh\alpha(2L - x) + \alpha^2 \cos\beta(2L - x) \cosh\alpha x - \alpha^2 \cos\beta x \cosh\alpha x \\ &- \beta^2 \cos\beta x \cosh\alpha x + \alpha\beta \sin\beta x \sinh\alpha(2L - x) - \alpha\beta \sin\beta(2L - x) \sinh\alpha x]/ \\ &(-\alpha^2 - \beta^2 + \alpha^2 \cos 2\beta L + \beta^2 \cosh 2\alpha L) \end{aligned} \tag{11}$$

For $A = 2\sqrt{B}$:

$$\begin{aligned} N_1 = &\frac{e^{(2L-x)s} xs(-1 + e^{2Ls} + 2Ls) + e^{(2L+x)s} x(s - se^{-2Ls} + 2Ls^2)}{(1 - 2e^{2Ls} + e^{4Ls} - 4e^{2Ls} L^2 s^2)} + \\ &\frac{e^{(2L+x)s}(-1 + e^{-2Ls} - 2Ls - 2L^2 s^2) + e^{(2L-x)s}(-1 + e^{2Ls} + 2Ls - 2L^2 s^2)}{(1 - 2e^{2Ls} + e^{4Ls} - 4e^{2Ls} L^2 s^2)} \end{aligned} \tag{12}$$

where

$$s = \sqrt[4]{\frac{k}{EI}} \tag{13}$$

For $A > 2\sqrt{B}$:

$$\begin{aligned} N_1 = &\cosh\alpha x \cosh\beta x + [\beta \cosh\beta x(-\beta \sinh 2\alpha L - \alpha \sinh 2\beta L) \sinh\alpha x + \\ &\alpha \cosh\alpha x(\beta \sinh 2\alpha L + \alpha \sinh 2\beta L) \sinh\beta x + \alpha\beta(-\cosh 2\alpha L + \cosh 2\beta L) \sinh\alpha x \sinh\beta x)]/ \end{aligned}$$

$$(\alpha^2-\beta^2+\beta^2\cosh 2\alpha L-\alpha^2\cosh 2\beta L) \tag{14}$$

We can obtain shape functions for beams on Winkler type foundations if k_1 approaches zero in any of Eqs. (11), (12) and (14). In all instances N_1 becomes

$$N_1=[\cos\lambda x\cosh\lambda(2L-x)+\cos\lambda(2L-x)\cosh\lambda x-2\cos\lambda x\cosh\lambda x+\\ \sin\lambda x\sinh\lambda(2L-x)-\sin\lambda(2L-x)\sinh\lambda x]/(-2+\cos 2L\lambda+\cosh 2L\lambda) \tag{15}$$

If we extend this process in such a way that we allow also λ to approach zero in Eq. (15), we obtain the Hermitian polynomial of the first kind

$$N_1=1-3\xi^2+2\xi^3 \tag{16}$$

in which $\xi=x/L$. This provides a check in that we can reduce successively Eqs. (11), (12), (14) to Eq. (16) by letting both k and k_1 approach zero. The solution is general, covering also beams on Winkler foundation as well as conventional beams, i.e., frame elements supported by no foundation, because Eq. (1) is general.

Shape functions for beams on Winkler or Pasternak-type foundations have been derived for part or the whole range of parameters (e.g., Wang 1983, Zhaohua and Cook 1983, Williams and Kennedy 1987, Razaqpur and Shah 1991) or unwittingly rediscovered and utilized in research, e.g., Eisenberger and Yankelevsky (1985).

It is instructive to indicate the variation of the shape function as a function of k or k_1, and to compare them with the corresponding Hermitian polynomials. This can be done either by fixing k and letting k_1 vary in a two-dimensional plot, or to describe their interaction in the form of a three-dimensional surface graph. For either purpose it is convenient to introduce two new nondimensional variables p and z such that for $A<2\sqrt{B}$:

$$\alpha=\frac{P}{L}\sqrt{1+z}\qquad \beta=\frac{P}{L}\sqrt{1-z} \tag{17}$$

and for $A>2\sqrt{B}$:

$$\alpha=\frac{P}{L}\sqrt{1+z}\qquad \beta=\frac{P}{L}\sqrt{z-1} \tag{18}$$

where

$$z=\frac{\delta}{\lambda^2}\quad\text{and}\quad p=\lambda L \tag{19}$$

The four frames of Fig. 2 show the variation of N_1 through a range of values for z at p=0.1, 1, 5 and 10. The shapes N_3 are simply the symmetric mirror images of these functions.

Two observations are in order:

1. For small values of k characterized approximately by the interval $0\leq p\leq 0.4$ the resulting shape is virtually the same as the Hermitian shape. This implies that for "short" beams or beams "weakly" bonded to the foundation ordinary beam elements in conjunction with lumped springs can be used. This explains convincingly that many problems involving beams on elastic foundations can be solved using traditional frame analysis software at the expense of added refinement of the physical model.

2. The Pasternak parameter k_1 (characterized by z) exhibits an increasingly dominant presence only for large values of k (characterized by p). This is of course not surprising because this

