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Analytical wave dispersion modeling in advanced piezoelectric double-layered nanobeam systems

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Abstract. This research deals with the wave dispersion analysis of functionally graded double-layered nanobeam systems (FG-DNBSs) considering the piezoelectric effect based on nonlocal strain gradient theory. The nanobeam is modeled via Euler-Bernoulli beam theory. Material properties are considered to change gradually along the nanobeams' thickness on the basis of the rule of mixture. By implementing a Hamiltonian approach, the Euler-Lagrange equations of piezoelectric FG-DNBSs are obtained. Furthermore, applying an analytical solution, the dispersion relations of smart FG-DNBSs are derived by solving an eigenvalue problem. The effects of various parameters such as nonlocality, length scale parameter, interlayer stiffness, applied electric voltage, relative motions and gradient index on the wave dispersion characteristics of nanoscale beam have been investigated. Also, validity of reported results is proven in the framework of a diagram showing the convergence of this model's curve with that of a previous published attempt.

Keywords: wave propagation; functionally graded double-layered nanobeam systems (FG-DNBSs); smart materials; non-local strain gradient piezoelectricity

1. Introduction

Functionally graded materials (FGMs), a newly developed type of composites initiated by a group of Japanese scientists in the mid-1980 (Koizumi *et al*. 1995), have been created by controlling the volume fraction of the mixture of two or more materials for the purpose of obtaining desired mechanical specifications by taking the desired properties of constituent phases. Such materials are isotropic and nonhomogeneous in common and possess continuously variable material properties about spatial coordinate and thus are able to remove the stress concentration found in laminated composites. Due to these applications, recently, FGMs engrossed great deal of attention of the researchers and scientific society to investigate these material's mechanical behavior (Javaheri and Eslami 2002, Chakraborty *et al*. 2003, Zenkour and Sobhy 2010, Zenkour 2013).

The physical, chemical and mechanical specifications of structures in the nanorealm, present obvious size effects that allow nanostructures to demonstrate precious mechanical and thermal behaviors which generates their superiority in comparison with the conventional macrostructures. Hence, nanostructures have gained wide application in micro- and nano-scale machineries because of their desirable specifications. Moreover, the interatomic bonds have an essential function on the nanostructure's organization, hence, the classical continuum theory which neglects such a pivotal fact is not a suitable alternative for research and study on the mechanical characteristics of nanostructures. Accordingly, the nonlocal elasticity theory (NET) introduced by Eringen (1972, 1983) is applicable for nanoscale devices due to a size-dependent parameter in which takes into consideration the small scale influences. Eringen's aforesaid theory explains that the stress state at any reference point shall be regarded as a function of the strains of all neighbor points in addition to strain state at that desired reference point. Thereafter, many authors applied NET to probe mechanical behavior of nanostructures. For instance, various beam theories including classical, first-order and higher-order beam theories are incorporated with the Eringen's differential fundamental equations by Reddy (2007) for the goal of examining the static and dynamic behaviors of nanoscale beams. Narendar and Gopalakrishnan (2009) highlighted the effect of nonlocal scale parameter on wave dispersion answers of multi-walled carbon nanotubes (MWCNTs) based on the NE. Also, a nonlocal Timoshenko beam model is utilized by Yang *et al*. (2011) in order to survey the demeanor of wave diffusion in double-walled CNTs (DWCNTs). Fotouhi *et al*. (2013) studied the effect of elastic foundation on the free dynamic behavior of embedded nanocones using a nonlocal continuum shell model. Another endeavor is performed by Naderi and Saidi (2014) applying NET to examine the postbuckling characteristics of graphene sheets (GSs) embedded on a nonlinear polymer foundation on the basis of an orthotropic nanoplate model. Reddy and El-Borgi (2014) reported the moderate rotation of nanobeams employing Euler-Bernoulli and Timoshenko beam models for deflection analysis of tiny beams. On the other hand, contrary to this fact that NET is broadly applied by lots of authors to account for size-dependency problem of small devices, this theory is not efficient enough to anticipate the stiffness-enhancement indicated by the experimental consequences

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and strain gradient elasticity (Fleck and Hutchinson 1993, Stölken and Evans 1998 and Yang *et al*. 2002). To this reason, a general size-dependent theory shall be able to predict both of these discussed influences. Thus, mixing these two impacts, nonlocal strain gradient theory (NSGT) is developed by Lim *et al*. (2015). Up to now, some of the authors tried to employ the NSGT while investigating the size-dependent mechanical behaviors of nanosize elements. For example, Euler-Bernoulli beam theory is used and mixed with the NSGT in order to study the stability responses of size-dependent nonlinear beams by Li and Hu (2015). Li *et al*. (2016) probed wave dispersion analysis of viscoelastic single-walled CNTs (SWCNTs) subjected to magnetic field employing a NSG surface elasticity. Ebrahimi and Dabbagh (2017) explored the wave dispersion answers of smart rotating composite nanoscale plates. Also, thermo-elastic stability analysis of orthotropic embedded nanoplates is reported by Farajpour *et al*. (2016) applying NSGT. Mahinzare et al. (2017) tried to clarify the influences of nonlocal and length scale parameters on the dynamic answers of a fluid conveying SWCNT. Most recently, a NSG based plate model is applied in order to perform dynamic analysis of GSs resting on an orthotropic elastic substrate tolerating hygro-thermal-magnetic loadings by Ebrahimi and Barati (2017). In addition, it is worth mentioning that the scale effects in the thickness direction of the tiny elements are not covered in the aforementioned theories. However, this can be a significant issue which has been lately regarded in a research by Li *et al.* (2018).

