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Thermal post-buckling analysis of a laminated composite beam

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Abstract. The purpose of this study is to investigate thermal post-buckling analysis of a laminated composite beam subjected under uniform temperature rising with temperature dependent physical properties. The beam is pinned at both ends and immovable ends. Under temperature rising, thermal buckling and post-buckling phenomena occurs with immovable ends of the beam. In the nonlinear kinematic model of the post-buckling problem, total Lagrangian approach is used in conjunction with the Timoshenko beam theory. Also, material properties of the laminated composite beam are temperature dependent: that is the coefficients of the governing equations are not constant. In the solution of the nonlinear problem, incremental displacement-based finite element method is used with Newton-Raphson iteration method. The effects of the fibber orientation angles, the stacking sequence of laminates and temperature rising on the post-buckling deflections, configurations and critical buckling temperatures of the composite laminated beam are illustrated and discussed in the numerical results. Also, the differences between temperature dependent and independent physical properties are investigated for post-buckling responses of laminated composite beams.

Keywords: thermal post-buckling; composite laminated beams; temperature dependent physical properties; Timoshenko Beam Theory; total lagragian finite element method

1. Introduction

Laminated composite structures have been used many engineering applications, such as aircrafts, space vehicles, automotive industries, defence industries and civil engineering applications because these structures have higher strength-weight ratios, more lightweight and ductile properties than classical materials. With the great advances in technology, the using of the laminated composite structures is growing in applications.

Laminated composite structures have many practical applications in high temperature systems as thermal barrier systems, for example, reactor vessels, aircrafts, space vehicles, defense industries and other engineering structures. As nuclear power plants, aerospace vehicles, thermal power plants etc. are subjected to large thermal loadings and laminated composite structures have found extensive applications in these applications. The design of laminated composite structural elements (beams, plates, shells etc.) in the high thermal environments is very important. Especially, in the case of structural elements with immovable ends, temperature rise causes compressible forces end therefore buckling and post-buckling phenomena occurs. Understanding the buckling and post-buckling mechanism of laminated composite structural elements is very important. It is known that buckling and post-buckling problems are nonlinear problems. Buckling or post-buckling is occurred by a sudden failure of a structural member subjected to high temperature values.

In the literature, nonlinear and post-buckling studies of Laminated composite beams are has not been investigated broadly. In the open literature, studies of the post-buckling and nonlinear behavior of laminated composite beams are as follows; Sheinman and Adan (1987) investigated effect of shear deformation on the post-buckling of laminated beams. Ghazavi and Gordaninejad (1989) studied geometrically nonliner static of laminated bimodular composite beams by using mixed finite element model. Singh *et al*. (1992) investigated nonlinear static responses of laminated composite beam based on higher shear deformation theory and von Karman's nonlinear type. Pai and Nayfeh (1992) presented three-dimensional nonlinear dynamics of anisotropic composite beams with von Karman nonlinear type. Di Sciuva and Icardi (1995) investigated larfe deflection of anisotropic laminated composite beams with Timoshenko beam theory and von Karman nonlinear strain-displacement relations by using Euler method. Donthireddy and Chandrashekhara (1997) investigated thermoelastic nonlinear static and dynamic analysis of laminated beams by using finite element method. Fraternali and Bilotti (1997) analyzed nonlinear stress of laminated composite curved beams. Ganapathi *et al*. (2009) studied nonlinear vibration analysis of laminated composite curved beams. Patel *et al*. (1999) examined nonlinear post-buckling and vibration of laminated composite orthotropic beams/columns resting on elastic foundation with Von-Karman's strain-displacement relations. Oliveira and Creus (2003) investigated flexure and buckling behaviors of thin-walled composite beams with nonlinear viscoelastic model. Valido and Cardoso (2003) developed a finite element model for optimal desing of laminated composite thin-eslled beams with geometrically nonlinear effects. Machado

