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Function space formulation of the 3-noded distorted Timoshenko metric beam element

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Abstract. The 3-noded metric Timoshenko beam element with an offset of the internal node from the element centre is used here to demonstrate the best-fit paradigm using function space formulation under locking and mesh distortion. The best-fit paradigm follows from the projection theorem describing finite element analysis which shows that the stresses computed by the displacement finite element procedure are the best approximation of the true stresses at an element level as well as global level. In this paper, closed form best-fit solutions are arrived for the 3-noded Timoshenko beam element through function space formulation by combining field consistency requirements and distortion effects for the element modelled in metric Cartesian coordinates. It is demonstrated through projection theorems how lock-free best-fit solutions are arrived even under mesh distortion by using a consistent definition for the shear strain field. It is shown how the field consistency enforced finite element solution differ from the best-fit solution by an extraneous response resulting from an additional spurious force vector. However, it can be observed that when the extraneous forces vanish fortuitously, the field consistent solution coincides with the best-fit strain solution.

Keywords: metric element; function spaces; symmetric formulation; element distortion; best-fit paradigm; variational correctness; orthogonal projections

1. Introduction

Distortion sensitivity of isoparametric elements is an important issue in finite element analysis, especially in stress analysis problems near cut-outs or corners where significant element distortion and high stress gradients are inevitable, leading to large errors in stress values. While the finite element method is credited with the ability to negotiate arbitrary external boundaries of elements as often necessary, the isoparametric element formulations, which depend on the mathematical mapping of the physical domains of metric co-ordinates onto those of parametric co-ordinates, do indeed suffer the consequences of extreme distortions.

Studies on the deterioration of performance of isoparametric elements under mesh distortions have been made previously by many researchers (Stricklin *et al.* 1977, Backlund 1978, Gifford 1979, Lee and Bathe 1993). It has been observed that isoparametric elements based on identical sets of Lagrangian shape functions in parametric coordinates for both geometry and displacement interpolations, performed extremely well for regular meshes but degraded rapidly under mesh distortions. To reduce distortion effects in isoparametric elements, Rajendran and co-workers (Rajendran and Liew 2003, Rajendran and Subramanian 2004, Rajendran 2010) have proposed the use of dual shape functions for the displacement field, viz., the compatibility/continuity enforcing parametric functions and completeness enforcing metric functions (denoted as parametric-metric or PM element in this study). The improvement in the performance of the element is possible because completeness in the metric functions as trial functions helps to reduce errors in the stresses, while continuity is enforced through the test functions. Recently, improved versions of the distortion immune unsymmetric elements have been reported in literature (Cen *et al.* 2012, Cen *et al.* 2015, Zhou *et al.* 2017, Shang and Ouyang 2017), by using analytical trial functions introduced by Fu *et al.* (2010), Cen *et al.* (2011a, b), using analytical trial solutions in terms of quadrilateral area/hexahedral volume coordinates and enriching the test functions with drilling degrees of freedom respectively. High performance distortion immune quadrilateral Mindlin-Reissner plate elements have been proposed recently by Cen *et al.* (2014) based on a hybrid displacement function element method and Shang *et al.* (2015) by combining the same displacement function with a generalized conforming approach.

The 'field consistency' paradigm determines the correct form of the assumed strain field interpolations in constrained media elasticity problems (Prathap 1993). Elements lock because they inadvertently enforce spurious constraints that arise from inconsistencies in the strains developed from the assumed displacement functions. The coefficients of the field consistent assumed strain fields are

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determined from an orthogonality condition that arises from the equilibrium equations derived from the mixed approaches based on Hellinger Reissner or Hu-Washizu variational theorems. The assumed strain function computations are hence performed in a 'variationally correct' way for smoothing the shear strain field to a field consistent level to avoid locking (Simo *et al.* 1986 and Prathap 1993). The finite element strain vector is thus computed as the best-fit or as an orthogonal projection of the analytical strain vector onto the strain displacement function subspace.

The various formulations of the 3-noded bar and beam elements have been examined in the light of variational correctness (Prathap *et al.* 2007, Prathap and Mukherjee 2003, Prathap and Naganarayana 1992, Mukherjee and Jafarali 2010, Kumar and Prathap 2008), that is based on the best-fit paradigm which essentially springs from the projection theorems founded in the first principles of the finite element method (Strang and Fix 1973, Norrie and De Vries 1978). They have observed that the conventional 3-noded isoparametric or parametric-parametric (PP) elements satisfy the best-fit rule for regular geometry, but violate it when the internal node is at an offset from the element centre. In the classical isoparametric formulation, the Jacobian appears as a function of the parametric coordinates in the denominator of the algebraic expression for the strain interpolations inside the distorted element. The performance of the element then becomes dependent on the characteristics of the Jacobian itself. Furthermore, under such circumstances, the best-fit paradigm in the parametric domain gets violated, indicating that under distortion, isoparametric elements are not really variationally correct. Furthermore, the symmetric metric-metric (MM) elements do not violate the best-fit rule in the metric domain (Prathap *et al.* 2006, Kumar and Prathap 2008, Mukherjee and Manju 2011).

The locking phenomena in finite elements have been extensively studied (Hughes 1987, Zienkiewicz 1991, Bathe 1996, Prathap 1993). The source of locking problem was explained using the field consistency paradigm (Prathap 1993), that predicts the existence of locking and *a priori* error estimates due to spurious stiffening and the unrealistic stress oscillations. Prathap and Naganarayana (1992) and Kumar and Prathap (2008) studied various versions of the Timoshenko beam element under distortion and locking. Mukherjee and Prathap (2001, 2002a, b) have further investigated shear locking in beams through function space approach and projection theorems. The concept of spurious forces for checking variational correctness in field consistency enforced finite element solutions has been introduced by Mukherjee and Prathap (2002b).

The function space investigation of the finite element method has been introduced by Strang and Fix (1973). Function space finite element formulation portrays finite element solutions as "shadows" or orthogonal projections on predetermined function subspaces of the analytical results (Mukherjee and Prathap 2001). Through function space formulation closed form best-fit strains, stresses and errors can be arrived by combining field consistency requirements and mesh distortion effects in the symmetric metric-metric (MM) and parametric-parametric (PP) Timoshenko beam elements.

In this paper, the function space formulation of the distortion immune 3-noded metric Cartesian Timoshenko beam element with field consistent shear strain has been studied. A single metric beam element subjected to various kinds of loading has been employed to demonstrate the performance of the element. The superiority of the present element over the conventional isoparametric element has also been demonstrated here. Owing to the symmetric nature of the metric formulation, closed form explicit algebraic expressions for the finite element/best-fit strains and the errors have been derived from a function space formulation under mesh distortion and shear locking. The extraneous response in finite element solution, wherever evoked from spurious extraneous forces due to 'stress smoothing' formulations, have been computed from the function space model and shown to be responsible for the slight deviation of the finite element solution from the best-fit one. In the case where the solution deviates from the best-fit, closed form algebraic expressions have been derived for both finite element and best-fit strains separately. However, it can be observed that when the extraneous forces vanish fortuitously, the field consistent solution coincides with the best-fit strain solution. The function space formulation of the metric beam element that computes the finite element strain vector as the best-fit or as an orthogonal projection of the analytical strain vector onto the strain displacement function subspace is detailed in Section 4.0.