In addition, a type of intelligent materials, named piezoelectric materials, have received an unbelievable interest from research and industry societies because of their superior electro-mechanical specifications, simple manufacturing, wide field of employment, and their ability of converting mechanical energy into electrical one, vice versa. Sensors indicate direct effect of piezoelectric materials that converts the electrostatic repercussion to a mechanical energy, whereas, actuators operating cycle is completely different and they use mechanical excitations to generate electric field. Due to this coupling nature, piezoelectric materials are appropriate choices in order to satisfy the requirements of applications in micro-/nano-electro-mechanical systems (MEMSs/NEMSs) as sensors, actuators and convertors**.** Moreover, because of practical attributes of these materials many researchers used such materials in their works dealing with the electro-mechanical answers of beams or plates (Xu and Zhou 2011, Elshafei and Alraiess 2013, Mareishi 2014, Alibeigloo and Liew 2015, Phung-Van *et al*. 2015).

Recently, investigation of mechanical behavior of piezoelectric and FG nanoscale constructions engrossed great deal of interest in the research society. Yan and Jiang (2011) presented Euler-Bernoulli beam theory and surface elasticity to analyze the electro-mechanical dynamic and stability treatment of smart nanoscale beams. Also, Liang and Shen (2011) surveyed bending deflection answers of piezoelectric nanobeams with respect to influences of electrostatic force. It is revealed that the generated effect due to the electrostatic force is more remarkable in small thickness values. Timoshenko beam model and NE are combined by Ke *et al*. (2012) to study the nonlinear dynamic characteristics of piezoelectric nanobeams. Asemi *et al*. (2014) utilized differential quadrature method (DQM) to examine the initial stress influences on the natural frequencies of piezoelectric double-layered nanoplates rested on a two-parameter medium. Liu *et al*. (2014) introduced a nonlocal beam theory to perform buckling and post-buckling analyses of piezoelectric nanobeams once sustaining electro-thermo-elastic loadings. Nonlinear dynamic characteristics of a piezoelectric nanoplate resting on Winkler medium is reported using NE by Liu *et al*. (2015). On the other hand, Ebrahimi *et al*. (2015, 2016) could show the thermo-electro-mechanical stability behaviors of FG piezoelectric nanosize beams. Then, a generalized differential quadrature method (GDQM) is incorporated with the NE to perform the thermo-electro-mechanical dynamic analysis of postbuckled smart nanobeams by Ansari *et al*. (2016). NSGT is applied to investigate the damping vibration treatment of smart piezoelectric nanoscale plate resting on viscoelastic foundation by Ebrahimi and Barati (2017). Lately, third order beam model is applied by Ebrahimi and Barati (2017) to probe stability characteristics of FG piezoelectric embedded nanobeams based on the NET.

It is observable that, most of the endeavors are dedicated to buckling, static bending and vibration characteristics of FG nano-beams, and just a few number of them are devoted to the field of wave propagation of FG small-scale beams. Coupled influences of size-dependency and surface piezoelectricity on the wave propagation behaviors of intelligent nanosize plates are highlighted in the recent years (Zhang *et al*. 2014a, Zhang *et al*. 2014b). On the other hand, NSGT is incorporated with classical beam model by Li et al. (2015) in order to analyze the transverse wave dispersion properties of FG nanobeams. Ebrahimi *et al*. (2016a, 2016b, 2016c, 2017) presented various nonlocal continuum theories for thermo-mechanical wave dispersion characteristics of compositionally graded nanoscale beams and plates. Ebrahimi and Dabbagh (2017) surveyed wave dispersion problem of FG piezoelectric rotating nanobeams based upon the nonlocal piezoelectricity. Barati (2017) introduced a general nonlocal strain gradient porous beam model to analyze the wave dispersion answers of functionally graded double-layered nanobeam systems (FG-DNBSs).

On the other hand, there are many solution methods to solve the mechanical problems of continuous systems. These different approaches can be generally divided to two major types. The first group contains the analytical solutions like Navier, Galerkin, Exponential and so on; whereas, the second group includes numerical methods such as differential transformation method (DTM), DQM, discrete singular convolution method (DSCM) and finite element method (FEM). Majorly, the numerical methods are utilized in the cases of solving a nonlinear problem. Herein, a brief review of some of the former papers dealing with numerical solution methods is presented. Baltacıoglu *et al*. (2010) used the DSCM to analyze the nonlinear bending characteristics of composite plates. Civalek (2013) tried to combine both DQ and DSC methods for the goal of solving the nonlinear dynamic deflection behaviors of laminated plates while rested on nonlinear substrate. In another attempt, Mercan and Civalek (2016) extended the DSCM to reach the critical buckling limit of boron nitride nanotubes.

This research deals with the wave dispersion specifications of FG-DNBSs on elastic substrate studying on the basis of an Euler-Bernoulli beam model mixed with NSGT. Material properties are presumed to change continuously about the thickness of nanobeam utilizing Power-law model. The governing equations are developed applying a Hamiltonian approach. Thereafter, the obtained equations are analytically solved to catch the eigenvalue of the equation which stands for the circular frequency of propagated waves. It is clear that responses of FG-DNBSs are extremely affected by wave number, nonlocal parameter, strain gradient, interlayer stiffness, applied electric voltage, relative motions and gradient index.

2. Theory and formulation

2.1 Functionally graded materials (FGMs)

The volume fraction of functionally graded double-layered nanobeam systems FG-DNBSs is assumed to vary based on Power-law function

$$V_f = (\frac{z}{h} + \frac{1}{2})^p \tag{1}$$

where p is gradient index which controls the percentage of each phase in each desired thickness. Therefore, the equivalent mechanical and electrical properties of a FG nanobeam based on Power-law model can be expressed as follows

$$P_f(z) = V_f(z)P_1 + (1 - V_f(z))P_2 \qquad for \quad -h/2 \le z \le h/2 \tag{2}$$

The lower and upper layers of FG nanobeam are made of *PZT-5H* and *PZT*-4, respectively; and their material properties are tabulated in Table 1.