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(2007) studied nonlinear buckling and vibration of thin-walled composite beams. Cardoso *et al*. (2009) investigated geometrically nonlinear behavior of the laminated composite thin-walled beam structures with finite element solution. Emam and Nayfeh (2009) investigated post-buckling of the laminated composite beams with different boundary conditions. Malekzadeh and Vosoughi (2009) studied large amplitude free vibration of laminated composite beams resting on elastic foundation by using differential quadrature method. Gupta *et al*. (2010) studied post-buckling analysis of composite beams with different boundary conditions by using Ritz method. Chang *et al*. (2011) investigated thermal buckling and post-buckling of laminated composite beams with higher order beam theories. Akgöz and Civalek (2011), Civalek (2013) examined nonlinear vibration laminated plates resting on nonlinear-elastic foundation. Baghani *et al*. and Jafari-Talookolaei (2011) examined large amplitude free vibration and post-buckling of laminated beams resting on elastic foundation by using the variational iteration method. Youzera *et al*. (2012) presented nonlinear dynamics of laminated composite beams with damping effect. Gunda and Rao (2013) investigated post-buckling analysis of composite beams based on von-Karman nonlinear type. Patel (2014) examined nonlinear static of laminated composite plates with the Green-Lagrange nonlinearity. Akbaş (2014, 2015a, b, c) investigated geometrically nonlinear of cracked and functionally graded beams. Stoykov and Margenov (2014) studied Nonlinear vibrations of 3D laminated composite Timoshenko beams. Cunedioğlu and Beylergil (2014) examined vibration of laminated composite beams under thermal loading. Li and Qiao (2015a, b), Shen *et al*. (2016, 2017), Li and Yang (2016) investigated nonlinear postbuckling analysis of composite laminated beams. Asadi and Aghdam (2014), Mareishi *et al*. (2014), Kurtaran (2015), Mororó *et al*. (2015), Pagani and Carrera (2017) analyzed large deflections of laminated composite beams. Benselama *et al*. (2015), Liu and Shu (2015), Topal (2017) investigated buckling behavior of composite laminate beams. Latifi *et al*. (2016), Ebrahimi and Hosseini (2017) presented nonlinear dynamics of laminated composite structures. Akbaş (2018a, 2018b, 2018c) analyzed nonlinear and post-buckling behaviors of composite beams. Also, there are many nonlinear, vibration, buckling studies of other type composite structures such as functionally graded materials, sandwich, nano composites etc. in the literature (Ebrahimi *et al*. 2015, Ebrahimi and Salari 2015a, 2015b, 2015c, 2016a, 2016b), Ebrahimi and Shafiei 2016, Akbaş and Kocatürk 2012, Ebrahimi and Barati 2016a, 2016b, 2016c, 2016d, 2016e, 2016f, 2016g, 2017a, 2017b, Kocatürk and Akbaş 2010, 2011, 2012, 2013, Ebrahimi and Farazmandnia 2016, 2017, Akbaş 2013, 2017, Ebrahimi and Hosseini 2016, Ebrahimi and Jafari 2016).

In the most of the post-buckling studies of laminated composite beams, the von-Karman strain displacement approximation is used. In the von-Karman strain, full geometric non-linearity cannot be considered because of neglect of some components of strain, satisfactory results can be obtained only for large displacements but moderate rotations. In the present study, the thermal post-buckling analysis of a laminated Timoshenko beams is studied with

Fig. 1 A pinned-pinned supported laminated beam subjected to uniform temperature rising ΔT and cross-section

temperature dependent physical properties by using total Lagrangian finite element method in which full geometric nonlinearity which can be considered as distinct from the studies by using von-Karman nonlinearity. In addition, the considered problem has material nonlinearity properties because the material properties are dependent the temperature. Hence, the problem is both geometrically and material nonlinearities. The considered higher nonlinear problem is solved by using incremental displacement-based finite element method with Newton-Raphson iteration method. The main purpose of this paper is to fill this gap for laminated composite beams for thermal post-buckling behavior. However, the material nonlinearity is not considered. It would be interesting to demonstrate the ability of the procedure through a wider campaign of investigations. The effects of the fibber orientation angles, the stacking sequence of laminates and temperature rising on the post-buckling deflections, configurations and critical buckling temperatures of the composite laminated beam are examined and discussed. Also, the differences between temperature dependent and independent physical properties are investigated for thermal post-buckling responses of laminated composite beams.

2. Theory and formulation

A pinned-pinned supported laminated composite beam with three layers of length L, width b and height h, as shown in Fig. 1. The beam is subjected to uniform temperature rising ΔT as seen from Fig. 1. It is assumed that the layers are located as symmetry according to mid-plane axis. The height of each layer is equal to each other.

In this study, the material properties are temperature-dependent. The effective material properties of the laminated beam, Young's modulus, coefficient of thermal expansion and shear modulus are a function of temperature T as follows (Shen 2001, Li and Qiao 2015a)

$$E_{11}(T) = E_{01}(1 - 0.5\ 10^{-3}T)\ \text{GPa} \tag{1a}$$

$$E_{22}(T) = E_{02}(1 - 0.2\ 10^{-3}T)\ \text{GPa} \tag{1b}$$

$$G_{12}(T) = G_{13} = G_{012}(1 - 0.2\ 10^{-3}T)\ \text{GPa} \tag{1c}$$

$$G_{23}(T) = G_{023}(1 - 0.2\ 10^{-3}T)\ \text{GPa} \tag{1d}$$

$$\alpha_{11}(T) = \alpha_{011}(1 + 0.5\ 10^{-3}T)\ 1/\ {}^0C \tag{1e}$$

$$\alpha_{22}(T) = \alpha_{022}(1 + 0.5\ 10^{-3}T)\ 1/\ {}^0C \tag{1f}$$

where, E_{11} and E_{22} indicate the Young's modulus in the longitudinal and transverse directions, respectively, G_{12},

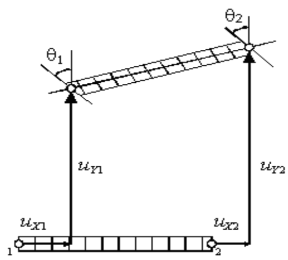


Fig. 2 Two-node beam element

G_{13}, G_{23} are the shear modulus, α_{11} and α_{22} are the thermal expansion coefficients in the longitudinal and transverse directions, respectively. $m=\cos\theta$ and $n=\sin\theta$, θ indicates the fiber orientation angle. E_{01}, E_{02}, G_{012}, G_{023}, α_{011}, α_{022} are the material constants at initial temperature value. T is final temperature; $T=T_0+\Delta T$. where,T_0 is installation temperature and ΔT is the temperature rising.