The exactly integrated versions of the field inconsistent and consistent isoparametric (PP), parametric-metric (PM) and metric (MM) elements are denoted as PP4I, PP3I, PM4I, PM4C, MM4I and MM4C respectively by Kumar and Prathap (2008) and hence named accordingly in the paper. Here, *I* and *C* stands for the field inconsistent and consistent versions of the element using 3- and 4-point integration rules. In the case of isoparametric elements, the simplest way to eliminate locking is the reduced integration with a *2*-point integration of the shear stiffness matrix and hence denoted as PP2R in the paper.

2. The 3-noded distorted Timoshenko beam element in the metric domain

The distorted quadratic 3-noded metric Timoshenko beam element of length L with two degrees of freedom per node is shown in Fig. 1. The internal node (of nodal index 2) is not necessarily in the middle position of the beam, and a non-dimensional distortion parameter T is used to indicate the relative offset of this node from the central position.

The geometry and displacement interpolations for the 3-noded distorted beam element are as given below

Nodal positions

$$x_1, x_2, x_3 = -L/2, TL/2, L/2 \tag{1}$$

Transverse Deflection

$$w^h(x) = \sum_{i=1}^{3} M_i w_i \tag{2}$$

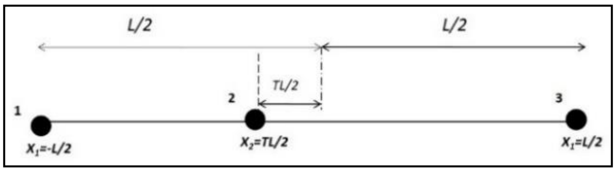


Fig. 1 The quadratic beam element with offset in internal node position

Rotation of the plane originally normal to the neutral axis

$$\theta^h(x) = \sum_{i=1}^{3} M_i\,\theta_i \tag{3}$$

where w_i and θ_i are the exact nodal displacement components. Note that a superscript 'h' is employed for finite element solutions; for example, $w(x)$ and $w^h(x)$ denote respectively the exact and finite element solutions for transverse displacements of the beam. The displacement vector is given by

$$\begin{Bmatrix} w^h \\ \theta^h \end{Bmatrix} = \begin{bmatrix} M_1 & 0 & M_2 & 0 & M_3 & 0 \\ 0 & M_1 & 0 & M_2 & 0 & M_3 \end{bmatrix} \{\delta^e\} = [M]\{\delta^e\} \tag{4}$$

where $\{\delta^e\}$ denotes the nodal displacement vector, i.e., $\{\delta^e\} = [w_1, \theta_1, w_2, \theta_2, w_3, \theta_3]^T$. Here M_i are the metric shape functions given by

$$M_1 = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)};\; M_2 = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}; \tag{5}$$

$$M_3 = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

It has been well established (Prathap and Naganarayana 1992) that the *conventional isoparametric element* (PP3I) locks in its original form with exact integration. The version based on reduced integration (PP2R) is free of locking if the element is undistorted, but locking reappears in the distorted element. The conventional element is made free of locking under distortion (PP3C) only by a non-trivial approach by ensuring consistency of the shear strains in the parametric space (Prathap and Naganarayana 1992).

Kumar and Prathap (2008) have demonstrated that the use of a consistent definition of the constrained strain field for the unsymmetric parametric-metric (PM4C) and the symmetric metric-metric (MM4C) elements and concludes that though both approaches are viable for a one dimensional Timoshenko beam element under distortion and locking, only the MM4C element computes the best possible finite element solutions. Mukherjee and Prathap (2002a, b) have shown that the field inconsistent 3-noded element (PP3I) (with full Gauss quadrature integration) locks weakly and is variationally correct, i.e., the finite element strain vector is the best-fit of the analytical strain vector. For any field consistent lock-free formulation that employs smoothened solutions or reduced integrations, the finite element solutions do not always give the best-fit strain solutions, i.e., they are not necessarily variationally correct. The quadratic element MM4C gives variationally correct best-fit solutions for single element test cases with exact analytical strains up to quadratic order but departs slightly from the best-fit due to the extraneous forces for the single element test case when the exact strains are of cubic order.

The field inconsistent and field consistency enforced metric finite element formulations are detailed in Sections 2.1 and 2.2 respectively.

2.1 The field inconsistent metric formulation (MM4I)

The element strain vector is given as

$$\{\varepsilon^h\} = \begin{bmatrix} d\theta^h/_{dx} \\ \theta^h - dw^h/_{dx} \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{\partial M_1}{\partial x} & 0 & \dfrac{\partial M_2}{\partial x} & 0 & \dfrac{\partial M_3}{\partial x} \\ \dfrac{\partial M_1}{\partial x} & M_1 & \dfrac{\partial M_2}{\partial x} & M_2 & \dfrac{\partial M_3}{\partial x} & M_3 \end{bmatrix} \{\delta^e\} = [B_M]\{\delta^e\} \tag{6}$$

where $[B_M]$ is the strain displacement matrix and $\{\delta^e\}$ is the nodal displacement vector.

The original strain displacement matrix *$[B_{MD}]$* for the distorted element is given by

$$[B_{MD}] = \begin{Bmatrix} 0 & \dfrac{-L - LT + 4x}{L^2(1+T)} & 0 & \dfrac{8x}{L^2(-1+T^2)} & 0 & \dfrac{-L + LT - 4x}{L^2(-1+T)} \\ \dfrac{L + LT - 4x}{L^2(1+T)} & -\dfrac{(L-2x)(-LT+2x)}{2L^2(1+T)} & -\dfrac{8x}{L^2(-1+T^2)} & \dfrac{-L^2+4x^2}{L^2(-1+T^2)} & \dfrac{L - LT + 4x}{L^2(-1+T)} & -\dfrac{(L+2x)(-LT+2x)}{2L^2(-1+T)} \end{Bmatrix} \tag{7}$$

from which the strain displacement matrix for the undistorted element can be obtained by eliminating the distortion parameter, i.e., T=0 in Eq. (7).

The Rayleigh-Ritz principle of minimum total potential energy (i.e., $\delta\pi = 0$) is used to determine the element stiffness matrix $[K^e]$ that appear in the equilibrium equation $[K^e]\{\delta^e\} = \{F^e\} + \{R^{he}\}$ where $\{F^e\}$ is the variationally consistent nodal force vector for a distributed load vector $\{q\}$ and $\{R^{he}\}$ is the nodal reaction vector acting on the element. These matrices are given by the following expressions

$$[K^e] = \int_{-\frac{L}{2}}^{\frac{L}{2}} [B_{MD}]^T\,[D][B_{MD}]dx\,; \tag{8a}$$

$$\{F^e\} = \int_{-L/2}^{L/2} [M]^T\{q\}dx \tag{8b}$$

The FE stress resultant vector is given by $\{\sigma^h\} = [B_{MD}]\,[D]\,\{\delta^e\}$ where the rigidity matrix is $[D] = \begin{bmatrix} EI & 0 \\ 0 & KGA \end{bmatrix}$; where EI and KGA represent respectively the bending and shear rigidities.

With a single element discretization, the equilibrium equations in finite element computation can be solved by prescribing nodal loads and the boundary conditions for at least two of the six degrees of freedom so that rigid body displacements can be eliminated. If the stiffness matrix $[K^e]$ is integrated exactly and used to calculate displacements, strains and stresses, it is observed that the element shows spurious stiffness and the displacements and strains are several times lower than the true values. This phenomenon, called shear locking is accompanied by violent shear stress oscillations due to field inconsistency in the shear strains.

