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A refined functional and mixed formulation to static analyses of fgm beams

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Abstract. In this study, an alternative solution procedure presented by using variational methods for analysis of shear deformable functionally graded material (FGM) beams with mixed formulation. By using the advantages of Gâteaux differential approaches, a refined complex general functional and boundary conditions which comprises seven independent variables such as displacement, rotation, bending moment and higher-order bending moment, shear force and higher-order shear force, is derived for general thick-thin FGM beams via shear deformation beam theories. The mixed-finite element method (FEM) is employed to obtain a beam element which have a 2-nodes and total fourteen degrees-of-freedoms. A computer program is written to execute the analyses for the present study. The numerical results of analyses obtained for different boundary conditions are presented and compared with results available in the literature.

Keywords: finite element method; high order shear deformation beam theory; functionally graded material beam; static analysis

1. Introduction

The functionally graded materials (FGM) which are known one of the next generation of engineering materials, renovate interface problems due to graded structures and provide high strength and thermal resistance in modern engineering applications. The characteristic properties of typical FGM are high bending-stretching coupling and thermal resistance due to their mechanical properties such as the volume fraction of constituent materials changes gradually. FGMs are usually made of a mixture of ceramic materials and metal materials. Because of the constant change in material properties, it is not possible to accumulate stress accumulations in the material.

A considerable number of researchers employed different beam theories (Li 2008, Li *et al*. 2010, Li *et al*. 2013, Bellifa *et al*. 2016, Hadji *et al*. 2016) and mathematical models (Chakraborty *et al*. 2003, Sina *et al*. 2009, Kim and Reddy 2013) for analyses of FGM beams therefore they have been received great attention.

In the literature, the Euler-Bernoulli beam theory (EBT) has been applied analyses of FGM beams by authors (Sankar 2001, Aydogdu and Taskin 2007, Lee and Lee 2017). The theory disregards the effect of the transverse shear deformations and normal stresses. Because, the theory based on these assumptions; straight lines perpendicular to the transverse normal before deformation remain straight after deformation, the transverse normals are inextensible. So that it is only applied for thin beams, inapplicable for thick beams.

In recent years it has been understood that EBT is not sufficient in modeling static and dynamic behaviors of newly developed modern materials. To overcome this shortcoming, shear deformation beam theories have been developed and used for analyses of FGM beams. One of them the first order shear deformation beam theory (FSDT) has been used for thick FGM beams (Menaa *et al*. 2012, Murin *et al*. 2013, Nguyen *et al*. 2013, Kahya and Turan 2017). The FSDT does not neglect the transverse shear deformation effects that consider a uniform transverse shear stress distribution through the beam thickness with using a shear correction factor. Therefore, it does not satisfy the condition of zero transverse shear stress at the surfaces of beam.

In the last years, The high order shear deformation beam theories (HOBT) without the use of the shear correction factor have been developed to overcome all this deficiencies which are the function of the vertical coordinate components of the vertical displacement components. The HOBT does not neglect the effect of the transverse shear deformations and also satisfy the zero transverse shear stress on the surfaces of the beam because it includes non-linear shear stress distributions provided by cubic, parabolic, trigonometric shear strain shape functions along the beam thickness. To mention some of works based on HOBT in the literature: The exact solutions for FGM beam analyses proposed by Zenkour (2006) based on HOBT using sinusoidal shape functions. Vo *et. al*. (2014) presented the refined shear deformation theory which does not require shear correction factor for static and dynamic analysis of FGM beams. Hadji *et. al*. (2016) developed a new HOBT model and obtained analytical solutions for static and dynamic analyses of FGM beams. Thai and Vo (2012) investigated bending and free vibration analyses of FGM beams by using various HOBTs. The study proposed satisfy the stress-free boundary conditions on the surfaces of the beam. Kadoli *et. al*. (2008) carried out displacement field based on HOBT for static behavior of FGM beams. They

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derived two stiffness matrices. Filippi *et al*. (2015) developed the 1D Carrera unified formulation by using exponential and trigonometric functions for static analyses of FGM beams.

The finite element method (FEM) is a well-known and highly effective technique for the computation of approximate solutions of complex and boundary value problems (Zienkiewicz *et al*. 1977). With the development and widespread use of FGMs, the FEM has begun to be used in the analysis of FGMs. For instance, Alshorbagy *et al*. (2011) used FEM to detect the free vibration characteristics of a FGM beam based on Euler-Bernoulli beam theory. Both axial and transverse material graduations based on a power-law are considered. Shahba *et al*. (2011) investigated free vibration and stability analysis of axially FGM tapered beams using classical and non-classical boundary conditions through FEM. The exact shape functions for uniform homogeneous Timoshenko beam elements are used to formulate the proposed element. Mohanty *et al*. (2011) presented an investigation of the dynamic stability of FGM ordinary beam and sandwich beam on Winkler's elastic foundation using FEM. Niguyen *et. al*. (2016) investigated free vibration of thin-walled FGM open-section beams. Governing differential equations were derived by means of Hamilton's principle. A FEM was developed to formulate the problem.

The main scope of this study, by obtaining a functional of shear deformable FGM beams with general boundary conditions, it is aimed to provide a new contribution to the researches and to gain a different point of view for FGM beam. In this study, an alternative solution method proposed for analyses of FGM beams by using general shear deformation theories, in particularly HOBT, and FEM with general boundary conditions. The FSDT and HOBT used in same kinematic relations and variational equations and solution procedure. FGM structures, differential rather than being expressed in terms of equations and boundary conditions, analysis with energy methods it is much easier to make. The differential field equations was obtained by using the virtual displacement principle in energy principle. In the classical FEM solution, the stress-displacement and strain-displacement relationships are satisfied exactly. However, the differential equations in the boundary conditions are satisfied only in the limit as the number of element increases and the approximation spaces for the different set of unknowns does not obtained independently. The difficulties in developing compatible displacement-based FEM that are computationally effective and the realization that by using variational approaches many more FEM discretization can be developed, led to large research efforts. In these activities various classes of new types of elements have been proposed (Bathe 2006). In this context, the mixed-type FEM model is far more efficient because of variables can be chosen independently and more sensitive. The mathematical analysis and applications of mixed-FEM have been widely developed since the seventies. Nevertheless, in the analysis of the FGM problems, studies using mixed-FEM method are rarely tried in the literature. One needs a method for the differential field equations used to transform to the functional in the mixed-type FEM. In

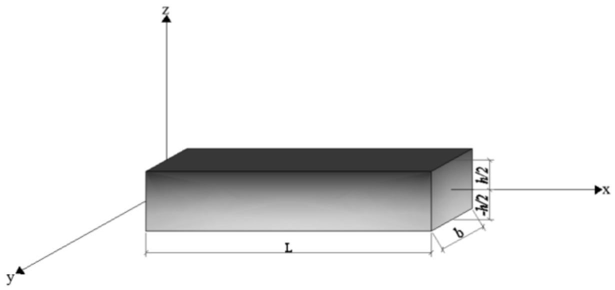


Fig. 1 Geometry and coordinates of FGM beam

this work, the partial differential field equations of FGM beam based on general shear deformation theories successfully transformed a complex refined functional with dynamic and geometric boundary conditions by using Gâteaux differential method (GDM). The mixed-FEM developed for analysis of FGM beams based on refined functional. The element has two nodes and total fourteen unknowns which are the displacement, rotation, shear force higher-order forces, normal moments and higher-order moments on the per-node. Using this method advantage, this freedoms are calculated independently. The element matrices transformed to system matrices by developed analyses program in FORTRAN language. The effects of material distributions, span-depth ratios, boundary conditions are presented and discussed. The performed of the derived element for FSDT and HOBT for static analysis of FGM beams is compared with numerical results of other studies.

