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The use of generalized functions modeling the concentrated loads on Timoshenko beams

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Abstract. An incongruity is underlined about the analysis of Timoshenko beams subjected to concentrated loads modelled through the use of generalized functions. While for Euler-Bernoulli beams this modeling always leads to effective results, on the contrary, the contemporary assumptions of concentrated external moment, interpreted as a generalized function (doublet), and of shear deformation determine inconsistent discontinuities in the deflection laws. A physical/theoretical explanation of this not-neglecting incongruity is given in the text.

Keywords: Timoshenko beam; concentrated loads; generalized functions; physical incongruity

1. Introduction

Finding efficient solutions for discontinuous Euler-Bernoulli beams (EBBs) is of great interest in engineering applications. In particular, shear force and bending moment discontinuities arise where concentrated forces and moments are applied, given either as external loads or reactions of along axis essential constraints, i.e., external roller and rotational supports.

In a classical approach, a 4th-order differential equilibrium equation shall be written for each of the beam portions singled out by distinct discontinuity locations. In this manner, however, the computational effort can become significant when the number of discontinuity locations increases.

An alternative analytical solution method involves reformulating the bending problem in the space of generalized functions. The first contribution in this sense has been the singularity function method due to Macaulay (1919), where point loads are treated as continuous by the introduction of appropriate generalized functions. These functions include the so-called bracket functions, introduced by the same Macaulay, and the distribution functions (Pilkey 1964, Schwartz 1966) The solution is built through integrations made in the generalized sense, that is following the Macaulay bracket formulation or the rules of the distribution functions, respectively. In this way the beam deflection is a generalized function itself and it is always given in terms of only four integration constants, to be determined by imposing the four boundary conditions (Falsone 2002). Macaulay's method has been later extended by Brungraber (1965) to EBBs with discontinuities due to along axis essential and natural constraints. Specifically, Brungraber has shown that each discontinuity results in an equivalent load, modelled by an appropriate generalized function involving an unknown response variable at the discontinuity location. An exhaustive review on Macaulay's and Brungraber's solutions may be found in Falsone (2002).

The birth of Schwarz (1966) distribution theory provided a rigorous justification for a number of very common formal manipulations in the engineering literature. Indeed, certain types of distributions, in particular, the Dirac delta function and its derivatives, were used in engineering problems years before the development of distribution theory. The delta function dates back to the first half of the 19th century. Dirac (1930) introduced this function in quantum mechanics and since then the function has been known as the Dirac delta function.

A more recent work on discontinuous beams, also based on the use of generalized functions, is due to Kanwal (1983). It considers mixed-type discontinuous EEBs; the discontinuities are due both to along axis essential and natural constraints, and to flexural-stiffness jumps. The response of the original beam is obtained as linear superposition of the responses of a uniform reference beam to loading conditions given by: (i) the external loads; (ii) the generalized loads, each given in terms of one or more unknown response variables at each discontinuity location. The solutions are sought in the space of classical functions. Obviously, this method offers no computational advantage as compared to the previous heuristic approaches. However, it is a first attempt to apply the theory of generalized functions to discontinuous beams with flexural-stiffness jumps.

As an improvement to Kanwal's method, Yavari *et al.* (2000) proposed the so-called auxiliary beam method. The auxiliary beam is a uniform reference beam, subjected to the external loads and equivalent generalized loads depending on the flexural-stiffness jump parameters. The original beam response is then expressed as the auxiliary beam response, corrected by a number of additional generalized functions depending on the unknown response

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variables at the discontinuity locations. The solution is built in the space of classical functions.

Arbabi (1991) generalized the singularity function method for a beam with an internal hinge and a beam with jump discontinuities in flexural stiffness, starting the analysis from the bending moment expression.

In a recent paper (Failla and Santini 2007) an analytical solution for arbitrary mixed-type discontinuous EBBs is presented. It implicitly satisfies the boundary conditions and depends on unknown parameters to be computed by appropriate conditions at the discontinuity locations. This objective is achieved by reformulating the bending problem in the space of generalized functions and computing the beam response in terms of the Green's functions of a uniform reference beam.

In the last years, the generalized functions have been used also for the study of discontinuous EBBs on elastic foundations (Yavari *et al*. 2001, Colajanni *et al*. 2009) and in the field of dynamical analyses (Ecsedi and Dluh 2004, Failla and Santini 2008, Cheng *et al*. 2014, Failla 2014). The generalized function have been used in the stochastic EBB, both in the case of random concentrated loads (Falsone and Settineri 2013) and in the case of stochastic beam properties (Failla *et al*. 2005). Finally, in the theory of cracked beams, the generalized functions have been advantageously taken into account for representing the arising discontinuities (Grossi and Raffo 2016).