2.2 The field consistent metric formulation (MM4C)

The field consistent metric element is formulated using the substitute shape function for the interpolation of the field variable θ^h in the shear strain expression as shown by Kumar and Prathap (2008). Here θ^h which is quadratic in *x* is replaced with $\bar{\theta}$ which is linear in *x*, making $\bar{\theta}$ consistent with $dw/_{dx}$ in the Cartesian space. The substitute shear strain field γ^h will then be field consistent even under severe distortion. Let the metric interpolation for the section rotation be written as

$$\theta^h = \frac{\theta_1(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} + \frac{\theta_2(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} + \frac{\theta_3(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} \tag{9}$$

With $\theta^h = \sum_{i=1}^{3}(a_0 + a_1x + a_2x^2)_i\,\theta_i$ and $\bar{\theta} = \sum_{i=1}^{3}(b_0 + b_1x)_i\,\theta_i$, Prathap (1993) has shown that the variationally correct manner to determine the coefficients is to use the orthogonality condition given by,

$$\int_0^L \delta\bar{\theta}(\bar{\theta} - \theta^h)\,dx = 0 \tag{10}$$

The smoothed shape functions are used for interpolating the slope to derive the field consistent metric formulation for the 3-noded Timoshenko beam element studied in this paper. The field consistent element strain vector is then given as

$$\{\varepsilon^h{}_C\} = \begin{bmatrix} d\theta^h/_{dx} \\ \bar{\theta} - dw^h/_{dx} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial M_1}{\partial x} & 0 & \frac{\partial M_2}{\partial x} & 0 & \frac{\partial M_3}{\partial x} \\ \frac{\partial M_1}{\partial x} & SM_1 & \frac{\partial M_2}{\partial x} & SM_2 & \frac{\partial M_3}{\partial x} & SM_3 \end{bmatrix}\{\delta^e\} = [B_{MCD}]\{\delta^e\} \tag{11}$$

where *SM1*, *SM2* and *SM3* are the substitute shape functions and $[B_{MCD}]$ is the field consistent strain displacement matrix.

The field consistent B matrix for the distorted element is given by

$$[B_{MCD}] = \begin{Bmatrix} 0 & \frac{-L(1+T)+4x}{L^2(1+T)} & 0 & \frac{8x}{L^2(-1+T^2)} & 0 & \frac{L(-1+T)-4x}{L^2(-1+T)} \\ \frac{L=(1+T)-4x}{L^2(1+T)} & \frac{1+3T}{6(1+T)} - \frac{x}{L} & -\frac{8x}{L^2(-1+T^2)} & -\frac{2}{3(-1+T^2)} & \frac{-L(-1+T)+4x}{L^2(-1+T)} & \frac{-1+3T}{6(-1+T)} + \frac{x}{L} \end{Bmatrix} \tag{12}$$

from which the field consistent strain displacement matrix for the undistorted element can be obtained by eliminating the distortion parameter i.e., *T*=0 in Eq. (12).

As a comparison, the conventional isoparametric 3-noded Timoshenko beam element with a reduced integration (PP2R) of the shear stiffness matrix is being taken up in Section 5. The non-uniform mapping of the quadratic beam element has been studied rigorously by Prathap and Naganarayana (1992). The details of the formulation are hence not given in this paper. The shape functions N_1, N_2 and N_3, the strain vector $\varepsilon^h{}_P$, the strain displacement matrix $[B_P]$ of the conventional element are as given in Eqs. (13)-(14).

$$N_1 = 0.5*\xi(-1+\xi);\;\; N_2 = (1+\xi^2);\;\; N_3 = 0.5*\xi(1+\xi) \tag{13}$$

The element strain vector for the conventional isoparametric element is given as

$$\{\varepsilon^h{}_P\} = \begin{bmatrix} d\theta^h/_{dx} \\ \theta^h - dw^h/_{dx} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial N_1}{\partial \xi}/J & 0 & \frac{\partial N_2}{\partial \xi}/J & 0 & \frac{\partial N_3}{\partial \xi}/J \\ \frac{\partial N_1}{\partial \xi}/J & N_1 & \frac{\partial N_2}{\partial \xi}/J & N_2 & \frac{\partial N_3}{\partial \xi}/J & N_3 \end{bmatrix}\{\delta^e\} = [B_P]\{\delta^e\} \tag{14}$$

where ξ is the natural coordinate with values of -1, 0 and 1 at the nodes, *[B_P]* is the strain displacement matrix, *J* is the Jacobian and $\{\delta^e\}$ is the nodal displacement vector, where $\{\delta^e\} = [w_1, \theta_1, w_2, \theta_2, w_3, \theta_3]^T$.

3. Spurious forces from best-fit projections in field consistency enforced, variationally incorrect formulation

In this section, the reason for the occasional slight departure of the field consistent finite element induced strains from the corresponding best-fit solutions is given, as introduced by Mukherjee and Prathap (2002b) for checking variational correctness in field consistency enforced finite element solutions.

It has been shown by these authors that a field inconsistent element, despite its inherent tendency of locking, is variationally correct. This is because the generalized nodal force vector, given by Eq. (8), can be shown to satisfy the following condition

$$\{F^e\} + \{R^{he}\} = \int_{-L/2}^{L/2} [M]^T\{q\}\,dx + \{R^{he}\} = \int_{-L/2}^{L/2} [B_{MD}]^T[D]\{\varepsilon\}\,dx \tag{15}$$

where $\{R^{he}\}$ is the nodal reaction vector on the element which matches exactly with the exact reaction vector. This condition gives the best-fit strain. This is because the equilibrium condition for the field inconsistent finite element,

$$[K^e]\{\delta^e\} = \{F^e\} + \{R^{he}\} \tag{16}$$

yields the following normal equation for the finite element strain $\{\varepsilon^h\}$ to be the best-fit of the analytical strain $\{\varepsilon\}$,

$$\int_{-L/2}^{L/2} [B_{MD}]^T[D][B_{MD}]dx\,\{\delta^e\} = \int_{-L/2}^{L/2} [B_{MD}]^T[D]\{\varepsilon\}\,dx \tag{17}$$

In a field consistency enforced formulation, one employs a field consistent strain displacement matrix $[B_{MCD}]$ given by Eq. (12). The lock-free, field consistent beam element responds to the same applied force vector $\{F^e\}$ as the locked, field inconsistent solution,

$$[K^{e*}]\{\delta^{e*}\} = \{F^e\} + \{R^{he}\} = \int_{-L/2}^{L/2} [B_{MD}]^T[D]\{\varepsilon\}\,dx \tag{18}$$

wherein the stiffness matrix for the lock-free, field consistent element is given by

$$[K^{e*}] = \int_{-L/2}^{L/2} [B_{MCD}]^T[D][B_{MCD}]dx \tag{19}$$

One can rewrite the equilibrium Eq. (18) as

$$[K^{e*}]\{\delta^{e*}\} = \int_{-L/2}^{L/2} [B_{MCD}]^T[D]\{\varepsilon\}\,dx + \int_{-L/2}^{L/2} \big[[B_{MD}] - [B_{MCD}]\big]^T[D]\{\varepsilon\}\,dx \tag{20}$$

or $[K^{e*}]\{\delta^{e*}\} = \int_{-L/2}^{L/2} [B_{MCD}]^T[D]\{\varepsilon\}\,dx + \{F_E^e\}$

where

$$\{F_E^e\} = \int_{-L/2}^{L/2} \big[[B_{MD}] - [B_{MCD}]\big]^T[D]\{\varepsilon\}\,dx \tag{21}$$

is the extraneous, self-equilibrating nodal force vector that acts on the finite element over and above the otherwise variationally correct force vector $\int_{-L/2}^{L/2} [B_{MCD}]^T[D]\{\varepsilon\}\,dx$.

