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Effective moment of inertia for rectangular elastoplastic beams

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Abstract. An effective moment of inertia is developed for a rectangular, prismatic elastoplastic beam with elastic, linear-hardening material behavior. The particular solution for a beam with elastic, perfectly plastic material behavior is also presented with applications for beam bending in closed-form. Equations are presented for the direct application of the virtual work method for elastoplastic beams with concentrated and distributed loads. Comparisons are made between the virtual work method deflections and the deflections obtained by using an average effective moment of inertia over two lengths of the beam in the elastoplastic region.

Key words: elastoplastic beams; effective moment of inertia; deflections; elastic, linear-hardening; elastic, perfectly plastic; virtual work.

1. Introduction

Over the years, the analysis of prismatic and non-prismatic beams with elastoplastic behavior has been studied extensively and investigated by many researchers. Explicit procedures, often referred to as plastic zone or distributed plasticity, have followed the gradual spread of yielding throughout the member. Plastification of members was modeled by discretization of members into several elements that included the effects of residual stresses, geometric imperfections, and strain-hardening. However, these procedures can be cumbersome and computationally intensive for general use. Consequently, engineers have often used simplified plastic-hinge models where the plasticity is concentrated at the hinge.

Inelastic analysis procedures are necessary for predicting the behavior of structural members stressed above the yield limit of the material and for determining displacements in both the elastic and plastic zones of the member. Most of the current inelastic procedures have addressed the reduced rigidity of the member by making use of a reduced modulus of elasticity, commonly referred to as the tangent modulus of elasticity, E_t.

Fertis, Taneja, and Lee (1991) and Fertis and Taneja (1991) showed that elastic and inelastic analysis of members with both constant and variable rigidity could be approximated using equivalent linear systems and applying known methods of linear elementary mechanics. For inelastic members, an expression was developed for a reduced modulus of elasticity, which was

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later solved for by using substitution of variables and trial-and-error procedures. The authors concluded that members could be analyzed for both elastic and inelastic response, all the way to failure, thus permitting the observation of progressive deterioration of a member's ability to resist load and deformation.

Fertis and Lee (1991) extended the method of equivalent linear systems to include the analysis of elastic and inelastic members with axial restraints resulting from both compressive and tensile forces, and the solution of statically indeterminate problems was also presented. Fertis and Schubert (1994) later showed that the method of equivalent linear systems for the analysis of elastic and inelastic members could be applied to aluminum members as well.

Liew, White, and Chen (1993) refined an existing plastic-hinge approach for the second-order analysis of beam-to-column connections to include the degradation of the element stiffness as the second-order forces at critical locations approach the cross-section plastic strength. In addition, a column-tangent modulus expression was used to represent the effective stiffness of the element with large axial loads. This procedure, without considering the effects of strain-hardening, approximated the effects of distributed plasticity along the element length caused by residual stresses as well as large bending moments and axial forces.

Romano, Ganduscio, and Zingone (1993) provided analytical solutions for beam deflections beyond the elastic limit for both prismatic and nonprismatic beams and for various support and loading conditions. The analytical solutions provide by differentiation the rotation, curvature, moment, and shear force. This procedure requires the determination of several constants of integration using the boundary conditions on the solution and its derivatives.

This paper describes a new procedure that reduces the rigidity of a beam stressed beyond the elastic limit of the material by using an effective moment of inertia in the elastoplastic regions. Closed-form equations are given for determining the effective moment of inertia for rectangular, prismatic beams having elastic, linear-hardening material behavior. The particular condition of a rectangular beam with elastic, perfectly plastic material behavior is further developed for predicting deflections that are "exact" and closed-form. This relationship to reduce the rigidity in the elastoplastic regions provides the means by which a solution can be found using classical methods of linear analysis that do not require the use of equivalent systems as presented by Fertis *et al.* (1991, 1994) or complex integrations as shown by Romano *et al.* (1993). Example problems are provided that illustrate the ease with which the effective moment of inertia can be used to predict elastoplastic beam deflections. This paper also presents a means by which the linear direct stiffness method can be used to approximate beam deflections by using an average moment of inertia over the element lengths in the elastoplastic regions.

2. Beam bending

2.1. Rectangular prismatic elastoplastic beams

The second-order differential equation that is routinely used for the determination of the curvature and deflection of elastic beams can be used for the same purpose for elastoplastic beams provided the proper reduction in rigidity is made for moments beyond the specified yield moment. This can be done by modifying the modulus of elasticity with the use of a tangent modulus, E_t, or by using a reduced moment of inertia, which in this paper will be referred to as the effective moment of inertia, I_{eff}. Eq. (1) shows the moment-curvature relationship with the effective moment of inertia for elastoplastic moments, M_{ep}, above the yield moment, M_y.

$$\frac{d^2y}{dx^2} = \frac{M_{ep}}{EI_{eff}} \tag{1}$$

It is assumed that the elastoplastic beam has small slopes and deflections with respect to the beam depth, and that shear deformations are negligible, the longitudinal beam dimensions are large compared to the dimensions of the cross-section resulting in small shear stresses compared to longitudinal stresses, and that plane sections remain plane after bending.

This paper is concerned with the development of I_{eff} for elastoplastic beams with elastic, linear-hardening and elastic, perfectly plastic material behavior. The stress-strain behavior of the elastic, linear-hardening material is illustrated in Fig. 1(a), where the σ_y is the stress at yield, ε_y is the strain at yield, σ_{ep} is the stress at some level above yield, and ε_{ep} is the strain corresponding to σ_{ep}. The increase in stress above yield is based on the strain-hardening coefficient, α, where $0 < \alpha < 1$. The elastic, perfectly plastic material behavior is represented by letting α=0 in Fig. 1(a).

