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Buckling analysis of tapered BDFGM nano-beam under variable axial compression resting on elastic medium

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Abstract. The current study presents a new technique in the framework of the nonlocal elasticity theory for a comprehensive buckling analysis of Euler-Bernoulli nano-beams made up of bidirectional functionally graded material (BDFGM). The mechanical properties are considered by exponential and arbitrary variations for axial and transverse directions, respectively. The various circumstances including tapering, resting on two-parameter elastic foundation, step-wise or continuous variations of axial loading, various shapes of sections with various distribution laws of mechanical properties and various boundary conditions like the multi-span beams are taken into account. As far as we know, for the first time in the current work, the buckling analyses of BDFGM nano-beams are carried out under mentioned circumstances. The critical buckling loads and mode shapes are calculated by using energy method and a new technique based on calculus of variations and collocation method. Fast convergence and excellent agreement with the known data in literature, wherever possible, presents the efficiency of proposed technique. The effects of boundary conditions, material and taper constants, foundation moduli, variable axial compression and small-scale of nano-beam on the buckling loads and mode shapes are investigated. Moreover the analytical solutions, for the simpler cases are provided in appendices.

Keywords: buckling analysis; nonlocal elasticity theory; bidirectional functionally graded material; two-parameter elastic foundation; variable axial compression; tapered nano-beam; multi-span nano-beam

1. Introduction

Having various benefits over the conventional composite laminates, such as withstanding very high temperature gradients, less stress concentrations, further corrosive resistance, higher toughness and higher fracture resistance attract many researchers for investigating behaviors of structures made up of functionally graded materials (FGMs) (Shariati 2008, Heydari and Kazemi 2009, Heydari 2011, 2013, 2015, Heydari and Kazemi 2015, Roshan and Neha 2015, Ranganathan *et al.* 2016, Heydari 2017, Lal and Ahlawat 2017, Shojaeefard *et al.* 2017, Sachdeva and Padhee 2018, Shaterzadeh and Foroutan 2016, Yang and Yu 2017). These materials have continuous variation of thermo-mechanical properties due to varying the microstructure or atomic order with a specific gradient. Therefore, inventing FGMs make it possible to have incompatible properties such as thermal resistance and ductility in a material simultaneously. These kinds of materials are designed to achieve specific properties for specific applications such as electrical devices, diodes, sensors, inner wall of nuclear reactors, car engine cylinders, turbine blades, optical fibers, computer circuit boards, biomedical engineering, energy transformation, optics, etc. (Suresh 1998, Shariati *et al.* 2011, Mohammadhassani *et al.* 2013, Nguyen and Gan 2014, Wang and Gu 2014, Rezaiee-Pajand *et al.* 2015, Khorami *et al.* 2017, Khorramian *et al.* 2017, She *et al.* 2017, Shariati *et al.* 2018, Toghroli *et al*. 2018, Li *et al*. 2017, Moradi and Arwade 2014).

Sun *et al*. (2015) study the buckling of a standing tapered Timoshenko column subjected to tip force and self-weight. Tossapanon and Wattanasakulpong (2016) study buckling and vibration of FG sandwich beams with the homogeneous core rested on two-parameter elastic foundation by using Chebyshev collocation method. Top and bottom layers of sandwich beam are made up of FGM. The parameters of foundation model are Winkler and shear layer springs. They assumed that the FG layers are composed of ceramic and metal phases. In their research the shear deformation and rotary inertia effects are considered for constructing the governing equations of motion by employing the Timoshenko beam theory (TBT). They investigate the effects of various boundary conditions, foundation parameters, thickness ratio and material volume fraction index. They show that the parameters of elastic foundation have significant influence on the buckling and vibration behavior of these types of beams. Buckling

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analysis of homogeneous nano-beams using finite element method in the frame work of size dependent nonlocal integral elasticity theory is carried out by Taghizadeh *et al*. (2015). Buckling analysis of bidirectional FG nano-beams with uniform section subjected to constant axial compression based on nonlocal elasticity theory is performed by Nejad *et al*. (2016). The similar solutions for bending (Nejad and Hadi 2016) and free vibration (Nejad and Hadi 2016) problems are conducted. The governing equations are obtained by employing the minimum potential energy and thereafter the Generalized Differential Quadrature Method (GDQM) is used in order to calculate the critical buckling load. In special case by taking scale coefficient equal to zero, their model is degenerated into the classical model. Şimşek (2016) studied the buckling behavior of beams composed of two-dimensional functionally graded materials (2D-FGMs) having variable mechanical properties in axial and through-thickness directions according to the power-law form. In his research by employing TBT and after using the Ritz method, the critical buckling loads of local 2D-FG beams are calculated numerically. Various classical boundary conditions by adding auxiliary functions to the displacement functions are considered and trial functions in polynomial forms are used to calculate the critical loads. By using nonlocal strain gradient theory a Timoshenko FG beam model, which accounts for through-thickness power-law variation is derived by Li *et al*. (2016). They employed the Hamilton principle for obtaining the equations of motion. In their research for considering the effects of strain gradient stress field and nonlocal elastic stress field, the material length scale parameter and nonlocal parameter are introduced respectively. They found that the transverse grading of beam can be used to control the natural frequencies. In addition, they discovered that the vibration frequencies increased by the increasing of material length scale parameter or decreasing of nonlocal parameter in general. They observed the stiffness-softening and stiffness-hardening effects on beam by using smaller and larger material characteristic parameter with respect to the nonlocal parameter, respectively. The buckling analysis of axially functionally graded and non-uniform Timoshenko beams were investigated by Huang *et al*. (2016). They obtained a system of linear algebraic equations by using the auxiliary functions and power series from the coupled governing equations. Also, by considering various boundary conditions, they derived the characteristic polynomial equations corresponding to the buckling loads for axially heterogeneous local beams. They calculate higher and lower-order eigenvalues from the multi-roots simultaneously. Nguyen *et al*. (2016) proposed an analytical modeling for thin-walled open mono-symmetric channel section and I-section made up of FGM. There is a distance between center of gravity and the centroid of section in mono-symmetric I-section. The mechanical properties are varied through-thickness of section based on the power law distribution. They derived the locations of shear center and center of gravity for these kinds of beams. They considered restrained warping for the thin-walled FG beam by using Vlasov's assumptions. The exact solutions are achieved by solving the general governing equations, directly. They investigate the effects of thickness ratios and gradual law of metal or ceramic on the behavior of beams. The stability of non-uniform and axially FG Euler-Bernoulli beams rested on elastic foundation with general form of classical boundary conditions is considered by Shvartsman and Majak (2016). They avoid using infinity values of the stiffness coefficients by transforming end conditions into convenient form. The initial parameters in differential form are used for conducting numerical solutions. In their research, the homogeneous linear algebraic system of order two and iterative solution of two initial value problems are obtained. For improving the accuracy of outcomes, they used the Richardson extrapolation method. Arbitrarily in-plane large transverse displacements of planner curved FG beams are studied by Eroglu (2016). For deriving the governing differential equation system the axial and shear deformations are considered. Also, across the section material properties distribution is considered as an arbitrary function. He used Variational Iterational Method (VIM) by explicitly steps to solve the equations and examined snap-through and bifurcation buckling of pinned-pinned FG circular arches. Based on Euler-Bernoulli beam theory and modified strain gradient theory the buckling behavior of size-dependent microbeams made up of FGM for different end conditions is investigated by Akgöz and Civalek (2013). They used a variational statement to obtain the higher-order governing differential equation after regarding all possible classical and non-classical end conditions. The effects of the slenderness ratio, ratio of additional material length scale parameters for two constituents, ratio of beam thickness to additional material length scale parameter, material volume fraction index and boundary conditions are investigated on the buckling behavior of FG microbeams. They compared the outcomes of their model with results of the modified couple stress and classical continuum models. Compared to many other techniques, the Differential Transformation (DT) based on Dynamic Stiffness approach is a versatile method. Rajasekaran (2013) studied the buckling and free vibration of axially FG non-uniform local beams with various end conditions using DT based on Dynamic Stiffness approach. The mentioned method is capable of modeling any beams whose mechanical and geometrical properties vary along the beam. The buckling phenomenon is an important issue in structural designs, especially in braced frames (Bazzaz *et al.* 2011, Bazzaz *et al.* 2012, Jalali *et al.* 2012, Andalib *et al.* 2014, Bazzaz *et al.* 2014, Bazzaz *et al.* 2015, Bazzaz *et al.* 2015, Mansouri *et al.* 2017).