2.2 Nonlocal strain gradient piezoelectricity theory

NSG elasticity enumerates the stress for both nonlocal elastic stress and strain gradient stress fields by introducing two parameters taking into consideration for small size. Hence, according to this theory, stress state can be formulated as

$$\sigma_{ij} = \sigma_{ij}^{(0)} - \frac{\sigma_{ij}^{(1)}}{dx} \tag{3}$$

in which the stress $\sigma_{xx}^{(0)}$ and $\sigma_{xx}^{(1)}$ are related to strain ε_{xx} and strain gradient $\varepsilon_{xx,x}$, respectively as

$$\sigma_{ij}^{(0)} = \int_0^L C_{ijkl}\,\alpha_0(x, x', e_0a)\varepsilon'_{kl}(x')dx' \tag{4}$$

$$\sigma_{ij}^{(1)} = l^2 \int_0^L C_{ijkl}\,\alpha_1(x, x', e_1a)\varepsilon'_{kl,x}(x')dx' \tag{5}$$

in which C_{ijkl} are the elastic coefficients and e_0a and e_1a are responsible for nonlocal effects and l is introduced to cover the strain gradient effects. Once choosing proper nonlocal kernel functions, the constitutive relations of NSGT can be reformulated as

Table 1 Material properties of PZT-4 and PZT-5H

Properties	PZT-4	PZT-5H
c_{11} (GPa)	81.3	60.6
e_{31} (cm^{-2})	-10	-16.604
k_{11} ($C^2m^{-2}N^{-1}$)	0.6712e-8	1.5027e-8
k_{33} ($C^2m^{-2}N^{-1}$)	1.0275e-8	2.554e-8
ρ (kgm^{-3})	7500	7500

$$[1-(e_1a)^2\nabla^2][1-(e_0a)^2\nabla^2]\sigma_{ij} = C_{ijkl} \times [1-(e_1a)^2\nabla^2]\varepsilon_{kl} - C_{ijkl}l^2[1-(e_0a)^2\nabla^2]\nabla^2\varepsilon_{kl} \tag{6}$$

where ∇^2 is the Laplacian operator. Assuming e_1=e_0=e, Eq. (6) can be rewritten as

$$[1-(ea)^2\nabla^2]\sigma_{ij} = C_{ijkl}[1-l^2\nabla^2]\varepsilon_{kl} \tag{7}$$

Also, NSGT can be developed for smart materials as

$$(1-(ea)^2\nabla^2)\sigma_{ij} = (1-l^2\nabla^2)[C_{ijkl}\,\varepsilon_{kl} - e_{mij}E_m] \tag{8}$$

$$(1-(ea)^2\nabla^2)D_i = (1-l^2\nabla^2)[e_{ikl}\,\varepsilon_{kl} - s_{im}E_m] \tag{9}$$

For a piezoelectric nanobeam, Eqs. (8) and (9) may be rearranged as

$$(1-\mu^2\nabla^2)\sigma_{xx} = (1-l^2\nabla^2)(c_{11}\varepsilon_{xx} - e_{31}E_x) \tag{10}$$

$$(1-\mu^2\nabla^2)D_x = (1-l^2\nabla^2)k_{11}E_x \tag{11}$$

$$(1-\mu^2\nabla^2)D_z = (1-l^2\nabla^2)(e_{31}\varepsilon_{xx} + k_{33}E_z) \tag{12}$$

2.3 Euler-Bernoulli beam model

Based on the Euler-Bernoulli beam theory, the displacement fields are considered to be in the following form

$$u_x(x,z,t) = u(x,t) - z\frac{\partial w(x,t)}{\partial x} \tag{13}$$

$$u_z(x,z,t) = w(x,t) \tag{14}$$

in which, u and w are displacement components in the mid-plane along the coordinates x and z, respectively. On the other hand, Maxwell's equation should be satisfied choosing a proper function for the electric potential. It has been proven that the assumption of a linear flexural profile for the electric potential incorporated with a uniform profile in the longitudinal direction cannot reveal that discussed satisfaction (Wang 2002). Later, a half linear and half trigonometric function is used for the electric potential and it has been shown that the results obtained by this profile are in a very good agreement with those achieved by the FEM. Thus, herein, this function is used again for the electric potential of piezoelectric materials as (Wang *et al.* 2001)

$$\Phi(x,z,t) = -\cos(\xi z)\phi(x,t) + \frac{2z}{h}V_E \tag{15}$$

where $\beta=\pi/h$. Also, V_E is the electric voltage applied to the piezoelectric FG-DNBS; and $\phi(x,t)$ is the spatial function of the electric potential in the x direction. Moreover, the only nonzero strain can be written as

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} \tag{16}$$

where ε_{xx} is axial strain tensor. Also, the following equations can be regarded for electric field in the x and z directions

$$E_x = -\Phi_{,x} = \cos(\xi z)\frac{\partial \phi}{\partial x}, \quad E_z = -\Phi_{,z} = -\xi\sin(\xi z)\phi - \frac{2V_E}{h} \tag{17}$$

To achieve equations of motion, Hamilton's principle can be stated as

$$\int_0^t \delta(\Pi_s - \Pi_k + \Pi_w)dt = 0 \tag{18}$$

Here, Π_s is strain energy, Π_k is kinetic energy and Π_w is work done by external applied forces. The first variation of strain energy Π_s can be calculated as

$$\delta\Pi_s = \int_0^L\int_{-h/2}^{h/2}(\sigma_{xx}\delta\varepsilon_{xx} - D_x\delta E_x - D_z\delta E_z)dzdx \tag{19}$$