When a beam is in flexure with an elastic-plastic material, the bending can be classified into two general cases. Case I involves elastic bending of a beam where stresses remain below or equal to the elastic limit of the material. For rectangular sections, the maximum elastic bending moment, M_y, is expressed by Eq. (2) where h is the depth and I is the moment of inertia of the rectangular cross-section.

$$M_y = \left(\frac{2\sigma_y}{h}\right)I \tag{2}$$

Case II involves elastoplastic bending of a beam, or bending in which the normal stresses throughout the beam depth are above and below the elastic limit of the material, as shown in Fig. 1(b). The elastoplastic bending moment, M_{ep}, is the moment that causes the rectangular beam section to have normal stresses that go beyond the yield stress of the material. The elastoplastic bending moment is determined by considering the internal couple consisting of forces C_1, C_2, T_1, and T_2 multiplied by their respective distance from the neutral axis. Since C_1=T_1 and C_2=T_2 for the

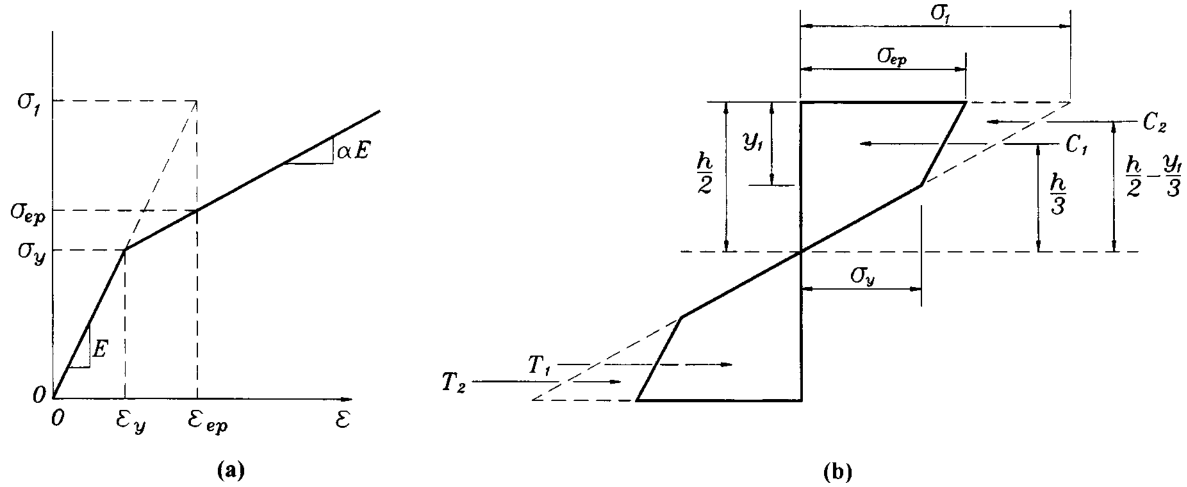


Fig. 1 (a) Elastic, linear-hardening material behavior, (b) Elastoplastic stress distribution rectangular cross-section, then

$$M_{ep} = 2\left[C_1\frac{h}{3} - C_2\left(\frac{h}{2} - \frac{y_1}{3}\right)\right] \tag{3}$$

where y_1 is the distance from the extreme fiber to the yield stress, σ_y, Eq. (3) can be expanded and rewritten into the form

$$M_{ep} = 2\left[\left(\frac{1}{2}\, b\, \frac{h}{2}\sigma_1\right)\frac{h}{3} - \left(\frac{1}{2} b y_1(\sigma_1 - \sigma_{ep})\right)\left(\frac{h}{2} - \frac{y_1}{3}\right)\right] \tag{4}$$

where b is the width of the beam. The elastoplastic normal stress, σ_{ep}, is given by Eq. (5), where ε_{ep} is the normal strain as shown in Fig. 1(a).

$$\sigma_{ep} = \sigma_y + \alpha E(\varepsilon_{ep} - \varepsilon_y) \tag{5}$$

The normal stress σ_1 is the stress that would develop at the strain ε_{ep} if the material remained elastic and is given by the expression

$$\sigma_1 = \left(\frac{h}{h - 2y_1}\right)\sigma_y \tag{6}$$

Using $\varepsilon_{ep}=\sigma_1/E$ and $\varepsilon_y=\sigma_y/E$, and substituting Eq. (6) into Eq. (5), then

$$\sigma_{ep} = \sigma_y\left[1 + \alpha\frac{2y_1}{h - 2y_1}\right] \tag{7}$$

The substitution of Eqs. (6) and (7) into Eq. (4) allows the expression for M_{ep} to be further developed as

$$M_{ep} = b\,\sigma_y\left[\frac{h^2}{6}\left(\frac{h}{h - 2y_1}\right) - 2y_1^2\left(\frac{1-\alpha}{h - 2y_1}\right)\left(\frac{h}{2} - \frac{y_1}{3}\right)\right] \tag{8}$$

2.2. Effective moment of inertia

Eq. (8) is written in the form given in Eq. (9) so that the new, effective moment of inertia can be determined. This equation is written in a form similar to that found in Eq. (2) so that the term in brackets is shown to be I_{eff} as given in Eq. (10).

$$M_{ep} = \left(\frac{2\sigma_y}{h}\right)\left[\frac{bh^3}{12}\left(\frac{h}{h - 2y_1}\right) - y_1^2 bh\left(\frac{1-\alpha}{h - 2y_1}\right)\left(\frac{h}{2} - \frac{y_1}{3}\right)\right] \tag{9}$$

$$I_{eff} = \frac{bh^3}{12}\left(\frac{h}{h - 2y_1}\right) - y_1^2 bh\left(\frac{1-\alpha}{h - 2y_1}\right)\left(\frac{h}{2} - \frac{y_1}{3}\right) \tag{10}$$

The bending moment, M_1, that would develop if the material remained elastic up to the stress level σ_1 is given as

$$M_1=\frac{bh^2}{6}\sigma_1 \tag{11}$$

and can be written in terms of σ_y and the unknown y_1 using Eq. (6) as

$$M_1=\frac{bh^3}{6(h-2y_1)}\sigma_y \tag{12}$$

Eq. (12) can also be written in a form similar to that found in Eq. (9) and, again, it is evident that the term in brackets is simply I, because by the definition of M_1, the material is elastic at σ_1 and y_1=0.