2. Theoretical formulations

2.1 Material properties of FGM beam

FGM structures, which are advanced composite materials, can be made of two or more materials. The variation coefficient is expressed by the formula expressed as "power-law". In Fig. 1. shows the geometry model of the FGM beam composed of ceramic and metal. The x-y-z are global coordinates, and length is indicated by "L", width is "b" and thickness is "h". The continuous variation of material properties of FGM beam constituents in power-law which introduced by Reddy (2000) can be defined as given

$$\mathscr{P}_{(z)} = \mathscr{P}_t\left(\frac{z}{h}+\frac{1}{2}\right)^n + \mathscr{P}_b\left[1-\left(\frac{z}{h}+\frac{1}{2}\right)^n\right] \tag{1}$$

In Eq. (1), "$\mathscr{P}_t$" and "$\mathscr{P}_b$" denote the values of the mechanical properties of the top at "$z = h/2$" and bottom at "$z = -h/2$" respectively, such as Young's modulus, Poisson's ratio, material densities etc. of the FGM beam, which are often metal and ceramic materials.

Variable "z" represents the distance to the mid-plane of the beam varying from "$-h/2$" to "$h/2$". The effective material properties are evaluated using Eq. (1), the Young's Modulus "$\mathcal{E}_{(z)}$", Poisson's ratio "$\upsilon(z)$" and shear modulus "$\mathcal{G}(z)$" can be defined by

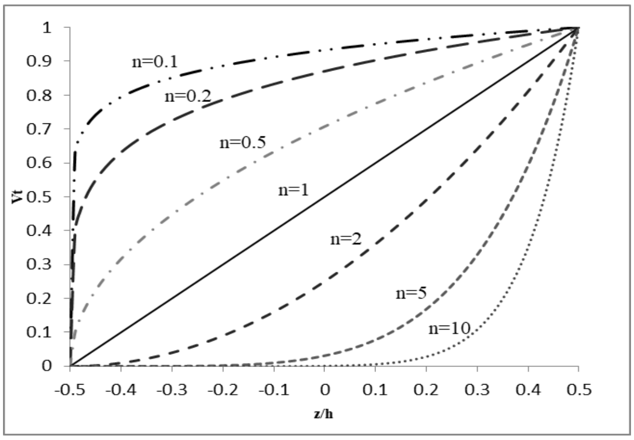


Fig. 2 Distribution of volume fraction exponent of FGM beam

$$\mathcal{E}_{(z)} = (E_t - E_b)(\frac{z}{h}+\frac{1}{2})^n + E_b \tag{2a}$$

$$\upsilon_{(z)} = (\upsilon_t - \upsilon_b)(\frac{z}{h}+\frac{1}{2})^n + \upsilon_b \tag{2b}$$

$$\mathcal{G}(z) = (G_t - G_b)(\frac{z}{h}+\frac{1}{2})^n + G_b \tag{2c}$$

In power-law variation, "n" is a power-law exponent and it is a non-negative variable parameter (n ≥ 0). In Eq. (1) the parameter "$(\frac{z}{h}+\frac{1}{2})^n$" is called volume fraction and denoted "V_t". The variation of the volume fraction of FGM beam is shown in Fig. 2. In this study, the FGM beam is graded from zirconia oxide (ZrO_2) and alumina (Al_2O_3) top surfaces to aluminum (Al) bottom surfaces.

2.2 Kinematic relations based on FSDT and HOBT

According to a generated shear deformation beam theory is employed to FGM beam in this study with displacement field in Eq. (3) to account for the effect of transverse shear strain deformation. The displacement components with respect to the x-y-z directions of any point on the $\mathcal{U}$, $\mathcal{V}^o$ and $\mathcal{W}^o$, respectively.

$$\begin{aligned} \mathcal{U}(x,z) &= z\left[-w'_x\xi_0 + \phi_x\xi_1\right] + \mathcal{F}_{(z)}\,\gamma\,\xi_2 \\ \mathcal{V}^o(x,z) &= 0 \\ \mathcal{W}^o(x,z) &= w_0 \end{aligned} \tag{3}$$

where "$(.)'_x$" is the partial derivatives with respect to x axis, the unknown variables "w_0" is the transverse displacement, "ϕ_x" is the total bending rotation of the cross-sections and "γ" is the transverse shear strain of any point at any point on the neutral axis. The term "$\mathcal{F}_{(z)}$" which is shape function determining the non-linear distribution of the transverse shear effects along the thickness of the beam. In this study it assumed cubic distribution $\mathcal{F}_{(z)} = z(1-\frac{4z^2}{3h^2})$ as Reddy (1984). The displacement field of general shear deformation theory in Eq. (3) contains the kinematics of the beam theories which are FSDT and HOBT obtained by substituting constants such as for FSDT "$\xi_0=0,\ \xi_1=1,\ \xi_2=0$" and for HOBT "$\xi_0=1,\ \xi_1=0,\ \xi_2=1$".

The strain-displacement relationships are

$$\varepsilon_x = \mathcal{U}'_x = z\varepsilon_x^{(0)} + \mathcal{F}_{(z)}\varepsilon_x^{(1)} \tag{4a}$$

$$\gamma_{xz} = \mathcal{U}'_z + \mathcal{W}^{o\prime}_x = \gamma_{xz}^{(0)} + \mathcal{F}'_{(z)}\gamma_{xz}^{(1)} \tag{4b}$$

in where

$$\varepsilon_x^{(0)} = -w''_x\xi_0 + \phi'_x\xi_1 \tag{5a}$$

$$\varepsilon_x^{(1)} = \gamma'_x\xi_2 \tag{5b}$$

$$\gamma_{xz}^{(0)} = w'_x - w'_x\xi_0 + \phi_x\xi_1 \tag{5c}$$

$$\gamma_{xz}^{(1)} = \gamma\xi_2 \tag{5d}$$

The constitutive stress-strain relations for FGM beam by using Eqs. (4) based on a generalized Hooke's law form can be written as follows

$$\begin{Bmatrix}\sigma_x\\ \tau_{xz}\end{Bmatrix} = \begin{bmatrix}\mathcal{C}_{11} & 0\\ 0 & \mathcal{C}_{55}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \gamma_{xz}\end{Bmatrix} \tag{6}$$

where "σ_x" is normal stress and "τ_{xz}" is transverse shear stress and elastic constants "$\mathcal{C}_{ij}$" known as

$$\left(\mathcal{C}_{11},\mathcal{C}_{55}\right) = \left(\mathcal{E}_{(z)},\frac{\mathcal{E}(z)}{2(1+\upsilon_{(z)})}\right) \tag{7}$$