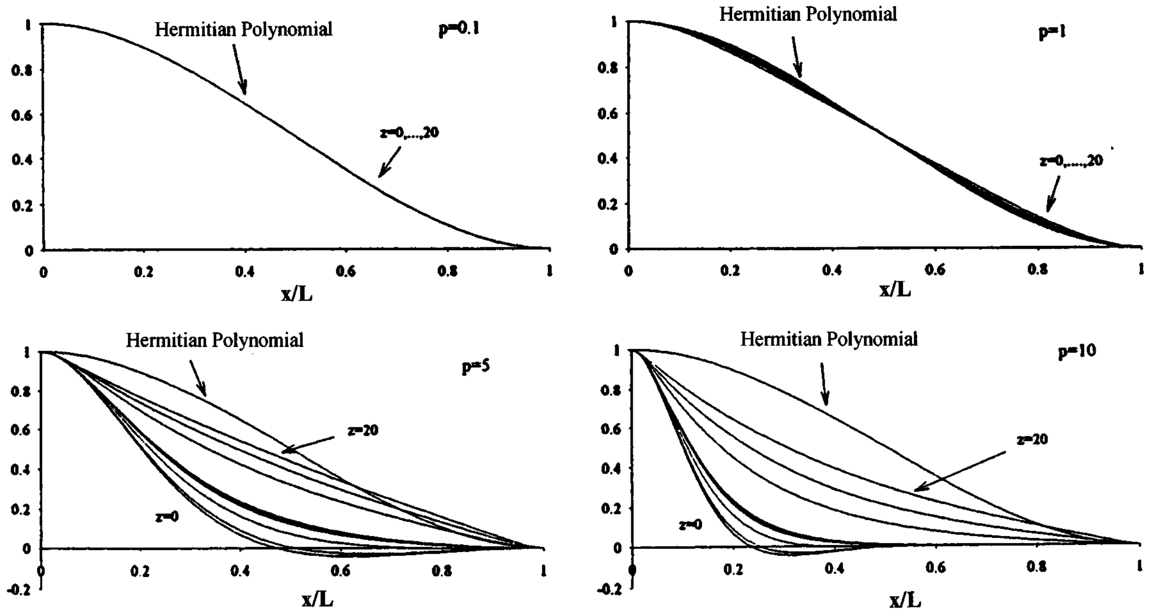

Fig. 2 Shape functions N_1

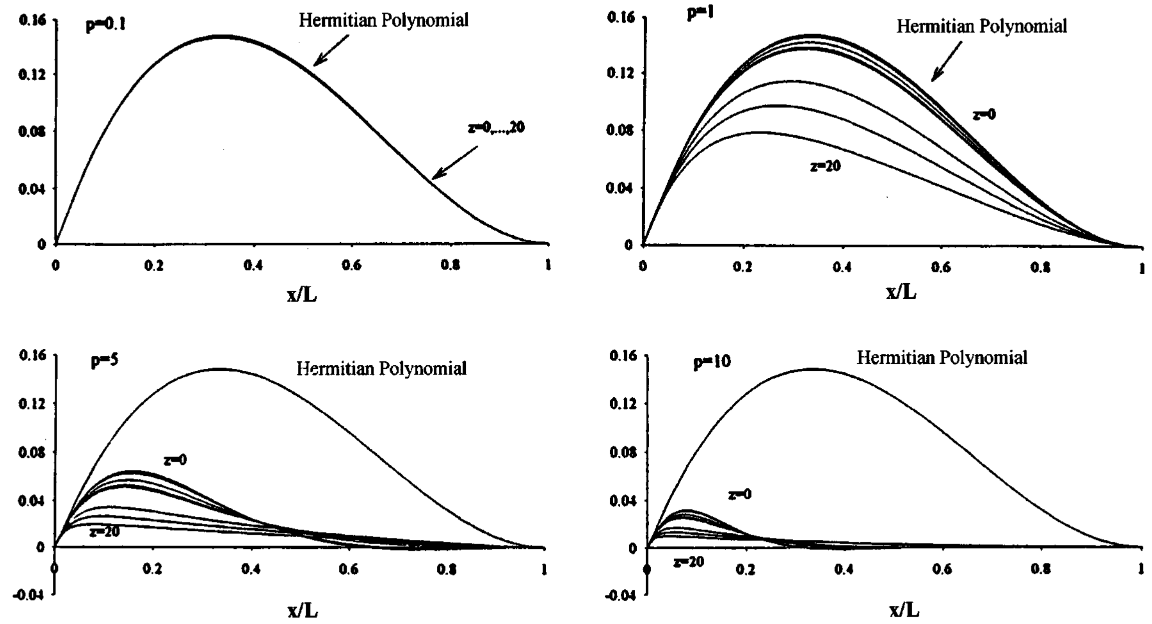

Fig. 3 Shape functions N_2

parameter is a reflection of the degree of interaction between the Winkler elements. With these elements missing or weakly represented, the concept of interaction between them becomes moot.

In Fig. 3 we show the variation of N_2 which corresponds to a unit rotation induced at the left end of the beam segment. The functions N_4 are antisymmetric mirror images of these functions. As prelude to similar subsequent graphs the surface plot renditions of N_1 and N_2 are shown in Figs. 4 and 5. We note that in this figure the variable *ks* represents the nondimensional axial coordinate ξ, and that for each distinct value of p, the first plot on the left corresponds to $A<2\sqrt{B}$, and that on the right to $A>2\sqrt{B}$. The vertical plane of termination of the former matches precisely the plane of initiation of the latter where $A=2\sqrt{B}$, so that the smooth transition required by heuristic reasoning and the physical reality is in fact achieved.