$$M_1=\left(\frac{2\sigma_y}{h}\right)\left[\frac{bh^3}{12}\frac{h}{(h-2y_1)}\right] \tag{13}$$

$$I_1=I=\frac{bh^3}{12}\frac{h}{(h-2y_1)} \tag{14}$$

Eqs. (2) and (8) are used to form Eq. (15), and Eqs. (10) and (14) are used to create Eq. (16).

$$\frac{M_{ep}}{M_y}=\frac{6}{h^2}\left[\frac{h^2}{6}\left(\frac{h}{h-2y_1}\right)-\frac{y_1^2}{3}\left(\frac{1-\alpha}{h-2y_1}\right)(3h-2y_1)\right] \tag{15}$$

$$\frac{I_{eff}}{I}=\frac{6}{h^3}(h-2y_1)\left[\frac{h^2}{6}\left(\frac{h}{h-2y_1}\right)-\frac{y_1^2}{3}\left(\frac{1-\alpha}{h-2y_1}\right)(3h-2y_1)\right] \tag{16}$$

Since the terms in brackets in Eqs. (15) and (16) are identical, a relationship between M_{ep}/M_y and I_{eff}/I can be found to be

$$\frac{M_{ep}}{M_y}=\frac{h}{(h-2y_1)}\frac{I_{eff}}{I} \tag{17}$$

It is noticed that the relationship between M_{ep}/M_y and I_{eff}/I is independent of α, and the form of the equation is similar to that found in Eq. (6). Eq. (17) is used to develop an expression for y_1 in terms of the known quantities h, I, M_{ep} and M_y, with I_{eff} being the only other unknown.

$$y_1=\frac{h}{2}\left(1-\frac{I_{eff}}{I}\frac{M_y}{M_{ep}}\right) \tag{18}$$

When y_1=0 in Eq. (18), then $M_{ep}/M_y=I_{eff}/I$ as expected for a completely elastic condition. When y_1= $h/2$, then either the ratio I_{eff}/I=0 or M_y/M_{ep}=0 (which is a trivial solution for non-zero M_y). For the case when I_{eff}/I=0, the beam is fully plastic and, as shown in Fig. 2, can only occur for an elastic, perfectly plastic material (α=0).

When Eq. (18) is substituted into either Eq. (15) or (16) to yield Eq. (19) a cubic equation is found in terms of the dependent variable I_{eff}/I and the independent variables M_{ep}/M_y and α.

$$\left[\frac{1}{2\left(\frac{M_{ep}}{M_y}\right)^3}(1-\alpha)\right]\left(\frac{I_{eff}}{I}\right)^3-\left[\frac{3}{2\left(\frac{M_{ep}}{M_y}\right)}(1-\alpha)-1\right]\left(\frac{I_{eff}}{I}\right)-\alpha=0 \tag{19}$$

Plots of this equation are shown in Fig. 2 where it is noticed that for any $\alpha>0$, and as M_{ep}/M_y approaches infinity, the ratio of I_{eff}/I approaches α. Also, the traditionally used tangent modulus of elasticity, E_t, can be substituted into Eq. (19), because it is recognized that $E_t/E=I_{eff}/I$.

Eq. (19) can be shown to have two solutions that have real roots and are given by Eqs. (20) and (21). Eq. (21) is used to find I_{eff} when $\alpha<1-(2/3)M_{ep}/M_y$ for $1<M_{ep}/M_y<1.5$; for all other conditions of α and M_{ep}/M_y Eq. (20) is used to calculate I_{eff}.

$$I_{eff}=\left[\sqrt[3]{-\frac{b}{2}+\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}}-\sqrt[3]{\frac{b}{2}+\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}}\right]I \tag{20}$$

$$I_{eff}=\left[\sqrt[3]{-\frac{b}{2}+\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}}+\sqrt[3]{-\frac{b}{2}-\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}}\right]I \tag{21}$$

where a and b are given as

$$a=\frac{2}{(1-\alpha)}\left(\frac{M_{ep}}{M_y}\right)^3-3\left(\frac{M_{ep}}{M_y}\right)^2 \tag{22}$$

$$b=-\frac{2\alpha}{(1-\alpha)}\left(\frac{M_{ep}}{M_y}\right)^3 \tag{23}$$

A particular solution is found for $\alpha=0$ in Eq. (19), which represents an elastic, perfectly plastic material behavior. Eq. (24) is the solution for this case and, because of its relative simplicity, it is

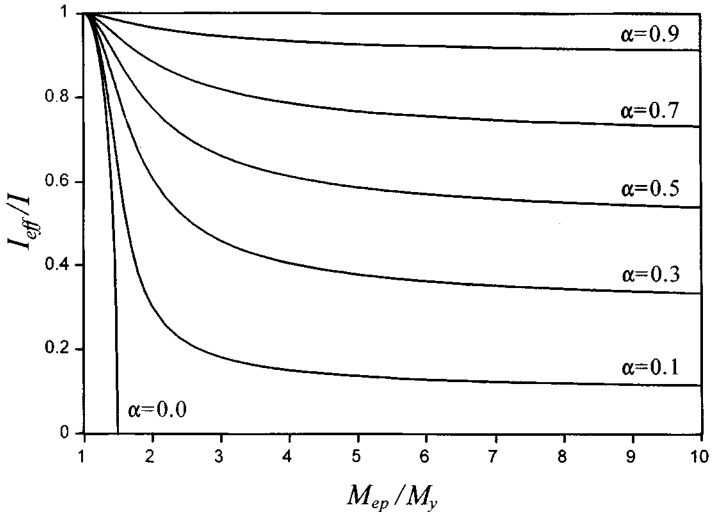


Fig. 2 Relationship between I_{eff}/I and M_{ep}/M_y for various α

particularly well suited to solutions in closed-form of elasto-perfectly plastic beam flexure.

$$I_{eff} = \left[\frac{M_{ep}}{M_y} \sqrt{3 - 2\frac{M_{ep}}{M_y}} \right] I \tag{24}$$

This equation will now be used with the moment-curvature relationship in Eq. (1) to determine the deflections in closed-form of rectangular, prismatic beams with elastic, perfectly plastic material behavior.