Eq. (21) highlights the fact that the field consistency enforced finite element solution will differ from the field consistent best-fit solution by an extraneous response resulting from the additional, spurious force vector $\{F_E^e\}$. However, when the extraneous forces vanish fortuitously, i.e., $\{F_E^e\} = 0$, the field consistent solution coincides with the field consistent best-fit strain solution.

4. Best-fit strain solutions of the 3-noded field consistent beam element as orthogonal projections onto the strain-displacement function subspace

For the distorted beam element, the best-fit paradigm of finite element analysis in the metric formulation can be realized if one obtains finite element strain solutions that agree with the orthogonal projections of the analytical strains onto the proper strain-displacement subspace B_{MCD} (linear in the metric *x* domain for a consistent formulation).

4.1 Orthogonal basis vectors spanning the metric B_{MCD} subspace

Eq. (12) provides the expression for the strain-displacement matrix $[B_{MCD}]$ in the metric domain. The inner product of the element in the B_{MCD} subspace in the *x* domain is defined as

$$\langle a, b\rangle_{MCD} = \int_{x=x1}^{x3} \{a\}^T[D]\{b\}\,dx = \int_{-L/2}^{L/2} \{a\}^T[D]\{b\}\,dx \tag{22}$$

where $[D]$ is the element rigidity matrix.

Using the Gram-Schmidt process, the *m*-numbers of the non-zero orthogonal basis vectors spanning the *m*-dimensional *B* subspace can be generated. In the case of the 3-noded distorted beam element, it can be shown that the 4-dimensional B_{MCD} subspace for the distorted element can be spanned by the following 4 orthogonal basis vectors,

$$\nu_1 = \left\{\left(\begin{matrix} 0 \\ \frac{L+LT-4x}{L^2(1+T)} \end{matrix}\right)\right\}; \quad \nu_2 = \left\{\begin{matrix} \frac{-L-LT+4x}{L^2(1+T)} \\ -\frac{(1+3T^2)(L+3x+3Tx)}{3L(7+13T+9T^2+3T^3)} \end{matrix}\right\};$$
$$\nu_3 = \left\{\begin{matrix} \frac{24\mathrm{KGA}(1+3T^2)(L+LT-4x)}{L(-1+T)(12\mathrm{EI}(7+6T+3T^2)^2+\mathrm{KGA}(L+3LT^2)^2} \\ -\frac{96\mathrm{EI}(7+6T+3T^2)(L+3(1+T)x)}{L^2(-1+T)(12\mathrm{EI}(7+6T+3T^2)^2+\mathrm{KGA}(L+3LT^2)^2} \end{matrix}\right\}; \tag{23}$$
$$\nu_4 = \left\{\begin{matrix} \frac{8(L+3x+3Tx)}{L^2(-7+T+3T^2+3T^3)} \\ 0 \end{matrix}\right\}$$

For the undistorted element, the four-dimensional B_{MC} subspace (with T=0) can be spanned by the following 4 orthogonal basis vectors,

$$\nu_1 = \left\{\begin{matrix} 0 \\ \frac{L-4x}{L^2} \end{matrix}\right\}; \quad \nu_2 = \left\{\begin{matrix} \frac{-L+4x}{L^2} \\ -\frac{1}{21} - \frac{x}{7L} \end{matrix}\right\};$$
$$\nu_3 = \left\{\begin{matrix} -\frac{24\mathrm{KGA}(L-4x)}{588\mathrm{EI}L+\mathrm{KGA}L^3} \\ \frac{672\mathrm{EI}(L+3x)}{L^2(588\mathrm{EI}+\mathrm{KGA}L^2)} \end{matrix}\right\}; \quad \nu_4 = \left\{\begin{matrix} -\frac{8(L+3x)}{7L^2} \\ 0 \end{matrix}\right\} \tag{24}$$

which satisfy the orthogonality condition $\langle \nu_i, \nu_j\rangle = 0$ for *i≠j*.

4.2 Orthogonal projections of analytical strain onto the metric B_{MCD} subspace

According to the best-fit paradigm, the FEM strain $\bar{\varepsilon}^h$ can be obtained as the orthogonal projection of the analytical strain ε onto the four-dimensional function subspace B_{MCD} as

$$\bar{\varepsilon} = \sum_{i=1}^{4} \frac{\langle \varepsilon,\ \nu_i\rangle}{\langle \nu_i, \nu_i\rangle} \nu_i \tag{25}$$

The geometrical projection is presented in simplified form in Fig. 2. Any variationally correct finite element solution for element strain appears as the orthogonal projection of the corresponding analytical strain onto the strain-displacement function space (Prathap and Mukherjee 2001-2003, Mukherjee and Jafarali 2010). The analytical strain ε is derived from the exact solution of the same differential equation upon which the finite element formulation is being based through the weak form. The following norm square values of the basis vectors that can be obtained from the definition of the inner products are useful for evaluating the best-fit strains as given in Eq. (25),

$$\langle \nu_1, \nu_1 \rangle = \int_{-\frac{L}{2}}^{\frac{L}{2}} \nu_1^{\mathrm{T}} D\, \nu_1 dx = \frac{KGA(3 + \frac{4}{(1+T)^2})}{3L}$$

$$\langle \nu_2, \nu_2 \rangle = \int_{-L/2}^{L/2} \nu_2^{\mathrm{T}} D\, \nu_2 dx =$$

$$\frac{12EI(7+6T+3T^2)^2 + KGA(L+3LT^2)^2}{36L(1+T)^2(7+6T+3T^2)}$$

$$\langle \nu_3, \nu_3 \rangle = \int_{-L/2}^{L/2} \nu_3^{\mathrm{T}} D\, \nu_3 dx \tag{26}$$

$$= \frac{192EIKGA(7+6T+3T^2)}{L(-1+T)^2(12EI(7+6T+3T^2)^2 + KGA(L+3LT^2)^2)}$$

$$\langle \nu_4, \nu_4 \rangle = \int_{-L/2}^{L/2} \nu_4^{\mathrm{T}} D\, \nu_4 dx$$

$$= \frac{16\mathrm{EI}}{L(-1+T)^2\big(7+3T(2+T)\big)}$$

The orthogonal projection $\bar{\varepsilon}$ of the analytical strain ε given by Eq. (25) naturally satisfies the following Pythagorean Theorem or the error-energy rule for best-fit finite element solutions

$$\|\varepsilon - \bar{\varepsilon}\|^2 = \|\varepsilon\|^2 - \|\bar{\varepsilon}\|^2 \tag{27}$$

i.e., *the energy of the error = the error of the energy*

In Eq. (27), the norms are given by

$$\|\varepsilon\| = \sqrt{\langle \varepsilon, \varepsilon \rangle}; \quad \|\bar{\varepsilon}\| = \sqrt{\langle \bar{\varepsilon}, \bar{\varepsilon} \rangle};$$
$$\|\varepsilon - \bar{\varepsilon}\| = \sqrt{\langle (\varepsilon - \bar{\varepsilon}), (\varepsilon - \bar{\varepsilon}) \rangle} \tag{28}$$