Substituting Eqs. (16) and (17) into Eq. (19) yields

$$\begin{aligned}\delta\Pi_s &= \int_0^L\left(N_x\delta\left(\frac{\partial u}{\partial x}\right) - M_x\delta\left(\frac{\partial^2 w}{\partial x^2}\right)\right)dx \\ &+\int_0^L\int_{-h/2}^{h/2}\left(-D_x\cos(\xi z)\delta\left(\frac{\partial\phi}{\partial x}\right) + D_z\xi\sin(\xi z)\delta\phi\right)dAdx\end{aligned} \tag{20}$$

In which, N_x and M_x are the axial force and bending moment resultants, respectively. Relations between the stress resultants and stress component used in Eq. (20) are defined as

$$N_x = \int_{-h/2}^{h/2}\sigma_{xx}dz, \qquad M_x = \int_{-h/2}^{h/2}\sigma_{xx}zdz \tag{21}$$

In addition, the kinetic energy Π_k for graded piezoelectric nanobeam is formulated as

$$\begin{aligned}\delta\Pi_k &= \int_V\left(\frac{\partial u_x}{\partial t}\frac{\partial\delta u_x}{\partial t} + \frac{\partial u_z}{\partial t}\frac{\partial\delta u_z}{\partial t}\right)\rho(z)dV + \\ &\int_0^L\left[I_0\left(\frac{\partial u}{\partial t}\frac{\partial\delta u}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial\delta w}{\partial t}\right) - I_1\left(\frac{\partial u}{\partial t}\frac{\partial^2\delta w}{\partial t\partial x} + \frac{\partial\delta u}{\partial t}\frac{\partial^2 w}{\partial t\partial x}\right) + I_2\frac{\partial^2 w}{\partial t\partial x}\frac{\partial^2\delta w}{\partial t\partial x}\right]dx\end{aligned} \tag{22}$$

where, mass moments of inertias can be expressed by the following relations

$$(I_0, I_1, I_2) = \int_{-h/2}^{h/2}\rho(z)(1, z, z^2)dz \tag{23}$$

Hence, the work can be formulated as below

$$\delta\Pi_w = \int_0^L\left[-k_w\delta w + \left(k_p - N_E\right)\frac{\partial^2\delta w}{\partial x^2}\right]dx \tag{24}$$

where, N_E are the normal forces induced by various external electric voltage V_E, which can be expressed as

$$N_E = -\int_{-h/2}^{h/2}e_{31}\frac{2V_E}{h}dz \tag{25}$$

Inserting Eqs. (20), (22) and (24) in Eq. (18) and integrating by parts, and collecting the coefficients of δ_u, δ_w and $\delta\phi$, the following local governing equations are obtained

$$\frac{\partial N_x}{\partial x} = I_0\frac{\partial^2 u}{\partial t^2} - I_1\frac{\partial^3 w}{\partial x\partial t^2} \tag{26}$$

$$\frac{\partial M_x}{\partial x} - N_E\frac{\partial^2 w}{\partial x^2} = I_0\frac{\partial^2 w}{\partial t^2} + I_1\frac{\partial^3 u}{\partial x\partial t^2} - I_2\frac{\partial^4 w}{\partial x^2\partial t^2} + k_w\delta w - k_p\frac{\partial^2\delta w}{\partial x^2} \tag{27}$$

$$\int_{-h/2}^{h/2}\left(\cos(\xi z)\frac{\partial D_x}{\partial x} + \xi\sin(\xi z)D_z\right)dz = 0 \tag{28}$$

By integrating Eqs. (10)-(12), the force–strain, moment-strain relations and other necessary nonlocal relations within the piezoelectric FG nanobeam structure are achieved as

$$N_x = (1-\lambda^2\nabla^2)(A_{xx}\frac{\partial u}{\partial x} - B_{xx}\frac{\partial^2 w}{\partial x^2} + A_{31}^e\phi - N_E) + \mu^2\frac{\partial^2 N_x}{\partial x^2} \tag{29}$$

$$M_x = (1-\lambda^2\nabla^2)(B_{xx}\frac{\partial u}{\partial x} - D_{xx}\frac{\partial^2 w}{\partial x^2} + E_{31}\phi) + \mu^2\frac{\partial^2 M_x}{\partial x^2} \tag{30}$$

$$\int_{-h/2}^{h/2}\left\{D_x - \mu\frac{\partial^2 D_x}{\partial x^2}\right\}\cos(\xi z)dz = (1-\lambda^2\nabla^2)F_{11}\frac{\partial\phi}{\partial x} \tag{31}$$

$$\int_{-h/2}^{h/2}\left\{D_z - \mu\frac{\partial^2 D_z}{\partial x^2}\right\}\xi\sin(\xi z)dz = (1-\lambda^2\nabla^2)(A_{31}^e\frac{\partial u}{\partial x} - E_{31}\frac{\partial^2 w}{\partial x^2} - F_{33}\phi) \tag{32}$$

where $\mu=ea$ and $\lambda=l$ are nonlocal and length scale parameters, respectively and other quantities are defined as

$$\{A_{xx}, B_{xx}, D_{xx}\} = \int_{-h/2}^{h/2}\{c_{11}, zc_{11}, z^2c_{11}\}dz \tag{33}$$

$$\{A_{31}^e, E_{31}\} = \int_{-h/2}^{h/2}\{e_{31}\xi\sin(\xi z), ze_{31}\xi\sin(\xi z)\}dz \tag{34}$$

$$\{F_{11}, F_{33}\} = \int_{-h/2}^{h/2}\{k_{11}\cos^2(\xi z), k_{33}\xi^2\sin^2(\xi z)\}dz \tag{35}$$