In some cases, the approaches before cited for the EBBs have been extended to the discontinuous Timoshenko beams (TBs), in the static case (Yavari *et al*. 2001), for the multi-stepped beams (Caddemi *et al.* 2013) and in the dynamic case (Ghannadiasl and Mofid 2014). But, however, the discontinuities treated in these works are always jumps in slope, deflection, bending stiffness and shear stiffness and they are considered in the space of generalized functions. No case of concentrated force or moment has been highlighted.

In the present work, after having recalled the expressions of the generalized functions, distinguishing the Macaulay functions from the distributions, giving the differentiation and integration rules for both the types, in Sec.3 the EBB excited by a concentrated force and by a concentrated moment is taken into consideration. The space of the generalized functions allows the definition of all the response characteristics (deflection, slope, bending moment and shear force) at the discontinuity abscissa. In Sec.4, the attempt to consider the TB loaded by the same concentrated actions is presented with the goal to find an explanation, if there is, of the lack of this kind of applications in the literature.

2. EBB under concentrated actions

It is well known that the transversal force equilibrium of a loaded EBB is governed by the following continuum differential equation

$$\frac{dT(x)}{dx} = -p(x); \qquad 0 \le x \le l \tag{1}$$

$T(\cdot)$ being the internal shear force and $p(\cdot)$ the distributed transversal load. The integration of this equation gives the law of the internal shear force as follows

$$T(x) = -\int p(x)\,dx + C_1 \tag{2}$$

where C_1 is an integration constant depending on the boundary conditions.

If the transversal load is concentrated at the abscissa x_0, Eqs. (1), (2) can be still used if the generalized functions are used and the integrations are made in the appropriate generalized sense. In particular, the load can be represented by a Dirac delta function (DDF), that, using the notation reported in Falsone (2002), can be written as

$$p(x) = P\delta(x - x_0) \equiv PR_{-1}(x - x_0) \tag{3}$$

The generalized integration of this equation leads to

$$T(x) = -PU(x - x_0) + C_1 \equiv -PR_0(x - x_0) + C_1 \tag{4}$$

$U(x-x_0)\equiv R_0(x-x_0)$ being the unit step function (USF) placed at x_0, that is considered as the integral function of the DDF in the generalized sense. The presence of this generalized function in the shear law implies a jump of amplitude $-P$ in the corresponding diagram. This is obviously a classical result in the beam theory.

The moment equilibrium equation implies that the bending moment law $M(x)$ can be obtained by integrating $T(x)$ that is

$$M(x) = -PR_1(x - x_0) + C_1 x + C_2 \tag{5}$$

where $R_1(x-x_0)$ is a generalized function, usually called linear ramp, that is the integral of the USF in the generalized sense and determining a change of slope at x_0 in the bending moment diagram, in accordance with the classical EBB theory.

Considering the moment-curvature constitutive equation and integrating the curvature-rotation congruence differential equation, at last the following expression is obtained for the rotation law $\varphi(x)$

$$\varphi(x) = -\frac{P}{EI} R_2(x - x_0) + C_1 \frac{1}{EI}\frac{x^2}{2} + C_2 \frac{1}{EI} x + C_3 \tag{6}$$

EI being the beam flexural stiffness, considered to be constant along the axis, while $R_2(x-x_0)$ is the so-called 2nd order ramp and that can be considered as the integral of the linear ramp in the generalized sense. It is important to note that this expression guarantees the continuity of the $\varphi(x)$ law.

The last step for characterizing the EBB response is the evaluation of the transversal deflection $w(x)$ which is obtained by the integration of the congruence rotation-deflection equation valid for the EBB, that is $\varphi(x)=-dw(x)/dx$. It has the following expression

$$w(x) = \frac{P}{EI} R_3(x - x_0) - C_1 \frac{1}{EI}\frac{x^3}{6} - C_2 \frac{1}{EI}\frac{x^2}{2} - C_3 x + \\ + C_4 \tag{7}$$

where $R_3(x-x_0)$ is the 3rd order ramp. Also this expression guarantees the continuity of the deflection law. The four integration constants depend on the boundary conditions and, hence, on the constrains acting on the beam extremes.