Mukherjee and Prathap (2001-2003) have shown how a variationally correct finite element solution actually conforms to the best-fit rule by yielding an approximate strain vector that is exactly the orthogonal projection or the best-fit of the analytical strain vector. In other words, a variationally correct finite element solution should satisfy the following equality,

$$\varepsilon^h = \bar{\varepsilon} \tag{29}$$

This is shown in Fig. 2 wherein it is obvious that this condition ensures that, of all the strains in the strain-displacement subspace, the best-fit strain (which agrees with the variationally correct finite element strain) actually guarantees a solution of *minimum* dispersion, or error, of the strain in the element. Any variationally incorrect finite element solution (from reduced integrations or stress smoothening) is subjected to spurious extraneous forces as discussed earlier. In such situations, the finite element suffers additional strain response $\Delta\varepsilon^h$ over and above the best-fit strain, and can be generically given by the following expression,

$$\varepsilon^h = \bar{\varepsilon} + \Delta\varepsilon^h \tag{30}$$

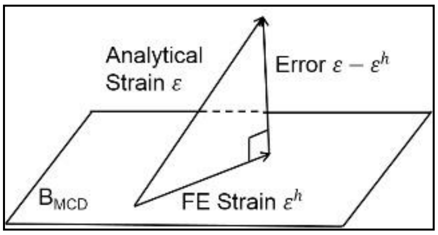


Fig. 2 The finite element (FE) strain as an orthogonal projection of the analytical strain onto the strain-displacement subspace B_{MCD} for variationally correct formulations

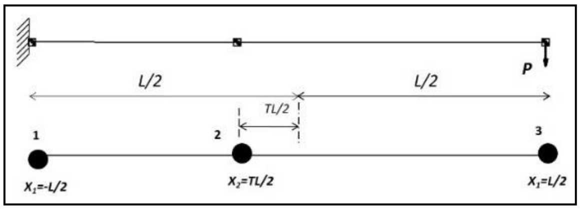


Fig. 3 Analysis of a cantilever beam subjected to tip transverse load using a single 3-noded Timoshenko beam element

5. Demonstrative problems

The single element test problems in this section are taken up for the cantilever beam with length L = 10, Young's modulus E = 1500, width of beam section b = 2 and depth d = 0.2.

5.1 Test example-1: Beam subjected to transverse load at tip

A uniform cantilever beam with a tip transverse load P = 1.0 is analyzed using the metric 3-noded distorted beam element. The beam is modeled as a single 3-noded element of length L with distortion of the internal node. The loaded bar and its finite element model are shown in Fig. 3.

The analytical strain vector, viz., the bending strain and shear strain of the beam is given by

$$\{\varepsilon\} = \begin{Bmatrix} \frac{P(L-2x)}{2\mathrm{EI}} \\ \frac{P}{\mathrm{KGA}} \end{Bmatrix} \; for\ x \to -\frac{L}{2} to \frac{L}{2} \tag{31}$$

The projection formula is used here to determine the best-fit strain, which exactly agrees with the finite element strain distribution as obtained by the metric distorted

Table 1 Deflections of a single element cantilever distorted beam with tip load

Distortion parameter T =-0.2									
Displacement		PP4I	PP3I	PP2R	PM4I	MM4I	PM4C	MM4C	Exact
Node 2	w_2	15.36	38.19	35.49	27.82	20.08	36.68	36.68	34.67
	θ_2	6.42	16.002	15.65	13.9	10.02	16.0	16.0	16.0
Node 3	w_3	64.21	160.07	162.7	173.67	125.19	166.7	166.7	166.7
	θ_3	9.16	22.87	25.0	34.69	25.0	25.0	25.0	25.0

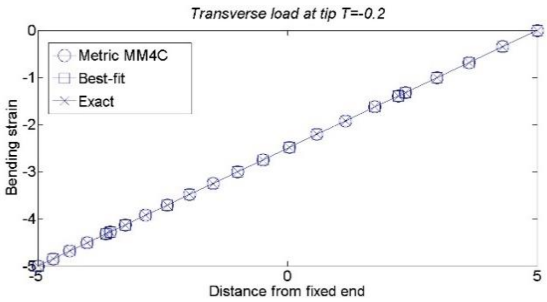


Fig. 4 Bending strain variation along the length of the 3-noded metric Timoshenko beam element subjected to a transverse load at the tip

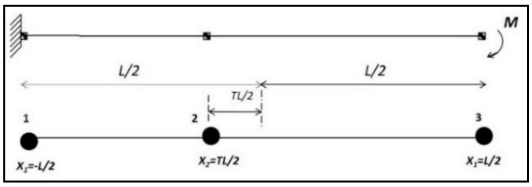


Fig. 5 Analysis of a cantilever beam subjected to tip moment using a single 3-noded Timoshenko beam element

Table 2 Deflections of a single element cantilever distorted beam with end moment

Distortion parameter T =-0.2									
Displacement		PP4I	PP3I	PP2R	PM4I	MM4I	PM4C	MM4C	Exact
Node 2	w_2	2.18	5.45	4.08	4.0	4.0	4.0	4.0	4.0
	θ_2	0.92	2.29	2.0	2.0	2.0	2.0	2.0	2.0
Node 3	w_3	9.16	22.86	25.0	25.0	25.0	25.0	25.0	25.0
	θ_3	1.31	3.27	5.0	5.0	5.0	5.0	5.0	5.0

element (see Fig. 4).

$$\{\varepsilon^h\} = \{\bar{\varepsilon}\} = \sum_{i=1}^{4} \frac{\langle \varepsilon,\ v_i \rangle}{\langle v_i, v_{i,} \rangle} v_i = \begin{Bmatrix} \dfrac{P(L-2x)}{2\mathrm{EI}} \\ \dfrac{P}{\mathrm{KGA}} \end{Bmatrix} = \{\varepsilon\} \quad (32)$$

Table 1 gives a comparison of the nodal displacements with the field inconsistent and consistent versions of the PP, PM and MM elements. It is obvious that the nodal displacements computed from integrations of the strains at the boundary are exact.

5.2 Test example-2: Beam subjected to end moment

A uniform cantilever beam with a tip moment of M = 1.0 is analyzed using the metric 3-noded distorted beam element. The beam is modeled as a single 3-noded element of length L with distortion of the internal node. The loaded bar and its finite element model are shown in Fig. 5.