3. Virtual work applications

With the I_{eff} equation for the elastic, perfectly plastic case being of a simple form, its use with the customary methods of linear structural analysis can be achieved with relative ease. The effective moment of inertia can be used with any commonly used elastic beam deflection method; however, the virtual work method was selected to illustrate the use of I_{eff} in this paper. The virtual work expression in the elastoplastic range is given as

$$\delta W = \int_0^{L_{ep}} \frac{M_{ep}}{EI_{eff}} \delta M \, dx \tag{25}$$

where L_{ep} is the length of the elastoplastic zone, M_{ep} is the bending moment in the elastoplastic range due to the real loading and δM is the virtual moment due to the virtual load. Eq. (25) can also be expressed as Eq. (26) where the displacement is equal to the summation of the M_{ep}/EI_{eff} area, given as A_{ep}, and the virtual moment, $\overline{\delta M}$, at the centroid of each A_{ep}.

$$\delta W = 1(\Delta) = \sum_{i=1}^{n} A_{ep_i} \overline{\delta M_i} \tag{26}$$

The area A_{ep} is found by integrating M_{ep}/EI_{eff} between the limits of 0 and L_{ep}, given by the expression

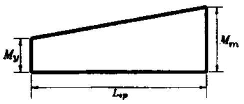
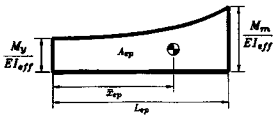
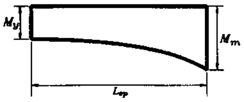
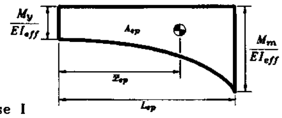
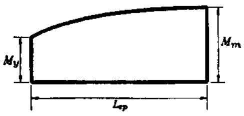
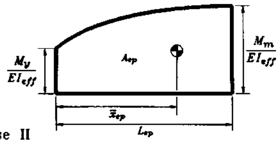
$$A_{ep} = \int_0^{L_{ep}} \frac{M_{ep}}{EI_{eff}} dx \tag{27}$$

The centroid of A_{ep}, referred to as $\bar{x}_{ep}$, is found by integrating the product of M_{ep}/EI_{eff} and x between 0 and L_{ep} and dividing by A_{ep}, given as

$$\bar{x}_{ep} = \frac{\int_0^{L_{ep}} \left(\frac{M_{ep}}{EI_{eff}} \right) x \, dx}{A_{ep}} \tag{28}$$

In order for the virtual work method to be practical and convenient when determining elastoplastic beam deflections, the solutions to Eqs. (27) and (28) have been developed for concentrated and distributed loading conditions. Three conditions of moment variation must be considered for these two loading conditions; the concentrated loads produce linear moments, and the distributed loads produce negative and positive quadratic moments. The three moment variation conditions are presented in Table 1.

Table 1 Area and centroid formulas for linear and quadratic moments in the elastoplastic region

Moment	$\dfrac{M}{EI_{eff}}$	Area Aep	Centroid $\bar{x}_{ep}$
Moment  Linear Moment Region	$\dfrac{M}{EI_{eff}}$ 	$A_{ep}=\dfrac{L_{ep}M_y^2(1-K_0)}{(M_m-M_y)EI}$	$\bar{x}_{ep}=\dfrac{L_{ep}(M_y-K_0M_m)}{3(1-K_0)(M_m-M_y)}$
 Quadratic Moment Region: Case I		$A_{ep}=\dfrac{\sqrt{2}}{2}\dfrac{M_y(K_2-K_3)}{K_1EI}$	$\bar{x}_{ep}=\dfrac{\sqrt{2}}{4}\dfrac{L_{ep}^2[2K_1M_y(K_0-1)+\sqrt{2}V_n(K_2-K_3)]}{(M_y-M_m+V_nL_{ep})(K_2-K_3)}$
 Quadratic Moment Region: Case II		$A_{ep}=\dfrac{\sqrt{2}}{4}\dfrac{M_yK_6}{K_5EI}$	$\bar{x}_{ep}=\dfrac{\sqrt{2}}{2}\dfrac{L_{ep}[-2L_{ep}M_yK_5(1-K_0)+\sqrt{2}K_6(M_m-M_y)]}{K_6(M_m-M_y)}$

Constants: $K_0=\sqrt{3-2\dfrac{M_m}{M_y}}$, $K_1=\dfrac{1}{L_{ep}}\sqrt{1-\dfrac{M_m}{M_y}+\dfrac{V_nL_{ep}}{M_y}}$,

$K_2=\ln[K_4(M_y-M_m)+\sqrt{2}K_1V_nL_{ep}^2+2K_0(M_y-M_m+V_nL_{ep})]$,

$K_3=\ln[2(M_y-M_m+V_nL_{ep})-\sqrt{2}K_1V_nL_{ep}^2]$,

$K_4=2\sqrt{2}K_1L_{ep}$, $K_5=\dfrac{1}{L_{ep}}\sqrt{\dfrac{M_m}{M_y}-1}$, $K_6=K_8-\ln(M_y)-2K_7$, $K_7=\ln(1-\sqrt{2}K_5L_{ep})$,