The equivalent Young's modulus of kth layer in the X direction (E_x^k) as is used the following formulation as a function of temperature T (Vinson and Sierakowski, 2002):

$$\frac{1}{E_x^k(T)} = \frac{cos^4(\theta_k)}{E_{11}(T)} + \left(\frac{1}{G_{12}(T)} - \frac{2\nu_{12}}{E_{11}(T)}\right) cos^2(\theta_k)\, sin^2(\theta_k) + \frac{sin^4(\theta_k)}{E_{22}(T)} \tag{2}$$

The temperature rising ΔT is governed by heat transfer equation of

$$-\frac{d}{d(Y)} = \left[K(Y)\frac{dT(Y)}{d(Y)}\right] = 0 \tag{3}$$

By integrating Eq. (3) using boundary conditions for kth layer $T(h_k/2)=\Delta T_T^k$ and $T(-h_k/2)=\Delta T_B^k$, the following expression can be obtained

$$\Delta T^k(Y) = \frac{\Delta T_B^k + \Delta T_T^k}{2} + \frac{\Delta T_T^k{}_{-}\Delta T_B^k}{h_k} \quad -0.5h_k \leq Y \leq 0.5h_k \tag{4}$$

where, h_k is the thickness of kth layer, ΔT_T^k is the temperature value of top surface in the kth layer and $\Delta T_B^k$is the temperature value of bottom surface in the kth layer.

It is known that the post-buckling case is a geometrically nonlinear problem. The Lagrangian formulations of the problem are developed for laminated composite beam by using the formulations given by Felippa (2018) for isotropic and homogeneous beam material. The finite beam element of the problem is derived by using a two-node beam element shown in Fig. 2, of which each node has three degrees of freedom, i.e., two displacements u_{xi} and u_{yi} and one rotation θ_i about the Z axis. The coordinates of the beam at the current C configuration are

$$x = x_c - Y(sin\psi + sin\gamma\, cos\psi) = x_c - Y[\sin(\psi+\gamma) + (1-cos\psi)sin\psi] = x_c - Ysin\theta \tag{5}$$

$$y = y_c + Y(cos\psi - sin\gamma\, sin\psi) = y_c + Y[\cos(\psi+\gamma) + (1-cos\gamma)cos\psi] = y_c + Ycos\theta \tag{6}$$

where, $x_c = X + u_{XC}$ and $y_c = u_{YC}$. Consequently, $x = X + u_{XC} - Ysin\theta$ and $y = u_{YC} + Ycos$. From now on, we shall call u_{XC} and u_{YC} simply u_X and u_Y, respectively. Thus the Lagrangian representation of the coordinates of the generic point at C is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} X + u_X - Ysin\theta \\ u_Y + Ycos\theta \end{bmatrix} \tag{7}$$

in which u_X, u_Y and θ are functions of X only. For a two-node element, it is natural to express the displacements and rotation as linear functions of the node degrees

$$\begin{bmatrix} u_X(X) \\ u_Y(X) \\ \theta(X) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1-\xi & 0 & 0 & 1+\xi & 0 & 0 \\ 0 & 1-\xi & 0 & 0 & 1+\xi & 0 \\ 0 & 0 & 1-\xi & 0 & 0 & 1+\xi \end{bmatrix} \begin{bmatrix} u_{X1} \\ u_{Y1} \\ \theta_1 \\ u_{X2} \\ u_{Y2} \\ \theta_2 \end{bmatrix} = \mathbf{Nu} \tag{8}$$

in which $\xi=(2X/L_0)-1$ is the isoparametric coordinate that varies from $\xi=-1$ at node 1 to $\xi=1$ at node 2. The Green-Lagrange strains are given as follows Felippa (2018)

$$[\boldsymbol{e}] = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} e_{XX} \\ 2e_{XY} \end{bmatrix} = \begin{bmatrix} (1+u_X')cos\theta + u_Y'\, sin\theta - Y\theta' - 1 \\ 2e_{XY} \end{bmatrix} = \begin{bmatrix} e - Y\kappa - -\alpha 11(Y,T)\Delta T(Y,T) \\ \gamma \end{bmatrix} \tag{9}$$

$$e=(1+u_x')\cos\theta + u_Y'\sin\theta - 1 - \alpha_{11}(Y,T)\Delta T(Y,T) \tag{10a}$$

$$\gamma = -(1+u_x')\sin\theta + u_Y'\sin\theta - 1,\ \kappa=\theta' \tag{10b}$$

where e is the axial strain, γ is the shear strain, and κ is curvature of the beam, $u_X' = X/dX$, $u_Y' = du_Y/dX$, $\theta' = d\theta/dX$. The axial force N, shear force V and bending moment M are given as follows

$$N = A_{11}e + B_{11}\kappa - N_T \tag{11a}$$

$$V = A_{55}\gamma \tag{11b}$$

$$M = B_{11}e + D_{11}\kappa - M_T \tag{11c}$$

where A_{11}, B_{11}, D_{11} and A_{55} are the extensional, coupling, bending, and transverse shear rigidities respectively. N_T and M_T are the thermal axial force and the bending moment, respectively, and their expressions are defined as