When the load is a concentrated moment $\hat{M}$ applied at x_0, it can be modeled, as a distributed action $p(x)$, by a doublet (Shames 1989, Falsone 2002, Chalishajar *et al*. 2016), that is the formal derivative, in the generalized sense, of the DDF

$$p(x)=\hat{M}R_{-2}(x-x_0)\equiv\hat{M}\delta'(x-x_0) \tag{8}$$

This representation is justified by the consideration that a concentrated moment can be considered as a couple of concentrated forces whose arm tends to zero. Each force can be represented by a DDF, whose intensity, which is already ∞, must be multiplied again for ∞ when the arm tends to zero. The generalized function deriving from this concept is just the doublet.

Recalling the same previous sequence of operations made for the concentrated force, it is easy to verify that

$$T(x)=-\hat{M}R_{-1}(x-x_0)+C_1\equiv-\hat{M}\delta(x-x_0)+C_1 \tag{9a}$$

$$M(x)=-\hat{M}R_0(x-x_0)+C_1x+C_2 \tag{9b}$$

$$\varphi(x)=-\frac{\hat{M}}{EI}R_1(x-x_0)+C_1\frac{1}{EI}\frac{x^2}{2}+C_2\frac{1}{EI}x+C_3 \tag{9c}$$

$$w(x)=\frac{\hat{M}}{EI}R_2(x-x_0)-C_1\frac{1}{EI}\frac{x^3}{6}-C_2\frac{1}{EI}\frac{x^2}{2}-C_3x+C_4 \tag{9d}$$

It is worth noting that Eq. (9a) shows the presence of a DDF in the internal shear law. This means that the value of $T(x)$ remains unchanged passing from the section placed immediately before the abscissa x_0 ($x=x_0^-$) to that one placed immediately after x_0 ($x=x_0^+$). The presence of the USF into Eq. (9b) implies the correct jump in the bending moment law. The rotation law (Eq. (9c)), showing the presence of the linear ramp function, implies a jump in the derivatives of rotations and, hence, the correct jump in the curvature. At last, the form of Eq. (9d) guarantees the continuity of the deflection law $w(x)$.

Alternatively, the concentrated moment load could be modeled as a distributed moment load $m(x)$ by considering the DDF, that is $m(x)=\hat{M}R_{-1}(x-x_0)$. In this case, the second member of the equilibrium Eq. (1) is zero and its integration implies $T(x)=C_1$. The internal moment differential equation writes

$$\frac{dM(x)}{dx}=-m(x)+T(x)$$
$$\Rightarrow\quad\frac{dM(x)}{dx}=-\hat{M}R_{-1}(x-x_0)+C_1 \tag{10a-b}$$

whose integration has exactly the same expression of Eq. (9a). The further operation is just the same of that made before and leading to Eqs. (9b)-(d). This implies that this procedure is perfectly equivalent to the previous one.

The results obtained in this section confirm the correctness of using the generalized functions for representing the concentrated loads on the EBB, with the corresponding computational advantage of having always four integration constants to be evaluated, against the $4n$ necessary if the traditional approach of dividing the beam in n parts where, in each, the response is continuous.

3. TB under concentrated loads

When the shear deformability of the beam is taken into account, the Timoshenko (1922) beam theory is usually considered (Elishakoff *et al*. 2015 and references herein). While the transversal and moment internal equilibrium differential equations, the compatibility rotation-curvature differential equation and the constitutive curvature-bending moment equation remain unchanged respect to the corresponding ones considered in the EBB, in the compatibility rotation-deflection differential equation must take into account the presence of the shear deformability and another constitutive equation between the internal shear force and the corresponding deformation $\gamma(x)$ must be considered, that are

$$\varphi(x)=\gamma(x)-\frac{dw(x)}{dx};\qquad\gamma(x)=\frac{\chi}{GA}T(x) \tag{11a-b}$$

where χ/GA is the shear deformability of the beam, here considered constant along the axis.