2.3 Variational formulation

In this work, the governing equations are derived by using the principle of virtual displacement which can be expressed as (Reddy 2002)

$$\delta W = \delta W_I + \delta W_E = 0 \tag{8}$$

where the delta "δ" is called the variational operator, "δW_I" is the virtual work due to internal forces and "δW_E" is the virtual work due to the external forces which are obtained as

$$\begin{aligned} \delta W_I &= \int_\Omega\left[\delta\varepsilon_x\sigma_x + \delta\gamma_{xz}\tau_{xz}\right]d\Omega \\ &= \int_0^L\int_A\begin{bmatrix}\sigma_x\,(-z\xi_0\delta w''_x + z\xi_1\delta\phi'_x + \xi_2\mathcal{F}_{(z)}\delta\gamma'_x)\\ +\tau_{xz}\,(\delta w'_x - \xi_0\delta w'_x + \xi_1\delta\phi_x\\ +\xi_2\,\mathcal{F}'_{(z)}\delta\gamma)\end{bmatrix}dxdA \end{aligned} \tag{9a}$$

$$\delta W_E = -\int_0^L q_{(x)}\ \delta w\, dx \tag{9b}$$

The strain relations from Eq. (4) and stress resultants from Eq. (6) are replaced into Eqs. (9) and using Eq. (8), the Euler-Lagrange equations are obtained as

$$\partial w: 0 \rightarrow \quad -\xi_0 M_x'' + \xi_0 Q_{xz}' - Q_{xz}' - q = 0 \qquad 0 < x < L \tag{10a}$$

$$\delta\phi_x: 0 \rightarrow \quad -\xi_1 M_x' + \xi_1 Q_{xz} = 0 \qquad 0 < x < L \tag{10b}$$

$$\delta\gamma: 0 \rightarrow \quad -\xi_2 \mathcal{M}_x' + \xi_2 \mathcal{Q}_{xz} = 0 \qquad 0 < x < L \tag{10c}$$

where "M_x" and "Q_{xz}" are the bending moment and shear force while "$\mathcal{M}_x$" and "$\mathcal{Q}_{xz}$" are the higher-order bending moment and shear force which are expressed as

$$\begin{Bmatrix} M_x \\ \mathcal{M}_x \end{Bmatrix} = \int_{-h/2}^{+h/2} \sigma_x \begin{Bmatrix} z \\ \mathcal{F}_{(z)} \end{Bmatrix} dz \tag{11a}$$

$$\begin{Bmatrix} Q_{xz} \\ \mathcal{Q}_{xz} \end{Bmatrix} = \int_{-h/2}^{+h/2} \tau_{xz} \begin{Bmatrix} 1 \\ \mathcal{F}_{(z)}' \end{Bmatrix} dz \tag{11b}$$

By substituting Eq. (6) and Eq. (4) into Eqs. (11), the constitutive equations can be expressed following

$$\begin{bmatrix} M_x \\ -\xi_2\mathcal{M}_x \\ Q_{xz} \\ -\xi_2\mathcal{Q}_{xz} \end{bmatrix} = \begin{bmatrix} D_{11} & F_{11} & 0 & 0 \\ F_{11} & H_{11} & 0 & 0 \\ 0 & 0 & A_{55} & 0 \\ 0 & 0 & 0 & \mathcal{A}_{55} \end{bmatrix} \begin{Bmatrix} -\xi_0 w_x'' + \xi_1 \phi_x \\ \xi_2 \gamma_x' \\ w_x' - \xi_0 w_x' + \xi_1 \phi_x \\ \xi_2 \gamma \end{Bmatrix} \tag{12}$$

where the rigidities "D_{11}" denote the bending stiffness, "F_{11}" and "H_{11}" the high order stiffnesses, "A_{55}" and "$\mathcal{A}_{55}$" normal and high order extensional stiffnesses defined as follows

$$\left(D_{ij}, F_{ij}, H_{ij}, A_{ij}, \mathcal{A}_{ij}\right) = \int_{-h/2}^{h/2} \mathcal{C}_{ij}\left(z^2,\ z\mathcal{F}_{(z)},\ (\mathcal{F}_{(z)})^2, 1, (\mathcal{F}_{(z)}')^2\right) dz \tag{13}$$

2.4 The functional

In the FEM based on the energy method, an appropriate energy expression is required from the field equations. Using the GDM, all field equations are enforced to the functional by systematic way and boundary conditions can be constructed (Eratli and Akoz 2002). Similar solution procedure was applied to laminated composite beams based many beam theories in a recent study (Özütok and Madenci 2013, Ozutok *et al.* 2014, Özütok and Madenci 2017) by one of the co-authors.

The field equations which are Eqs. (10) and Eqs. (12) including boundary conditions for FGM beam can be written in operator form as "$\mathbf{P} = \mathbf{L}\mathbf{y} - \mathbf{f}$" where L is the differential operator, y is the unknown freedoms and f is the external influence vectors as summarized by Oden and Reddy (1976). This form can be written in matrices form as follows

$$\left[\begin{array}{ccccccc:cccc} 0 & 0 & 0 & L_{1,4} & L_{1,5} & L_{1,6} & L_{1,7} & & & & \\ 0 & 0 & 0 & L_{2,4} & L_{2,5} & L_{2,6} & 0 & & & & \\ 0 & 0 & 0 & L_{3,4} & L_{3,5} & 0 & L_{3,7} & & & & \\ L_{4,1} & L_{4,2} & L_{4,3} & L_{4,4} & L_{4,5} & 0 & 0 & & 0 & & \\ L_{5,1} & L_{5,2} & L_{5,3} & L_{5,4} & L_{5,5} & 0 & 0 & & & & \\ L_{6,1} & L_{6,2} & 0 & 0 & 0 & L_{6,6} & 0 & & & & \\ 0 & 0 & L_{7,3} & 0 & 0 & 0 & L_{7,7} & & & & \\ \hdashline & & & & & & & 0 & 0 & 0 & 1 \\ & & & & & & & 0 & 0 & 1 & 0 \\ & & & 0 & & & & 0 & -1 & 0 & 0 \\ & & & & & & & -1 & 0 & 0 & 0 \end{array}\right] \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ \hdashline \mathbf{w} \\ \Omega \\ \mathbf{M} \\ \mathbf{R} \end{Bmatrix} = \begin{bmatrix} q \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hdashline \hat{\mathbf{R}} \\ \hat{\mathbf{M}} \\ -\hat{\Omega} \\ -\hat{\mathbf{w}} \end{bmatrix} \tag{14}$$