$K_8=\ln(3M_y-2M_m)$

3.1. Linear moment variation

A linear varying moment in the elastoplastic region is given as

$$M_{ep}(x)=M_y+\left(\frac{M_m-M_y}{L_{ep}}\right)x \tag{29}$$

where M_m is the maximum bending moment in the elastoplastic region. The area A_{ep} for this condition is calculated by substituting Eqs. (24) and (29) into (27) and integrating the expression between 0 and L_{ep}. The solution of the integral is

$$A_{ep}=\frac{L_{ep}M_y^2(1-K_0)}{EI\,(M_m-M_y)} \tag{30}$$

where the constant K_0 is given as

$$K_0=\sqrt{3-2\frac{M_m}{M_y}} \tag{31}$$

The centroid of area, $\bar{x}_{ep}$, is found by substituting Eqs. (24) and (29) into (28) and integrating between 0 and L_{ep}, then dividing by A_{ep} from Eq. (30). The solution is given as

$$\bar{x}_{ep} = \frac{L_{ep}\,(M_y - K_0 M_m)}{3\,(1-K_0)\,(M_m - M_y)} \tag{32}$$

3.2. Quadratic moment region - Case I

A negative quadratic moment in the elastoplastic region is given as

$$M_{ep}(x) = M_y + V_n x + \left(\frac{M_m - M_y}{L_{ep}^2} - \frac{V_n}{L_{ep}}\right) x^2 \tag{33}$$

where V_n is the shear force at the location of M_y. The area A_{ep} is calculated by substituting Eqs. (24) and (33) into (27) and integrating between 0 and L_{ep}. The solution of the integral is

$$A_{ep} = \frac{\sqrt{2}}{2}\,\frac{M_y\,(K_2 - K_3)}{EI\,K_1} \tag{34}$$

where the constants K_1 through K_3 are given as

$$K_1 = \frac{1}{L_{ep}}\sqrt{1 - \frac{M_m}{M_y} + \frac{V_n L_{ep}}{M_y}} \tag{35}$$

$$K_2 = \ln\left[K_4(M_y - M_m) + \sqrt{2}K_1 V_n L_{ep}^2 + 2K_0(M_y - M_m + V_n L_{ep})\right] \tag{36}$$

$$K_3 = \ln\left[2(M_y - M_m + V_n L_{ep}) - \sqrt{2}K_1 V_n L_{ep}^2\right] \tag{37}$$

and the constant K_4 in Eq. (36) is given as

$$K_4 = 2\sqrt{2}K_1 L_{ep} \tag{38}$$

The centroid of area, $\bar{x}_{ep}$, is found by substituting Eqs. (24) and (33) into (28) and integrating between 0 and L_{ep}, then dividing by A_{ep} from Eq. (34). The solution is given as

$$\bar{x}_{ep} = \frac{\sqrt{2}}{4}\,\frac{L_{ep}^2\left[2K_1 M_y (K_0 - 1) + \sqrt{2}V_n (K_2 - K_3)\right]}{(M_y - M_m + V_n L_{ep})(K_2 - K_3)} \tag{39}$$

3.3. Quadratic moment region - Case II

A positive quadratic moment in the elastoplastic region is given as

$$M_{ep}(x) = M_y + 2\left(\frac{M_m - M_y}{L_{ep}}\right)x + \left(\frac{M_y - M_m}{L_{ep}^2}\right)x^2 \tag{40}$$

where a shear force is not needed for this expression because it is assumed the shear force is zero at M_m. Again the area A_{ep} is calculated by substituting Eqs. (24) and (40) into (27) and integrating between 0 and L_{ep}. The solution of the integral is given as

$$A_{ep} = \frac{\sqrt{2}}{4}\frac{M_y K_6}{EI\,K_5} \tag{41}$$

where the constant K_5 and K_6 are given as

$$K_5 = \frac{1}{L_{ep}}\sqrt{\frac{M_m}{M_y} - 1} \tag{42}$$

$$K_6 = K_8 - \ln(M_y) - 2K_7 \tag{43}$$

and the constants K_7 and K_8 in Eq. (43) are given as

$$K_7 = \ln\left(1 - \sqrt{2}K_5 L_{ep}\right) \tag{44}$$

$$K_8 = \ln\left(3M_y - 2M_m\right) \tag{45}$$

The centroid of area, $\bar{x}_{ep}$, is found by substituting Eqs. (24) and (40) into (28) and integrating between 0 and L_{ep}, then dividing by A_{ep} from Eq. (41). Eq. (48) gives the solution of the integral in Eq. (29) where the constants K_5 through K_8 are given by Eq. (42) through Eq. (45).

$$\bar{x}_{ep} = \frac{\sqrt{2}}{2}\frac{L_{ep}\left[-2L_{ep}M_y K_5(1-K_0) + \sqrt{2}K_6(M_m - M_y)\right]}{K_6(M_m - M_y)} \tag{46}$$

The results for A_{ep} and $\bar{x}_{ep}$ for all three moment variation conditions are summarized in Table 1.

4. Applied results

Two example problems are presented using the effective moment of inertia, I_{eff}, for the determination of elastoplastic beam deflections. The first example problem consists of a cantilevered beam with a concentrated load at the free end. Three solutions for bending of the cantilevered beam are presented using the virtual work method, the direct integration method, and an approximate direct stiffness method using average values of I_{eff} in the elastoplastic region. The second example problem consists of a continuous, statically determinate beam with concentrated and distributed loads. Two solutions are provided for the continuous beam using the virtual work method and the approximate direct stiffness method.

4.1. Cantilevered beam example

The cantilevered beam, as shown in Fig. 3, consists of a beam of length L, whereby the elastic region of the beam is L_1 and the elastoplastic region is L_{ep}. The concentrated load P at the left end of the beam is greater than the load necessary to cause yielding.