If a transversal concentrated load P is applied at the abscissa x_0 of a TB, as the internal equilibrium conditions do not change respect to what said for the EBB in the previous section, then Eqs. (1)-(5) are still valid for the TB case. Moreover, as the rotation -curvature compatibility equation remains unchanged, too, then Eq. (6) is still valid also. Hence, taking into account Eq. (11a), where Eqs. (6) and (11b) have been replaced, then the following expression is obtained for the deflection derivative

$$\begin{aligned}\frac{dw(x)}{dx}&=F\left[\frac{\chi}{GA}R_0(x-x_0)-\frac{1}{EI}R_2(x-x_0)\right]+\\&\quad+\left(\frac{\chi}{GA}-\frac{1}{EI}\frac{x^2}{2}\right)C_1-\frac{1}{EI}xC_2-C_3\end{aligned} \tag{12}$$

that, integrated, gives

$$\begin{aligned}w(x)&=F\left[\frac{\chi}{GA}R_1(x-x_0)-\frac{1}{EI}R_3(x-x_0)\right]+\\&\quad+\left(\frac{\chi}{GA}x-\frac{1}{EI}\frac{x^3}{6}\right)C_1-\frac{1}{EI}\frac{x^2}{2}C_2-C_3x+C_4\end{aligned} \tag{13}$$

As the generalized functions $R_i(x-x_0)$, with $i\geq2$, can be considered continuous at x_0 together with their first order derivatives, it is easy to recognize that Eq. (12) implies a jump in the derivative of deflection law, which is confirmed by Eq. (13) where the deflection law shows, for the presence of the generalized function $R_1(x-x_0)$, a tangent jump. Nevertheless, if the rotations $\varphi(x)$ are considered, the application of Eq. (11a) shows that no jumps arise in their diagram representation. It is not difficult to verify that the results so obtained are coincident with those obtained by the classical approaches, but with a sure computational effort advantage.

What before exposed shows that when the TB is loaded by concentrated transversal forces, then the generalized functions can be used for the load representation in such an efficient way as for the EBB case.

When a concentrated moment $\hat{M}$ is considered applied at the abscissa x_0 of the TB, even in this case the laws of the internal shear force, the bending moment and the rotations are just the same of those defined for the EBB in the previous section. Then Eqs. (9a)-(c) remain valid for the TB, too. Now, taking into account Eqs. (9a), (c) and (11b), Eq. (11a) gives

$$\frac{dw(x)}{dx} = \hat{M}\left[\frac{1}{EI}R_1(x-x_0) - \frac{\chi}{GA}R_{-1}(x-x_0)\right] + \left(\frac{\chi}{GA} - \frac{1}{EI}\frac{x^2}{2}\right)C_1 - \frac{1}{EI}xC_2 - C_3 \tag{14}$$

whose integration leads to

$$w(x) = \hat{M}\left[\frac{1}{EI}R_2(x-x_0) - \frac{\chi}{GA}R_0(x-x_0)\right] + \left(\frac{\chi}{GA}x - \frac{1}{EI}\frac{x^3}{6}\right)C_1 - \frac{1}{EI}\frac{x^2}{2}C_2 - C_3x + C_4 \tag{15}$$

The presence of the USF $R_0(x-x_0)$ in this expression implies a jump of intensity $-\hat{M}\chi/(GA)$ in the deflection law, that is obviously incongruent for the beam continuity.

If the concentrated moment $\hat{M}$ is represented as a distributed moment $m(x) = \hat{M}R_{-1}(x-x_0)$ the Eqs. (10a, b), considered in the previous section for the EBB, are still valid for the TB, together with the condition on the shear force, that is $T(x)=C_1$. Then, it is not difficult to verify that the compatibility equation on the shear deformability writes as follows

$$\frac{dw(x)}{dx} = \hat{M}\frac{1}{EI}R_1(x-x_0) + \left(\frac{\chi}{GA} - \frac{1}{EI}\frac{x^2}{2}\right)C_1 - \frac{1}{EI}xC_2 - C_3 \tag{16}$$

That, integrated, gives the following expression for the deflection law

$$w(x) = \hat{M}\frac{1}{EI}R_2(x-x_0) + \left(\frac{\chi}{GA}x - \frac{1}{EI}\frac{x^3}{6}\right)C_1 - \frac{1}{EI}\frac{x^2}{2}C_2 - xC_3 + C_4 \tag{17}$$

It is easy to recognize that in this expression no term determines discontinuity on the deflection law, implying the congruence of the assumption of modeling the concentrated moment as a distributed moment through the use of the DDF, instead of modeling it as a distributed transversal load through the use of a doublet. The difference in the deflection law between the two assumptions, observable in Eqs. (15) and (17), consists just only in the presence of the term determining the jump in Eq. (15).

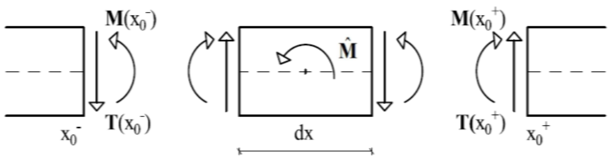

Fig. 1 Infinitesimal element loaded by the concentrated moment

4. Physical significance of the various approaches

In this section some physical implications about the choice of using one of the approaches rather than another will be given, trying to better explain the incongruence evidenced in the previous section when a concentrated moment load is considered acting on a TB.