where boundary conditions are written in symbolic form such as $\mathbf{M} - \hat{\mathbf{M}} = 0$ and $\mathbf{R} - \hat{\mathbf{R}} = 0$ are dynamic boundary conditions terms, $-\mathbf{w} + \hat{\mathbf{w}} = 0$ and $-\Omega + \hat{\Omega} = 0$ are geometric boundary conditions terms and "y_i"and "L_i" are obtained following

$$\left\{\begin{array}{c} y_1 = w\xi_0; \quad y_2 = \phi_x \xi_1; \quad y_3 = \gamma\xi_2; \\ y_4 = \xi_0 \left[\dfrac{F_{11}\mathcal{M}_x - H_{11}M_x}{D_{11}H_{11} - F_{11}^2}\right]; \quad y_5 = \xi_0 \left[\dfrac{D_{11}\mathcal{M}_x - F_{11}M_x}{D_{11}H_{11} - F_{11}^2}\right]; \\ y_6 = \xi_1 \left[\dfrac{Q_{xz}}{A_{55}}\right]; \quad y_7 = \xi_2 \left[\dfrac{\mathcal{Q}_{xz}}{\mathcal{A}_{55}}\right] \end{array}\right\} \tag{15a}$$

$$\left.\begin{array}{l} L_{1,4} = D_{11}\dfrac{\partial^2}{\partial x^2};\ L_{1,5} = -F_{11}\dfrac{\partial^2}{\partial x^2};\ L_{1,6} = -A_{55}\dfrac{\partial}{\partial x}; \\ L_{1,7} = \dfrac{\mathcal{A}_{55}}{\xi_2}\dfrac{\partial}{\partial x};\ L_{2,4} = \dfrac{\xi_1}{\xi_0}D_{11}\dfrac{\partial}{\partial x}\ L_{2,5} = -\dfrac{\xi_1}{\xi_0}F_{11}\dfrac{\partial}{\partial x}; \\ L_{2,6} = A_{55};\ L_{3,4} = \dfrac{\xi_2}{\xi_0}F_{11}\dfrac{\partial}{\partial x};\ L_{3,5} = -\dfrac{\xi_2}{\xi_0}H_{11}\dfrac{\partial}{\partial x}; \\ L_{3,7} = \mathcal{A}_{55} L_{4,1} = D_{11}\dfrac{\partial^2}{\partial x^2};\ L_{4,2} = -D_{11}\dfrac{\partial}{\partial x};\ L_{4,3} = -F_{11}\dfrac{\partial}{\partial x}; \\ L_{4,4} = -\dfrac{D_{11}}{\xi_0};\ L_{4,5} = \dfrac{F_{11}}{\xi_0} L_{5,1} = -F_{11}\dfrac{\partial^2}{\partial x^2};\ L_{5,2} = F_{11}\dfrac{\partial}{\partial x}; \\ L_{5,3} = H_{11}\dfrac{\partial}{\partial x};\ L_{5,4} = F_{11}\dfrac{\xi_2}{\xi_0};\ L_{5,5} = H_{11}\dfrac{\xi_2}{\xi_0}; \\ L_{6,1} = -\dfrac{A_{55}}{\xi_0}\dfrac{\partial}{\partial x} + A_{55}\dfrac{\partial}{\partial x};\ L_{6,2} = -A_{55};\ L_{6,6} = \dfrac{A_{55}}{\xi_1}; \\ L_{7,3} = \mathcal{A}_{55};\ L_{7,7} = -\mathcal{A}_{55} \end{array}\right\} \tag{15b}$$

The mathematical procedure of GDM is explained in detail by references (Aköz and Kadioğlu 1996, Aköz and Özütok 2000, Özütok and Madenci 2017). If the operator **P** is a potential, then the functional corresponding to the field equations will be given as Eq. (16)

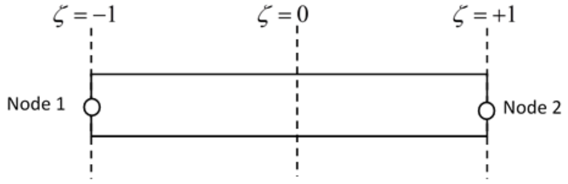


Fig. 3 Dimensionless coordinate system of FGM beam for shape functions

$$\mathbf{I}(\mathbf{y})=\int_0^1[\mathrm{P}(s\,\mathbf{y},\mathbf{y}),\mathbf{y}]\,ds \tag{16}$$

For FGM beam based on general shear deformation beam theory, the functional is obtained with the appropriate boundary conditions as Eq. (17a)

$$\left.\begin{aligned}I_{[y]}=&\left[(\xi_0 M_x'+\xi_1 Q_{xz}),w_x'\right]+\xi_1\left[Q_{xz},\phi_x'\right]+\xi_2\left[\mathscr{M}_x,\gamma_x'\right]\\&+\xi_1\left[Q_{xz},\phi_x\right]+\xi_2\left[\mathscr{Q}_{xz},\gamma\right]-\Delta\left[M_x,\mathscr{M}_x\right]-\\&\frac{\mathrm{B}}{2}\left[M_x,M_x\right]-\frac{A}{2}\left[\mathscr{M}_x,\mathscr{M}_x\right]-\frac{\mathrm{E}}{2}\left[Q_{xz},Q_{xz}\right]-\\&\frac{\mathrm{E}}{2}\left[\mathscr{Q}_{xz},\mathscr{Q}_{xz}\right]+\left[(\hat{\mathrm{w}}-\mathrm{w}),\mathrm{R}\right]_\varepsilon+\left[(\hat{\Omega}-\Omega),\mathrm{M}\right]_\varepsilon\\&-\left[\mathrm{R},\mathrm{w}\right]_\sigma-\left[\hat{\mathrm{M}},\Omega\right]_\sigma-\left[q,w\right]\end{aligned}\right\} \tag{17a}$$

where parentheses with the subscripts notations "σ" and "ε" indicate the boundary conditions, and coefficients are given as Eq. (17b)

$$\left.\begin{aligned}&\mathrm{A}=\frac{\xi_2 D_{11}}{D_{11}H_{11}-F_{11}^2};\quad \mathrm{B}=\frac{\xi_1}{D_{11}}+\frac{\xi_2 H_{11}}{D_{11}H_{11}-F_{11}^2};\\&\Delta=\frac{\xi_2 F_{11}}{D_{11}H_{11}-F_{11}^2};\quad \mathrm{E}=\frac{\xi_1}{A_{55}}+\frac{\xi_2}{\mathscr{A}_{55}}\end{aligned}\right\} \tag{17b}$$

2.5 The mixed-FEM and element matrices

In this part, the mixed element matrix is presented for FGM beam based on FSDT and HOBT. Using the interpolation functions the variable expressed such as "$\mathbf{u}=\sum u_i\psi_i$" at the FGM element nodes. Where "$\psi_{i,j}$" denote the interpolation functions. The variables can be written in vector form for one FGM beam element as

$$\{\mathbf{u}\}_i^e=\left\{\{\mathbf{u}\}_1^e;\{\mathbf{u}\}_2^e\right\} \tag{18a}$$

where

$$\{\mathbf{u}\}_i^e=\left\{w\quad w_x'\quad \phi_x'\quad \gamma\quad \gamma_x'\quad M_x\quad M_x'\quad \mathscr{M}_x\quad Q_{xz}\quad \mathscr{Q}_{xz}\right\} \tag{18b}$$

It is possible to express element coordinates and element unknowns in isoparametric finite element definition by using natural coordinate system.