In current work, the neutral stability equation is obtained by setting the total potential energy equal to zero. By regarding the principle of the minimum potential energy a new semi-analytical technique based on calculus of variation and collocation method is obtained for buckling analysis of tapered arbitrary bidirectional functionally graded material (BDFGM) Euler-Bernoulli nano-beam subjected to variable axial compression. This scheme has a good compatibility for tapered FG plates (Heydari *et al.* 2017). The BDFGM nano-beam is rested on the Winkler (one-parameter) or Pasternak (two-parameter) foundation.

The effects of material and taper constants, foundation parameters, continuous or step-wise variations of axial compression, various boundary conditions like the multi-span conditions and small-scale of the nano-beam on the buckling loads and mode shapes are investigated. After simplifying the governing equations, the exact transverse displacement of one-span tapered local FG beam rested on shear layer by ignoring Winkler stiffness parameter is obtained in terms of Bessel functions of the first and second kinds. For this case, by considering nontrivial solution after applying various boundary conditions, the dimensionless first mode shapes and corresponding critical buckling loads are obtained. The exact analytical results are obtained from solution of stability differential equation directly; therefore the results can be used for verifying the numerical outcomes based on collocation method. Moreover, the governing differential equation of current work after simplifying for the uniform nonlocal nano-beam subjected to constant axial force rested on Pasternak elastic foundation is same as the nonlinear vibration equation of motion of nano-beam after neglecting time dependent terms and axial deformation (Ghannadpour *et al.* 2013, Mohammadhassani *et al.* 2014, Togun and Bağdatlı 2016). Meanwhile, excellent agreement between current work outcomes with the known data in literature, wherever possible, is observed (Ghannadpour *et al.* 2013, Nejad *et al.* 2016, Shahabi *et al.* 2016) and validity of current new work is proved.

2. Formulation

The stress tensor in nonlocal elasticity theory depends to strain tensor at all points in domain of material. The one dimensional nonlocal differential constitutive equation that includes the small scale effect is as follows (Ghannadpour *et al.* 2013)

$$\sigma_{xx} - \eta^2 \frac{d^2}{(dx)^2}\sigma_{xx} = E\varepsilon_{xx} \tag{1}$$

in which σ_{xx}, ε_{xx} and E are axial stress, axial strain and modulus of elasticity respectively. The small scale effect is considered by using the scale coefficient η. In the Euler-Bernoulli beam theory the axial strain is equal to $-yw''(x)$, in which w is transverse displacement and $w''(x)$ denotes second derivative of $w(x)$ with respect to x. The x axis lies on neutral axis (N.A.) and y is taken along the thickness of BDFGM beam. After multiplying Eq. (1) by ydA and integrating over the cross section, one has

$$M - \eta^2 \frac{d^2}{(dx)^2} M = -EI_{eq} \frac{d^2}{(dx)^2} w \tag{2}$$

The parameter EI_{eq} is equivalent through-thickness bending rigidity of BDFGM nano-beam. By using power law distribution, the modulus of elasticity for rectangular section is defined as follows

$$E(x,Y) = e^{\alpha\left(\frac{x}{L}\right)}\left((E_c - E_m)\left(\frac{1}{2}+\frac{Y}{h}\right)^n + E_m\right) \tag{3}$$

in which E_m and E_c present modulus of elasticity for metallic and ceramic constituents. Also, the parameters n and α are material volume fraction index and material constant, respectively ($\alpha \geq 0, n \geq 0$). The parameters h and L are depth and length of the beam. The origin of Y ordinate is located at mid-axis ($-h/2 \leq Y \leq h/2$). The geometrical properties for rectangular section with positive taper constant are presented in Fig. 1. The parameters h_0, e, ξ and b are depth at left end, distance between mid-axis and neutral axis (e=Y-y), taper constant and width of the rectangular section respectively. The variation of thickness in BDFGM nano-beam with rectangular section is linear ($h(x)/h_0=1+\xi x/L$).

Because of through-thickness variation of mechanical properties, there exists a distance between neutral axis (N.A.) and centroid of the rectangle. The axial pressure load is imposed on N.A. which induces non-uniform pressure distribution due to distance between N.A. and mid-axis. Small changes of pressure, prebuckling forces or other imperfections are equivalent to prebuckling deformations imposed on the original BDFGM tapered beam's configuration. The experiments performed on columns and flat plates under in-plane compressive forces showed that they are relatively insensitive to slight imperfections (Ventsel and Krauthammer 2001). The sum of all infinitesimal axial forces must be vanished due to pure bending. Eq. (4) presents the sum of all infinitesimal axial forces in Cartesian coordinate. The parameter ρ denotes the radius of curvature caused by bending.

$$\sum f_x = -\int\int \frac{yE}{\rho} dA \tag{4}$$

By considering Eq. (4), the distance between N.A. and mid-axis in rectangular section is obtained.

$$e = \int_{-h(x)/2}^{h(x)/2} EYdY \Big/ \int_{-h(x)/2}^{h(x)/2} EdY = \left(\frac{n(E_c - E_m)}{(E_c + nE_m)(2n+4)}\right)\left(1+\xi\frac{x}{L}\right)h_0 \tag{5}$$

The equivalent flexural rigidity for rectangular section is calculated as follows

$$EI_{eq}(x) = b\int_{-\frac{h(x)}{2}-e}^{\frac{h(x)}{2}-e} Ey^2dy = be^{\alpha\left(\frac{x}{L}\right)} \int_{-\frac{h(x)}{2}}^{\frac{h(x)}{2}} \left((E_c - E_m)\left(\frac{1}{2}+\frac{Y}{h}\right)^n + E_m\right)(Y-e)^2 dY \tag{6}$$