$$A_{11} = \sum_{k=1}^{n} bE_x^k(T)(y_{k+1} - y_k) \tag{12a}$$

$$B_{11} = \frac{1}{2}\sum_{k=1}^{n} bE_x^k(T)(y_{k+1}^2 - y_k^2) \tag{12b}$$

$$D_{11} = \frac{1}{3}\sum_{k=1}^{n} bE_x^k(T)(y_{k+1}^3 - y_k^3) \tag{12c}$$

$$N_T = \sum_{k=1}^{n} bE_x^k(T)\alpha_{11}^k(T)\left(\frac{T_B^k + T_T^k}{2}\right)(y_{k+1} - y_k) \tag{12d}$$

$$M_T = \frac{1}{3}\sum_{k=1}^{n} bE_x^k(T)\alpha_{11}^k(T)\left(\frac{T_T^k{}_{-}T_B^k}{h_k}\right)(y_{k+1}^3 - y_k^3) \tag{12e}$$

Expression of the transverse shear rigidity A_{55} given as follows (Vinson and Sierakowski 2002)

$$A_{55} = \frac{5}{4}\sum_{k=1}^{n} bQ_{55}^k(T)\left(y_{k+1} - y_k - \frac{4}{3h^2}(y_{k+1}^3 - y_k^3)\right) \tag{13}$$

where Q_{55}^k is given below

$$Q_{55}^{k}(T) = G_{13}(T)cos^2(\theta_k) + G_{23}(T)sin^2(\theta_k) \qquad (14)$$

In the solution of the nonlinear problem, Newton-Raphson iteration method is used. The solution for the n+1 st load increment and ith iteration is performed using the following relation

$$d\boldsymbol{u}_n^i = \left(\boldsymbol{K}_T^i\right)^{-1}\boldsymbol{R}_{n+1}^i \qquad (15)$$

where $\boldsymbol{K}_T^i$ is the tangent stiffness matrix of the system at the i th iteration, $d\boldsymbol{u}_n^i$ is the displacement increment vector at the i th iteration and n+1 st load increment, $(\boldsymbol{R}_{n+1}^i)_S$ is the residual vector in i th iteration and n+1 st load increment. A series of successive iterations at the n+1 st incremental step gives

$$\boldsymbol{u}_{n+1}^{i+1}\ \boldsymbol{u}_{n+1}^i + d\boldsymbol{u}_{n+1}^i = \boldsymbol{u}_n + \Delta\boldsymbol{u}_n^i \qquad (16)$$

where

$$\Delta\boldsymbol{u}_n^i = \sum_{k=1}^{i} d\boldsymbol{u}_n^k \qquad (17)$$

The residual vector $\boldsymbol{R}_{n+1}^i$ for the structural system is given as follows

$$\boldsymbol{R}_{n+1}^i = \mathbf{f} - \mathbf{p} \qquad (18)$$

Where **f** is the vector of total external forces and **p** is the vector of total internal forces, as given in the Appendix.

The element tangent stiffness matrix for the total Lagrangian Timoshenko beam element as given by Felippa (2018) is

$$\boldsymbol{K}_T = \boldsymbol{K}_M + \boldsymbol{K}_G \qquad (19)$$

where $\boldsymbol{K}_G$ is the geometric stiffness matrix, and $\boldsymbol{K}_M$ is the material stiffness matrix given as follows

$$\boldsymbol{K}_M = \int_{L_0} B_m^T S\, B_m dX \qquad (20)$$

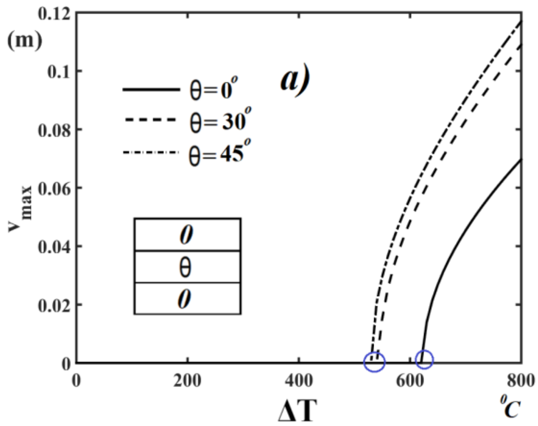
The explicit expressions of the terms in Eq. (22) are given in the Appendix. After integration of Eq. (23), the matrix $\boldsymbol{K}_M$ can be expressed as follows

$$\boldsymbol{K}_M = \boldsymbol{K}_M^a + \boldsymbol{K}_M^c + \boldsymbol{K}_M^b + \boldsymbol{K}_M^s \qquad (21)$$

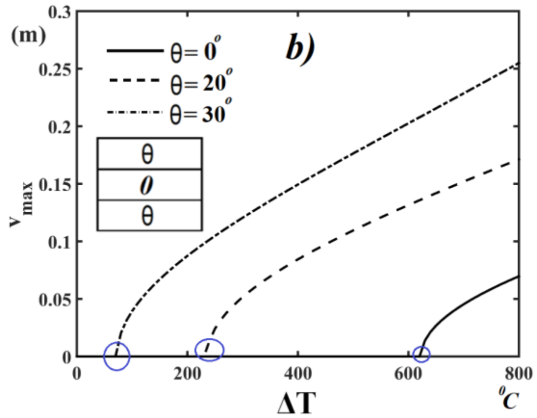
where $\boldsymbol{K}_M^a$ is the axial stiffness matrix, $\boldsymbol{K}_M^c$ the coupling stiffness matrix, $\boldsymbol{K}_M^b$ the bending stiffness matrix, and $\boldsymbol{K}_M^s$ the shearing stiffness matrix. These expressions are given in the Appendix.