4.1.1. Virtual work method

The virtual work expressions developed for a linear varying moment are used to determine the

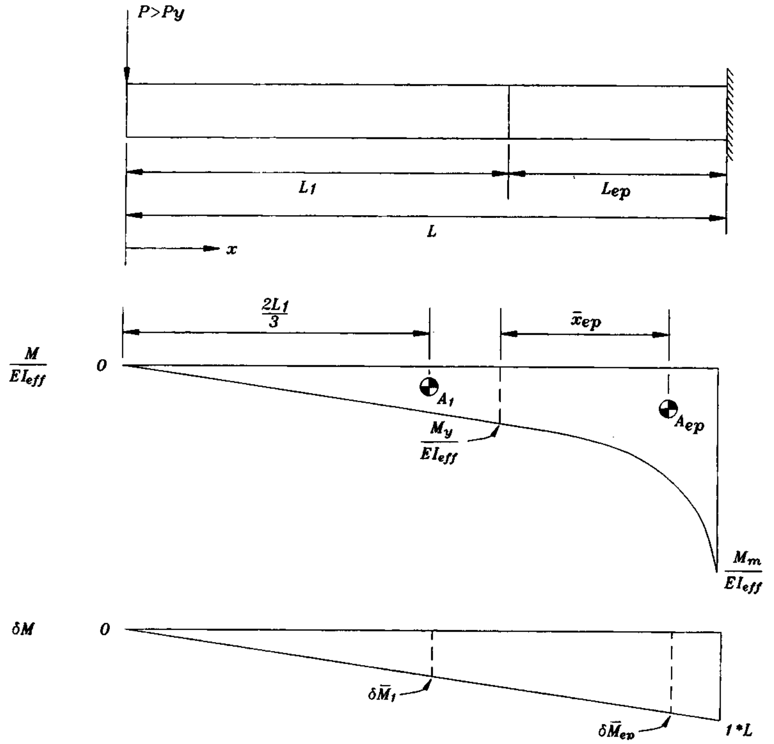


Fig. 3 Cantilevered beam example problem

elastoplastic beam deflection at the left-end. As shown in Fig. 3, the area under the elastic region of the curvature diagram is denoted as A_1 and the area under the elastoplastic region as A_{ep}. The corresponding virtual moments at the centroids of the areas, δM_1 and δM_{ep} are also shown. The following equation is all that is needed to find the deflection at the left-end of the beam.

$$y_{x=0}=A_1\,\delta\overline{M}_1+A_{ep}\,\delta\overline{M}_{ep}=A_1\left(\frac{2L_1}{3}\right)+A_{ep}\left(L_1+\overline{x}_{ep}\right) \tag{47}$$

where A_{ep} and $\overline{x}_{ep}$ are found by using Eqs. (30) and (32), respectively. Substitution into Eq. (47), and after some simplification, yields the left-end beam deflection equation as

$$y_{x=0}=\frac{PL_1^3}{3EI}\left[5-\left(3+\frac{L}{L_1}\right)\sqrt{3-2\frac{L}{L_1}}\right] \tag{48}$$

If P=2624.45 kN, L=2413 mm, M_y=4.2369×10^6 kN-mm, E=199.948 kN/mm^2, and I=5.4197×10^9 mm^4, then L_1=M_y/P=1614.41 mm and substitution into Eq. (48) yields the elastoplastic deflection at the left-end to be 15.41 mm.

4.1.2. Direct integration method

The direct integration method for elastic beams involves writing an expression for M/EI as a function of x along the beam. The expression is then integrated twice to obtain the expressions for the deflections of the elastic curve, and the constants of integration are determined from the boundary conditions. For the elastoplastic cantilever beam, the integrations must be conducted over the elastic and elastoplastic regions of the beam. This introduces two additional constants of integration that are evaluated using the conditions of continuity. Thus, expressions for the slope and deflection must be developed for both the elastic and elastoplastic regions.

Elastoplastic region

Expressions for the slope and deflection of the elastoplastic region of the cantilever beam are determined by integrating Eq. (1). The expression for M_{ep} used in Eq. (1) is $M_y+P(x-L_1)$ for $L_1 \leq x \leq L$, where $M_y=PL_1$. The equation for the effective moment of inertia is found by substituting the equations for M_{ep} and M_y into Eq. (24). The slope of the beam in the elastoplastic region is

$$\frac{dy}{dx_{ep-p}} = \frac{PL_1^2}{EI}\left[-\sqrt{3-2\frac{x}{L_1}}+\sqrt{3-2\frac{L}{L_1}}\right] \tag{49}$$

where the constant of integration is found using $dy/dx_{ep\text{-}p}=0$ at $x=L$. The deflection of the elastoplastic region is obtained by integrating Eq. (49) and is given as

$$y_{ep-p} = \frac{PL_1^2}{EI}\left[\frac{L_1}{3}\left(3-2\frac{x}{L_1}\right)^{3/2}+x\sqrt{3-2\frac{L}{L_1}}-\frac{L_1}{3}\left(3-2\frac{L}{L_1}\right)^{3/2}-L\sqrt{3-2\frac{L}{L_1}}\right] \tag{50}$$

where the constant of integration is found using $y_{ep\text{-}p}=0$ at $x=L$.

Elastic region

Expressions for the slope and deflection of the elastic region of the cantilever beam are determined by integrating the elastic moment-curvature relationship. The slope of the beam in the elastic region is given as

$$\frac{dy}{dx_{ep-e}} = \frac{P}{EI}\left[\frac{x^2}{2}-\frac{3L_1^2}{2}+L_1^2\sqrt{3-2\frac{L}{L_1}}\right] \tag{51}$$

where the constant of integration is determined using $dy/dx_{ep\text{-}p}=dy/dx_{ep\text{-}e}$ at $x=L_1$. The equation for the deflection in the elastic region is given as

$$y_{ep-e} = \frac{P}{EI}\left[\frac{x^3}{6}+xL_1^2\left(-\frac{3}{2}+\sqrt{3-2\frac{L}{L_1}}\right)+\frac{5L_1^3}{3}-\frac{L_1^3}{3}\left(3-2\frac{L}{L_1}\right)^{3/2}-L_1^2L\sqrt{3-2\frac{L}{L_1}}\right] \tag{52}$$

where the constant of integration is determined using $y_{ep\text{-}p}=y_{ep\text{-}e}$ at $x=L_1$. If the same constants used in 4.1.1 are substituted in Eq. (52), the deflection at the left-end is identical to that found using the virtual work method.