The equivalent bending rigidity for rectangular section is simplified as follows

$$EI_{eq}(x) = be^{\alpha\left(\frac{x}{L}\right)} h_0^3 \left(1+\xi\frac{x}{L}\right)^3 \left(\frac{12E_c^2 + (n^2+4n+7)(4nE_cE_m + n^2E_m^2)}{12(n+2)^2(n+3)(E_c+nE_m)}\right) \tag{7}$$

The modulus of elasticity for annular section is defined

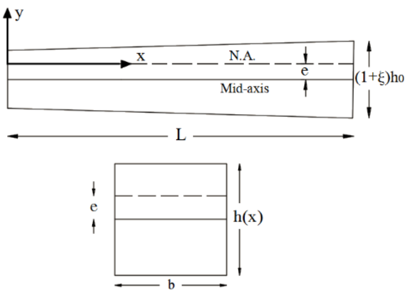

Fig. 1 Geometrical properties of rectangular section

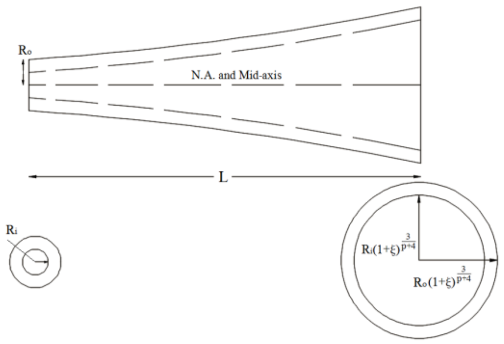

Fig. 2 Geometrical properties of annular section

as follows

$$E(x,r)=e^{\alpha\left(\frac{x}{L}\right)}E_0r^p \tag{8}$$

in which α, E_0 and p are material constants and p takes the non-positive real values. Fig. 2 shows the geometrical properties for annular section for positive ξ and $p\neq -4$. The parameters r_i and r_o are inner and outer radiuses at any section of the beam and the parameters R_i and R_o are inner and outer radiuses at left end, respectively.

By substituting Eq. (8) into Eq. (4) and after using coordinate transformation, the equilibrium of axial forces for annular section is observed. Therefore, in annular section N.A. is located at mid-axis (see Fig. 2).

$$\sum f_x=-\iint\frac{(r\sin\theta)E(r)}{\rho}dA=\\ -\frac{E_0}{\rho}\int_{r_i(x)}^{r_o(x)}r^{p+2}dr\int_0^{2\pi}\sin\theta d\theta=0 \tag{9}$$

By assuming nonlinear thickness variation pattern for inner and outer radiuses, an analytical solution for equivalent through-thickness bending rigidity is obtained.

$$\frac{r_o(x)}{R_o}=\frac{r_i(x)}{R_i}=\left(1+\xi\frac{x}{L}\right)^{\frac{3}{p+4}} \tag{10}$$

Equivalent through-thickness flexural rigidity for annular section is calculated with respect to the center line as follows

$$EI_{eq}(x)=\iint E(r\sin\theta)^2 rdrd\theta=\\ E_0e^{\alpha\left(\frac{x}{L}\right)}\int_{r_i(x)}^{r_o(x)}r^{p+3}dr\int_0^{2\pi}\sin^2\theta d\theta \tag{11}$$

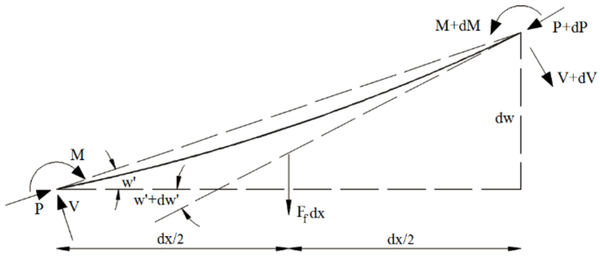

Fig. 3 Equilibrium of an infinitesimal BDFGM beam element

The equivalent through-thickness flexural rigidity for annular section is obtained as follows

$$EI_{eq}(x)=\\ \begin{cases}\dfrac{\pi e^{\alpha\left(\frac{x}{L}\right)}E_0\left(R_o^{p+4}-R_i^{p+4}\right)}{p+4}\left(1+\xi\dfrac{x}{L}\right)^3 & p\neq-4\\ \pi e^{\alpha\left(\frac{x}{L}\right)}E_0(p+4)\left(R_o^{p+3}-R_i^{p+3}\right)\left(1+\xi\dfrac{x}{L}\right)^3 & p=-4\end{cases} \tag{12}$$

Fig. 3 illustrates increment of internal forces and moment at right end of the beam element with an infinitesimal length, in which F_f, P, V, M and w' are response of foundation, axial load (compression taken to be positive), shear force, bending moment and slope at left end (first derivative of transverse displacement with respect to x) respectively. The equilibrium of forces in vertical direction is calculated as follows

$$P\sin(w')-(P+dP)\sin(w'+dw')+V\cos(w')-\\ (V+dV)\cos(w'+dw')-F_f dx=0 \tag{13}$$

The symbols ‘ $'$ ’ and ‘ $''$ ’ denote first and second derivatives with respect to x, respectively. For small amount of w' we have $\sin(w')=w'$ and $\cos(w')=1$. After neglecting differential terms with the orders greater than or equal to two, the equilibrium equation in vertical direction is obtained.

$$V'=-Pw''-P'w'-F_f \tag{14}$$

In Fig. 3, the equilibrium of bending moment after ignoring differential terms with the orders greater than or equal to two, implies that v = dM/dx. The bending moment in Fig. 3 and bending moment in Eq. (2) have the opposite signs. Therefore, the right hand-side of Eq. (14) after applying a minus sign is equal to the second derivative of bending moment with respect to x.

By considering Eqs. (2) and (14), the bending moment of nonlocal BDFGM nano-beam is obtained.

$$M=-EI_{eq}w''+\eta^2(Pw''+P'w'+F_f) \tag{15}$$

In two parameter elastic foundation models, the continuity of independent, discrete, linear elastic springs (supporting Winkler medium) is presumed by connecting them to a thin elastic membrane under a constant tension (Filanenko Borodich model) or connecting them to an isotropic layer of incompressible vertical elements with the shear interaction among the elements (Pasternak model). The response of Pasternak model for rectangular beam is presented mathematically as follows

$$F_f = k_0 bw - Gw'' \tag{16}$$

in which F_f is the response of the foundation model with the unit N/m, k_0 is Winkler modulus with the unit N/m^3 and G is Pasternak modulus (or constant tension in Filanenko Borodich model) with the Newton unit in SI.