3. Numerical results

In the numerical examples, post-buckling deflections and configurations of the simply supported laminated beam are calculated and presented for different fiber orientation angles and the stacking sequence of laminates under uniform temperature rising in both temperature dependent physical properties and temperature independent physical properties. The difference between the temperature dependent and temperature independent physical properties of the laminated beam is examined discussed for post-buckling case. Using the conventional assembly procedure for the finite elements, the tangent stiffness matrix and the residual vector are obtained from the element stiffness matrices and residual vectors in the total Lagrangian sense for finite element model of the laminated Timoshenko beams. After that, the solution process outlined in the preceding section is used to obtain the solution for the problem of concern. In obtaining the numerical results, graphs and solution of the nonlinear finite element model, MATLAB program is used. Numerical calculations of the integrals seen in the rigidity matrices will be performed by using five-point Gauss rule.



(a) for the stacking sequence [0/θ/0]

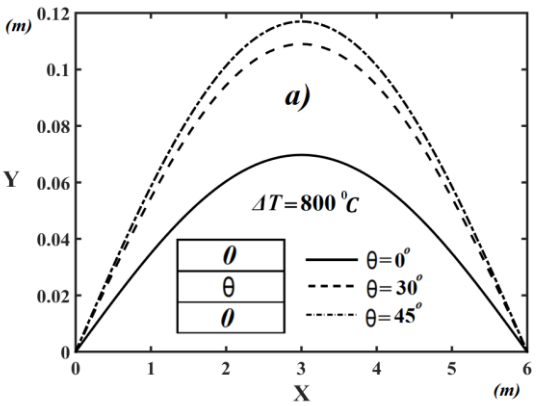


(b) for the stacking sequence [θ/0/θ]

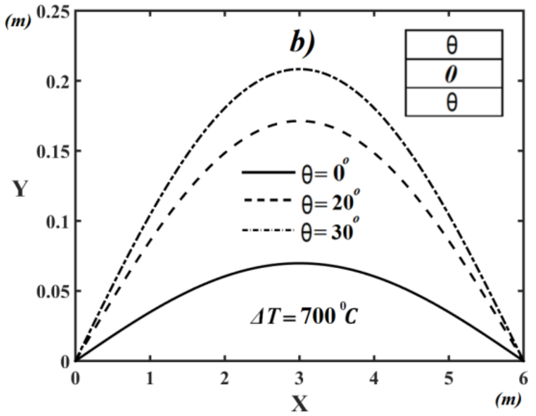
Fig. 3 Temperature rising ΔT- maximum displacements ($v_{\max}$) curves for different values of the fiber orientation angles (θ)

The laminated composite beam considered in numerical examples is made of Graphite/Epoxy. The material properties of Graphite/Epoxy are temperature dependent. For Eq. (1), the material constants of Graphite/Epoxy are expressed as follows (from Wang *et al*. 2002, Oh *et al*. 2000); E_{01}=150 GPa, E_{02}=9 GPa, G_{012}=7,1 GPa, G_{023}=2,5 GPa, α_{011}=1,1x10^{-6} $1/\,^0C$, α_{022}= 25,2x10^{-6} $1/\,^0C$ at 30 ^0C. In this study, the Possion's ratio is taken as ν=0.3. The geometry properties of the beam are considered as follows: b=0.2 m, h=0.2 m and L=6 m. It is mentioned before that the thickness of layers is equal to each other. The number of finite elements is taken as 100 in the numerical calculations. In the numerical results, the installation temperature is T_0=30 ^0C.

In Fig. 3, the maximum vertical displacements (at the midpoint of the beam) versus temperature rising ΔT are



(a) for the stacking sequence [0/θ/0]



(b) for the stacking sequence [θ/0/θ]

Fig. 4 Thermal Post-buckling configuration of the laminated beam for different values of the fiber orientation angles (θ)

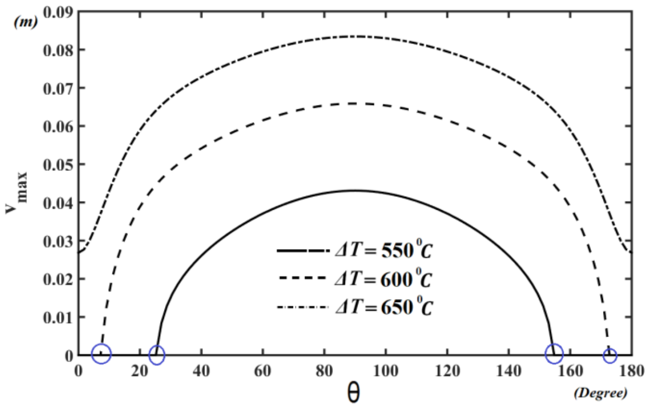


Fig. 5 The relationship between fiber orientation angles and thermal post-buckling displacements for different temperature values for [0/θ/0]

presented for different values of the fiber orientation angles (θ) for the stacking sequences [*0/θ/0*] and [θ/0/θ] in the temperature dependent physical properties.

