Structural Engineering and Mechanics, *Vol. 68, No. 6 (2018) 713-720*
DOI: https://doi.org/10.12989/sem.2018.68.6.713

Numerical method to determine the elastic curve of simply supported beams of variable cross-section

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(Received August 22, 2017, Revised December 22, 2017, Accepted November 7, 2018)

Abstract. In this paper a new numerical method to determine the elastic curve of the simply supported beams of variable cross-section is demonstrated. In general case it needs to solve linear or small nonlinear second order differential equations with prescribed boundary conditions. For numerical solution the initial values of the slope and the deflection of the end cross-section of the beam is necessary. For obtaining the initial values a lively procedure is developed: it is a special application of the shooting method because boundary value problems can be transformed into initial value problems. As a result of these transformations the initial values of the differential equations are obtained with high accuracy. Procedure is applied for calculating of elastic curve of a simply supported beam of variable cross-section. Results of these numerical procedures, analytical solution of the linearized version and finite element method are compared. It is proved that the suggested procedure yields technically accurate results.

Keywords: elastic curve; simply supported beams of variable cross-section; initial guess for slope and deflection

1. Introduction

Many parts of different machines and structures are modeled as simply supported beams of variable cross-section. The loading of the beam is assumed to be concentrated or continually distributed along the beam, but very often it is an assembly of two or more loadings. Due to static loading, the axis of the beam bends. Timoshenko and Goodier (1952) gave the formula for calculating the elastic curves, $y(z)$, of the supported beams of variable cross-section and loaded with bending moment $M(z)$ in nonlinear case

$$EI(z)\frac{y''(z)}{\left(1+y'^2(z)\right)^{\frac{3}{2}}}=-M(z), \tag{1a}$$

and in linear case

$$EI(z)y''(z)=-M(z), \tag{1b}$$

where $EI(z)$ is flexural rigidity of the beam, E is modulus of elasticity, $I(z)$ is moment of inertia of the cross section about its neutral axis, $M(z)$ represents the bending moment function of the beam, z is the position coordinate, while (')=d/dz and (")= $\mathrm{d}^2/\mathrm{d}z^2$. Unfortunately, the relation (1a) is nonlinear and not easy to be solved.

In case of analytical solution, to overcome the problem for practical reasons, the relation is simplified and the nonlinearity is neglected. The obtained results are valid for the systems with small nonlinearity. Nowadays, when the beams are made from composites or metamaterials for example, the approximation of the linear type is not correct.

Besides, the real systems require us to include the additional deformation effects like compression and shear into calculation of the elastic curve. Thus, in publications (Rojas 2014, Rojas and Espino 2015, Rojas *et al.* 2016), mathematical models of elastic curves for simply supported beams subjected to a uniformly distributed load and a concentrated load located anywhere along length of beam, where the shear effect is also considered, are presented. As it is well known, the traditional models of elastic curves and equations of slopes for tangents to the elastic curves for simply supported beams did not include the shear deformations. The developed models are more appropriate and realistic, but more complex. Usually, they are strong nonlinear and their analytical consideration is a heavy task. To simplify the nonlinear bending problem various linearization techniques are introduced and specified for solving differential equation of elastic curve: the locally transversal linearization method (Ramachandra and Roy 2002, Roy and Kumar 2005), the multi-step linearization techniques (Kumar *et al.* 2004, 2006), the multi-step transversal and tangential linearization methods (Viswanath and Roy 2007, Merli *et al.* 2010).

However, these nonlinear equations of elastic curves need to be solved numerically. The problem is a boundary value one (Ramachandra and Roy 2001, 2002), and the solution has to satisfy the boundary conditions. Very often the solutions are obtained using the shooting procedure. It

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ISSN: 1225-4568 (Print), 1598-6217 (Online)

Fig. 1 Sketch of arbitrary loaded simply supported beam

seems to be a simple method, but is quite stochastic and selection of the accurate initial conditions depends on the experience of the engineer. For this reason, its practical application is limited.

To escape the mathematical barrier connected with the problem, we suggest a method which transforms the boundary value problem into the initial value problem which is much simpler for treating. The main part of the method is concentrated on determination of the initial conditions necessary for calculation. Then some direct integration schemes such Euler, Runge-Kutta, Adams-Bashfort or Newmark which are incorporated into some software like the MATLAB finite difference solver, ANSYS and COMSOL with space discretization, Mathematica (Fertis 2006, Thankane and Stys 2009), etc. can be applied.

In this paper the Euler method was applied. As it is well-known for the numerical solution we need two initial values: the slope y'_o and the displacement y_o. The effective solution of boundary problems requires accurate values of initial conditions. In case of incorrect initial values the solution could diverge.

In this paper applying a special kind of transformation the boundary value problem is transformed into the initial value problem. The essence of the matter is to calculate the initial values which satisfy the boundary conditions with high accuracy. Bíró and Cveticanin (2016) suggested a simple procedure for transformation of the boundary to initial value problem. The beam was supported on the left side and the initial deflection y_o equals zero, because the support on the left side and the origin of the frame fixed to the beam coincide with each other. For this reason, a simple rotational transformation was enough to calculate the initial slope to start the numerical procedure.

The paper contains five sections. In Sec.2, the corollary about transformation from the boundary to initial value problem for the equation of elastic curve is introduced and proved. In Sec.3 the procedure for numerical calculation of the initial conditions according to the corollary is specified. In Sec.4 two examples are discussed: the first one is a simply supported beam of variable cross-section and the second one is a beam of uniform strength having circular cross-section. The numerically calculated initial conditions for the linear elastic curve equations are compared with analytically obtained ones for the linearized elastic curve equations and results of finite element analysis. Paper ends with conclusions. This of method is applicable in case of small nonlinearity and small deformation.