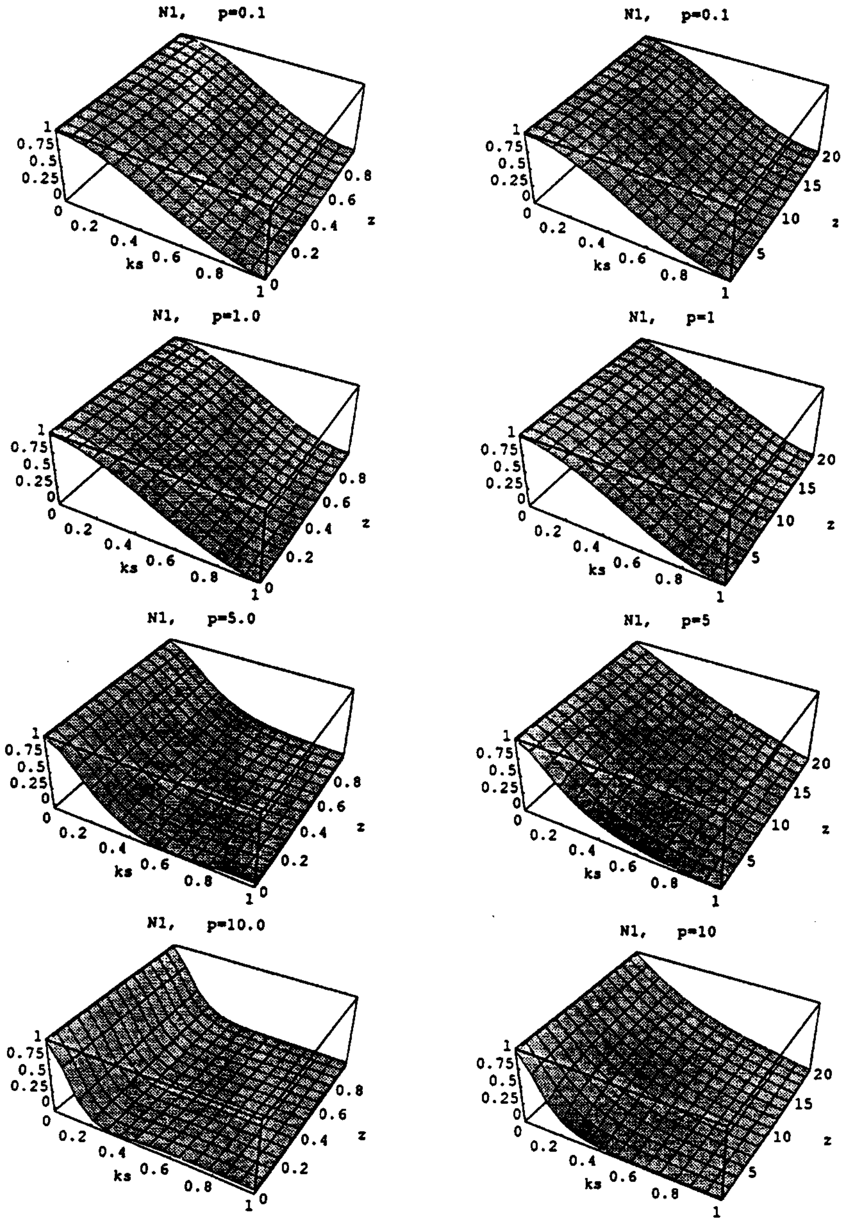

Fig. 4 Surface plots for N_1

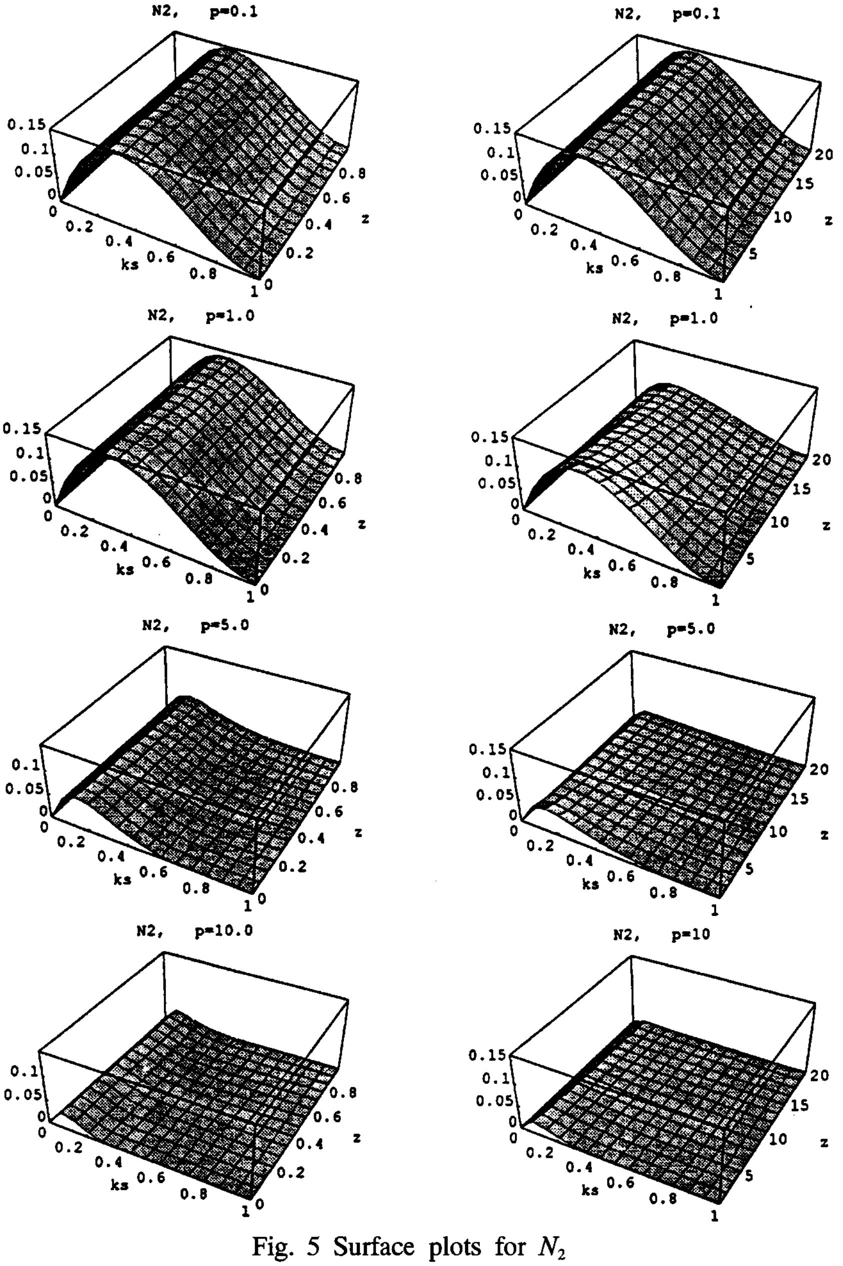

Fig. 5 Surface plots for N_2

3. The element stiffness matrix

The element stiffness matrix $\boldsymbol{k}_e$ which relates the nodal forces to nodal displacements (see Fig. 1) can be obtained from the minimization of the strain energy functional U which is

$$U = \frac{1}{2} EI \int_o^L \left(\frac{d^2w}{dx^2} \right)^2 dx + \frac{1}{2} k_1 \int_o^L \left(\frac{dw}{dx} \right)^2 dx + \frac{1}{2} k \int_o^L w^2 dx - \int_o^L wq(x)\, dx \tag{20}$$