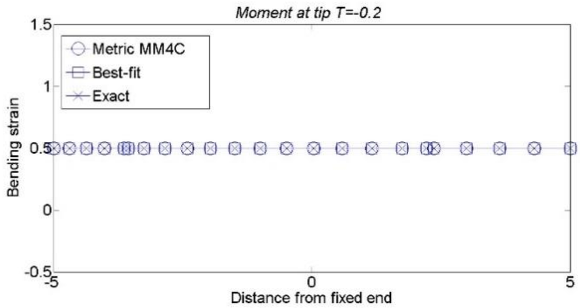


Fig. 6 Bending strain variation along the length of the 3-noded metric Timoshenko beam element subjected to end moment

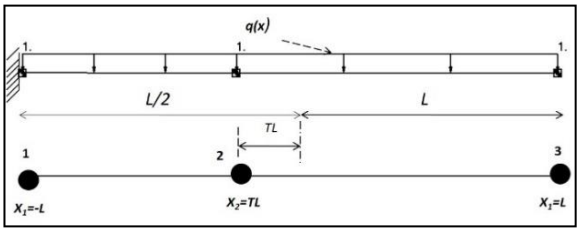


Fig. 7 Analysis of a cantilever beam subjected to uniformly distributed load q using a single 3-noded Timoshenko beam element

For the case of pure bending moment distribution, the analytical strain vector is given by

$$\{\varepsilon\} = \begin{Bmatrix} M/EI \\ 0 \end{Bmatrix} \quad (33)$$

Again, the 3-noded beam element exactly captures the best-fit and exact strain (see Fig. 6).

$$\{\varepsilon^h\} = \{\bar{\varepsilon}\} = \sum_{i=1}^{4} \frac{\langle \varepsilon, v_i \rangle}{\langle v_{i,} v_{i,} \rangle} v_i = \begin{Bmatrix} M/EI \\ 0 \end{Bmatrix} \quad (34)$$

Table 2 gives a comparison of the nodal displacements with the field inconsistent and consistent versions of the PP, PM and MM elements

5.3 Test example-3: Beam subjected to uniformly distributed load

A uniform cantilever beam with uniformly distributed load q = 0.16 is analyzed using the metric 3-noded distorted beam element. The beam is modeled as a single 3-noded element of length L with distortion of the internal node. The loaded beam and its finite element model are shown in Fig. 7.

The analytical strain vector, viz., the bending strain and shear strain of the beam is given by

$$\{\varepsilon\} = \begin{Bmatrix} \dfrac{q\ (L-2x)^2}{8EI} \\ \dfrac{q\ (L-2x)}{2KGA} \end{Bmatrix}\ for\ x = -\frac{L}{2} to \frac{L}{2} \quad (35)$$

Note that in the exact strain vector as given in Eq. (35), the bending strain is a quadratic function of the metric co-ordinate x and the shear strain is a linear function of the metric co-ordinate x. The bending and shear strains are solved through the projection formula and is given by

$$\{\varepsilon^h\} = \{\bar{\varepsilon}\} = \sum_{i=1}^{4} \frac{\langle \varepsilon,\ v_i \rangle}{\langle v_{i,}\ v_{i,} \rangle} v_i = \begin{Bmatrix} \dfrac{qL(L-3x)}{6\mathrm{EI}} \\ \dfrac{q(L-2x)}{2\mathrm{KGA}} \end{Bmatrix} \tag{36}$$

The beam element gives the finite element strain as the best-fit strain to the exact strain distribution, confirming that the extraneous force vector vanishes (see Appendix for its derivation). The shear strain is exactly captured in this example. Fig. 8 shows the bending and shear strain variations along the length of the 3-noded metric Timoshenko beam element subjected to uniformly distributed load. Fig. 9 presents a comparison of the bending and shear strains of the metric 3-noded element MM4C with the conventional isoparametric element PP2R. It can be observed that the conventional PP2R element captures the bending strains more closely than the MM4C but shows curious shear strain oscillations. The investigations of the conventional isoparametric element for various versions are critically examined by Prathap (1993) and hence omitted here.

If $\{\varepsilon\} - \{\varepsilon^h\}$ is the error in strain, then its norm squared, known as the 'energy of the error' is given by,

$$\|\varepsilon - \varepsilon^h\|^2 = \int_{-L/2}^{L/2} (\varepsilon - \varepsilon^h)^T\, [D] (\varepsilon - \varepsilon^h)\, dx = \frac{q^2\, L^5}{720\mathrm{EI}} \tag{37}$$

The error of the energy is given by

$$\|\varepsilon\|^2 - \|\varepsilon^h\|^2 = \frac{q^2\, L^5}{720\mathrm{EI}} \tag{38}$$

Since the finite element strain matches the best fit solution, it automatically satisfies the following Pythagorean Theorem or the error-energy rule,

$$\|\varepsilon - \varepsilon^h\|^2 = \|\varepsilon\|^2 - \|\varepsilon^h\|^2 \tag{39}$$

which can be interpreted as '*energy of the error = error of the energy*'. Hence from this it can be concluded that the solution for a single element of a cantilever beam with uniformly distributed load satisfies the projection theorem.

The nodal displacements are consequences of the integrals of the strains which are given by the areas covered under the strain curves. Area covered by the finite element strain (best-fit strain) within the element is given by

$$A_{BF} = \int_{-L/2}^{L/2} \varepsilon^h\, dx = \begin{Bmatrix} \dfrac{qL^3}{6\mathrm{EI}} \\ \dfrac{qL^2}{2\mathrm{KGA}} \end{Bmatrix} \tag{40}$$

Area covered by the exact strain distribution will be

$$A = \int_{-L/2}^{L/2} \varepsilon\, dx = \begin{Bmatrix} \dfrac{qL^3}{6\mathrm{EI}} \\ \dfrac{qL^2}{2\mathrm{KGA}} \end{Bmatrix} \tag{41}$$

Since the spatial integrals (with respect to metric coordinate *x*) of the best-fit strain and the analytical strains are identical for arbitrary distortions, it can be concluded that for this problem the exact displacements at the boundary nodes are recovered by the finite element computations. Table 3 gives a comparison of the nodal displacements with the field inconsistent and consistent versions of the PP, PM and MM elements. It can be observed that, exact displacements are computed only at node-3 and not at node-2 for the MM4C element. This is because the equality of area under the exact and finite

Table 3 Deflections of a single element cantilever distorted beam with uniformly distributed load

Distortion parameter T =-0.2									
Displacement		PP4I	PP3I	PP2R	PM4I	MM4I	PM4C	MM4C	Exact
Node 2	w_2	9.66	23.99	23.25	17.43	10.74	25.08	24.02	24.32
	θ_2	4.04	10.05	9.99	8.71	5.35	10.52	10.13	10.45
Node 3	w_3	40.35	100.49	98.61	108.72	66.83	102.69	100.03	100.03
	θ_3	5.75	14.35	13.69	21.71	13.33	13.33	13.33	13.33

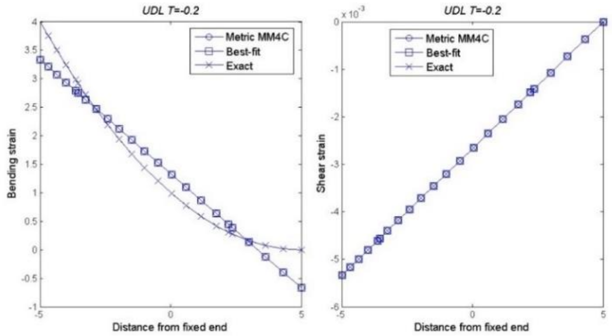


Fig. 8 Bending strain and shear strain variation along the length of the 3-noded metric Timoshenko beam element subjected to uniformly distributed load