2. Transformation procedure of boundary value problems into initial value problems

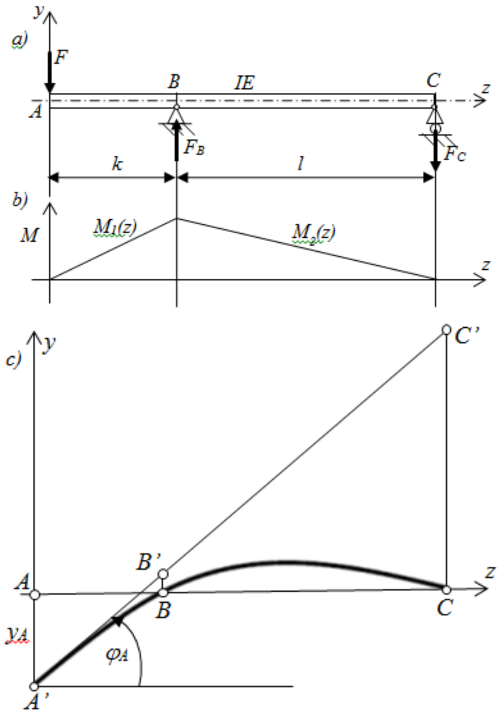

Fig. 2 Sketch of overhanging beam loaded on the end of the cantilever, the bending moment and the form of the elastic curve of the beam

Corollary: The elastic curve of the arbitrary loaded simply supported beam in case of small deformation (Fig. 1) can be determined according to Eq. (1b) as the initial value problem if the initial displacement and the corresponding slope of the end point of the beam are obtained based on the known boundary conditions.

Active loading of the beam can be separated into two groups: forces acting on the cantilever of the beam or between its supports.

Proof. To prove the Corollary two different types of simple supported beam are considered. Let us assume a simply supported overhanging beam (Fig. 2(a)) where the supports are in B and C and the active force F is located at the end A. The distance between supports is l and of the load to the support in B is k. The bending rigidity of the beam is EI.

The bending moment and the form of the elastic curve in case of uniform cross-section can be seen in Fig. 2(b)) and 2(c)) in case of linearized model.

As it can be seen in Fig. 2(c), y_A is the initial deflection and $y'_A=\varphi_A$ is the initial slope of end cross section. Drawing the tangent line to point A' the length $\overline{A'C'}$ can be obtained.

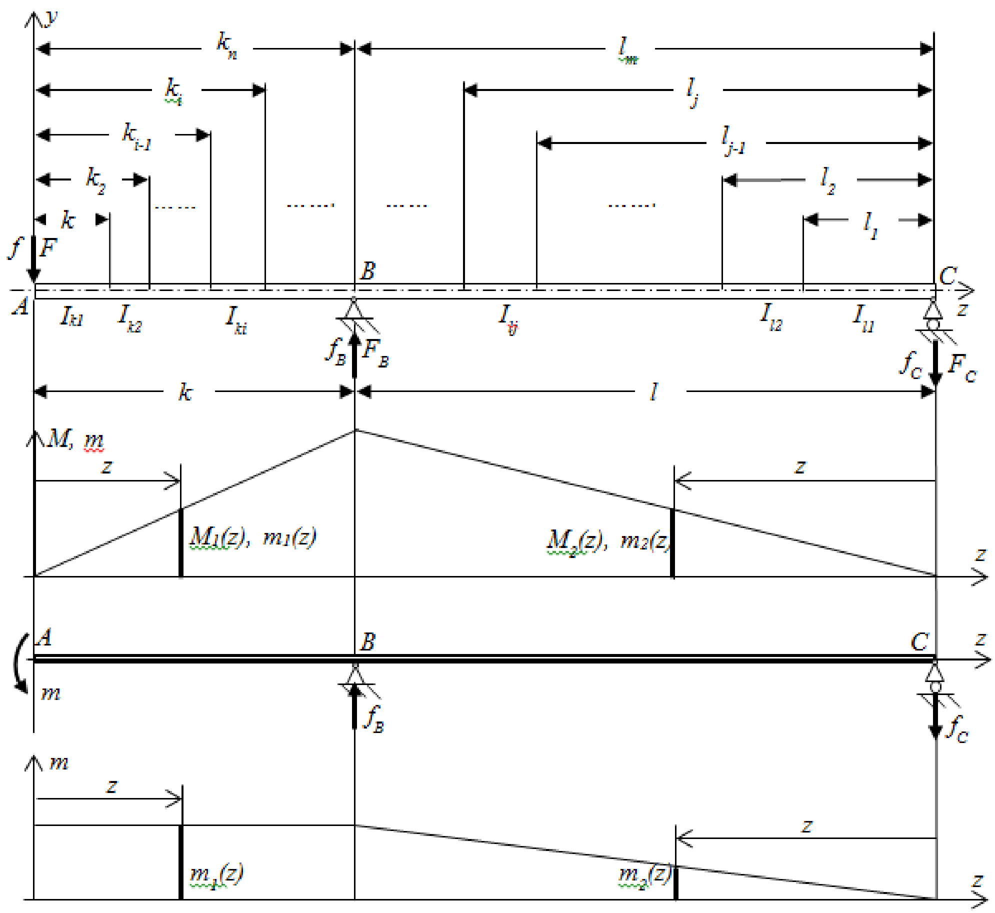

Fig. 3 Sketch of simple supported beam of sectional variable cross-section loaded on the end of the cantilever ($k_o=l_o=0$)