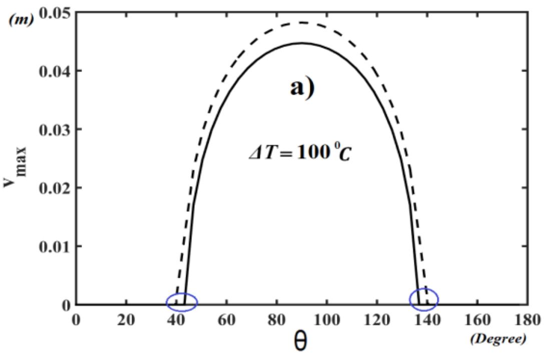
In Fig. 3, bifurcation points can be seen (see circle). It is known that buckling case occurs at the bifurcation points. Hence, these points give the value of critical buckling temperature. It is seen from figure 3 that increase in the fiber orientation angle (θ) causes a decrease in the critical buckling temperature (see bifurcation points) in both [0/θ/0] and [θ/0/θ] because the equivalent Young's modulus and bending rigidity decrease according to the Eq. (2). As a result, the strength of the material decreases and the critical buckling temperature decreases naturally. It is observed from Fig. 3 that the critical buckling temperature in [θ/0/θ] are smaller than [0/θ/0]'s. Thermal post-buckling displacements and critical buckling temperature in [θ/0/θ] change quickly with increasing the fiber orientation angles in contrast with [0/θ/0]]. The thermal post-buckling responses are very sensitive in the [θ/0/θ] in contrast with [0/θ/0]]. It is shows that the stacking sequence plays very important role on the thermal post-buckling behavior of the laminated beams.

In Fig. 4, the effect of the stacking sequences on the thermal post-buckling configuration of the laminated beam for the constant temperature rising ΔT for the stacking sequences [0/θ/0] and [θ/0/θ] in the temperature dependent physical properties. As seen from Fig. 4, the displacement shapes of the laminated beams change significantly with change the fiber orientation angles. With the increase in the fiber orientation angles, the displacements increase significantly as temperature is constant.

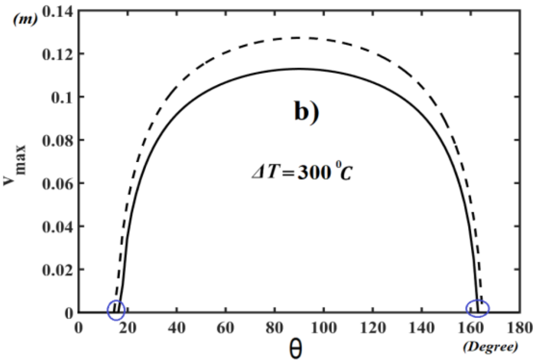
Fig. 5 shows that the effects of the fiber orientation angles on the maximum vertical displacements are plotted for the stacking sequence [0/θ/0], for different values of temperature rising ΔT in the temperature dependent physical properties.

It is observed from Fig. 5 that increasing the fiber orientation angles to 0° from 90°, the thermal post-buckling deflections increase significantly. At the fiber orientation angle θ=90°, the deflections are the greatest value for each layer arrangements because the equivalent Young's modulus and bending rigidity is the smallest values at θ=90°. As seen from Fig. 5 that the fiber orientation angles have a great influence on the buckling and post-buckling behaviour of the laminated beam. Decreasing the fiber orientation angles to 90° from 0° or to 90° from 180°, the buckling occurs in more temperatures (see circle) and critical buckling temperatures (see circle) increase seriously. Increasing the fiber orientation angles, the displacements increase significantly. It can be concluded from here: to choice suitable the fiber orientation angles is very important for safe design of laminated composite structures.

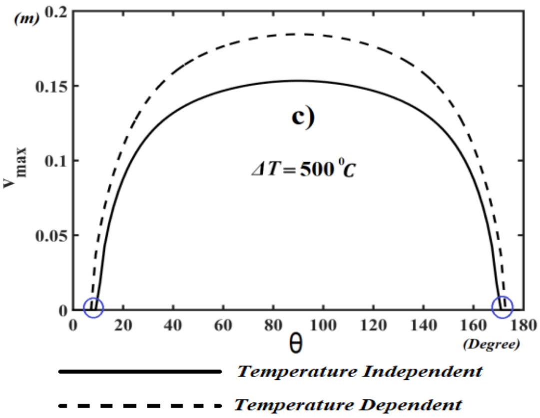
In Fig. 6, the difference between the temperature dependent and temperature independent physical properties is investigated in the temperature rising ΔT-maximum displacements ($v_{\max}$) curves in [θ/0/θ]. It is observed from Fig. 6 that the thermal post-buckling displacements for the temperature-dependent physical properties are greater than those for the temperature-independent physical properties. Critical buckling temperatures in temperature-dependent physical properties are smaller than those for the temperature-independent physical properties. This situation may be explained as follows: in the temperature-dependent physical properties, with the temperature increase, the intermolecular distances of the material increase and intermolecular forces decrease. As a result, the strength of the material decreases. Hence, the beam becomes more flexible in the case of the temperature-dependent physical