4.1.3. Approximate direct stiffness method

The direct stiffness method was chosen to show the ease with which the effective moment of inertia can be used with a linear stiffness matrix to approximate the bending of elastoplastic beams. A model of the cantilever beam is shown in Fig. 4, where two beam elements are used in

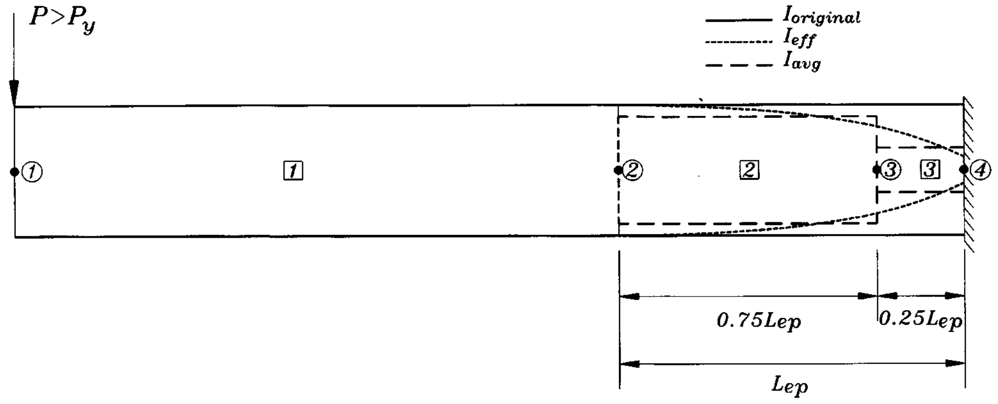


Fig. 4 Model of cantilevered beam

the elastoplastic region and one element is used in the elastic region. The elastoplastic region is divided into two, unequal length elements consisting of 25% and 75% of L_{ep}, where the shortest element is closest to the maximum moment. For this example problem, analyses with various element lengths showed that a 1:3 proportion produced results that were the closest to the "exact" solutions. Since the beam is determinate, the bending moments above yield are known *a priori*, and the effective moment of inertia can be determined at each node in the elastoplastic region. The effective moments of inertia at each end of the elements in the elastoplastic region are then averaged and used as a constant I_{avg} in the linear stiffness matrix. Fig. 4 shows the reduction in the moment of inertia for the two elements in the elastoplastic region.

If this procedure is applied to the cantilevered beam example, using the same constants as in 4.1.1, the end deflection is found to be 15.80 mm. This results in a 2.53 percent error when compared with the "exact" solution of 15.41 mm. Fig. 5 shows the relative percent error of the end deflections from the onset of yielding (M_{ep}/M_y=1) up to full plasticity at the fixed-end (M_{ep}/M_y

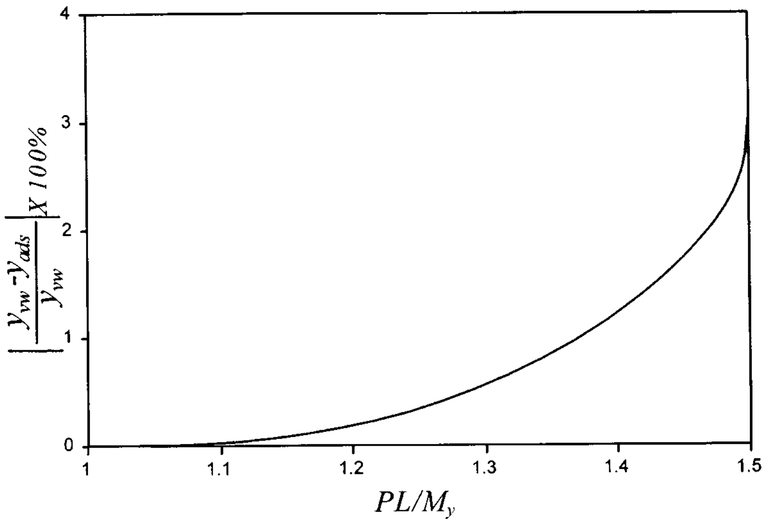


Fig. 5 Relative percent error of the cantilevered beam end deflections

=1.5). The relative error increases as the load is increased, and the maximum error is 3.03 percent just before full plasticity is reached. From this example, it is evident that the average effective moment of inertia method overestimates the end deflection, however the relative error is reasonable for most practical purposes.

4.2. Determinate continuous beam example

Elastoplastic beam deflections are also found for a statically determinate, continuous beam with both concentrated and distributed loads, as shown in Fig. 6. The shear and moment diagrams for the continuous beam example are not shown for brevity, however they can be found by simple structural analysis because the structure is determinate. The curvature diagram, M/EI_{eff}, is shown in Fig. 6, where it is noticed that the diagram is divided into 14 areas, with the six shaded areas representing the elastoplastic regions. Two solutions are provided for the continuous beam example using the virtual work method and the approximate direct stiffness method. The virtual work method was used to determine the deflections at various locations along the length of the beam so that the versatility of the method could be shown for a more complex problem and the deflections could be used for comparison with the approximate direct stiffness method results.

4.2.1. Virtual work method

The deflections of the beam are found by using Eq. (26) and the A_{ep} and $\bar{x}_{ep}$ equations given in Table 1 for all three moment conditions. The area and location of the centroid for each of the 14 regions in Fig. 6 are first calculated, then the virtual moments are determined at the centroid of