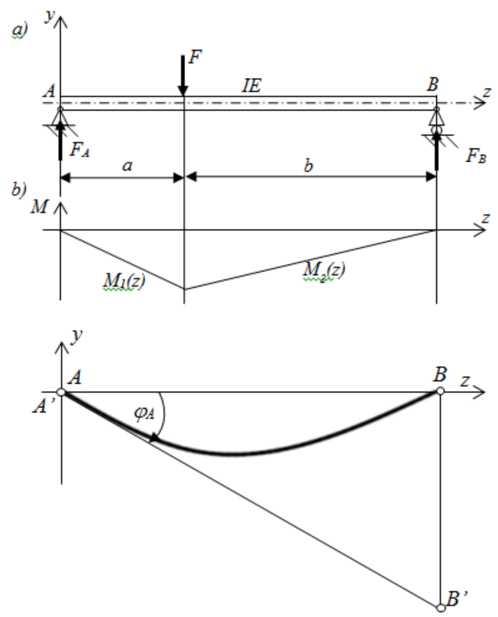

Fig. 4 Sketch of simple supported beam loaded between its supports, moment functions and the form of the elastic curve

This tangent line is axis z itself rotated by angle φ_A and translated by distance y_A. On the basis of Fig. 2(c) in case of small deformation and linearized model

$$\overline{BB'} = \varphi_A k - y_A, \quad \overline{CC'} = \varphi_A (k + l) - y_A. \tag{2}$$

After rearrangement

$$\frac{\overline{CC'} - \overline{BB'}}{z_C - z_B} = \varphi_A, \tag{3}$$

i.e.,

$$\varphi_A (k + l) - y_A - (\varphi_A k - y_A) = \varphi_A l. \tag{4}$$

As it can be noticed (4) is an identical equation. It is true for any values of dimensions k and l moreover for any values of initial deflection y_A and slope $y'_A=\varphi_A$ in case of small deformation and linearized model calculated for overhanging beam of any continuously variable cross-section.

Similarly to Fig. 2, in Fig. 3, the sketch of a simply supported overhanging beam of sectional variable cross-section can be seen. The beam is loaded on the end of the cantilever.

Applying notations of Fig. 3, the moment functions of real loading are

$$M_1(z) = Fz \ , M_2(z) = \frac{Fk}{l} z \tag{5}$$

loaded by unit force

$$m_1(z) = fz \ , m_2(z) = \frac{fk}{l} z \tag{6}$$

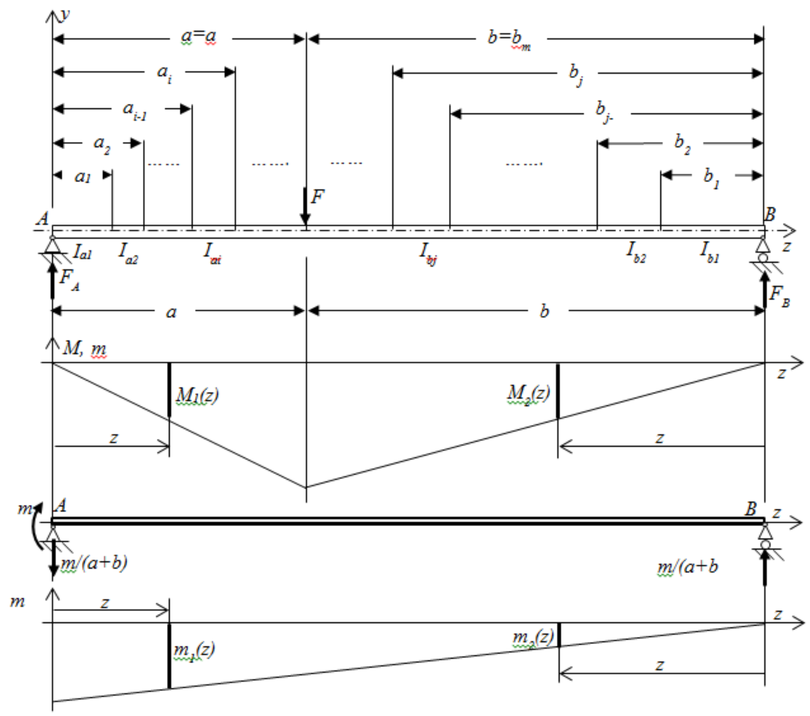

Fig. 5 Sketch of a simply supported beam of sectional variable cross-section loaded between its supports and moment functions ($a_o=b_o=0$)

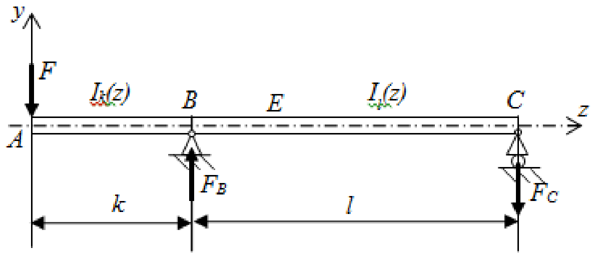

Fig. 6 Sketch of simple supported overhanging beam loaded on the end of the cantilever. The moment of inertia of the cross section about its neutral axis is continuous function of position coordinate z