The interpolation functions used to obtain element matrices can be defined by using a dimensionless and normalized coordinate system (in Fig. 3) as Eq. (19)

Table 1 Material properties

Material No	Top Material E_t (GPa)	Bottom Material E_b (GPa)	Poisson's ratio $\nu_t=\nu_b$	L/h
Material 1	200 (ZrO_2)	70 (Al)	0.3	4-16
Material 2	380 (Al_2O_3)	70 (Al)	0.3	5-20

$$\{\psi_1,\psi_2\}=\frac{1}{2}\{1\pm\zeta\} \tag{19}$$

where "ζ" is the dimensionless local coordinate. The interpolation function expressions and their partial derivatives are required for the calculation of the integration element matrix on the FGM beam element. To give an explicit form of the element matrix for FGM beams, the following sub matrices are defined as Eq. (20a-c)

$$[k_1]=\int_{-1}^{1}\left\{\psi_i\ \psi_j\right\}dx=\begin{bmatrix}\frac{L}{3}&\frac{L}{6}\\\frac{L}{6}&\frac{L}{3}\end{bmatrix}\qquad (i,j=1,2) \tag{20a}$$

$$[k_2]=\int_{-1}^{1}\left\{\psi_i'\ \psi_j'\right\}dx=\begin{bmatrix}\frac{1}{L}&-\frac{1}{L}\\-\frac{1}{L}&\frac{1}{L}\end{bmatrix}\qquad (i,j=1,2) \tag{20b}$$

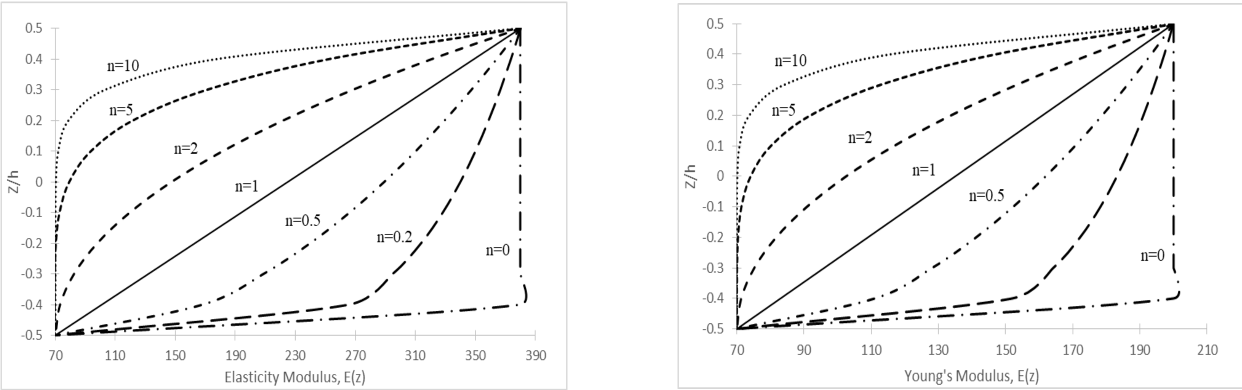
$$[k_3]=\int_{-1}^{1}\left\{\psi_i\ \psi_j'\right\}dx=\begin{bmatrix}-\frac{1}{2}&\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}\end{bmatrix}\qquad (i,j=1,2) \tag{20c}$$

Thus, the mixed-FEM matrices of FGM beam element based on generation shear deformation which has seven degree-of-freedom at the per node is obtained as

$$[k_{el}]=\begin{bmatrix}[w]&[\phi_x]&[\gamma]&[M_x]&[\mathscr{M}_x]&[Q_{xz}]&[\mathscr{Q}_{xz}]\\\hline [0]&[0]&[0]&\xi_2[k_3]&[0]&\xi_1[k_2]^T&[0]\\&[0]&[0]&\xi_1[k_2]&[0]&\xi_1[k_1]&[0]\\&&[0]&[0]&\xi_2[k_2]^T&[0]&\xi_2[k_1]\\&&&-\mathrm{B}[k_1]&\Delta[k_1]&[0]&[0]\\&&&&-\mathrm{A}[k_1]&[0]&[0]\\&&sym.&&&-\mathrm{E}[k_1]&[0]\\&&&&&&-\mathrm{E}[k_1]\end{bmatrix}\begin{Bmatrix}q[k_1]\\[0]\\[0]\\[0]\\[0]\\[0]\\[0]\end{Bmatrix} \tag{21}$$

3. Numerical results and discussion

In this section, the performance of the functional and element matrices of the mixed-FEM solutions based on FSDT and HOBT are evaluated with several numerical examples in bending of the FGM beams. The FGM beam is taken to be made of ceramic and metal riches with the following material properties which are change through the



(a) ZrO_2/Al (b) Al_2O_3/Al

Fig. 4 Variation of the Young's modulus of ZrO_2/Al (a) and Al_2O_3/Al (b) along the FGM beam with power-law exponent

Table 2 Maximum non-dimensional transverse deflections of the S-S FGM beams by FSDT

Power-law index	L/h=4		L/h=16	
	Present FSDT	Vo *et al.* (2014)	Present FSDT	Vo *et al.* (2014)
$n=0$ (Full Ceramic)	0.40465	0.40460	0.35382	0.35341
$n=0.2$	0.46862	0.46874	0.40833	0.41133
$n=0.5$	0.53165	---	0.46634	---
$n=1$	0.64127	0.64281	0.56418	0.56698
$n=2$	0.73477	0.73516	0.64051	0.64483
$n=5$	0.82332	0.82401	0.71158	0.71232
$n=10$	0.89765	0.89517	0.77238	0.77004
Full Metal	1.06427	1.06500	1.00503	1.00406

Table 3 Maximum non-dimensional transverse deflections of the C-C FGM beams by FSDT

Power-law index	L/h=4		L/h=16	
	Present FSDT	Vo *et al.* (2014)	Present FSDT	Vo *et al.* (2014)
$n=0$ (Full Ceramic)	0.62623	0.62300	0.37050	0.36706
$n=0.2$	0.71356	0.71366	0.42656	0.42663
$n=0.5$	0.81418	---	0.48740	---
$n=1$	0.96750	0.96628	0.58858	0.58711
$n=2$	1.12050	1.12044	0.66908	0.66879
$n=5$	1.29475	1.30041	0.74638	0.74200
$n=10$	1.42787	1.42898	0.80168	0.80335
Full Metal	1.78137	1.78000	1.05786	1.04875

thickness of the beam according to power-law given in Table 1.