(a) for ΔT=20 ^0C

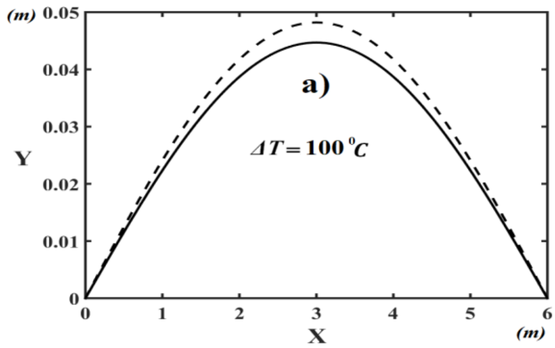


(b) for ΔT=200 ^0C

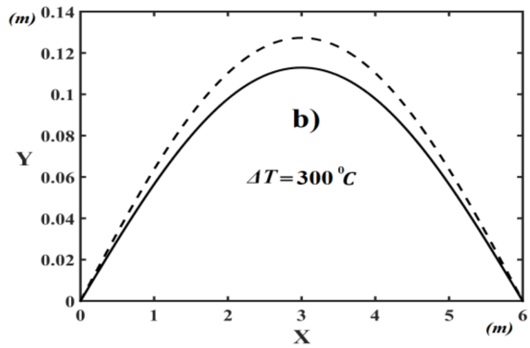


(c) for ΔT=500 ^0C

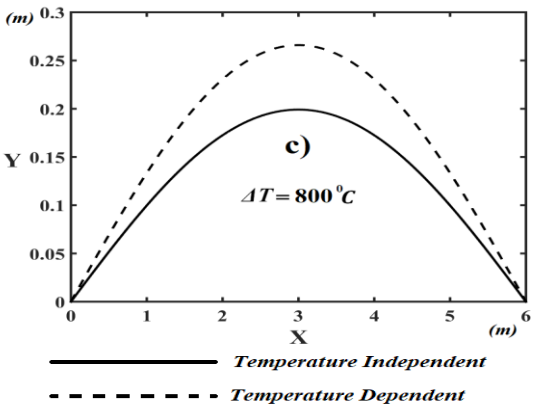
Fig. 6 The relationship between fiber orientation angles and thermal post-buckling displacements with temperature dependent and independent physical properties for [θ/0/θ]



(a) for ΔT=100 ^0C



(b) for ΔT=300 ^0C



(c) for ΔT=800 ^0C

Fig. 7 The difference between temperature dependent and independent physically properties on the thermal post-buckling deflected configuration for the stacking sequence 90/0/90

properties. Increasing the fiber orientation angles to 0° from 90°, the differences between the temperature dependent and temperature independent physical properties increase significantly. It shows that the fiber orientation angles play important role on the difference between the temperature dependent and temperature independent physical properties.

Fig. 7 shows that the difference between the temperature dependent and independent physical properties on the thermal post-buckling configuration is shown for different temperature rising for the stacking sequence 90/0/90. It is seen from Fig. 7 that increasing temperature leads to increase the difference between the temperature dependent and independent physical properties considerably. In the higher temperature values, the temperature dependent physical properties must be taken into account in the mechanical analysis of composite laminated beams and for obtaining more realistic results.

4. Conclusions

Thermal post-buckling analysis of a laminated composite beam is investigated under uniform temperature rising by using total Lagrangian finite element model with the Timoshenko beam theory with temperature dependent physical properties. The considered non-linear problem is solved by using incremental displacement-based finite element method in conjunction with Newton-Raphson iteration method. The effects of the fibber orientation angles, the stacking sequence of laminates and temperature rising on the post-buckling deflections, configurations and critical buckling temperatures of the composite laminated beam are studied. The differences between temperature dependent and independent physical properties are examined in the thermal post-buckling case.

It is concluded from numerical results that the fibber orientation angles and the stacking sequence play important role on the thermal post-buckling behaviour of the laminated composite beams. The fibber orientation angle is very effective to change the critical buckling temperatures and the thermal post-buckling responses. In higher temperature values, there are significant differences of the analysis results for the temperature dependent and independent physical properties in the thermal post-buckling case. The fiber orientation angle is very effective on the difference between the temperature dependent and temperature independent physical properties.

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Appendix

In this Appendix, the entries of the following matrices are given: axial stiffness matrix $\boldsymbol{K_M^a}$, coupling stiffness matrix $\boldsymbol{K_M^c}$, bending stiffness matrix $\boldsymbol{K_M^b}$, and shearing stiffness matrix $\boldsymbol{K_M^s}$ are developed for laminated composite beam by using the formulations given by Felippa (2018) for isotropic and homogeneous beam material.