By substituting Eqs. (26) and (27), into Eqs. (29) and (30), the following explicit relations can be derived as

$$N_x = (1-\lambda^2\nabla^2)(A_{xx}\frac{\partial u}{\partial x} - B_{xx}\frac{\partial^2 w}{\partial x^2} + A_{31}^e\phi - N_E) + \mu^2\left(I_0\frac{\partial^3 u}{\partial x\partial t^2} - I_1\frac{\partial^4 w}{\partial x^2\partial t^2}\right) \tag{36}$$

$$M_x = (1-\lambda^2\nabla^2)(B_{xx}\frac{\partial u}{\partial x} - D_{xx}\frac{\partial^2 w}{\partial x^2} + E_{31}\phi) + \mu^2\left(I_0\frac{\partial^2 w}{\partial t^2} + I_1\frac{\partial^3 u}{\partial x\partial t^2} - I_2\frac{\partial^4 w}{\partial x^2\partial t^2} + N_E\frac{\partial^2 w}{\partial x^2}\right) \tag{37}$$

Herein, by substituting Eqs. (31) and (32) in Eq. (28) and also substituting Eqs. (36) and (37) in Eqs. (26) and (27), the governing equations of functionally graded nanobeam are obtained in the following form

$$\left(1-\lambda^2\nabla^2\right)\left(A\frac{\partial^2 u}{\partial x^2}-B\frac{\partial^3 w}{\partial x^3}+A_{31}^e\frac{\partial\phi}{\partial x}\right)+\left(1-\mu^2\nabla^2\right)\left(-I_0\frac{\partial^2 u}{\partial t^2}+I_1\frac{\partial^3 w}{\partial x\,\partial t^2}\right)=0 \quad (38)$$

$$\left(1-\lambda^2\nabla^2\right)\left(B\frac{\partial^3 u}{\partial x^3}-D\frac{\partial^4 w}{\partial x^4}+E_{31}^e\frac{\partial^2\phi}{\partial x^2}\right)+\left(1-\mu^2\nabla^2\right)\begin{pmatrix}-I_0\frac{\partial^2 w}{\partial t^2}-I_1\frac{\partial^3 u}{\partial x\,\partial t^2}+I_2\nabla^2\frac{\partial^2 w}{\partial t^2}\\ -k_w w+k_p\nabla^2 w-N^E\nabla^2 w\end{pmatrix}=0 \quad (39)$$

$$\left(1-\lambda^2\nabla^2\right)\left(A_{31}^e\frac{\partial u}{\partial x}-E_{31}^e\frac{\partial^2 w}{\partial x^2}+F_{11}^e\frac{\partial^2\phi}{\partial x^2}-F_{33}^e\phi\right)=0 \quad (40)$$

Coupling these equations for DNBSs, the following relations can be derived

$$\left(1-\lambda^2\nabla^2\right)\left(A\frac{\partial^2 u_1}{\partial x^2}-B\frac{\partial^3 w_1}{\partial x^3}+A_{31}^e\frac{\partial\phi_1}{\partial x}\right)+\left(1-\mu^2\nabla^2\right)\left(-I_0\frac{\partial^2 u_1}{\partial t^2}+I_1\frac{\partial^3 w_1}{\partial x\,\partial t^2}\right)=0 \quad (41)$$

$$\left(1-\lambda^2\nabla^2\right)\left(A\frac{\partial^2 u_2}{\partial x^2}-B\frac{\partial^3 w_2}{\partial x^3}+A_{31}^e\frac{\partial\phi_2}{\partial x}\right)+\left(1-\mu^2\nabla^2\right)\left(-I_0\frac{\partial^2 u_2}{\partial t^2}+I_1\frac{\partial^3 w_2}{\partial x\,\partial t^2}\right)=0 \quad (42)$$

$$\left(1-\lambda^2\nabla^2\right)\left(B\frac{\partial^3 u_1}{\partial x^3}-D\frac{\partial^4 w_1}{\partial x^4}+E_{31}^e\frac{\partial^2\phi_1}{\partial x^2}\right)+\left(1-\mu^2\nabla^2\right)\begin{pmatrix}-I_0\frac{\partial^2 w_1}{\partial t^2}-I_1\frac{\partial^3 u_1}{\partial x\,\partial t^2}+I_2\nabla^2\frac{\partial^2 w_1}{\partial t^2}-k_w w_1\\ +k_p\nabla^2 w_1-N^E\nabla^2 w_1-K_0\left(w_1-w_2\right)\end{pmatrix}=0 \quad (43)$$

$$\left(1-\lambda^2\nabla^2\right)\left(B\frac{\partial^3 u_2}{\partial x^3}-D\frac{\partial^4 w_2}{\partial x^4}+E_{31}^e\frac{\partial^2\phi_2}{\partial x^2}\right)+\left(1-\mu^2\nabla^2\right)\begin{pmatrix}-I_0\frac{\partial^2 w_2}{\partial t^2}-I_1\frac{\partial^3 u_2}{\partial x\,\partial t^2}+I_2\nabla^2\frac{\partial^2 w_2}{\partial t^2}-k_w w_2\\ +k_p\nabla^2 w_2-N^E\nabla^2 w_2-K_0\left(w_2-w_1\right)\end{pmatrix}=0 \quad (44)$$

$$\left(1-\lambda^2\nabla^2\right)\left(A_{31}^e\frac{\partial u_1}{\partial x}-E_{31}^e\frac{\partial^2 w_1}{\partial x^2}+F_{11}^e\frac{\partial^2\phi_1}{\partial x^2}-F_{33}^e\phi_1\right)=0 \quad (45)$$

$$\left(1-\lambda^2\nabla^2\right)\left(A_{31}^e\frac{\partial u_2}{\partial x}-E_{31}^e\frac{\partial^2 w_2}{\partial x^2}+F_{11}^e\frac{\partial^2\phi_2}{\partial x^2}-F_{33}^e\phi_2\right)=0 \quad (46)$$