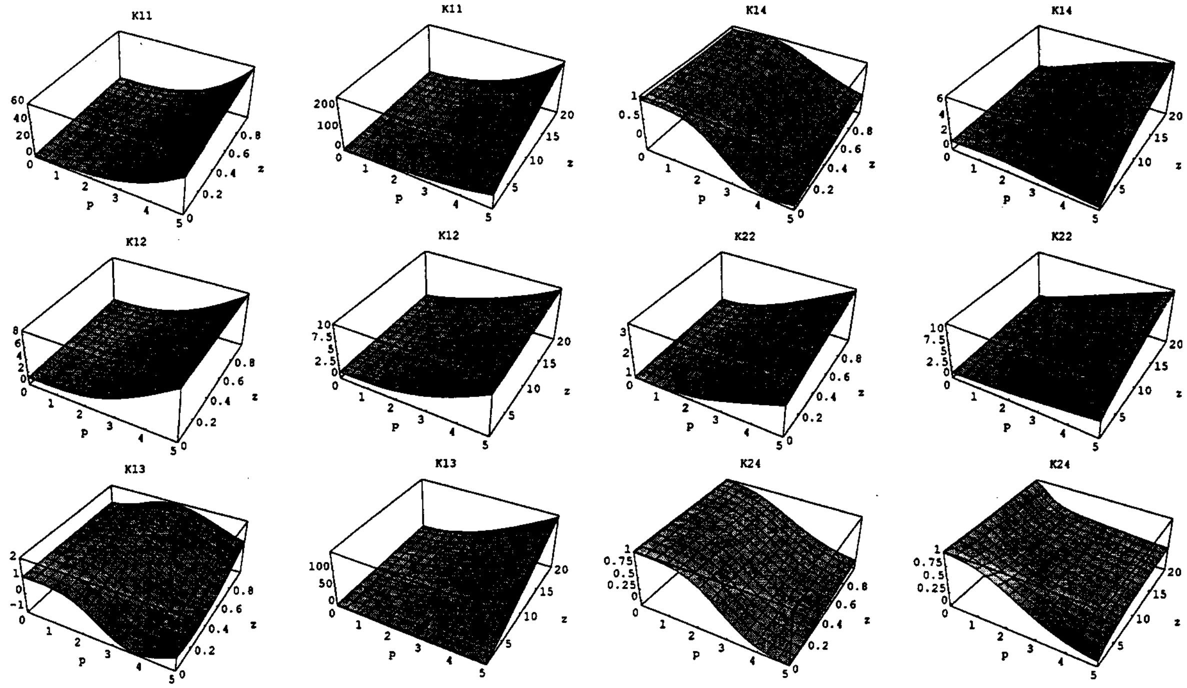

Fig. 6 Normalized stiffness matrix coefficients

From this equation the individual stiffness matrix terms are obtained as follows:

$$k_{ij} = EI\int_o^L N_i''N_j''dx + k_1\int_o^L N_i'N_j'\,dx + k\int_o^L N_iN_j\,dx \tag{21}$$

where N_i represents i'th shape function and primes denote differentiation with respect to x. Because there exist three sets of shape functions for the cases, $A<2\sqrt{B}$, $A=2\sqrt{B}$, $A>2\sqrt{B}$, we should evaluate Eq. (21) by inserting the appropriate shape functions. This yields three sets of stiffness matrix terms for the corresponding cases. Stiffness terms for the two-parameter foundation are presented here in both graphical form (Fig. 6) and in explicit list form (Appendix). The first entry of the stiffness matrix for a beam on a Winkler foundation when k_1 tends to zero is:

$$k_{11} = \frac{4EI\lambda^3(\sin 2L\lambda + \sinh 2L\lambda)}{-2+\cos 2L\lambda + \cosh 2L\lambda} \tag{22}$$

Eq. (22) tends to 12 EI/L^3 if k is allowed to vanish.

It is appropriate at this stage to observe the variation of the stiffness matrix terms as a function of the foundation parameters. We portray these variations graphically in Fig. 6 where the stiffness terms have been normalized with respect to the corresponding entry of the element stiffness matrix of an ordinary beam, i.e., the value which would have been obtained if the four Hermitian polynomials had been used in the first integral of Eq. (21). For example, the normalizing factor for k_{11} is $12EI/L_3$. The various frames of Fig. 6 display only distinct elements of the stiffness matrix coefficients. A general statement as to whether the first foundation parameter k or the second parameter k_1 govern a given stiffness matrix coefficient cannot be made because the interaction is very complex. We note that when the Winkler parameter vanishes (p=0) all normalized coefficients (whose order is given implicitly in Fig. 1) tend to unity regardless of the value of k_1, reflected by z.

4. Utilization of the shape functions for derivation of auxiliary quantities

Once we have been equipped with closed form expressions of the shape functions, we can extend our study to other areas such as work equivalent nodal loads for an arbitrarily distributed load $q(x)$:

$$\boldsymbol{P} = \int \boldsymbol{N}^T q(x)\, dx \tag{23}$$

The consistent mass matrix $\boldsymbol{m}$ for a uniform segment with mass per unit length μ is given by:

$$\boldsymbol{m} = \mu \int \boldsymbol{N}^T \boldsymbol{N}\, dx \tag{24}$$

and the consistent geometric stiffness matrix $\boldsymbol{k}_g$ for a constant compressive axial force of P is given by:

$$\boldsymbol{k}_g = \boldsymbol{P} \int \frac{d\boldsymbol{N}^T}{dx} \frac{d\boldsymbol{N}}{dx}\, dx \tag{25}$$

The terms in Eqs. (23)-(25) can also been obtained in closed form when the interpolation equations are derived from Eq. (10). We refrain, however, from presenting them in the interest of brevity in this paper.