For convenience, the following non-dimensional terms are used, the vertical displacement and stresses of FGM beams under the uniformly distributed load q

$$\left.\bar{w}=\frac{1}{K}\frac{w}{}\frac{E_{Al}I}{qL^4};\quad \bar{\sigma}_x=\frac{h}{qL}\sigma_x(L/2,h/2);\quad \bar{\tau}_{xz}=\frac{h}{qL}\tau_{xz}(0,0)\right\} \quad (22)$$

where "$I=bh^3/12$" and "$K=\{5/384;\ 1/8;\ 1/384\}$" for simply-supported (S-S), cantilever (C-F) and clamped-clamped (C-C) boundary conditions, respectively.

In the numerical examples, two type material properties are considered: Aluminum (Al) is as metal, Zirconia (ZrO_2) and Alumina (Al_2O_3) are as ceramic of FGM beam and other properties are given in Table 1. Using the relation in Eq. (2) it is possible to obtain an insight into the variation of the material properties across the thickness of the beam for different power-law exponents. Fig. 4. illustrates the variation of Young's modulus of Zirconia/Aluminum (ZrO_2/Al) and Alumina/Aluminum (Al_2O_3/Al) FGM beams, respectively.

The first example is about the non-dimensional maximum displacement of FGM beams according to different parameters which are boundary conditions, power-law exponent, shear correction factors and span-depth ratio.

Table 4 Maximum non-dimensional transverse deflections of the C-F FGM beams by FSDT

Power-law index	L/h=4		L/h=16	
	Present FSDT	Vo *et al.* (2014)	Present FSDT	Vo *et al.* (2014)
$n=0$ (Full Ceramic)	0.37315	0.37275	0.35191	0.35142
$n=0.2$	0.42966	0.43209	0.40567	0.40910
$n=0.5$	0.49091	---	0.46370	---
$n=1$	0.59270	0.59564	0.56117	0.56404
$n=2$	0.67450	0.67897	0.63685	0.64134
$n=5$	0.75364	0.75453	0.69710	0.70800
$n=10$	0.80968	0.81732	0.75771	0.76518
Full Metal	1.06604	1.06500	1.00474	1.00406

Material properties chosen as Material-1 (ZrO_2/Al), three different boundary conditions and the two span-depth ratio that thick (L/h = 4) and thin (L/h = 16) FGM beams are considered in the analysis. Present numerical results are compared with the finite element solutions based on FSDT and HOBT theories results by reference (Vo *et al*. 2014) are shown in Tables 2-7.

The non-dimensional vertical displacements along the length of the beam are plotted with simply-supported boundary conditions in Fig. 5(a)-5(b). for thick and thin

Table 5 Maximum non-dimensional transverse deflections of the S-S FGM beams by HOBT

Power-law index	L/h=4		L/h=16	
	Present FSDT	Vo *et al.* (2014)	Present FSDT	Vo *et al.* (2014)
$n=0$ (Full Ceramic)	0.41067	0.40452	0.35356	0.35341
$n=0.2$	0.46912	0.46805	0.41218	0.41129
$n=0.5$	0.52120	---	0.47364	---
$n=1$	0.65005	0.64269	0.56702	0.56698
$n=2$	0.73900	0.73884	0.64565	0.64507
$n=5$	0.83661	0.83544	0.71278	0.71305
$n=10$	0.91000	0.90566	0.77100	0.77071
Full Metal	1.06540	1.06321	1.00536	1.00403

Table 6 Maximum non-dimensional transverse deflections of the C-C FGM beams by HOBT

Power-law index	L/h=4		L/h=16	
	Present FSDT	Vo *et al.* (2014)	Present FSDT	Vo *et al.* (2014)
$n=0$ (Full Ceramic)	0.60854	0.60773	0.36704	0.36676
$n=0.2$	0.69502	0.69410	0.42777	0.42611
$n=0.5$	0.81121	---	0.51202	---
$n=1$	0.94416	0.94365	0.58712	0.58667
$n=2$	1.11100	1.11025	0.67000	0.66943
$n=5$	1.31887	1.31813	0.74496	0.74488
$n=10$	1.43888	1.43793	0.80656	0.80586
Full Metal	1.73638	1.73637	1.04810	1.04789

Table 7 Maximum non-dimensional transverse deflections of the C-F FGM beams by HOBT

Power-law index	L/h=4		L/h=16	
	Present FSDT	Vo *et al.* (2014)	Present FSDT	Vo *et al.* (2014)
$n=0$ (Full Ceramic)	0.37254	0.37212	0.35148	0.35141
$n=0.2$	0.43331	0.43209	0.41002	0.40907
$n=0.5$	0.51268	---	0.47962	---
$n=1$	0.59602	0.59471	0.56488	0.56402
$n=2$	0.68111	0.67937	0.64196	0.64141
$n=5$	0.75889	0.75773	0.70901	0.70827
$n=10$	0.82112	0.81997	0.76603	0.76543
Full Metal	1.06354	1.06321	1.00444	1.00403

FGM beams according to changes of power-law exponent. All the displacements increase with the increase of the power-law exponent value.

The effect of elasticity ratio (E_t/E_b) on the FGM beam are investigated base on power-law exponent. Tables 8-9. show the maximum non-dimensional displacement of simply-supported FGM beams are presented with effect of the Young’s modulus ratio for material-1. It can be seen that for a constant power-law exponent, the displacement