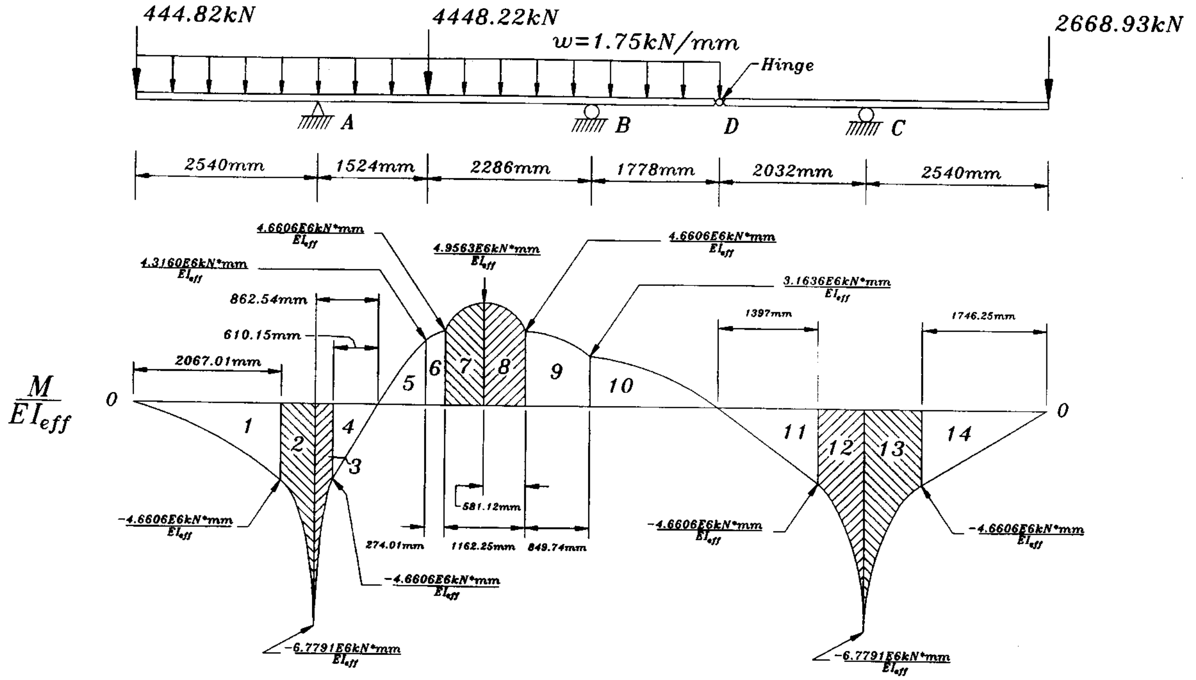


Fig. 6 Determinate continuous beam example problem

Table 2 Summary of continuous beam results

Node	Elastoplastic Deflection (mm)		Percent Error (%)
	Virtual Work Method (exact)	Approximate Direct Stiffness Method (approximate)	
1	8.402	8.708	3.64
9	5.214	5.208	0.115
13	4.116	4.113	0.073
15	14.408	14.402	0.042
16	5.981	6.031	0.836
21	44.169	44.628	1.039

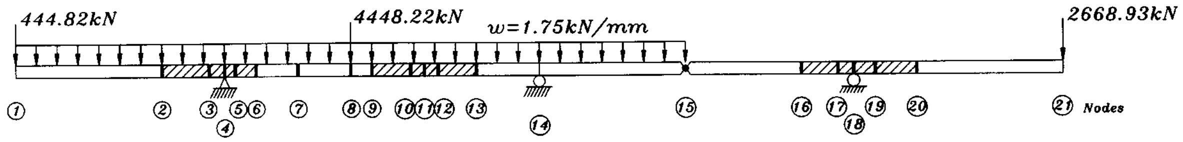


Fig. 7 Model of continuous beam

each region for each virtual loading condition corresponding to a desired deflection location. In the elastic regions, the areas and centroidal locations are calculated using well documented formulas found in most structural analysis references (Kassimali 1993).

If the loads and span lengths shown in Fig. 6 are used with M_y=4.6606$\times 10^6$ kN-mm, E=199.948 kN/mm^2, and I=5.9616$\times 10^9$ mm^4, the virtual work method yields the deflections at six locations along the beam as shown in Table 2. The six node numbers in Table 2 correspond to those used in Fig. 7.

4.2.2. Approximate direct stiffness method

The direct stiffness method is used to determine the deflections of the continuous beam using two beam elements with an average effective moment of inertia for each element in the elastoplastic regions. Fig. 7 illustrates the model used for the continuous beam example. It consists of 20 linear elastic beam elements, where each element between nodes 2 through 6, 9 through 13, and 16 through 20 has a moment of inertia that is the average between the two effective moments of inertia found on each end of the element. As with the previous example, each side of the elastoplastic region is divided into two, unequal length elements consisting of 25% and 75% of L_{ep}, where the smaller element length is closest to the maximum moment. For this determinate beam example M_{ep} does not change due to the reduced stiffnesses in the elastoplastic regions; therefore, Eq. (24) can be used to determine the effective moment of inertia at each node in the shaded regions. The average moment of inertia, I_{avg}, is calculated for each affected element and is used as a constant I in the beam element stiffness matrix.

The deflection results from this method are compared with those obtained from the virtual work method. The results in Table 2 indicate that the deflections calculated by the approximate direct stiffness method compare very closely with the "exact" deflections. The relative errors are highest at the two ends of the beam where the deflections are largest, although most of the errors are well below the maximum value of 3.64 percent. This loading condition produces moments nearing the full plasticity of the section at M_{ep}/M_y=1.4545, indicating the approximate direct stiffness method can be used with good accuracy up to the formation of plastic hinges and with relative ease compared with the other methods of analysis.

5. Conclusions

A new procedure was developed to reduce the rigidity of elastoplastic beams by reducing the moment of inertia using an effective moment of inertia, I_{eff}. Using the I_{eff} relationship for a rectangular beam with elastic, perfectly plastic material behavior, convenient virtual work equations were developed to find deflections of beams with concentrated and distributed loads that are "exact" and closed-form. The approximate direct stiffness method using an average effective moment of inertia produced results that agreed very closely to the "exact" results up to full plasticity.

This study shows that the effective moment of inertia can be accurately and conveniently used to represent the beam stiffness reduction for rectangular, elastoplastic beams. It is anticipated that the procedures outlined in this paper can be used to determine I_{eff} for other beam cross-sections with equal success, and can also be extended for predicting deflections of indeterminate structures in an iterative manner.

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