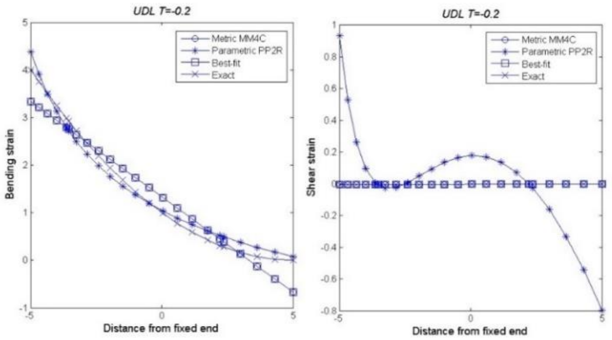


Fig. 9 Comparison of bending and shear strains of the metric 3-noded element with the conventional isoparametric element for the case of uniformly distributed load

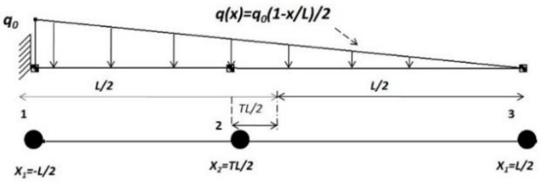


Fig. 10 Analysis of a cantilever beam subjected to uniformly varying load $q(x)=q_0.(1-x/L)/2$ using a single 3-noded Timoshenko beam element

element strain curves are with respect to the integration of the whole element between node-1 and node-3. The nodal displacements are still very closer to the exact solution as seen in Table 3.

Note that in the exact strain vector the bending strain is a cubic function and the shear strain is a quadratic function of the metric co-ordinate *x*. The best-fit strain vector is obtained through the projection formula and is given by

$$\{\bar{\varepsilon}\} = \sum_{i=1}^{4} \frac{\langle \varepsilon, \ v_i \rangle}{\langle v_{i,} \ v_{i,} \rangle} v_i = \begin{Bmatrix} \dfrac{q_0 L(5L - 18x)}{120\text{EI}} \\ \dfrac{q_0(L - 3x)}{6\text{KGA}} \end{Bmatrix} \tag{43}$$

This best-fit solution satisfies the Pythagorean rule,

$$\|\varepsilon - \bar{\varepsilon}\|^2 = \|\varepsilon\|^2 - \|\bar{\varepsilon}\|^2 = \frac{{q_0}^2 L^5}{2800\text{EI}} + \frac{{q_0}^2 L^3}{720\text{KGA}} \tag{44}$$

Fig. 11 shows the bending and shear strain variations along the length of the 3-noded metric Timoshenko beam element subjected to uniformly varying load. The bending strain computed from finite element analysis departs slightly from the best-fit strain obtained from projection formula. The slight departure in the bending strain is due to the non-vanishing extraneous force generated that has to be computed using Eq. (21). The shear strain is again exactly captured in this example.

Fig. 12 gives a comparison of bending and shear strains of the 3-noded metric element with the conventional isoparametric element. It can be observed that the conventional PP2R element captures the bending strains more closely than the MM4C element but at the cost of curious shear strain oscillations. The investigations of the conventional isoparametric element for various versions are critically examined by Prathap (1993) and Mukherjee and Prathap (2002a, b) and hence omitted here.

5.4.1 Finite element strains and the best-fit strain

Due to the non-vanishing extraneous force vector (over and above the variationally correct force) acting on the beam element for the linearly varying distributed loading example, the finite element strain solution deviates from the best-fit solution. This extraneous force (derived in the Appendix) is given by

$$\{F_E^e\} = \int_{-L/2}^{L/2} \left[[B_{MD}] - [B_{MCD}]\right]^T [D]\{\varepsilon\}\, dx = \begin{Bmatrix} 0 \\ \dfrac{q_0 L^2}{144} \\ 0 \\ -\dfrac{5 q_0 L^2}{432} \\ 0 \\ \dfrac{q_0 L^2}{216} \end{Bmatrix} \tag{45}$$

The finite element suffers additional strain response $\Delta\varepsilon^h$ over and above the best-fit strain, and can be generically given by the following expression,

$$\varepsilon^h = \bar{\varepsilon} + \Delta\varepsilon^h \tag{46}$$

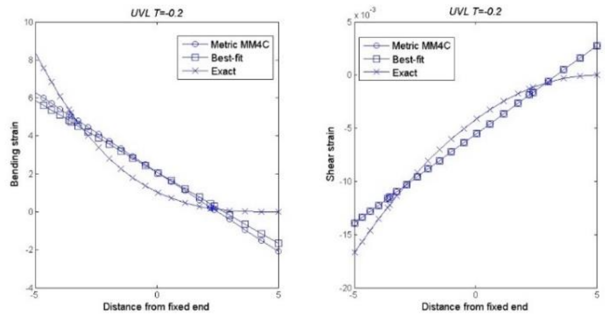


Fig. 11 Bending strain and shear strain variation along the length of the 3-noded metric Timoshenko beam element subjected to uniformly varying load

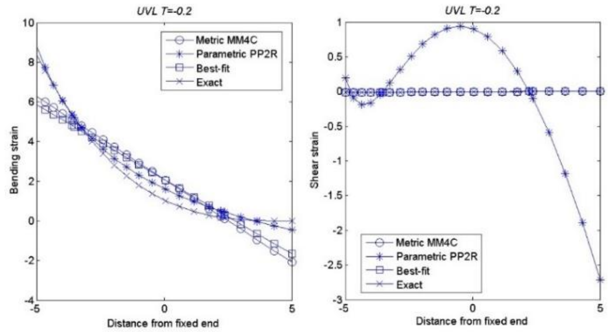


Fig. 12 Comparison of bending and shear strains of the metric 3-noded element with the conventional isoparametric element for the case of uniformly varying load

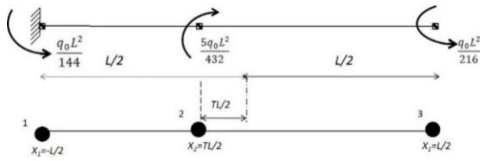
The extraneous nodal moments alone induce additional bending strain as response $\Delta\varepsilon^h$ which is explicitly derived in Appendix and shown graphically in Fig. 13. Hence from Eq. (46), the finite element strain vector is expected to be of the following form,

$$\{\varepsilon^h\} = \begin{Bmatrix} \dfrac{q_0 L(5L - 18x)}{120\text{EI}} \\ \dfrac{q_0(L - 3x)}{6\text{KGA}} \end{Bmatrix} + \begin{Bmatrix} -\dfrac{q_0 L x}{60\text{EI}} \\ 0 \end{Bmatrix} = \begin{Bmatrix} \dfrac{q_0 L(L - 4x)}{24\text{EI}} \\ \dfrac{q_0(L - 3x)}{6\text{KGA}} \end{Bmatrix} \tag{47}$$

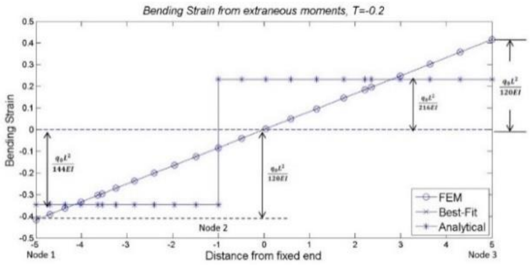
The validity of the best-fit paradigm is confirmed by the agreement of the actual finite element strain solutions obtained by the usual formulation with that expressed in Eq. (47). The finite element strain solution for this problem can be interpreted as the best-fit to that of a *stiffened* analytical solution, the stiffening being induced by the extraneous, self-equilibrating nodal moments (see Table 5 of Appendix). Since the finite element strain solution is not the best-fit to the original exact strain, it violates the area preservation rule. Consequently, exact nodal displacements are not recovered by the finite element solution for the MM4C element. This fact is shown in Table 4. The nodal displacement deviations can be attributed to the effects of the extraneous nodal moments. It can be observed from Table 5 that when the extraneous force effect is added, the