3. Governing differential equation

According to the Lyapunov's stability definition, the stability of system is classified with respect to the response to an infinitesimal perturbation. If system returns to its initial position then the system is stable. But unstable system buckles significantly. In the neutral equilibrium state the system remains in vicinity of initial position. For deriving neutral equilibrium equation of BDFGM nano-beam the total potential energy (Π) is set equal to zero. The bending strain energy of BDFGM nano-beam and strain energy of elastic foundation are the elastic strain energy of the system (U). The work done by axial force (P) is the potential energy associated to the external force ($\Omega = -W_e$), where the minus sign implies a loss of potential energy. The thin BDFGM nano-beam is subjected to axial compression without eccentricity. It is assumed that the BDFGM nano-beam is perfect without any initial transverse deformation in unloaded state. The work done by external force, after using two terms of Taylor series expansion is calculated as follows

$$W_e = \int_0^L \left(\sqrt{1+(w')^2} - 1\right) P dx = \frac{1}{2}\int_0^L (w')^2 P dx \tag{17}$$

The neutral stability equation of BDFGM nano-beam is obtained as follows ($\Pi = U + \Omega$)

$$\Pi = \frac{1}{2}\int_0^L (F_f w - Mw'')dx - \frac{1}{2}\int_0^L (w')^2 P dx = 0 \tag{18}$$

After substituting Eqs. (15) and (16) into Eq. (18), the stability equation is obtained as follows

$$\int_0^L \phi(x, w, w', w'')dx = 0 \tag{19}$$

in which ϕ is introduced in Eq. (20).

$$\begin{gathered}\phi = \left(EI_{eq}w'' - \eta^2(Pw'' + P'w')\right)w'' + \\ (k_0 bw - Gw'')(w - \eta^2 w'') - (w')^2 P\end{gathered} \tag{20}$$

The BDFGM nano-beam rested on elastic foundation is in a stable state if and only if its potential energy attains minimum. According to the Fermat's theorem at any point where a differentiable function attains a local extremum, its derivative is zero. This is analogous to concept of Euler's equation in calculus. First, we assume that the function w(x) is differentiable and the integral in Eq. (19) is twice continuously differentiable. If w(x) extremizes Eq. (19), then any slight perturbation of w(x) preserves the boundary values. An auxiliary function is the result of such a perturbation of w(x) as g(x)=w(x)+μλ(x) in which λ(x) is differentiable function satisfying the following equations λ(i)(0)=λ(i)(L)=0 (i∈{0,1}). For minimizing the value of P in Eq. (19) the following equation must be satisfied.

$$\lim_{\mu\to 0}\frac{\partial}{\partial\mu}\int_0^L \phi(x, g, g', g'')dx = 0 \tag{21}$$

After using integrations by parts the general form of modified Euler-Lagrange equation is obtained in which $w^{(i)}$ is the ith derivative of w(x) with respect to x.

$$\begin{gathered}\frac{\partial}{\partial w}\phi + \sum_{i=1}^{2}(-1)^i\frac{\partial^i}{(\partial x)^i}\frac{\partial}{\partial w^{(i)}}\phi = \\ \varphi_0 + \sum_{i=1}^{4}\varphi_i w^{(i)} = 0\end{gathered} \tag{22}$$

The coefficients φ_0 to φ_4 are as follows

$$\begin{gathered}\varphi_0 = -k_0 bwL^5 \\ \varphi_1 = P^{(3)}L^5\eta^2/2 - L^5P' \\ \varphi_2 = \frac{(2\eta^2 k_0 b + 3\eta^2 P'' + 2G - 2P)L^5}{2} - EI_0 e^{\frac{\alpha x}{L}} \\ [\alpha x^2\xi^2(\alpha x\xi + 3L(\alpha + 2\xi)) + 3L^2\xi \\ (\alpha^2 x + 4\alpha\xi x + 2x\xi^2 + 2L(\alpha + \xi)) + \alpha^2 L^3] \\ \varphi_3 = 2\eta^2(P')L^5 - 2LEI_0 e^{\frac{\alpha x}{L}} \\ [\xi(3L^2(\alpha + 2\xi) + x\xi(\alpha x\xi + 3\alpha L + 3L\xi))x \\ + L^3(\alpha + 3\xi)] \\ \varphi_4 = L^2\left(L^3\eta^2(P - G) - EI_0 e^{\frac{\alpha x}{L}}(\xi x + L)^3\right)\end{gathered} \tag{23}$$

in which the parameter $EI_0 = EI_{eq}(0)$ is bending rigidity at left end of the BDFGM nano-beam. The parameter EI_0 is calculated from Eqs. (7) and (12) for rectangular and annular sections respectively. The governing differential equation in Eq. (22) for the uniform nonlocal nano-beam subjected to constant axial force rested on Pasternak elastic foundation is same as the nonlinear vibration equation of motion of nano-beam after ignoring time dependent terms and axial deformation (Arabnejad Khanouki *et al.* 2011, Ghannadpour *et al.* 2013, Togun and Bağdatlı 2016).

4. Buckling analysis

For surveying buckling loads and mode shapes of tapered bidirectional FG nano-beam rested on Pasternak elastic foundation subjected to variable axial compression the collocation method is used. The Taylor series expansion of w(x) up to m degree is selected as the finite-dimensional space of candidate solution and the points $x_i = iL/m$ ($i = 1,2,..,m - c$) for perfect FG-nano beam are selected as the collocation points, in which c is the number of boundary condition equations. In the first step, some coefficients in candidate solution are obtained by satisfying boundary conditions then the polynomial of approximated deflection is substituted into Eq. (22) to find the other coefficients of

Taylor series expansion of w(x) at collocated points. After obtaining the coefficients, the approximated deflection has been substituted into Eq. (19) and the resultant has been solved with respect to P. The mentioned procedure results to solve an algebraic system of linear equations to find the coefficients of Taylor series expansion of transverse displacement. The unknown coefficients of Taylor series expansion of w(x) are b_0, b_1, …, b_m. The coefficients b_0 to b_{c-1} are determined in terms of the coefficients b_c to b_m by satisfying boundary conditions. For perfect BDFGM nano-beam the coefficients b_c to b_{m-1} are determined in terms of b_m by satisfying Eq. (22) at collocation points x_i as b_j=-$(B_{ji}A_{im})b_m$ where b_m is a scalar, b_j and A_{im} (m^th column of matrix A) are vector and B_{ji} is a second order tensor. The indices i and j are dummy and free indices, respectively ($i\in\{1,2,\ldots,m-c\}$ and $j\in\{c,c+1,\ldots,m-1\}$). By using Levi-Civita symbol concept, B_{ji} is calculated in terms of matrix A as follows

$$(-1)^{i+j}q\sum_{b_{q-1}=1}^{q-1}\cdots\sum_{b_2=1}^{q-1}\sum_{b_1=1}^{q-1}\sum_{a_{q-1}=1}^{q-1}\cdots\sum_{a_2=1}^{q-1}\sum_{a_1=1}^{q-1}\prod_{s=1}^{q-1}\prod_{t=s+1}^{q-1}\prod_{k=1}^{q-1}\mathrm{sign}(a_t-a_s)\,\mathrm{sign}(b_t-b_s)\widetilde{A}_{a_kb_k}/\left[\sum_{b_q=1}^{q}\cdots\sum_{b_2=1}^{q}\sum_{b_1=1}^{q}\sum_{a_q=1}^{q}\cdots\sum_{a_2=1}^{q}\sum_{a_1=1}^{q}\prod_{s=1}^{q}\prod_{t=s+1}^{q}\prod_{k=1}^{q}\mathrm{sign}(a_t-a_s)\,\mathrm{sign}(b_t-b_s)A_{a_kb_k}\right] \tag{24}$$

in which $q = m - c$. The matrix $\widetilde{A}$ is obtained from deleting row j and column i of matrix A. The notation 'sign' represents the sign function. The matrix A_{ij} is defined as follows

$$A_{ij}=\frac{2Pi^2jL^7}{m^2}\left(\frac{iL}{m}\right)^j\left(1-j-\frac{P'iL}{Pm}\right)+A_{ij}^{TB}+A_{ij}^{N} \tag{25}$$

in which A_{ij}^{TB} and A_{ij}^{N} are matrices corresponding to mechanical and geometrical properties of tapered bidirectional (superscript TB) and nonlocal (superscript N) nano-beam respectively. The matrix A_{ij}^{TB} includes the parameters ξ and α.