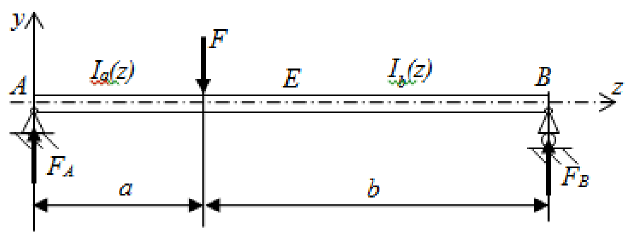

Fig. 7 Sketch of simple supported beam loaded between its supports. The moment of inertia of the cross section about its neutral axis is continuous function of position coordinate z

and finally, by unit moment

$$m_1(z)=m\ ,m_2(z)=\frac{m}{l}z \tag{7}$$

Applying notations of Fig. 3 deflection (8) and slope (9) of the end cross-section of the beam of sectional variable cross-section applying of Betti-theorem are

$$\begin{aligned} y_A=&\frac{F}{E}\left[\int_0^{k_1}\frac{z^2}{I_{k1}}dz+\int_{k_1}^{k_2}\frac{z^2}{I_{k2}}dz+\dots+\int_{k_{i-1}}^{k_i}\frac{z^2}{I_{ki}}dz+\dots+\int_{k_{n-1}}^{kn}\frac{z^2}{I_{kn}}dz\right]+\frac{Fk^2}{l^2E}\left[\int_0^{l_1}\frac{z^2}{I_{l1}}dz+\int_{l_1}^{l_2}\frac{z^2}{I_{l2}}dz\right.\\ &\left.+\dots+\int_{l_{j-1}}^{l_j}\frac{z^2}{I_{lj}}dz+\dots+\int_{l_{m-1}}^{l_m}\frac{z^2}{I_{lm}}dz\right]=\\ =&\frac{F}{3E}\left[\frac{k_1^3}{I_{k1}}+\frac{k_2^3-k_1^3}{I_{k2}}+\dots+\frac{k_i^3-k_{i-1}^3}{I_{ki}}+\dots+\frac{k_n^3-k_{n-1}^3}{I_{kn}}\right]+\frac{Fk^2}{3l^2E}\left[\frac{l_1^3}{I_{l1}}+\frac{l_2^3-l_1^3}{I_{l2}}+\dots+\frac{l_j^3-l_{j-1}^3}{I_{lj}}+\dots+\frac{l_m^3-l_{m-1}^3}{I_{lm}}\right],\end{aligned} \tag{8}$$

$$y_A=\frac{F}{3E}\sum_{i=1}^{n}\frac{k_i^3-k_{i-1}^3}{I_{ki}}+\frac{Fk^2}{3l^2E}\sum_{j=1}^{m}\frac{l_j^3-l_{j-1}^3}{I_{lj}},$$

$$\begin{aligned} \varphi_A=&\frac{F}{E}\left[\int_0^{k_1}\frac{z}{I_{k1}}dz+\int_{k_1}^{k_2}\frac{z}{I_{k2}}dz+\dots+\int_{k_{i-1}}^{k_i}\frac{z}{I_{ki}}dz+\dots+\int_{k_{n-1}}^{kn}\frac{z}{I_{kn}}dz\right]+\frac{Fk}{l^2E}\left[\int_0^{l_1}\frac{z^2}{I_{l1}}dz\right.\\ &\left.+\int_{l_1}^{l_2}\frac{z^2}{I_{l2}}dz+\dots+\int_{l_{j-1}}^{l_j}\frac{z^2}{I_{lj}}dz+\dots+\int_{l_{m-1}}^{l_m}\frac{z^2}{I_{lm}}dz\right]=\\ =&\frac{F}{2E}\left[\frac{k_1^2}{I_{k1}}+\frac{k_2^2-k_1^2}{I_{k2}}+\dots+\frac{k_i^2-k_{i-1}^2}{I_{ki}}+\dots+\frac{k_n^2-k_{n-1}^2}{I_{kn}}\right]+\frac{Fk}{3l^2E}\left[\frac{l_1^3}{I_{l1}}+\frac{l_2^3-l_1^3}{I_{l2}}+\dots+\frac{l_j^3-l_{j-1}^3}{I_{lj}}+\dots+\right.\\ &\left.\frac{l_m^3-l_{m-1}^3}{I_{lm}}\right],\end{aligned} \tag{9}$$

$$\varphi_A=\frac{F}{2E}\sum_{i=1}^{n}\frac{k_i^2-k_{i-1}^2}{I_{ki}}+\frac{Fk}{3l^2E}\sum_{j=1}^{m}\frac{l_j^3-l_{j-1}^3}{I_{lj}}.$$

As second case let us see the next simple supported beam loaded between its supports (Fig. 4). In Fig. 4 the form of the elastic curve of the beam and the tangent line to point A' can be seen. This tangent line is axis z itself rotated by angle φ_A. On the basis of Fig. 4 in case of small deformation and linearized model

$$\overline{BB'}=\varphi_A(a+b). \tag{10}$$

According to Eq. (10) the ratio $\frac{\overline{BB'}}{\varphi_A}$ is constant which equals to the distance between the supports. It can be noticed that the ratio is independent to dimensions a and b (position of loading) moreover it is true for any values of initial slope $y'_A=\varphi_A$ in case of small deformation and linearized model calculated in case of anyhow continuously variable cross-section.

In Fig. 5, the sketch of a simply supported beam of sectional variable cross-section loaded between its supports can be seen.