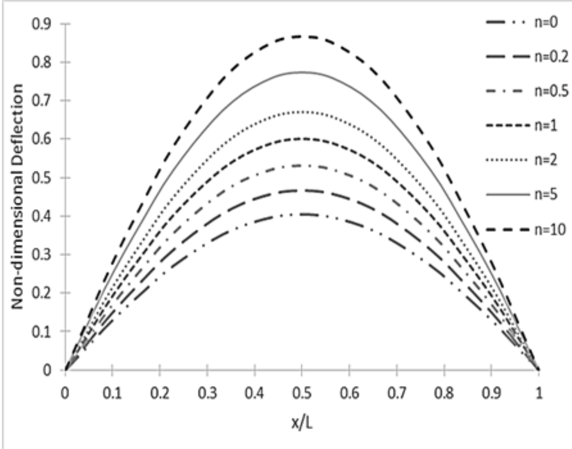


Fig. 5(a) Non-dimensional displacements along the FGM beam length with L/h = 4

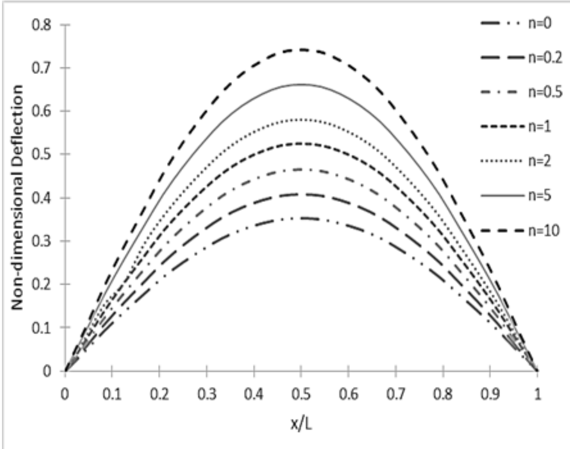


Fig. 5(b) Non-dimensional displacements along the FGM beam length with L/h = 16

decreases with increasing elasticity ratio. Also, the displacements decrease with the increase of the power-law exponent value for ($E_t/E_b<1$) while, the displacements increase with the increase of the power-law exponent value for ($E_t/E_b>1$), and the displacements does not change with the increase of the power-law exponent value for ($E_t/E_b=1$).

The non-dimensional axial stresses and transverse shear stresses in FGM beams are presented for material-2 in Table 10. Present results compared with other results of references. In Table 10., dimensionless axial stresses values of simply supported FGM beam at point ($x=L/2, z=h/2$), transverse shear stresses ($\bar{\tau}_{xz}$) at point ($x=0, z=0$) with changes of power-law exponent for thick ($L/h=5$) and thin ($L/h=20$) FGM beams presented and results are compared with FEM solutions based on FSDT and HOBT of references. The shear correction factor is not effective on the axial stress.

The Figs. 6(a)-6(b) show the non-dimensional axial stress distributions of simply-supported FGM beam based on FSDT and HOBT at mid-span and ($L/h=5$) for various

Table 8 Effects of Young’s Modulus ratio on non-dimensional mid-span displacements of S-S FGM beams with $L/h = 4$

E_t/E_b	Theory	Power-law exponent, n						
		$n=0$	$n=0.2$	$n=0.5$	$n=1$	$n=2$	$n=5$	$n=10$
0.25	Present FSDT	4.62500	2.89700	2.20620	1.85150	1.63820	1.45250	1.34100
	Present HOBT	4.64101	2.90156	2.23554	1.87654	1.64201	1.46321	1.34752
0.5	Present FSDT	2.31450	1.92970	1.69420	1.54300	1.43870	1.33820	1.27320
	Present HOBT	2.32125	2.00547	1.79854	1.56235	1.45215	1.35021	1.29203
1.0	Present FSDT	1.06427	1.06427	1.06427	1.06427	1.06427	1.06427	1.06427
	Present HOBT	1.06540	1.06540	1.06540	1.06540	1.06540	1.06540	1.06540
2.0	Present FSDT	0.57850	0.64270	0.70850	0.77150	0.83220	0.91250	0.98020
	Present HOBT	0.57901	0.65333	0.72004	0.79016	0.84845	0.92456	0.99016
4.0	Present FSDT	0.28920	0.34020	0.39900	0.46300	0.53300	0.64300	0.75320
	Present HOBT	0.29116	0.34202	0.41203	0.46895	0.54652	0.64752	0.75852
6.0	Present FSDT	0.19280	0.23130	0.27770	0.33070	0.39220	0.49720	0.61320
	Present HOBT	0.20120	0.24015	0.28121	0.33846	0.41053	0.50125	0.62123

Table 9 Effects of Young’s Modulus ratio on non-dimensional mid-span displacements of S-S FGM beams with $L/h = 16$

E_t/E_b	Theory	Power-law exponent, n						
		$n=0$	$n=0.2$	$n=0.5$	$n=1$	$n=2$	$n=5$	$n=10$
0.25	Present FSDT	4.04390	2.50780	1.91400	1.61720	1.44330	1.28510	1.18350
	Present HOBT	4.04562	2.53125	2.0112	1.63258	1.45236	1.30125	1.19235
0.5	Present FSDT	2.02140	1.67870	1.47460	1.34760	1.26360	1.17920	1.12010
	Present HOBT	2.12035	1.69045	1.49235	1.36254	1.26895	1.18562	1.132015
1.0	Present FSDT	1.00503	1.00503	1.00503	1.00503	1.00503	1.00503	1.00503
	Present HOBT	1.00536	1.00536	1.00536	1.00536	1.00536	1.00536	1.00536
2.0	Present FSDT	0.50540	0.56290	0.62060	0.67400	0.72240	0.78710	0.84620
	Present HOBT	0.51523	0.57201	0.63125	0.68420	0.74023	0.79684	0.85623
4.0	Present FSDT	0.25270	0.29840	0.35020	0.40440	0.45990	0.54560	0.63830
	Present HOBT	0.26321	0.30230	0.36895	0.42132	0.47235	0.55896	0.64251
6.0	Present FSDT	0.16840	0.20300	0.24390	0.28880	0.33740	0.41750	0.51260
	Present HOBT	0.17125	0.22145	0.25645	0.29865	0.34125	0.42785	0.52123

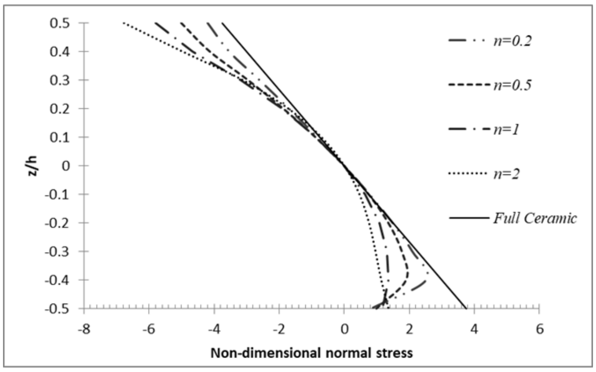


Fig. 6(a) Non-dimensional axial stress $\bar{\sigma}_x$ distributions of S-S FGM beam with the power-law exponent, FSDT

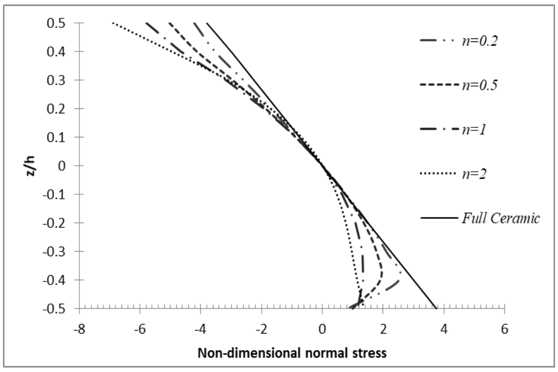


Fig. 6(b) Non-dimensional axial stress $\bar{\sigma}_x$ distributions of S-S FGM beam with the power-law exponent, HOBT

values of power-law exponent. The axial stress distributions of FGM beams are different from isotropic beam such as full ceramic.