5. Implementation

Expressions for all coefficients of the element matrices in Eqs. (21), (23)-(25) have been derived in closed form as in Appendix for the relative values of A and $2\sqrt{B}$, and inserted into the instructional software CAL-91 (Wilson 1991) although any other software for stress analysis can in principle be supplemented in this way. Its accuracy has been checked by means of numerous benchmark solutions of problems in stress analysis, vibration and stability.

When a finite-length beam is supported by a longer two-parameter foundation, special consideration needs to be given to force boundary conditions at the free ends because the shear force at a free end does not vanish unless the rotation there also vanishes. One way of avoiding incorrect boundary conditions is to introduce concentrated translational springs at such points of termination, and to assign them a stiffness equal to $\sqrt{kk_1}$, (Eisenberger and Bielak 1989). Another approach is to consider a virtual extension of the beam beyond its physical end, and to assign this extension very small section properties. The issue then becomes: how long should this extension be? If the problem domain with the beam segment is defined as $0 \leq x \leq L$ then for $x<0$ and $x>L$, Eq. (1) becomes:

$$-k_1 \frac{d^2 w}{dx^2} + k w_1 = 0 \tag{26}$$

where foundation deflection beyond the beam ends has been denoted by w_1. The solution of Eq. (26) is:

$$w_1(x) = c_1 e^{-\sqrt{\frac{k}{k_1}}x} + c_2 e^{\sqrt{\frac{k}{k_1}}x} \tag{27}$$

For $x \to -\infty$, $w_1 \to 0$ so that $c_1=0$, and for $x=0$ the condition $c_2=w_1(0)$ must hold. This implies that for $x<0$:

$$w_1(x) = w(0) e^{\sqrt{\frac{k}{k_1}}x} \tag{28}$$

A similar condition holds for $x > L$. In either case, the virtual length of the beam segment should be extended so that $w_1(x)$ becomes a small fraction of either $w(0)$ or $w(L)$. If this ratio is taken as 0.001, then the length of the foundation required beyond a free end can be calculated as $6.9\sqrt{k_1/k}$ as. A single element is sufficient for this representation.

As an example we consider a free-free beam segment resting successively on a one- or a two-parameter elastic foundation shown in Fig. 7 which has been the subject of many previous analyses (e.g., by Harr *et al.* 1969, Razaqpur and Shah 1991 and Chiwanga and Valsangkar 1988). The foundation beneath the beam terminates either abruptly at the ends of the 7.32 m long beam, for it is assumed to extend indefinitely beyond the ends in either direction. Four cases are considered in the present study. First, k_1 is assumed to be zero which ensures that the shear force at the ends of the segment also vanish for the Winkler foundation. Next k_1=1298.54 kN is assumed, which, given the other numerical information in Fig. 7, corresponds to the case $A<2\sqrt{B}$. This case is solved twice. First, it is assumed that the foundation terminates also at the beam ends, so that extensions in Fig. 7 do not exist. In this case the end shears vanish. When the two-parameter foundation does extend beyond the beam ends, the end shear does not vanish because rotations are not equal to zero at these points. In this case it is necessary to assume a virtual extension of the beam beyond its physical points of termination with a negligible flexural stiffness to obtain a better approximation of the deflected shape and internal forces. Finally k_1=14, 239.166 kN is assumed so that now the solution corresponding to $A>2\sqrt{B}$ must be utilized. In this case, the foundation does not extend beyond the beam ends. The number of elements utilized in the analyses was determined by the number of points where displacements were sought, so that the deflected shape (shown in Fig. 8) could be adequately reproduced.

The results in terms of deflections and moments are displayed in Figs. 8 and 9, respectively. It