5. Solution procedure

The solution of governing equations of piezoelectric FG-DNBSs can be expressed by

$$u(x,t)=U\exp[i(\beta x-\omega t)] \quad (47)$$

$$w(x,t)=W\exp[i(\beta x-\omega t)] \quad (48)$$

$$\phi(x,t)=\Phi\exp[i(\beta x-\omega t)] \quad (49)$$

where U, W, and Φ are the wave amplitudes; β and ω indicate the wave number and circular frequency, respectively. Inserting Eqs. (47)-(49) into Eqs. (41)-(46) gives

Table 2 Comparison of the first dimensionless frequency of S-S power-law FG piezoelectric nanobeams once subjected to a uniform thermal loading (L/h=15, p=2)

μ (nm^2)	V_E (V)	ΔT=10 (K)		ΔT=50 (K)	
		Present	Ebrahimi and Salari (2016)	Present	Ebrahimi and Salari (2016)
	-0.5	9.9520	9.9520	9.6316	9.6316
0	0	9.5909	9.5909	9.2581	9.2581
	+0.5	9.2159	9.2158	8.8690	8.8689
	-0.5	9.1443	9.1443	8.7947	8.7946
2	0	8.7502	8.7501	8.3840	8.3839
	+0.5	8.3375	8.3372	7.9520	7.9520
	-0.5	8.5185	8.5183	8.1417	8.1417
4	0	8.0937	8.0936	7.6964	7.6962
	+0.5	7.6453	7.6453	7.2235	7.2234

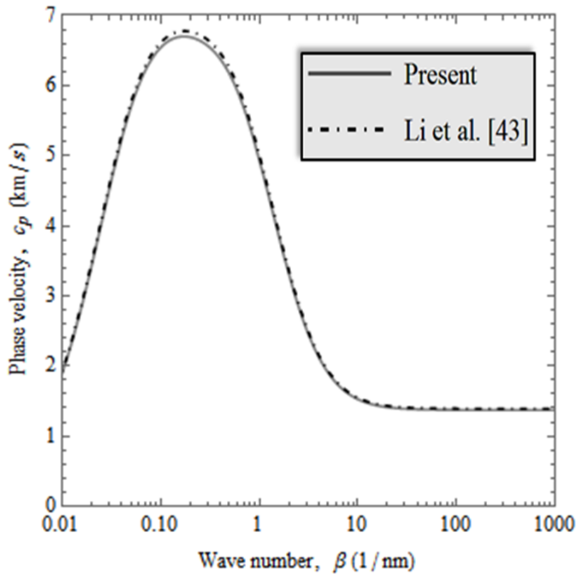


Fig. 2 Comparison of phase velocity variations of FG nanobeams with respect to wave number between two NSG based models (h=100 nm, p=1, μ=1 nm, λ=0.2 nm)

$$\left(\left[K\right]-\omega^2\left[M\right]\right)\begin{bmatrix}U\\W\\\Phi\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \quad (50)$$

By setting the determinant of above matrix to zero, the circular frequency ω can be obtained. Also, the phase velocity of waves can be calculated by the following relation

$$c_p=\frac{\omega}{\beta} \quad (51)$$

6. Results and discussion

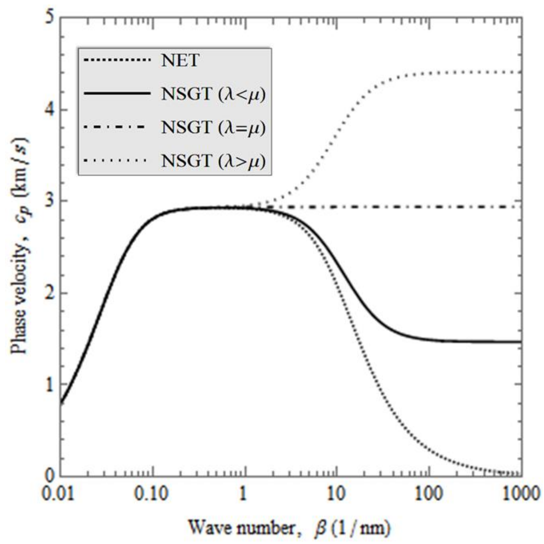
Wave dispersion characteristics of piezoelectric FG-

DNBSs on elastic substrate is figured out and this section is devoted to investigate the figures and outcomes. The mentioned nanobeam is modeled via Euler-Bernoulli beam theory. The material properties of such a FG nanobeam that is composed of *PZT*-4 and *PZT*-5*H* is presented in Table 1. Validity of the presented modelling is proven in both graphical and tabular ways. First, the comparison of the dimensionless frequency of simply-supported FG piezoelectric nanobeams is performed in Table 2 in the presence of a uniform thermal loading for various nonlocal parameters and applied electric voltages. Also, Fig. 2 is presented to highlight the variations of phase velocity curves of FG nanobeams achieved from this research and a recently published work performed by Li *et al*. (2015). It can be figured out that this model is a reliable model for mechanical analysis of such structures.