In this Appendix, the entries of the following matrices are given: axial stiffness matrix $\boldsymbol{K_M^a}$, coupling stiffness matrix $\boldsymbol{K_M^c}$, bending stiffness matrix $\boldsymbol{K_M^b}$, and shearing stiffness matrix $\boldsymbol{K_M^s}$ developed for laminated composite beam by using the formulations given by Felippa (2018)

$$\boldsymbol{K}_M^a = \frac{\mathrm{A}_{11}}{L_0}\begin{bmatrix} c_m^2 & c_m s_m & -c_m\gamma_m L_0/2 & -c_m^2 & -c_m s_m & -c_m\gamma_m L_0/2 \\ c_m s_m & s_m^2 & -s_m\gamma_m L_0/2 & -c_m s_m & -s_m^2 & -s_m\gamma_m L_0/2 \\ -c_m\gamma_m L_0/2 & -s_m\gamma_m L_0/2 & \gamma_m^2 L_0^2/4 & c_m\gamma_m L_0/2 & -s_m\gamma_m L_0/2 & \gamma_m^2 L_0^2/4 \\ -c_m^2 & -c_m s_m & c_m\gamma_m L_0/2 & c_m^2 & c_m s_m & c_m\gamma_m L_0/2 \\ -c_m s_m & -s_m^2 & s_m\gamma_m L_0/2 & c_m s_m & s_m^2 & s_m\gamma_m L_0/2 \\ c_m\gamma_m L_0/2 & -s_m\gamma_m L_0/2 & \gamma_m^2 L_0^2/4 & c_m\gamma_m L_0/2 & s_m\gamma_m L_0/2 & \gamma_m^2 L_0^2/4 \end{bmatrix} \tag{A1}$$

$$\boldsymbol{K}_M^c = \frac{\mathrm{B}_{11}}{L_0}\begin{bmatrix} 0 & 0 & -c_m & 0 & 0 & c_m \\ 0 & 0 & -s_m & 0 & 0 & s_m \\ -c_m & -s_m & \gamma_m L_0 & c_m & s_m & 0 \\ 0 & 0 & c_m & 0 & 0 & -c_m \\ 0 & 0 & s_m & 0 & 0 & -s_m \\ c_m & s_m & 0 & -c_m & -s_m & -\gamma_m L_0 \end{bmatrix} \tag{A2}$$

$$\boldsymbol{K}_M^b = \frac{\mathrm{D}_{11}}{L_0}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \tag{A3}$$

$$\boldsymbol{K}_M^a = \frac{\mathrm{A}_{55}}{L_0}\begin{bmatrix} s_m^2 & -c_m s_m & \frac{-s_m\alpha_1 L_0}{2} & -s_m^2 & c_m s_m & \frac{-s_m\alpha_1 L_0}{2} \\ -c_m s_m & c_m^2 & \frac{c_m\alpha_1 L_0}{2} & c_m s_m & -c_m^2 & \frac{c_m\alpha_1 L_0}{2} \\ \frac{-s_m\alpha_1 L_0}{2} & \frac{c_m\alpha_1 L_0}{2} & \frac{\alpha_1^2 L_0^2}{4} & \frac{s_m\alpha_1 L_0}{2} & \frac{-c_m\alpha_m L_0}{2} & \frac{\alpha_1^2 L_0^2}{4} \\ -s_m^2 & c_m s_m & \frac{s_m\alpha_1 L_0}{2} & s_m^2 & -c_m s_m & \frac{s_m\alpha_1 L_0}{2} \\ c_m s_m & -c_m^2 & -\frac{c_m\alpha_1 L_0}{2} & -c_m s_m & c_m^2 & -\frac{c_m\alpha_1 L_0}{2} \\ \frac{-s_m\alpha_1 L_0}{2} & \frac{c_m\alpha_1 L_0}{2} & \frac{\alpha_1^2 L_0^2}{4} & \frac{s_m\alpha_1 L_0}{2} & -\frac{c_m\alpha_1 L_0}{2} & \frac{\alpha_1^2 L_0^2}{4} \end{bmatrix} \tag{A4}$$

where m denotes the midpoint of the beam, ξ=0, and $\theta_m = (\theta_1 + \theta_2)/2$, $\omega_m = \theta_m + \varphi$, $c_m = cos\omega_m$, $s_m = sin\omega_m$, $e_m = \frac{Lcos(\theta_m - \psi)}{L_0} - 1$, $\alpha_1 = 1 + e_m$ and $\gamma_m = \frac{Lcos(\psi - \theta_m)}{L_0}$. The initial axis of the beam considered is taken as horizontal, therefore φ=0. The matrix **S** is defined as follows

$$\boldsymbol{S} = \begin{bmatrix} A_{11} & 0 & -B_{11} \\ 0 & A_{55} & 0 \\ -B_{11} & 0 & D_{11} \end{bmatrix} \tag{A5}$$

The geometric stiffness matrix $\boldsymbol{K}_G$, the matrix $\boldsymbol{B}_m$ and the internal nodal force vector **p** remains the same as those given by Felippa (2018). Interested reader can find the related formulations in Felippa (2018).