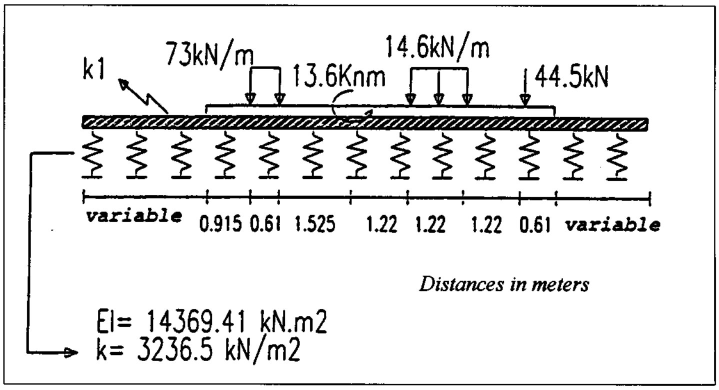

Fig. 7 Beam resting on two-parameter foundation

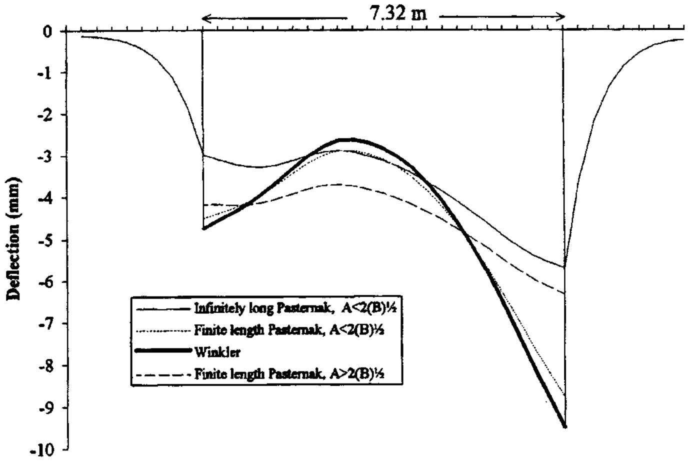

Fig. 8 Deflection profile

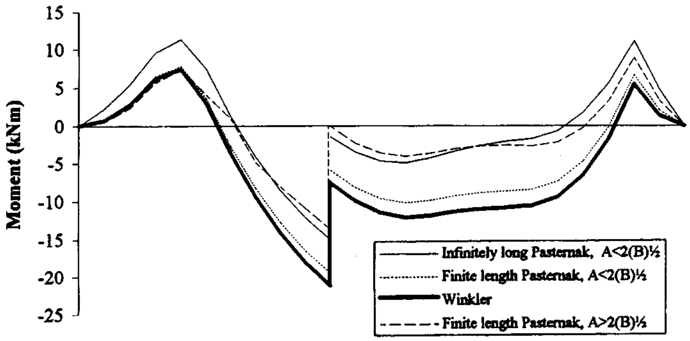

Fig. 9 Moment diagram

is noted that the Winkler foundation solution has the largest end displacements. The Pasternak foundation without extension exhibits a trend similar to the Winkler solution. When the foundation beyond the beam ends is considered, the deflections are substantially reduced. The finite-length Pasternak foundation for $A > 2\sqrt{B}$ represents an intermediate case.

The results shown in Figs. 8 and 9 match exactly, where meaningful, the published results cited earlier. The results for the case $A > 2\sqrt{B}$ are unique to this paper.

6. Conclusions

Study of beams on elastic foundation has been a fertile ground for many previous studies. The differential equation for a beam supported by a completely general two-parameter foundation can be simplified in successive stages to represent both a beam on a Winkler-type foundation and a transversely loaded beam supported by no foundation. Unsurprisingly, because of this continuity of the governing differential equation, its general solution masks the solutions for these more elementary cases. A general solution for the shape functions with no physical limits imposed on the magnitudes of the parameters characterizing the foundation has been shown to exhibit smooth transition across the numerical boundaries, and the solutions for stiffness matrix coefficients are also shown to blend continuously into one another across the mathematical limits governing the character of the solution. Three-dimensional plots of the interpolation functions in Fig. 4 display the effects of the foundation parameters.

Closed form expression of the shape functions enable the derivation of similar expressions for stiffness, equivalent nodal load, consistent mass or geometric stiffness matrix coefficients. These have been incorporated into an existing educational software.

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Notations

A	k_1 / EI
B	k / EI
$\boldsymbol{B}$	Column vector of transverse displacement functions
$\boldsymbol{C}$	Column vector containing the constants of integration c_i

c_i	Integration constant, i=1-4
$\boldsymbol{H}$	Matrix defined in Eq. (9)
EI	Flexural stiffness
$\boldsymbol{k}$	Beam element stiffness matrix
$\boldsymbol{k}_g$	Consistent geometric stiffness matrix
k	Winkler foundation parameter
k_1	Pasternak foundation parameter
L	Beam segment length
$\boldsymbol{m}$	Consistent beam mass matrix
$\boldsymbol{N}$	Row vector of shape functions N_i
$\boldsymbol{P}$	Vector of work equivalent nodal loads
P	Axial force
p	λL
$q(x)$	Transverse load
s	$\sqrt[4]{\frac{k}{EI}}$
U	Strain energy functional
$\boldsymbol{W}$	Column matrix of displacement boundary conditions
w	Transverse beam displacement
z	$\frac{\delta}{\lambda^2}$
α	$\sqrt{\lambda^2+\delta}=\frac{p}{L}\sqrt{1+z}$
β	$\sqrt{\lambda^2-\delta}$ or $\sqrt{\delta-\lambda^2}$ which correspond to $\frac{p}{L}\sqrt{1-z}$ or $\frac{p}{L}\sqrt{z-1}$, respectively
δ	$\frac{k_1}{4EI}$
λ	$\sqrt[4]{\frac{k}{4EI}}$
ξ	Dimensionless coordinate x/L