Table 4 Deflections of a single element cantilever distorted beam with uniformly varying load

Distortion parameter T =-0.2									
Displacement		PP4I	PP3I	PP2R	PM4I	MM4I	PM4C	MM4C	Exact
Node 2	w_2	16.75	41.49	44.65	30.08	16.82	46.76	44.49	44.91
	θ_2	6.99	17.37	18.89	15.01	8.37	18.96	18.33	18.13
Node 3	w_3	69.84	173.76	179.79	187.46	104.45	174.33	173.67	166.67
	θ_3	9.94	24.81	23.06	37.42	20.83	19.17	20.83	20.83



(a)



(b)

Fig. 13(a) Applied extraneous moments (b) Analytical and FE strains due to extraneous forces

displacement at Node-3 agrees closely with the exact displacement.

6. Conclusions

The paper examines, in the light of variational correctness through function space projections, the performance of the field consistency enforced 3-noded Timoshenko beam element under distortion and locking. Completeness, continuity, consistency and correctness in the metric formulation for a one-dimensional beam problem even under element distortion are attractive features for an element. These have been explored here for the lock-free, distortion immune 3-noded Timoshenko metric beam element and closed form algebraic expressions have been derived.

A single metric beam element subjected to various kinds of loading has been used to demonstrate the performance of the element. The projection formula has been invoked to estimate the best-fit strain from which the finite element strain solutions have been obtained. The best-fit strain is obtained through orthogonal projection of the analytical strain onto the strain displacement function space in the metric domain. The extraneous response in finite element solution, wherever evoked from spurious extraneous forces resulting from variational incorrectness due to 'stress smoothing' formulations, have been generated and shown to be responsible for the slight deviation of the finite element solution from the best-fit one.

It is now evident from the development of spurious extraneous nodal forces that the present field consistency enforced formulation is not fully variationally correct. However, the superiority of the present element over the conventional isoparametric element have been demonstrated by three significant features. Firstly, the present element is immune to distortions of the position of the internal node; it performs better than the isoparametric element under such distortions. Secondly, being field consistent, the present element eliminates any spurious shear strain oscillations that plague the conventional isoparametric element under distortion. Lastly, owing to the symmetric nature of the metric formulation, closed form explicit algebraic expressions for the finite element/best-fit stresses, strains and the errors could be derived from function space projections under mesh distortion and locking in all the test cases studied in the paper.

In the case of 2- and 3-dimensional problems under distortion, the requirements for continuity (for the test functions) and completeness (or the best-fit stress requirement for the trial functions) in the case of the metric element are difficult to achieve together. However, in the case of one-dimensional elements, as seen in the present study, a proper field consistent metric formulation preserves continuity and completeness even under distortion.

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Appendix

Calculation of extraneous forces in field consistency enforced single element test examples 3 and 4

The spurious force vector as given in Eq. (21) is calculated as

$$\{F_E^e\} = \int_{-L/2}^{L/2} \left[[B_{MD}] - [B_{MCD}]\right]^T [D]\{\varepsilon\}\, dx \tag{A1}$$

For a distortion of T=-0.2,

$$\left[[B_{MD}] - [B_{MCD}]\right]^T = \begin{Bmatrix} 0 & 0 \\ 0 & -\frac{5}{24} + \frac{x}{L} - \frac{(2L-5x)x}{2L^2} \\ 0 & 0 \\ 0 & \frac{25}{72} - \frac{25x^2}{6L^2} \\ 0 & 0 \\ 0 & -\frac{5}{36} - \frac{x}{L} + \frac{x(3L+5x)}{3L^2} \end{Bmatrix} \tag{A2}$$

For the test case 3 (with uniformly distributed load) the extraneous force vector vanishes, confirming that finite element strain is the best fit strain given by,

$$\{F_E^e\} = \int_{-L/2}^{L/2} \begin{Bmatrix} 0 & 0 \\ 0 & -\frac{5}{24} + \frac{x}{L} - \frac{(2L-5x)x}{2L^2} \\ 0 & 0 \\ 0 & \frac{25}{72} - \frac{25x^2}{6L^2} \\ 0 & 0 \\ 0 & -\frac{5}{36} - \frac{x}{L} + \frac{x(3L+5x)}{3L^2} \end{Bmatrix} \begin{bmatrix} EI & 0 \\ 0 & KGA \end{bmatrix} \left\{ \begin{pmatrix} \frac{q_0(L-2x)^2}{8EI} \\ \frac{q_0(L-2x)}{2KGA} \end{pmatrix} \right\} dx = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{A3}$$

For the test case 4 (with linearly varying load distribution) the non-vanishing extraneous force vector is given by,

$$\{F_E^e\} = \int_{-L/2}^{L/2} \begin{Bmatrix} 0 & 0 \\ 0 & -\frac{5}{24} + \frac{x}{L} - \frac{(2L-5x)x}{2L^2} \\ 0 & 0 \\ 0 & \frac{25}{72} - \frac{25x^2}{6L^2} \\ 0 & 0 \\ 0 & -\frac{5}{36} - \frac{x}{L} + \frac{x(3L+5x)}{3L^2} \end{Bmatrix} \begin{bmatrix} EI & 0 \\ 0 & KGA \end{bmatrix} \begin{Bmatrix} \frac{q_0(L-2x)^3}{48EIL} \\ \frac{q_0(L-2x)^2}{8KGAL} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{q_0L^2}{144} \\ 0 \\ -\frac{5q_0L^2}{432} \\ 0 \\ \frac{q_0L^2}{216} \end{Bmatrix} \tag{A4}$$

Extraneous response of field consistent finite element to the extraneous forces for test example 4

The extraneous response of the 3-noded beam element due to the spurious nodal forces in Eq. (A4) has been computed by loading the beam with the spurious moments. The finite element strain ε^h for test example 4 considering extraneous nodal moments (refer Section 5.4.1) is given as

$$\varepsilon^h = \bar{\varepsilon} + \Delta\varepsilon^h \tag{A5}$$

The additional bending strain response $\Delta\varepsilon^h$ in the finite element, from the extraneous nodal moments is

$$\Delta\varepsilon^h = -\frac{q_0L^2x}{120\mathrm{EI}} * \frac{1}{\mathrm{L}/2} = -\frac{q_0Lx}{60\mathrm{EI}} \tag{A6}$$

It can be observed from Fig. 13 that the additional linear bending strain response from finite element analysis is exactly a best-fit of the exact bending strains in the element. The finite element solution cannot sense the discontinuity in bending moment at node 2; it gives a linear bending strain

Table 5 Deflections of a single element cantilever distorted beam at Node-3 with uniformly varying load considering extraneous forces

Displacement		Distortion parameter T = - 0.2		
		3-noded beam element - MM4C	3-noded beam element with extraneous nodal moments	Exact
Node 3	w_3	173.67	166.72	166.67
	θ_3	20.83	20.83	20.83

as a best-fit to the analytical solution.

The nodal displacements of Node-3 given in Table 4 has been verified by considering the spurious nodal moments and is as given in Table 5.