The Figs. 7(a)-7(b) shows the non-dimensional shear

Table 10 Non-dimensional axial stress $\bar{\sigma}_x(L/2, h/2)$ and shear stress $\bar{\tau}_{xz}(0,0)$ of S-S FGM beam

Power-law index	Theory	L/h=5		L/h=20	
		$\bar{\sigma}_x$	$\bar{\tau}_{xz}$	$\bar{\sigma}_x$	$\bar{\tau}_{xz}$
$n=0$ (Full Ceramic)	Present FSDT	3.7500	0.6000	15.0000	0.6000
	Present HOBT	3.8017	0.7500	15.0101	0.7500
	FSDT (Vo *et al*. 2015)	3.7520	0.5850	15.0100	0.5850
	HOBT (Li *et al*. 2010)	3.8020	0.7500	15.0130	0.7500
	HOBT (Thai and Vo 2012)	3.8020	0.7332	15.0129	0.7451
	HOBT (Vo *et al*. 2015)	3.8040	0.7335	15.0200	0.7470
$n=0.2$	Present FSDT	4.2010	0.6211	17.4043	0.6211
	Present HOBT	4.2200	0.7204	17.4100	0.7204
$n=0.5$	Present FSDT	5.0002	0.6271	19.6210	0.6271
	Present HOBT	5.0202	0.7512	19.6875	0.7512
	HOBT (Li *et al*. 2010)	4.9925	0.7676	19.7005	0.7676
	HOBT (Thai and Vo 2012)	4.9924	0.7504	19.7004	0.7620
$n=1$	Present FSDT	5.8003	0.6000	23.1333	0.6000
	Present HOBT	5.8916	0.7500	23.2189	0.7500
	FSDT (Vo *et al*. 2015)	5.7990	0.5850	23.2000	0.5850
	HOBT (Li *et al*. 2010)	5.8837	0.7500	23.2054	0.7500
	HOBT (Thai and Vo 2012)	5.8836	0.7332	23.2053	0.7451
	HOBT (Vo *et al*. 2015)	5.8870	0.7335	23.2200	0.7470
$n=2$	Present FSDT	6.7653	0.5150	27.0814	0.5150
	Present HOBT	6.8985	0.6789	27.1105	0.6789
	FSDT (Vo et al. 2015)	6.7710	0.4978	27.0800	0.4978
	HOBT (Li *et al*. 2010)	6.8812	0.6787	27.0989	0.6787
	HOBT (Thai and Vo 2012)	6.8826	0.6706	27.0991	0.6824
	HOBT (Vo *et al*. 2015)	6.8860	0.6700	27.1100	0.6777
$n=5$	Present FSDT	7.9864	0.3929	31.6459	0.3929
	Present HOBT	8.1176	0.5800	31.8256	0.5800
	FSDT (Vo *et al*. 2015)	7.9470	0.3832	31.7900	0.3832
	HOBT (Li *et al*. 2010)	8.1030	0.5790	31.8112	0.5790
	HOBT (Thai and Vo 2012)	8.1106	0.5905	31.8130	0.6023
	HOBT (Vo *et al*. 2015)	8.1150	0.5907	31.8300	0.6039
$n=10$	Present FSDT	9.6028	0.4296	38.1115	0.4296
	Present HOBT	9.7201	0.6439	38.1363	0.6439
	FSDT (Vo *et al*. 2015)	9.5290	0.4189	38.1100	0.4189
	HOBT (Li *et al*. 2010)	9.7063	0.6436	38.1372	0.6436
	HOBT (Thai and Vo 2012)	9.7122	0.6467	38.1385	0.6596
	HOBT (Vo *et al*. 2015)	9.7170	0.6477	38.1600	0.6682

stress distributions of simply-supported FGM beam based on FSDT and HOBT at $x=0$ and $L/h=5$ for various power-law exponent. The non-dimensional shear stresses of the isotropic beam (the full ceramic beam) come across with each other, and they are symmetric about the mid-plane of the beam. Also, as is known from strength of materials, it can be observed from this figure that the value of the shear stresses is maximum on the neutral axis of the beam. But, the shear stress distributions of FGM beams are greatly influenced by the power-law exponent.

4. Conclusions

In this study, generalized shear deformation beam theory with the total fourteen unknowns is used to analyze the bending of FGM beams with mixed-FEM. The kinematic relations written based on FSDT and HOBT, together. By using the Gâteaux differential method to partial differential field equations, the refined complex general functional is obtained for thick-thin FGM beams. It provides the consistency of the field equations and does not

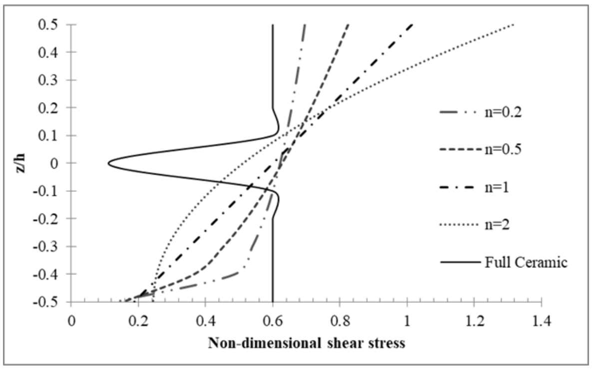


Fig. 7(a) Non-dimensional axial stress $\bar{\tau}_{xz}$ distributions of S-S FGM beam with the power-law exponent, FSDT

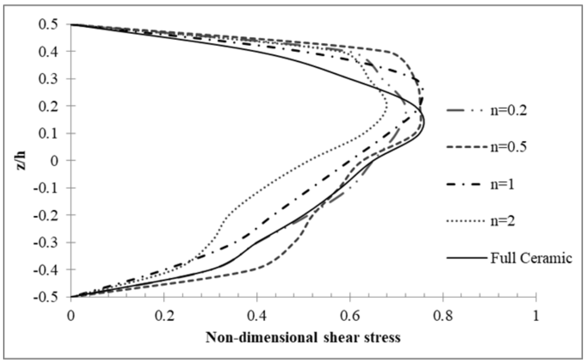


Fig. 7(b) Non-dimensional axial stress $\bar{\tau}_{xz}$ distributions of S-S FGM beam with the power-law exponent, HOBT

exhibits shear locking problem. By applying the mixed-FEM, the FGM beam element matrices was derived. The FGM beam element has a total fourteen degree-of-freedoms and they are calculated independently. This element matrices are provided convenience for analysis of FGM beams and add innovation in literature. Using the obtained mixed-FEM matrices of FGM beam, influence of power-law exponent, side to thickness ratio, shear correction factor, boundary conditions on displacement and stresses of FGM beams have been investigated and discussed. The obtained results show good agreement with those available in the literature.

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