Appendix: Elements of the stiffness matrix

```
C (*  Stiffness Matrix A<2Sqrt(B)          *)
C (*  XL = Length of the Element            *)
C (*  k = Winkler Parameter                 *)
C (*  k1 = Pasternak Parameter              *)
C (*  EI = Flexural Rigidity of the Element *)

ALAM=(k/(4.0*EI))**(0.25)
AGAM=(k1/(4.0*EI))

STF11 = (4*EI*ALAM**2*(-AGAM**2 + ALAM**4)*
* ((AGAM + ALAM**2)**(0.5)*SIN(2*XL*(-AGAM + ALAM**2)**(0.5)) +
* (-AGAM + ALAM**2)**(0.5)*SINH(2*XL*(AGAM + ALAM**2)**(0.5))))/
* ((-AGAM + ALAM**2)**(0.5)*(AGAM + ALAM**2)**(0.5)*
* (-2*ALAM**2 + AGAM*COS(2*XL*(-AGAM + ALAM**2)**(0.5)) +
* ALAM**2*COS(2*XL*(-AGAM + ALAM**2)**(0.5)) -
* AGAM*COSH(2*XL*(AGAM + ALAM**2)**(0.5)) +
```

```
* ALAM**2*COSH(2*XL*(AGAM + ALAM**2)**(0.5))))

STF22 = (2*EI*(AGAM**2 - ALAM**4)*(-((AGAM + ALAM**2)**(0.5)*
* SIN(2*XL*(-AGAM + ALAM**2)**(0.5))) +
* (-AGAM + ALAM**2)**(0.5)*SINH(2*XL*(AGAM + ALAM**2)**(0.5))))/
* ((-AGAM + ALAM**2)**(0.5)*(AGAM + ALAM**2)**(0.5)*
* (2*ALAM**2 - AGAM*COS(2*XL*(-AGAM + ALAM**2)**(0.5)) -
* ALAM**2*COS(2*XL*(-AGAM + ALAM**2)**(0.5)) +
* AGAM*COSH(2*XL*(AGAM + ALAM**2)**(0.5)) -
* ALAM**2*COSH(2*XL*(AGAM + ALAM**2)**(0.5))))

STF12 = (2*EI*ALAM**2*(-2*AGAM + AGAM*COS(2*XL*
* (-AGAM + ALAM**2)**(0.5)) +
* ALAM**2*COS(2*XL*(-AGAM + ALAM**2)**(0.5)) +
* AGAM*COSH(2*XL*(AGAM + ALAM**2)**(0.5)) -
* ALAM**2*COSH(2*XL*(AGAM + ALAM**2)**(0.5))))/
* (2*ALAM**2 - AGAM*COS(2*XL*(-AGAM + ALAM**2)**(0.5)) -
* ALAM**2*COS(2*XL*(-AGAM + ALAM**2)**(0.5)) +
* AGAM*COSH(2*XL*(AGAM + ALAM**2)**(0.5)) -
* ALAM**2*COSH(2*XL*(AGAM + ALAM**2)**(0.5)))

STF24 = (4*EI*(AGAM - ALAM**2)*(AGAM + ALAM**2)**(0.5)*
* (-((AGAM + ALAM**2)**(0.5)*COSH(XL*(AGAM + ALAM**2)**(0.5))*
* SIN(XL*(-AGAM + ALAM**2)**(0.5))) +
* (-AGAM + ALAM**2)**(0.5)*COS(XL*(-AGAM + ALAM**2)**(0.5))*
* SINH(XL*(AGAM + ALAM**2)**(0.5))))/
* ((-AGAM + ALAM**2)**(0.5)*(-2*ALAM**2 +
* AGAM*COS(2*XL*(-AGAM + ALAM**2)**(0.5)) +
* ALAM**2*COS(2*XL*(-AGAM + ALAM**2)**(0.5)) -
* AGAM*COSH(2*XL*(AGAM + ALAM**2)**(0.5)) +
* ALAM**2*COSH(2*XL*(AGAM + ALAM**2)**(0.5))))

STF14 = (8*EI*ALAM**2*(-AGAM**2 + ALAM**4)*SIN(XL*
* (-AGAM + ALAM**2)**(0.5))*
* SINH(XL*(AGAM + ALAM**2)**(0.5)))/
* ((-AGAM + ALAM**2)**(0.5)*(AGAM + ALAM**2)**(0.5)*
* (-2*ALAM**2 + AGAM*COS(2*XL*(-AGAM + ALAM**2)**(0.5)) +
* ALAM**2*COS(2*XL*(-AGAM + ALAM**2)**(0.5)) -
* AGAM*COSH(2*XL*(AGAM + ALAM**2)**(0.5)) +
* ALAM**2*COSH(2*XL*(AGAM + ALAM**2)**(0.5))))

STF13 = (8*EI*ALAM**2*(-AGAM + ALAM**2)**(0.5)*
* ((AGAM + ALAM**2)**(1.5)*COSH(XL*(AGAM + ALAM**2)**(0.5))*
* SIN(XL*(-AGAM + ALAM**2)**(0.5)) +
* AGAM*(-AGAM + ALAM**2)**(0.5)*COS(XL*
* (-AGAM + ALAM**2)**(0.5))*
* SINH(XL*(AGAM + ALAM**2)**(0.5)) +
* ALAM**2*(-AGAM + ALAM**2)**(0.5)*COS(XL*
* (-AGAM + ALAM**2)**(0.5))*
* SINH(XL*(AGAM + ALAM**2)**(0.5))))/
* ((AGAM + ALAM**2)**(0.5)*(2*ALAM**2 - AGAM*COS(2*XL*
* (-AGAM + ALAM**2)**(0.5)) -
```

```
* ALAM**2*COS(2*XL*(-AGAM + ALAM**2)**(0.5)) +
* AGAM*COSH(2*XL*(AGAM + ALAM**2)**(0.5)) -
* ALAM**2*COSH(2*XL*(AGAM + ALAM**2)**(0.5))))

STF33=STF11
STF34=-STF12
STF23=-STF14
STF44=STF22→

C (*    Stiffness Matrix A=2*Sqrt(B)    *)
ALAM=(k/(4.0*EI))**(0.25)
E2=EXP(1.0)

STF11 = (4*EI*ALAM**3*(-2**(0.5) +
* 2**(0.5)*E2**(2**(2.5)*XL*ALAM) +
* 8*E2**(2**(1.5)*XL*ALAM)*XL*ALAM))/
* (1 - 2*E2**(2**(1.5)*XL*ALAM) + E2**(2**(2.5)*XL*ALAM) -
* 8*E2**(2**(1.5)*XL*ALAM)*XL**2*ALAM**2)