Applying the notations of Fig. 5, the moment functions

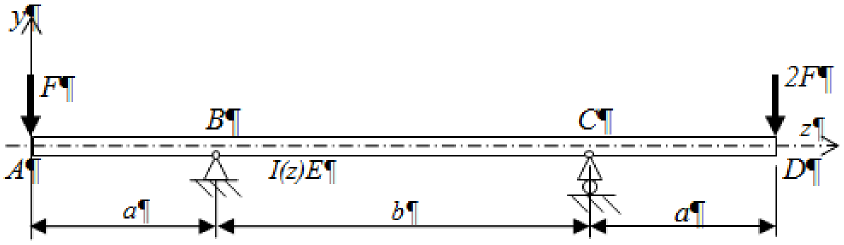

Fig. 8 Cantilevered simply supported beam

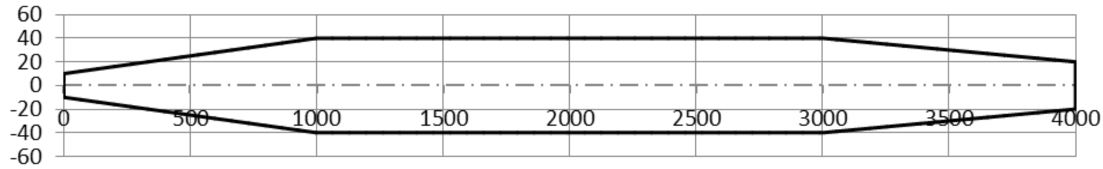

Fig. 9 Shape of the simply supported beam (top view). The height of cross-section is constant: h=140 *mm*

of the real loading are

$$M_1(z) = -\frac{Fb}{a+b}z, \qquad M_2(z) = -\frac{Fa}{a+b}z\,, \tag{11}$$

moreover, loaded by unit moment for two sections of the beam

$$m_1(z) = \frac{m\big(z-(a+b)\big)}{a+b}\,, m_2(z) = -\frac{m}{a+b}\,z. \tag{12}$$

Applying the Betti-theorem, with notations of Fig. 5, the slope of cross-section A of beam of sectional variable cross-section

$$\varphi_A = -\frac{Fb}{(a+b)^2E}\left[\int_0^{a_1}\frac{z^2-(a+b)z}{I_{a1}}dz + \dots + \int_{a_{i-1}}^{a_i}\frac{z^2-(a+b)z}{I_{ai}}dz + \dots + \int_{a_{n-1}}^{a_n}\frac{z^2-(a+b)z}{I_{an}}dz\right] + \frac{Fa}{(a+b)^2E}\left[\int_0^{b_1}\frac{z^2}{I_{b1}}dz + \int_{b_1}^{b_2}\frac{z^2}{I_{b2}}dz + \dots + \int_{b_{j-1}}^{b_j}\frac{z^2}{I_{bj}}dz + \dots + \int_{b_{m-1}}^{b_m}\frac{z^2}{I_{bm}}dz\right], \tag{13}$$

$$\varphi_A = -\frac{Fb}{3(a+b)^2E}\sum_{i=1}^{n}\frac{a_i^3-a_{i-1}^3}{I_{ai}} + \frac{Fb}{2(a+b)^2E}\sum_{i=1}^{n}\frac{(a+b)(a_i^2-a_{i-1}^2)}{I_{ai}} + \frac{Fa}{3(a+b)^2E}\sum_{j=1}^{m}\frac{b_j^3-b_{j-1}^3}{I_{bj}}.$$

In some cases, the moment of inertia of the cross section about its neutral axis can be described as continuous function of position coordinate z.

Applying the notations of Fig. 6 without going into details, according to Betti-theorem, the deflection and slope of cross-section A are

$$y_A = \frac{F}{E}\int_0^k \frac{z^2}{I_k(z)}dz + \frac{Fk^2}{l^2E}\int_0^l \frac{z^2}{I_l(z)}dz \tag{14}$$

$$\varphi_A = \frac{F}{E}\int_0^k \frac{z}{I_k(z)}dz + \frac{Fk}{l^2E}\int_0^l \frac{z^2}{I_l(z)}dz \tag{15}$$

In case of loading between its supports by the aid of Fig. 7, the slope of cross-section A is

$$\varphi_A = -\frac{Fb}{(a+b)^2E}\int_0^a \frac{z^2-(a+b)z}{I_a(z)}dz + \frac{Fa}{(a+b)^2E}\int_0^b \frac{z^2}{I_b(z)}dz \tag{16}$$

The Corollary is proved.

Remark: Using the previous results it can be concluded that for the accurate initial conditions, independently on position of loading and dimensions of the beam, a numerical procedure for obtaining of the elastic curve can be developed.

3. Procedure for numerical determination of the elastic curve in case of small deformation and linearized model

The suggested procedure for determination of the initial conditions for calculation of the elastic curve of the simple supported beams is as follows:

1. Constraining forces in the supports are calculated.
2. Based on external load and constraint forces in supports the moment-position functions $M(z)$ are determined.
3. Substituting the moment-position functions into Equation (1b) the differential equation of elastic curves for each loading section are formed.
4. The equations are solved numerically for assumed initial conditions

$$y_0 = 0, \qquad y'_0 = 0\,rad.$$

(These conditions correspond to a clamped-free beam.)