$$A_{ij}^{TB}=-2EI_0L^4e^{\frac{\alpha i}{m}}\left(\frac{iL}{m}\right)^j j(j-1)\left(\frac{\xi iL}{m}+L\right)\left[\frac{\xi^2i^2j^2}{m^2}+\frac{2\xi^2\alpha i^3j}{m^3}+\frac{\xi^2\alpha^2i^4}{m^4}+\frac{2\xi ij^2}{m}+\frac{4\xi\alpha i^2j}{m^2}+\frac{\xi^2i^2j}{m^2}+\frac{2\xi\alpha^2i^3}{m^3}+\frac{2\xi^2\alpha i^3}{m^3}+\frac{2\alpha ij}{m}-\frac{4\xi ij}{m}+\frac{\alpha^2i^2}{m^2}-\frac{2\xi\alpha i^2}{m^2}-\frac{4\alpha i}{m}+j(j-5)+6\right] \tag{26}$$

Also, the matrix A_{ij}^{N} includes the nonlocal parameter η.

$$A_{ij}^{N}=\eta^2L^5\left(\frac{iL}{m}\right)^j j\left[\frac{4P'ij^2L}{m}+\frac{3P''i^2jL^2}{m^2}+\frac{P^{(3)}i^3L^3}{m^3}-\frac{12P'ijL}{m}-\frac{3P''i^2L^2}{m^2}+\frac{8P'iL}{m}+2P(j^3-6j^2+11j-6)\right] \tag{27}$$

For FG nano-beam subjected to constant axial compression an eigenvalue problem is solved. For this case the determinant of matrix A in Eq. (25) is vanished and buckling load of first modes are obtained. By assuming $P=P_0f(x)$, the unknown coefficients of transverse displacement b_0 to b_m are obtained in terms of P_0, where P_0 is a constant and the function f(x) shows the variation of axial load in axial direction. After substituting the transverse displacement in stability equation a nonlinear equation in terms of P_0 will be obtained. The roots of the characteristic equation are corresponding to the buckling loads of the associated buckling modes. The buckling load of multi-span nano-beam can be obtained after considering the deflection of each span as an independent function. In first step, the dependent boundary and natural conditions (like the continuation of slope or moment at inner supports) are neglected and only independent boundary and natural conditions (like the zero deflection or free end conditions) are satisfied for each span separately. Thereafter, the Eq. (24) is employed to calculate the coefficients b_j=-$(B_{ji}A_{im})b_m$ ($i\in\{1,2,\ldots,m-c\}$ and $j\in\{c,c+1,\ldots,m-1\}$) of Taylor series expansion for transverse displacements of side-spans. The unknown coefficients of Taylor series expansion of deflection for inner spans are calculated as b_j=-$(B_{ji}A_{im})b_m$ ($i\in\{1,2,\ldots,m-c-1\}$ and $j\in\{c,c+1,\ldots,m-2\}$) in terms of P_0, b_{m-1} and b_m. After performing this step, the buckling loads are obtained by vanishing the sum of rotational stiffness (the ratio of moment to rotation) at inner supports. For this purpose, the following equations must be satisfied

$$\left(\frac{w''_{ql}}{w'_{ql}}\right)_{\tilde{x}_{q-1}=L_q}-\left(\frac{w''_{qr}}{w'_{qr}}\right)_{\tilde{x}_q=0}=0,\qquad q=1,2,\ldots,s \tag{28}$$

where s, w_{ql}, w_{qr} and L_q are number of inner supports, deflections of the left and right spans and length of left span at the inner support q, respectively (see Fig. 4). The variation of mechanical and geometrical properties are considered in local coordinate $\tilde{x}_q$. The local coordinate $\tilde{x}_q$ is defined for the right span of inner support q. The global coordinate is replaced by $x=x_q+\tilde{x}_q$, where x_q is the distance between the inner support q and the left end of the multi-span beam. It is worth mentioning that, after assigning a small amount to L_q in Eq. (28), the boundary condition for clamped end will be obtained. The natural condition for free end is obtained after considering the equilibrium of shear force at free end as follows

$$\left(Pw'(\tilde{x})-\frac{d}{d\tilde{x}}M(\tilde{x})\right)_{\tilde{x}=L_c}=0 \tag{29}$$

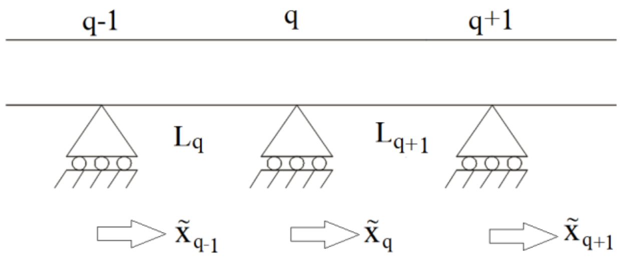

Fig. 4 Multi-span FG nano-beam

Table 1 Dimensionless critical buckling load for various size scale parameters and various boundary conditions

η/L	Ref.	Boundary condition			
		C-C	S-S	C-S	C-F
	Current work	39.4784	9.8696	20.1907	2.4674
0	(Nejad *et al.* 2016)	39.4784	9.8696	20.1907	-
	(Ghannadpour *et al.* 2013)	39.4784	9.8696	20.1907	2.4674
	Current work	15.3068	7.0760	11.1697	2.2457
0.2	(Nejad *et al.* 2016)	15.3068	7.0760	11.1697	-
	(Ghannadpour *et al.* 2013)	15.3068	7.0760	11.1697	2.2458
	Current work	0.9752	0.9080	0.9528	0.7115
1	(Nejad *et al.* 2016)	0.9753	0.9080	0.9528	-
	(Ghannadpour *et al.* 2013)	0.9753	0.9080	0.9528	0.7116

in which $\tilde{x}$ and x_c are the local coordinate of cantilever beam and the distance between support of the cantilever beam and left end of the multi-span beam respectively. The parameter L_c is the length of the cantilever beam.

5. Results and discussion

In this section the numerical exercises are conducted and the results for buckling analysis of tapered bidirectional nonlocal FG beam rested on two-parameter elastic foundation subjected to variable axial compression are presented. The dimensionless buckling loads of current work (P_0L^2/EI_0) for FG nano-beam with uniform section ($\xi = 0$) and transverse graduation ($\alpha = 0$) subjected to constant axial compression (f(x)=1) after neglecting parameters of elastic foundation ($k_0 = G = 0$) are compared with the known data in literature in Table 1. Table 1 presents the validity of current work outcomes for various boundary conditions and various scale coefficients. Moreover, the results of collocation method are same as the results of Eqs. (A6), (A7), (A9) and (A11) for α=η=0 when ξ approaches to zero.