STF22 = (2*EI*ALAM*(2**(0.5) -
* 2**(0.5)*E2**(2**(2.5)*XL*ALAM) +
* 8*E2**(2**(1.5)*XL*ALAM)*XL*ALAM))/
* (-1 + 2*E2**(2**(1.5)*XL*ALAM) - E2**(2**(2.5)*XL*ALAM) +
* 8*E2**(2**(1.5)*XL*ALAM)*XL**2*ALAM**2)


STF12 = (2*EI*ALAM**2*(1 - 2*E2**(2**(1.5)*XL*ALAM) +
* E2**(2**(2.5)*XL*ALAM) +
* 8*E2**(2**(1.5)*XL*ALAM)*XL**2*ALAM**2))/
* (1 - 2*E2**(2**(1.5)*XL*ALAM) + E2**(2**(2.5)*XL*ALAM) -
* 8*E2**(2**(1.5)*XL*ALAM)*XL**2*ALAM**2)

STF24 = (4*E2**(2**(0.5)*XL*ALAM)*EI*ALAM*
* (2**(0.5) - 2**(0.5)*E2**(2**(1.5)*XL*ALAM) + 2*XL*ALAM +
* 2*E2**(2**(1.5)*XL*ALAM)*XL*ALAM))/
* (1 - 2*E2**(2**(1.5)*XL*ALAM) + E2**(2**(2.5)*XL*ALAM) -
* 8*E2**(2**(1.5)*XL*ALAM)*XL**2*ALAM**2)

STF14 = (2**(7/2)*E2**(2**(0.5)*XL*ALAM)*
* (-1 + E2**(2**(1.5)*XL*ALAM))*EI*XL*ALAM**3)/
* (1 - 2*E2**(2**(1.5)*XL*ALAM) + E2**(2**(2.5)*XL*ALAM) -
* 8*E2**(2**(1.5)*XL*ALAM)*XL**2*ALAM**2)

STF13 = (8*E2**(2**(0.5)*XL*ALAM)*EI*ALAM**3*
* (-2**(0.5) + 2**(0.5)*E2**(2**(1.5)*XL*ALAM) + 2*XL*ALAM +
* 2*E2**(2**(1.5)*XL*ALAM)*XL*ALAM))/
* (-1 + 2*E2**(2**(1.5)*XL*ALAM) - E2**(2**(2.5)*XL*ALAM) +
* 8*E2**(2**(1.5)*XL*ALAM)*XL**2*ALAM**2)
STF33=STF11
STF34=-STF12
STF23=-STF14
STF44=STF22→
```

```
C (*    Stiffness Matrix A>2*Sqrt(B)    *)

ALAM=(k/(4.0*EI))**(0.25)
AGAM=(k1/(4.0*EI))

STF11 = (2*EI*ALAM**2*(AGAM - ALAM**2)**(0.5)*
* (AGAM + ALAM**2)**(0.5)*
* ((AGAM + ALAM**2)**(0.5)*SINH(2*XL*(AGAM - ALAM**2)**(0.5)) +
* (AGAM - ALAM**2)**(0.5)*SINH(2*XL*(AGAM + ALAM**2)**(0.5))))/
* (-((AGAM + ALAM**2)*SINH(XL*(AGAM - ALAM**2)**(0.5))**2) +
* (AGAM - ALAM**2)*SINH(XL*(AGAM + ALAM**2)**(0.5))**2)

STF12 = -2*EI*(AGAM + ((AGAM - ALAM**2)*(AGAM + ALAM**2)*
* (COSH(XL*(AGAM + ALAM**2)**(0.5))**2*SINH(XL*
* (AGAM - ALAM**2)**(0.5))**2 -
* COSH(XL*(AGAM - ALAM**2)**(0.5))**2*SINH(XL*
* (AGAM + ALAM**2)**(0.5))**2))/
* (-((AGAM + ALAM**2)*SINH(XL*(AGAM - ALAM**2)**(0.5))**2) +
* (AGAM - ALAM**2)*SINH(XL*(AGAM + ALAM**2)**(0.5))**2))

STF13 = (-4*EI*ALAM**2*(AGAM - ALAM**2)**(0.5)*
* (AGAM + ALAM**2)**(0.5)*
* ((AGAM + ALAM**2)**(0.5)*COSH(XL*(AGAM + ALAM**2)**(0.5))*
* SINH(XL*(AGAM - ALAM**2)**(0.5)) +
* (AGAM - ALAM**2)**(0.5)*COSH(XL*(AGAM - ALAM**2)**(0.5))*
* SINH(XL*(AGAM + ALAM**2)**(0.5))))/
* (-((AGAM + ALAM**2)*SINH(XL*(AGAM - ALAM**2)**(0.5))**2) +
* (AGAM - ALAM**2)*SINH(XL*(AGAM + ALAM**2)**(0.5))**2)

STF14 = (4*EI*ALAM**2*(AGAM - ALAM**2)**(0.5)*
* (AGAM + ALAM**2)**(0.5)*
* SINH(XL*(AGAM - ALAM**2)**(0.5))*SINH(XL*
* (AGAM + ALAM**2)**(0.5)))/
* (-((AGAM + ALAM**2)*SINH(XL*(AGAM - ALAM**2)**(0.5))**2) +
* (AGAM - ALAM**2)*SINH(XL*(AGAM + ALAM**2)**(0.5))**2)

STF22 = (2*EI*(AGAM - ALAM**2)**(0.5)*(AGAM + ALAM**2)**(0.5)*
* (-((AGAM + ALAM**2)**(0.5)*COSH(XL*(AGAM - ALAM**2)**(0.5))*
* SINH(XL*(AGAM - ALAM**2)**(0.5))) +
* (AGAM - ALAM**2)**(0.5)*COSH(XL*(AGAM + ALAM**2)**(0.5))*
* SINH(XL*(AGAM + ALAM**2)**(0.5))))/
* (-((AGAM + ALAM**2)*SINH(XL*(AGAM - ALAM**2)**(0.5))**2) +
* (AGAM - ALAM**2)*SINH(XL*(AGAM + ALAM**2)**(0.5))**2)

STF24 = (2*EI*(AGAM - ALAM**2)**(0.5)*(AGAM + ALAM**2)**(0.5)*
* ((AGAM + ALAM**2)**(0.5)*COSH(XL*(AGAM + ALAM**2)**(0.5))*
* SINH(XL*(AGAM - ALAM**2)**(0.5)) -
* (AGAM - ALAM**2)**(0.5)*COSH(XL*(AGAM - ALAM**2)**(0.5))*
* SINH(XL*(AGAM + ALAM**2)**(0.5))))/
* (-((AGAM + ALAM**2)*SINH(XL*(AGAM - ALAM**2)**(0.5))**2) +
* (AGAM - ALAM**2)*SINH(XL*(AGAM + ALAM**2)**(0.5))**2)
```

```
STF33=STF11
STF34=-STF12
STF23=-STF14
STF44=STF22→
```