5. Based on the obtained numerical values the initial slope φ is determined. It corresponds to the initial condition, i.e.,

$$y'_A = -\varphi.$$

6. Calculation of the elastic curve is repeated for initial conditions

$$y'_A = -\varphi, \qquad y_A = 0.$$

7. The obtained displacements for the supports are equal to y_S =const. and represent the negative value of the initial displacement

$$y_A = -y_S.$$

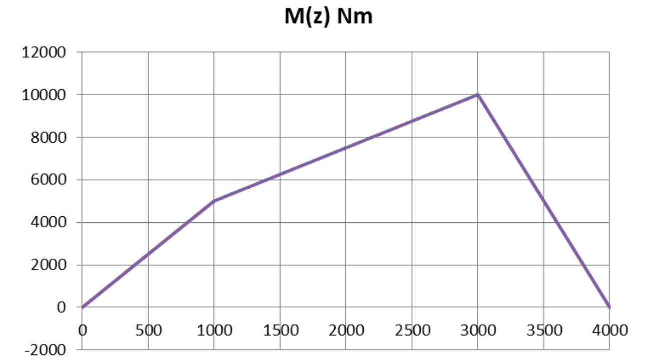

Fig. 10 Moment-position functions of the cantilevered simply supported beam

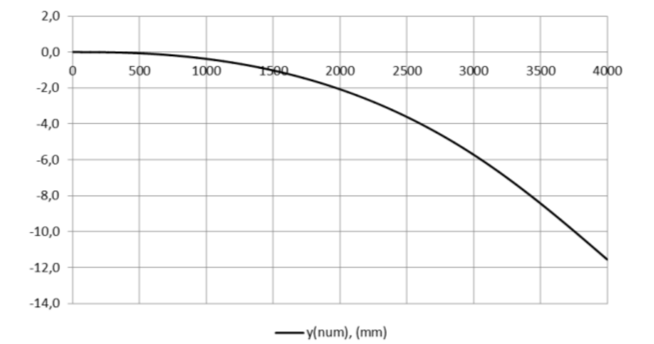

Fig. 11 Elastic curve of the cantilever for initial conditions y'_o=0 and y_o=0

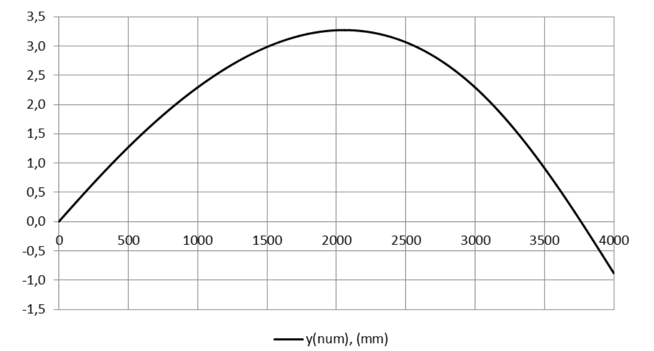

Fig. 12 Elastic curve for initial values: $y_A = 0\,mm$, $y'_A = 0{,}00266887\,rad$

8. Finally, the nonlinear differential equations calculated numerically with initial conditions

$$y'_A = -\varphi, \qquad y_A = -y_S,$$

give the elastic curve for the simple supported beam.

4. Examples

As an example, the task is to determine numerically the elastic curve of cantilevered simply supported beam shown in Fig. 8. The following numerical data are given: a=1000 *mm*, b=2000 *mm*, E=210 *GPa*, F=5000 *N*. Top view of the beam can be seen in Fig. 9.

For the given numerical values the constraining forces

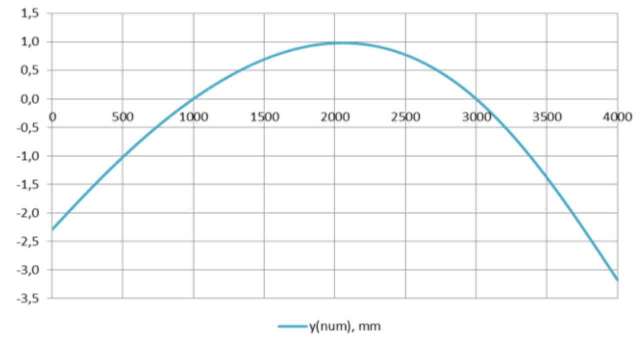

Fig. 13 Elastic curve determined based on numerically and analytically obtained initial conditions

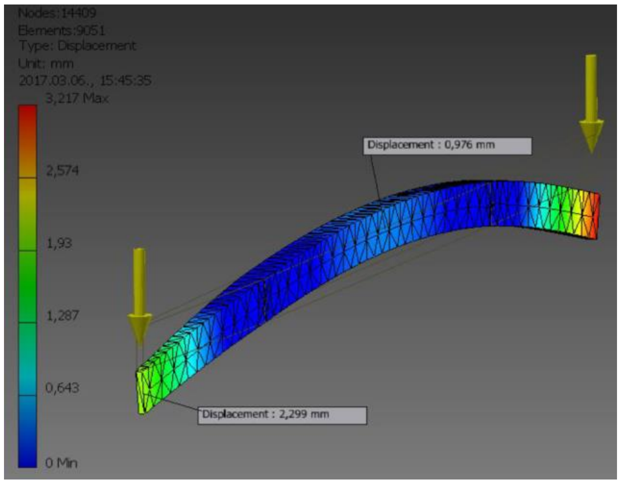

Fig. 14 Displacements of the simple supported beam as result of finite element analysis

of the supports

$$F_B = \frac{1}{2}F, \qquad F_C = \frac{5}{2}F, \tag{17}$$

and the moment-position functions are calculated and plotted in Fig. 10.

Three segments along the beam are evident and the differential equations according to (1b) of the elastic curve for each segment are formed.

The obtained relations are

$$0 \le z \le a, \qquad y_1'' = -\frac{Fz}{EI_1(z)}, \tag{18}$$

$$a \le z \le a+b, \qquad y_2'' = -\frac{F}{2\,EI_2(z)}(z+a), \tag{19}$$

$$a+b \le z \le 2a+b, \quad y_3'' = -\frac{F}{EI_3(z)}(-2z+3a+2{,}5b). \tag{20}$$

Let us solve the above equations numerically for initial values y'_o=0 and y_o=0. Namely, it is assumed that the left end of the beam is fixed and corresponds to a cantilever. Thereat, the moment-displacement function is not varied. The obtained result is plotted in Fig. 11.