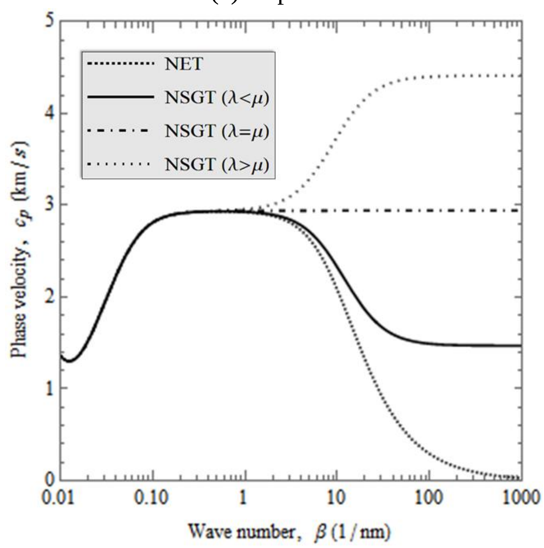
Variation of phase velocity of FG-DNBSs versus wave number for various continuum theories at (a) in-phase, (b) out-of-phase and (c) one nanobeam fixed motions is plotted in Fig. 3 It is observable that the phase velocity increases with a raise in the wave number for β≤0.1 range. However, in β≥1 there is a limitation for the case of nonlocal elasticity theory (μ≠0, λ=0) which results in a decrease in the phase velocity values. But, for the case of nonlocal strain gradient elasticity at a constant wave number in β≥1 by increasing in length scale parameter with respect to nonlocality parameter, phase velocity increases and tend to a constant value. On the other hand, for 0.1≤β≤1 and β≥100 the phase velocity has no sensible change with the change in wave number.

Fig. 4 shows the Coupled effects of applied electric voltage and gradient index on the phase velocity of FG-DNBSs in the case of out-of-phase relative motion whenever length scale parameter is smaller than nonlocal parameter. It is clear that with the increase in wave number the phase velocity increases for β≤0.1 but for 1≤ β≤100 the phase velocity decreases and tends to a constant value. However, in 0.1≤β≤10 no remarkable variation can be reported while wave number is supposed to be changed. Also, in a constant value of wave number, in the range of β≤0.1 it is observable that an increase in the applied electric voltage leads to a reduction the phase velocity values. However, this diagrams of different applied electric voltage are more distinguished in lower values of wave number. Moreover, with the increase in gradient index, in a constant value of wave number, the phase velocity will decrease. So, it can be concluded that both of the voltage and material distribution parameters play a decreasing role on the wave dispersion answers of FG-DNBSs.

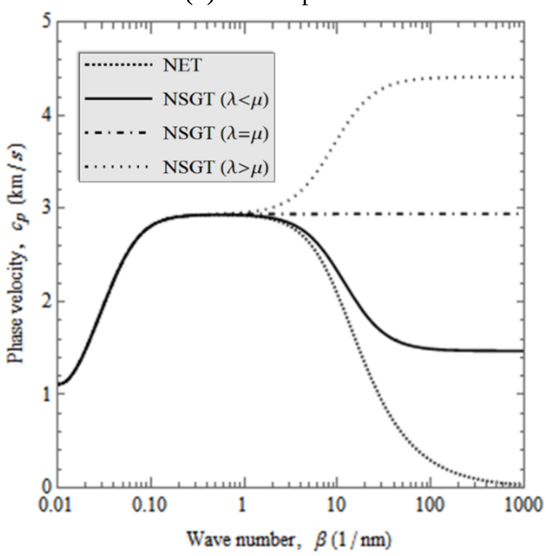
Besides, Fig. 5 is depicted to highlight the impact of interlayer stiffness on the phase velocity of FG-DNBSs at (a) out-of-phase and (b) one nanobeam fixed motions once length scale parameter is assumed to be smaller than nonlocality. It can be seen that, with an increase in wave number from β=0.01 at first, the phase velocity decreases a little and then raises and tends to a constant value in β≤0.1. Then, for β≥1 the phase velocity begins to decreases and tends to a constant value. Also, in a constant value of wave number, once higher values of interlayer stiffness are employed, greater phase velocities can be reached. However, the influences of different interlayer stiffnesses



(a) In-phase

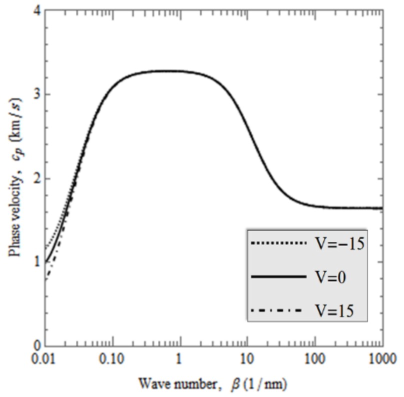


(b) Out-of-phase

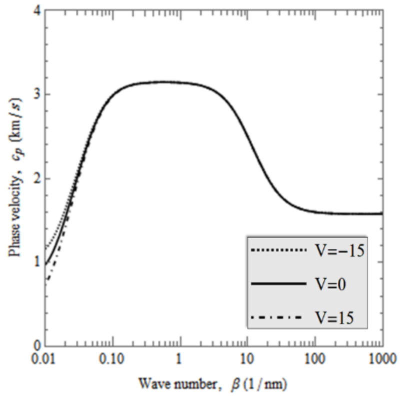


(c) One nanobeam fixed

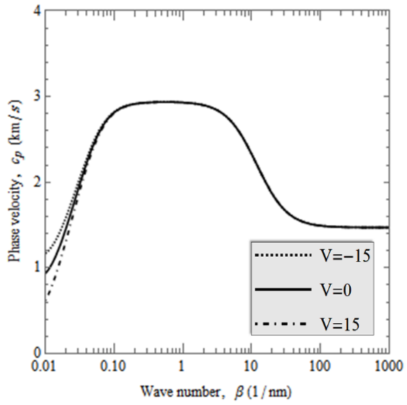
Fig. 3 Variation of phase velocity of FG-DNBSs versus wave number for various continuum theories at (a) in-phase, (b) out-of-phase and (c) one nanobeam fixed motions (p=1, k_w=10^{14}, k_p=1, K_0=5×10^{16})