The metallic and ceramic constituents are Zirconia and Aluminum, the material properties of which are E_c=200 GPa and E_m=70 GPa. For numerical solutions some numerical values are selected as $b = h_0 = r_o = \eta = 1$nm, r_i=0.5 nm, L=10 nm, ξ=β=0.1, n=2, p=-2. Furthermore, the material constant for annular FG nano-beam is selected as E_0= 7.75074 to have the same EI_0 with the rectangular section. Fig. 5 illustrates the ratio of nonlocal critical buckling load to local critical buckling load (P_R) for tapered

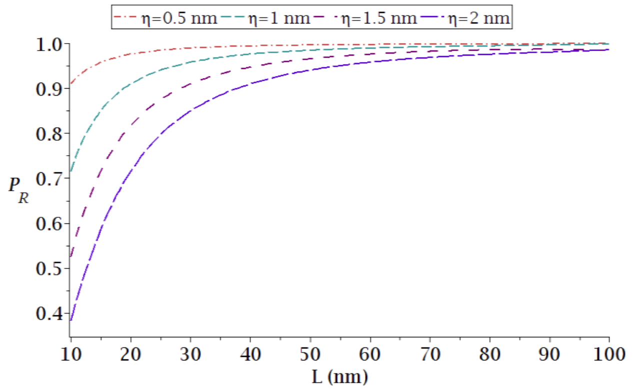

Fig. 5 The size scale parameter effect on the critical buckling load of tapered BDFGM (ξ=α=0.1)

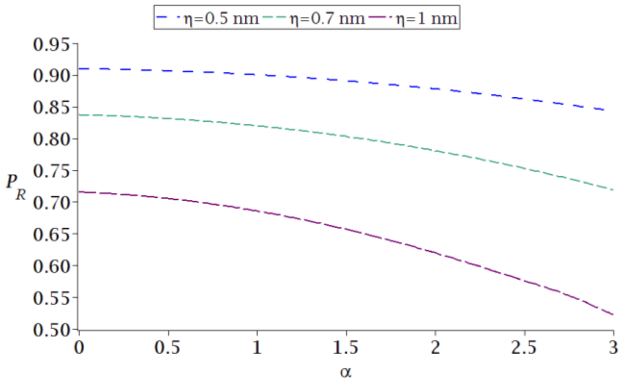

Fig. 6 Critical buckling load ratio versus α ($\xi = 0.1$)

rectangular and annular bidirectional FG nano-beams with the clamped-clamped (C-C) boundary condition against L for various amounts of η. This figure demonstrates the increase of buckling load ratio by decreasing scale coefficient or increasing length of the beam. The results of nonlocal elasticity theory approaches to the results of local elasticity theory by increasing the length of BDFGM nano-beam.

Fig. 6 presents the critical buckling load ratio of tapered rectangular and annular two-directional FG nano-beams with the end condition (C-C) against α for various amounts of η. The buckling load ratio decreases by increasing α. This figure shows that the difference between the results of nonlocal elasticity theory and local elasticity theory for bidirectional FG nano-beam is more than through-thickness graded FG nano-beam for the same amount of EI_0.

The dimensionless critical buckling load (P_0L^2/EI_0) of tapered bidirectional FG nano-beam for constant, sinusoidal and quadratic compression loads and for various end conditions is presented in Fig. 7. The minimum and maximum amounts of total applied compression load (area under the load curve for P_0=1) are for quadratic and sinusoidal loads respectively. However the total applied load for quadratic distribution is less than other loads, but the load concentration at vicinity of mid-span for quadratic distribution is more than other distributions that yields to maximize the dimensionless critical buckling load. Also, dimensionless buckling load for beam with clamped-

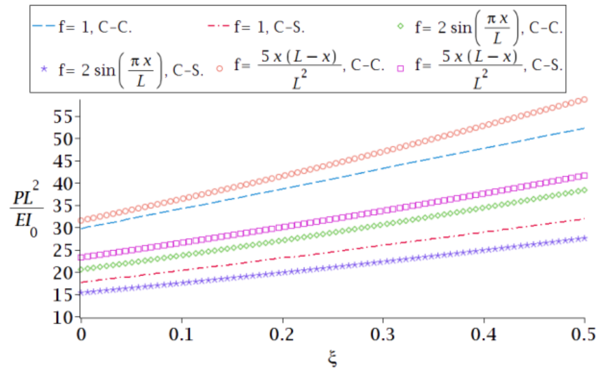

Fig. 7 The dimensionless critical buckling load for various patterns of distributed axial compression (η=1 nm, α=0.1)

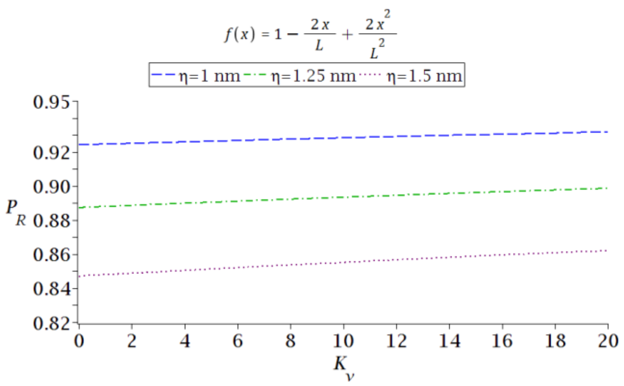

Fig. 8 Critical buckling load ratio against K_v ($\xi = \alpha = 0.1, K_s = 3$)

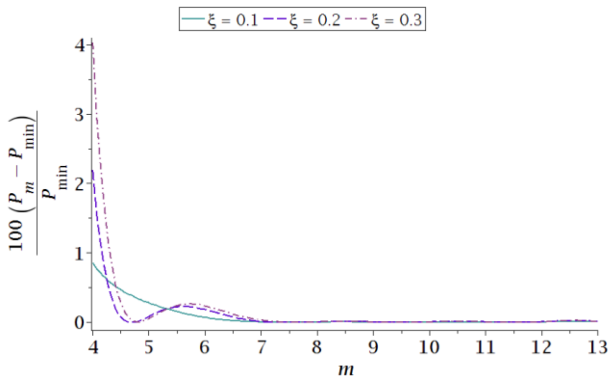

Fig. 9 The normalized error for various amounts of taper constant ($\alpha = 0.1, \eta = 1\text{nm}$)

clamped (C-C) end condition is more than beam with clamped-simply supported (C-S) end condition.

Fig. 8 presents the critical buckling load ratio for rectangular tapered BDFGM nano-beam subjected to variable axial compression with boundary condition $(S-S)$ for various values of scale coefficient (η) against K_v for $K_s = 3$ (where $K_v = k_0/(EI_0/bL^4)$ and $K_s = G/(EI_0/L^2)$). This figure demonstrates the increase of buckling load ratio due to increase of Winkler modulus. The difference between results of nonlocal elasticity theory and results of local elasticity theory decreased in the presence of the elastic foundation.