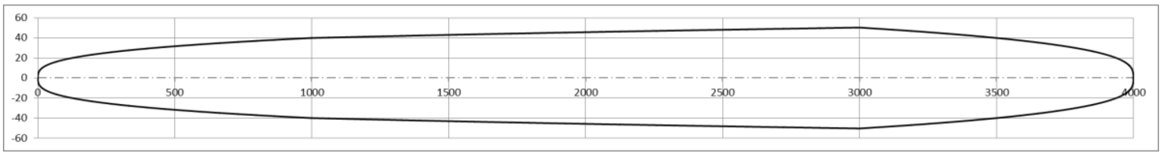

Fig. 15 Shape of the simply supported beam of uniform strength having circular cross-section (side view)

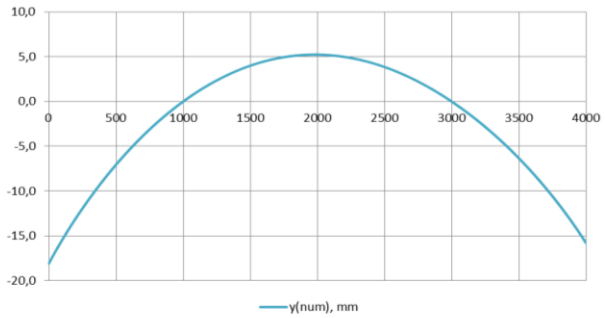

Fig. 16 Elastic curve determined based on numerically and analytically obtained initial conditions

Table 1 Comparison of deflections of cross-sections *A* and *D* obtained in different ways (Example 2)

	y_A, *mm*	y_D, *mm*
Numerical transformation method -rotation and translation -transformation of the boundary value problem into the initial value problem	-18.0716	-15.7878
Betti-theorem for beams of variable cross-section	-18.0633	-15.7871
Finite element analysis	-18.08	-15.85

Obviously, the shape of the elastic curve is not suitable to the real loading and the constraint relations. In order to get the accurate initial values let us carry out the following transformations.

Rotation around axis perpendicular to xy plane

Creating the ratio of differences between deflections of cross-sections *B* and *C* and between their positions coordinates an angle can obtained

$$\varphi = \frac{y_C - y_B}{z_C - z_B} = \frac{-5{,}7146733mm - (-0{,}3769307)mm}{3000\,mm - 1000\,mm} = -0{,}00266887\,rad. \quad (21)$$

This angle with opposite sign can be treated as initial slope of cross-section *A*, i.e.,

$$y_A' = -\varphi = 0{,}00266887\,rad. \quad (22)$$

The numerical calculation of differential equations of the elastic curves is repeated with initial values

$$y_A = 0, \qquad y_A' = 0{,}00266887\,rad. \quad (23)$$

The obtained elastic curve is plotted in Fig. 12.

It can be noticed that for initial conditions (23) the values of deflection at supports *B* and *C* are equal: $y_B = y_C = 4{,}83944\,mm$. After this recognition translation along axis *y* seems to be obvious.

Translation along axis y

Now, the curve is translated along *y* axis for the value $y_A = -y_B = -y_C = -2{,}29194771\,mm$, to move the supports in the position with zero deflection.

Starting with numerical procedure and applying the calculated initial values $y_A = -2{,}291937473\,mm$, $y_A' = 0{,}00266887\,rad$ the elastic curve of the beam are obtained and plotted in Fig 13. In order to check the obtained results the finite element analysis (Fig. 14) and Betti-theorem (Table 2) are applied.

The Finite Element Analysis was made by Autodesk Inventor 2012 which was produced for solving only linear problems (linear material properties, deflection and stress are linearly proportional) with static load and small deformation. During the analysis the Convergence was set to 10%, the Average Element Size 0.02, Minimum Element Size: 0.2, Grading Factor: 1.5, Maximum Turn Angle: 60.0 deg with Curved Elements as well. Refinement Threshold (0 to 1): 0.75.

Results obtained in different ways are compared to each other and summarized in Table 1.

Example 2 Cantilevered simply supported beam of uniform strength having circular cross-section

The sketch of the cantilevered simply supported beam can be seen in previous example. In this case there is a beam of uniform strength having circular cross-section, other input data are the same.

Starting with numerical procedure again and applying the calculated initial values $y_A = -18{,}071617\,mm$, $y_A' = 0{,}028785\,rad$ the elastic curve of the beam of uniform strength are obtained and plotted in Fig. 15. In order to check the obtained results the finite element method (Figs. 17-18) and Betti-theorem are applied.

Results obtained in different ways are compared to each other and summarized in Table 2. Comparing the results obtained numerically and analytically for the linearized system moreover applying finite element analysis it can be concluded that the difference between them is negligible

Table 2 Comparison of deflections of cross-sections *A* and *D* obtained in different ways (Example 2)

	y_A, *mm*	y_D, *mm*
Numerical transformation method -rotation and translation -transformation of the boundary value problem into the initial value problem	-18.0716	-15.7878
Betti-theorem for beams of variable cross-section	-18.0633	-15.7871
Finite element analysis	-18.08	-15.85