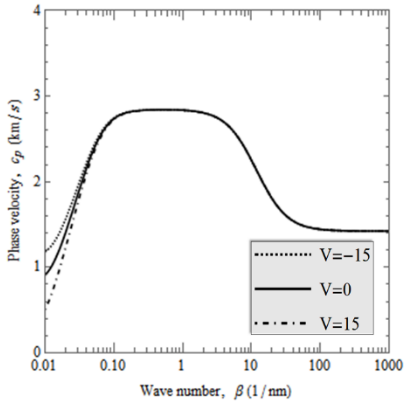
(a) p=0



(b) p=0.2

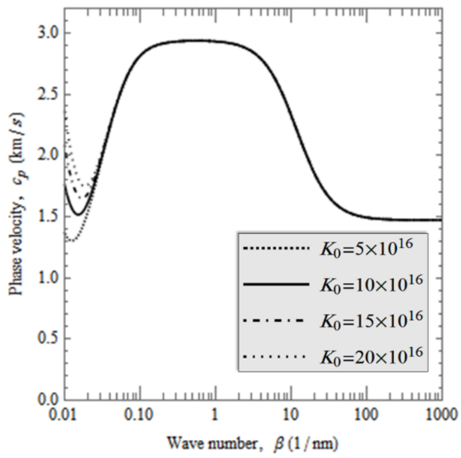


(c) p=1

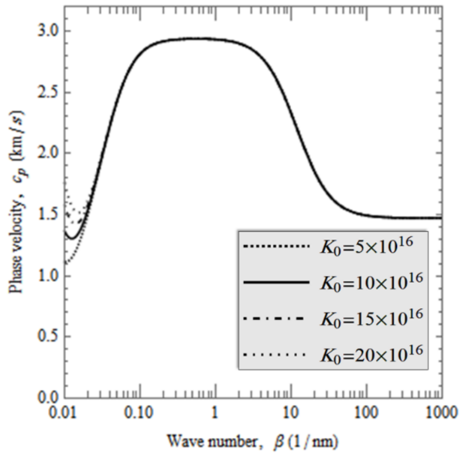


(d) p=5

Fig. 4 Coupled effects of applied electric voltage and gradient index on the phase velocity of FG-DNBSs in the case of out-of-phase relative motion ($\lambda<\mu$, $k_w=10^{14}$, $k_p=1$, $K_0=10^{16}$)



(a) Out-of-phase



(b) One nanobeam fixed

Fig. 5 Effect of interlayer stiffness on the phase velocity of FG-DNBSs at (a) out-of-phase and (b) one nanobeam fixed motions ($p=1$, $\lambda<\mu$, $k_w=10^{14}$, $k_p=1$)

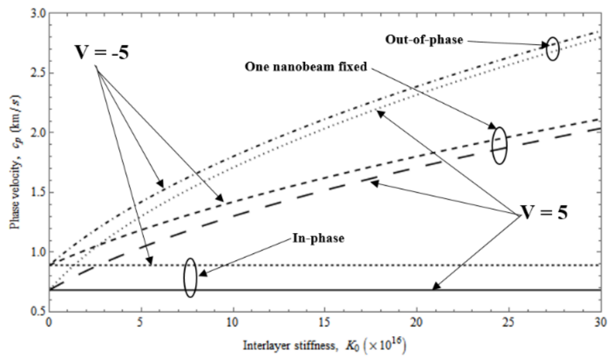


Fig. 6 Effects of electric voltage and interlayer stiffness on the phase velocity of FG-DNBSs for different types of relative motion of nanobeams ($\beta=0.01\times10^9$, $\lambda<\mu$, $p=1$, $k_w=10^{14}$, $k_p=1$)

are more distinguished in lower values of wave number and also are more distinguished in out-of-phase diagram.

Finally, mixed effects of electric voltage and interlayer stiffness on the phase velocity of FG-DNBSs for different types of relative motion of nanobeams (in-phase, one nanobeam fixed and out-of-phase) are shown in Fig. 6. It is obvious that an increase in interlayer stiffness can be answered by achieving higher phase velocity values. Also, it is shown that in a constant value of interlayer stiffness the phase velocity of out-of-phase motion is greater than one nanobeam fixed motion and much greater than in-phase motion. Also, once the electric voltage is added, the phase velocity decreases. It is of significance to point that any change in the amount of interlayer stiffness cannot affect phase velocity in the case of in-phase relative motion because in this situation the relative motion of nanobeams is zero and the system treats like a single nanobeam.

7. Conclusions

Wave dispersion of an embedded piezoelectric FG-DNBS is performed by employing Euler-Bernoulli beam theory. Nonlocal strain gradient piezoelectricity which contains two scale parameters is applied for taking into account the small size effects. Governing differential equations are derived by implementing Hamilton's principle. The constitutive relations of the smart materials are derived with the assumption that the constituent materials of the FG nanobeams are isotropic materials. In addition to the coupling interlayer spring, the lower nanobeam is considered to be embedded on a two-parameter elastic medium to present a more realistic mathematical model. Finally, through some parametric study, the effect of different parameters such as λ/μ ratio, wave number, phase velocity, different continuum theories, gradient index, elastic foundation constants, interlayer stiffness, applied electric voltage and relative motions of nanobeams on wave dispersion behavior of the system are covered. Herein, in order to put emphasize on the crucial role of involved parameters, the most important highlights of present article are reviewed as follows:

• Wave dispersion answers of FG-DNBSs can be aggrandized either employing higher length scale parameters or using smaller nonlocal parameters.

• Magnitude of phase velocity can be decreased whenever smaller gradient indexes are selected. In other words, the mechanical response is in its peak value whenever the power-law exponent is equal to zero.

• Phase velocity of DNBSs doesn't depend on the value of interlayer stiffness in the case of in-phase relative motion of nanobeams. However, a raise in the value of phase velocity can be observed once interlayer stiffness is increased for out of phase and one-nanobeam fixed cases.

• Based on the smart behaviors of the structure, smaller phase velocities can be obtained if the applied electric voltage is increased. Thus, applied voltage possesses a decreasing effect on the mechanical response of the structure.

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