Firstly, the classical approach of dividing the beam into continuous pieces, through a section at the abscissa x_0, where the concentrated external moment $\hat{M}$ is applied, is taken into account. This approach requires some further boundary conditions at $x=x_0$, besides of those at the beam extremes $x=0$ and $x=l$. These further conditions arise in order to impose the continuity of the deflection and the rotation at $x=x_0$

$$w(x_0^+) = w(x_0^-); \qquad \varphi(x_0^+) = \varphi(x_0^-) \tag{18a-b}$$

x_0^- and x_0^+ being the sections immediately before and after, respectively, of the abscissa $x=x_0$, belonging to the beam pieces I and II, respectively (Fig. 1), and the equilibrium of the infinitesimal beam element containing the section at x_0

$$T(x_0^+) = T(x_0^-); \qquad M(x_0^+) = M(x_0^-) - \hat{M} \tag{19a-b}$$

Hence, considering Eqs. (18) and (19) implies that the compatibility and the equilibrium conditions at $x=x_0$ are implicitly satisfied, without considering the behavior of the infinitesimal beam element. Moreover, Eq. (19a) shows that this last one is characterized by the presence of a constant internal shear force and, as a consequence, by a constant shear deformation $\gamma(x_0)=\chi T(x_0)/(GA)$. This last one determines a relative deflection $w(x_0^+) - w(x_0^-) = -\gamma(x_0)dx$ that, being infinitesimal, is not in contrast with Eq. (18a).

If the generalized functions are considered for modeling the concentrated moment load, then, as said in the previous section, it is possible to represent this load as a distributed transversal load by using the doublet, $p(x) = \hat{M}R_{-2}(x-x_0)$, or as a distributed moment load by using the DDF, $m(x) = \hat{M}R_{-1}(x-x_0)$. When the generalized functions are used, no beam partition is required and all the beam governing equations must be satisfied at every x. When the doublet is used for the load representation, it implies the presence of a particular couple in the infinitesimal beam element at $x=x_0$ (Fig. 2). Following the theory of the generalized functions, this couple is defined by transversal forces of intensity $\hat{M}/(dx)$, determining an internal shear force inside the infinitesimal element $T(x) = \hat{M}/(dx)$

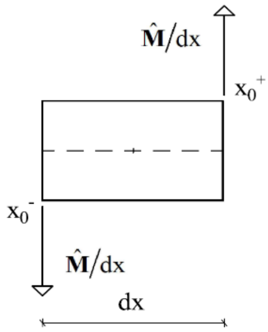

Fig. 2 Coupling of forces statically equivalent to the concentrated moment

with $x_0^- \le x \le x_0^+$. As a consequence, a shear deformation $\gamma(x) = \chi \hat{M} / (GAdx)$ is present, always in the same beam element, implying a relative deflection

$$w(x_0^+) - w(x_0^-) = -\gamma(x_0)\,dx = -\chi \hat{M} / (GA) \tag{20}$$

which is exactly the value of the jump that is evidenced by the presence of the USF in the deflection law given into Eq. (15). It is clear that this deflection jump is not admissible and, as a consequence, the load representation here considered cannot be accepted

It is not difficult to verify that, if the concentrated moment is represented through the DDF, $m(x) = \hat{M} R_{-1}(x - x_0)$, it does not imply any internal shear force in the beam element at x=x_0 and, hence, any shear deformation and deflection jump. Consequently, this load representation is admissible.

5. Conclusions

An application of the generalized functions (Macaulay brackets and distributions) for representing the concentrated loads on deflected beam has been presented. The use of these functions allows to avoid the division of the beam in various continuous pieces with a reduction of the computational effort that could be relevant in some cases.

It has been shown that in the case of EBBs this application gives the expected classical response results, in terms of shear forces, bending moments, rotations and deflections. On the contrary, when the same application is made on the TBs, a physical inconsistence arises. In particular, for the case of the concentrated moment load, an inconsistent jump on the deflections arises. What presented in this work leads one to think that these results are due to an inconsistence of the contemporary assumptions of shear deformability of the beam element and of concentrated moment represented as a double force, analytically defined by the doublet. This thought originates also from the fact that no inconsistence comes out when the concentrated moment is represented as distributed moment and, hence, analytically defined by a DDF.

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