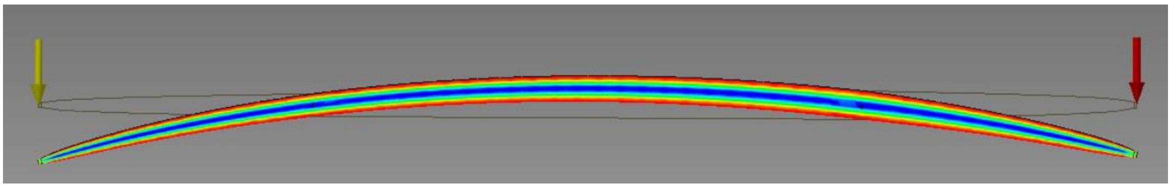

Fig. 17 Stress sheet of the simple supported beam as result of finite element analysis (side view)

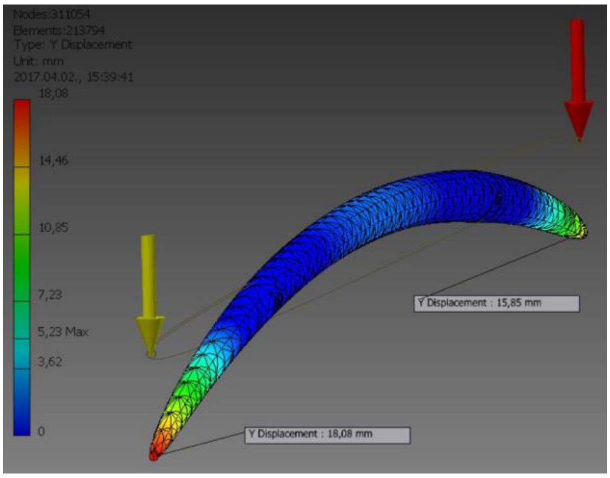

Fig. 18 Displacements of the simple supported beam of uniform strength as result of finite element analysis

(Tables 2-3).

5. Conclusions

Applying the proposed numerical transformation procedure the linearized boundary value problems of elastic simply supported beams of variable cross-section can be transformed into initial value problems. The procedure is a special and effective application of the shooting method to calculate the elastic curve.

Initial values of boundary problems of beams are deterministic ones. As a result of consequent application of rotation and translation the initial values (slope and deflection of end cross-section) of the beam can be obtained with high accuracy and using them the elastic curve can be determined in traditional numerical way. The method is stable and simple to use, and its applicability is proved in the paper.

Initial values calculated numerically for the differential equation of the elastic curve are compared with those obtained analytically for the linearized model of the elastic curve, moreover with results of finite element analysis. The difference between values is negligible.

The above method is applicable in nonlinear case (Eq. (1b)) and small deformation. In this case additional iteration step in numerical procedure is necessary to reach the demanded accuracy.

Acknowledgments

The investigation is financially supported by the Faculty of Technical Sciences, Novi Sad, Serbia (Proj.054/2016) and by Faculty of Engineering, University of Szeged, Hungary.

References

Bíró, I. and Cveticanin, L. (2016), “Procedures for determination of elastic curve of simply and multiple supported beams”, *Struct. Eng. Mech.*, **60**(1), 21-30.

Fertis, D.G. (2006), *Nonlinear Structural Engineering with Unique Theories and Methods to Solve Effectively Complex Nonlinear Problems*, Springer.

Kumar, R., Ramachandra, L.S. and Roy, D. (2004) “Techniques based on generic algorithms for large deflection analysis of beams”, *Sadhana*, **29**(6), 589-604.

Kumar, R., Ramachandra, L.S. and Roy, D. (2006), “A multi-step linearization techniques for a class of bending value problems in non-linear mechanics”, *Comput. Mech.*, **39**, 73-81.

Merli, R., Lázaro, S., Monleón, S. and Domingo, A. (2010), “Comparison of two linearization schemes for the nonlinear bending problem of a beam pinned at both ends”, *Int. J. Sol. Struct.*, **47**(6), 865-874.

Ramachandra, L.S. and Roy, D. (2001), “A new method for nonlinear two-point boundary value problems in solid mechanics”, *J. Appl. Mech.*, **68**(5), 776-785.

Ramachandra, L.S. and Roy, D. (2002), “The locally transversal linearization (LTL) method revisited: A simple error analysis” *J. Sound Vibr.*, **256**(3), 579-589.

Rojas, A.L. (2014), “A mathematical model of elastic curve for simply supported beams subjected to uniformly distributed load taking into account the shear deformations”, *Int. J. Innovat. Comput. Informat. Contr.*, **5**(3), 885-890.

Rojas, A.L. and Espino, J.V.R. (2015), “Fixed-end moments for beams subjected to a concentrated forced localized anywhere taking into account the shear deformations”, *Int. J. Innovat. Comput. Informat. Contr.*, **11**(2), 463-474.

Rojas, A.L., Chavarria, S.L., Elizonda, M.M. and Kalashnikov, V.V. (2016), “A mathematical model of elastic curve for simply supported beams subjected to a concentrated load taking into account the shear deformations”, *Int. J. Innovat. Comput. Informat. Contr.*, **12**(1), 41-54.

Roy, D. and Kumar, R. (2005), “A multi-step transversal linearization (MLT) method in non-linear structural dynamics”, *J. Sound Vibr.*, **287**(1-2), 203-226.

Thankane, K.S. and Stys, T. (2009), “Finite difference method for beam equation with free ends using mathematica”, *South. Afr. J. Pure Appl. Math.*, **4**, 61-78.

Timoshenko, S. and Goodier, J.N. (1952), *Theory of Elasticity*, McGraw-Hill.

Viswanath, A. and Roy, D. (2007) “Multi-step transversal and tangential linearization methods applied to a class of nonlinear beam equations”, *Int. J. Sol. Struct.*, **44**(14-15), 4872-4891.

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