Fig. 9 illustrates the convergence of critical buckling load for (S-S) boundary condition and various amounts of taper constant, where P_m is buckling load corresponding to Taylor series expansion of approximated transverse displacement up to m degree. The high rate of convergence is observed. The convergence rate is increased by

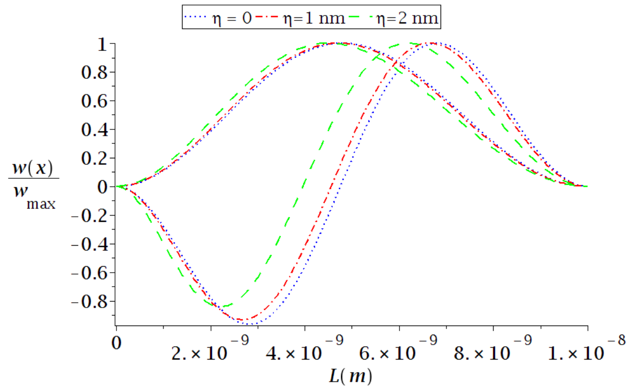

Fig. 10 First two mode shapes of BDFGM nano-beam for various amounts of scale coefficient (ξ=α=0.1)

Table 2 Dimensionless critical buckling loads for nonlocal beams (A), (B), (C) and (D)

η/L	Ref.	Mode	P_0L^2/EI A	B	C	D
0	Present work	1	83.9094	49.3812	9.50168	8.53913
		2	149.952	96.3978	61.0275	36.4239
		3	238.113	147.138	120.072	70.4409
	(Ghannadpour, Mohammadi *et al.* 2013)	1	83.911	-	9.5019	-
		2	149.96	-	61.028	-
		3	238.12	-	120.07	-
0.05	Present work	1	69.3596	39.9957	9.28121	8.32098
		2	109.065	66.8675	52.9491	31.3938
		3	149.260	89.2857	92.3503	52.9700
	(Ghannadpour, Mohammadi *et al.* 2013)	1	69.361	-	9.2814	-
		2	109.07	-	52.949	-
		3	149.26	-	92.35	-
0.1	Present work	1	45.6254	25.2751	8.67720	7.71986
		2	59.9924	33.8943	37.8988	22.1476
		3	70.4241	38.7446	54.5603	30.1192
	(Ghannadpour, Mohammadi *et al.* 2013)	1	45.626	-	8.6774	-
		2	59.994	-	37.899	-
		3	70.425	-	54.56	-

Table 3 The effect of additional supports on buckling load ratio (P_R) in BDFGM multi-span nano-beam

Number of spans (n)	Boundary conditions at end spans C-C	S-S	C-S	C-F
1	0.7141	0.9095	0.8305	0.9757
2	0.5472	0.7133	0.6579	0.9407
3	0.4202	0.5197	0.4900	0.8826

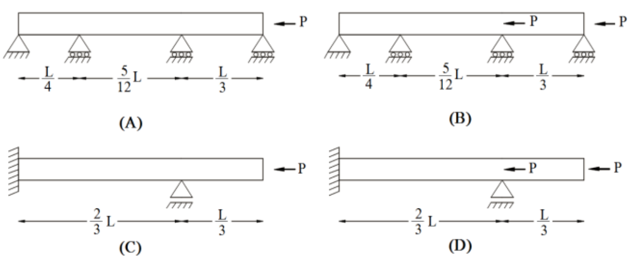

Fig. 11 Multi-span nonlocal beams (A), (B), (C) and (D)

decreasing taper constant. The approximated transverse displacement up to 10 degree is sufficient and using more terms in Taylor series expansion does not affect the accuracy of outcomes significantly. The first two normalized buckling mode shapes of tapered BDFGM nano-beam with (C-C) end condition for various amounts of η is presented in Fig. 10. The scale coefficient affects second mode shape more than first mode shape.

The results for first three modes of nonlocal beams (A) and (C) in Table 2 are verified by the Eqs. (B2) and (B3) in appendix B. Also the results for nonlocal beams (A) and (C) show an excellent agreement with the known data in literature (Ghannadpour *et al.* 2013). Table 3 presents the ratio of nonlocal ($\eta/L = 0.1$) critical buckling load to local critical buckling load (P_R) for tapered ($\xi = 0.1$), BDFGM ($\alpha = 0.1$) nano-beam by considering various end conditions. The length of each span is L/n, in which n is the number of spans. The results of Table 3 show that, using additional supports in mid-span increases the difference between the outcomes of local and nonlocal elasticity theories.

6. Conclusions

In current study for the first time, the buckling analysis of rectangular and annular tapered bidirectional FG Euler-Bernoulli nano-beam resting on two-parameter elastic foundation subjected to variable axial compression based on nonlocal elasticity theory is conducted. A new scheme is proposed to calculate the buckling loads of first modes of multi-span FG nano-beams subjected to stepwise variation of axial compression with various side-span end conditions. To the best of the authors' knowledge, all the previous works about buckling analysis of bidirectional FG nano-beams are limited to the BDFGM nano-beams having uniform thickness subjected to the constant axial force without considering the elastic foundation.

The derived governing differential equation in current work for the uniform nonlocal nano-beam subjected to constant axial force rested on Pasternak elastic foundation is same as the nonlinear vibration equation of motion of nano-beam after neglecting time dependent terms and axial deformation (Ghannadpour *et al.* 2013, Khorramian *et al.* 2015, Tahmasbi *et al.* 2016, Togun and Bağdatlı 2016). A novel technique based on calculus of variations and collocation method is used. For the simpler cases, the exact analytical solutions are obtained in appendices to validate the results of proposed technique. In addition, the excellent agreement between our outcomes with those of reported in the literature, wherever possible, is observed (Ghannadpour *et al.* 2013, Nejad *et al.* 2016) and validity of this new work is proved. The difference between results of nonlocal elasticity theory and results of local elasticity theory is decreased by increasing length, decreasing scale coefficient, increasing foundation moduli and decreasing material constant in length direction. The current work will be helpful for analyzing and developing tapered BDFGM thin nano-beams rested on elastic mediums subjected to variable axial compression.

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Appendix A - Analytical solution for local tapered FG beam rested on shear layer

For tapered local FG beam ($\eta = 0$) with transverse graduation ($\alpha = 0$), the ordinary differential equation (ODE) in Eq. (22) for $k_0 = 0$, has the analytical solution in terms of hypergeometric function. In mathematics, the Gaussian or ordinary hypergeometric function ${}_2F_1(a,b;c;z)$ is a special function represented by the hypergeometric series.

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{A1}$$

in which $(x)_n$ is Pochhammer symbol and represent the rising factorial as follows

$$(x)_n = x(x+1)(x+2)\ldots(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)} \tag{A2}$$

The ODE in Eq. (22) for $\eta = \alpha = k_0 = 0$ and constant P has the following solution

$$\begin{gathered} w(x) = c_0 + c_1 x + \\ c_2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2)_n} \frac{\left((P-G)L^3\right)^n}{\left(EI_0 \xi^2 (\xi x + L)\right)^n n!} + \\ c_3 K\left(1, 2i\sqrt{\frac{(P-G)L^3}{EI_0 \xi^2 (\xi x + L)}}\right)\sqrt{\xi x + L} \end{gathered} \tag{A3}$$

The parameter i represent one of the square roots of -1. $J(v,x)$ and $Y(v,x)$ are the Bessel functions of the first and second kinds respectively. They satisfy Bessel's equation $x^2 y'' + xy' + (x^2 - v^2)y = 0$. Also $I(v,x)$ and $K(v,x)$ are the modified Bessel functions of the first and second kinds, respectively. They satisfy the modified Bessel's equation $x^2 y'' + xy' - (v^2 + x^2)y = 0$. The Taylor series expansion for $I(1,2i\sqrt{z})/(i\sqrt{z})$ and $-J(1,-2\sqrt{z})/\sqrt{z}$ is presented in Eq. (A4).

$$\frac{I(1,2i\sqrt{z})}{i\sqrt{z}} = -\frac{J(1,-2\sqrt{z})}{\sqrt{z}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2)_n} \frac{z^n}{n!} \tag{A4}$$

The series in Eq. (A3) has the form of the series in Eq. (A4). Also the real part of $K(1,iz)$ is equal to $J(1,z) \times \mathrm{Re}\big(K(1,mi)\big)/J(1,m)$ and imaginary part of $K(1,iz)$ is equal to $Y(1,z) \times \mathrm{Im}\big(K(1,mi)\big)/Y(1,m)$, in which m is a positive real number and Re and Im denote the real and imaginary parts, respectively. After some manipulations the Eq. (A3) is simplified as follows

$$\begin{gathered} w(x) = c_0 + c_1 x + \\ c_2 J\left(1, 2\sqrt{\frac{(P-G)L^3}{EI_0 \xi^2 (\xi x + L)}}\right)\sqrt{L + \xi x} + \\ c_3 Y\left(1, 2\sqrt{\frac{(P-G)L^3}{EI_0 \xi^2 (\xi x + L)}}\right)\sqrt{L + \xi x} \end{gathered} \tag{A5}$$

It is noteworthy to mention that the parameter (P-G) in Eq. (A5) represent the increase of buckling load of FG beam caused by shear modulus of foundation. The amount of G will be added to the buckling load of local FG beam directly. In general, deflection of FG beam in Eq. (A5) has the four unknown coefficients to be determined by using natural and boundary conditions at the ends of FG beam. These equations for pinned (S-S), clamped (C-C), clamped and pinned (C-S) and pinned and clamped (S-C) are $w(0) = w(L) = w''(0) = w''(L) = 0$, $w(0) = w(L) = w'(0) = w'(L) = 0$, $w(0) = w(L) = w'(0) = w''(L) = 0$ and $w(0) = w(L) = w'(L) = w''(0) = 0$, respectively. The equations for free right end and clamped left end (C-F) are $w''(L) = w'(0) = 0$. Also, the equations for end condition (F-C) are $w''(0) = w'(L) = 0$. For boundary conditions (C-F) and (F-C) the coefficients c_0 and c_1 in Eq. (A5) are vanished. After satisfying above mentioned equations a system of algebraic homogeneous equations is obtained. It is obvious that for having nontrivial solution the determinant of coefficient matrix must be vanished. This leads to solve an eigenvalue problem. The buckling load for FG beam with rectangular or annular section is equal to $G + EI_0 z^2/L^3$ in which z for (S-S), (C-S), (S-C), (C-F), (F-C) and (C-C) boundary conditions is obtained after solving Eqs. (A6) to (A11) with respect to z for various numerical values of L and ξ, respectively.

$$J(1,\gamma)Y(1,\Gamma) - J(1,\Gamma)\,Y(1,\gamma) = 0 \tag{A6}$$

$$\begin{aligned} &Y(1,\Gamma)\big(J(1,\gamma)(1+\xi)L - J(0,\gamma)\sqrt{L}z\big) - \\ &Y(1,\gamma)J(1,\Gamma)(1+\xi)L + Y(0,\gamma)J(1,\Gamma)\sqrt{L}z = 0 \end{aligned} \tag{A7}$$

$$\begin{aligned} &-Y(1,\Gamma)\left(J(1,\gamma)\sqrt{(1+\xi)L}\right) - Y(0,\Gamma)(J(1,\gamma)z) + \\ &Y(1,\gamma)\left(J(1,\Gamma)\sqrt{(1+\xi)L} + J(0,\Gamma)z\right) = 0 \end{aligned} \tag{A8}$$

$$\begin{aligned} &Y(1,\Gamma)\big(J(1,\gamma)\xi L - J(0,\gamma)\sqrt{L}z\big) - \\ &Y(1,\gamma)J(1,\Gamma)\xi L + Y(0,\gamma)J(1,\Gamma)\sqrt{L}z = 0 \end{aligned} \tag{A9}$$

$$\begin{aligned} &Y(1,\Gamma)J(1,\gamma)\xi\sqrt{(1+\xi)L} - Y(0,\Gamma)z + \\ &Y(1,\gamma)\left(J(0,\Gamma)z - J(1,\Gamma)\xi\sqrt{(1+\xi)L}\right) = 0 \end{aligned} \tag{A10}$$

$$\begin{aligned} &Y(1,\Gamma)\sqrt{L(1+\xi)}\big(J(0,\Gamma) - J(0,\gamma)\big) + \\ &Y(1,\gamma)\sqrt{L}\left(\big(J(0,\gamma) - J(0,\Gamma)\big)(1+\xi)\right) + \\ &+Y(0,\Gamma)[J(1,\gamma)\xi\sqrt{L} - J(1,\Gamma)\sqrt{L(1+\xi)} + \\ &J(1,\gamma)\sqrt{L} - J(0,\gamma)z] + Y(0,\gamma) \\ &[J(1,\Gamma)\sqrt{L(1+\xi)} - J(1,\gamma)\xi\sqrt{L} \\ &-J(1,\gamma)\sqrt{L} + J(0,\Gamma)z = 0 \end{aligned} \tag{A11}$$

The parameters γ and Γ are defined in Eq. (A12).

$$\gamma = \frac{2z}{\xi\sqrt{L}},\quad \Gamma = \frac{2z}{\xi\sqrt{L(\xi+1)}} \tag{A12}$$

Appendix B - Analytical solution for non-local uniform FG beam

The ODE in Eq. (22) for ($\alpha = \xi = k_0 = G = 0$ and f(x)=1) has the following solution

$$\begin{aligned} &w = c_0 + c_1 x + \\ &c_3 \sin\left(\sqrt{\frac{P}{EI - P\eta^2}}\,x\right) + c_4 \cos\left(\sqrt{\frac{P}{EI - P\eta^2}}\right) \end{aligned} \tag{B1}$$

After considering Eq. (B1) for local FG beam ($\eta = 0$) the relation between the local buckling load (F) and nonlocal buckling load (P) for all buckling modes is obtained as follows

$$F = \frac{P}{1 - \frac{P}{EI}\eta^2} \tag{B2}$$

Also, the nonlocal buckling load (P) in terms of local buckling load (F) is calculated as follows

$$P = \frac{F}{1 + \frac{F}{EI}\eta^2} \tag{B3}$$