**Structural Engineering and Mechanics**, *Vol. 66, No. 6 (2018) 703-711*
DOI: https://doi.org/10.12989/sem.2018.66.6.703


# Timoshenko theory effect on the vibration of axially functionally graded cantilever beams carrying concentrated masses

Carlos A. Rossit[a], Diana V. Bambill* and Gonzalo J. Gilardi[b]

*Department of Engineering, Universidad Nacional del Sur(UNS) Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Avenida Alem 1253, (8000) Bahía Blanca, República Argentina*

*(Received October 26, 2017, Revised February 20, 2018, Accepted March 19, 2018)*

**Abstract.** In this paper is studied the effect of considering the theory of Timoshenko in the vibration of AFG beams that support ground masses. As it is known, Timoshenko theory takes into account the shear deformation and the rotational inertia, provides more accurate results in the general study of beams and is mandatory in the case of high frequencies or non-slender beams. The Rayleigh-Ritz Method is employed to obtain approximated solutions of the problem. The accuracy of the procedure is verified through results available in the literature that can be represented by the model under study. The incidence of the Timoshenko theory is analyzed for different cases of beam slenderness, variation of its cross section and compositions of its constituent material, as well as different amounts and positions of the attached masses.

**Keywords:** vibration of beams; AFG beam; tapered beam; Timoshenko beam; attached masses; Rayleigh-Ritz method

## 1. Introduction

Recently, the authors presented a paper (Rossit *et al.* 2017) regarding dynamic behavior of cantilever tapered beams carrying concentrated masses. For the authors knowledge it was the first paper on the matter since, there has not been any previous attempt to solve the problem of an axially functionally graded beam carrying an attached mass.

Subsequently, Nikolić (2017) analyzed a non-uniform axially functionally graded cantilever beam with a tip body, by means of the rigid element method.

In those papers, beams flexural deformation is described by means of Bernoulli-Euler theory.

According to a recent literature survey, apparently, there are no papers on Timoshenko AFG beams carrying attached masses. As it is known Timoshenko theory, posed in 1921 (Timoshenko 1921, 1922), provides more accurate results in the general study of beams and is mandatory in the case of high frequencies or non-slender beams. Remarks of historical character on the subject can be found in the paper by Elishakoff *et al.* (2015).

Certainly, even papers about bare AFG Timoshenko beams are scarce. Among them, mention must be made of the paper of Shahba *et al.* (2011) who studied free vibration and stability of AFG Timoshenko beams through a finite element approach. Huang *et al.* (2013) presented a new approach: by introducing an auxiliary function, they changed the coupled governing equations with variable coefficients for the deflection and rotation to a single governing equation. He *et al.* (2013) improved the traditional beam element to consider the variable axial parameters which were formulated in terms of a power series. Tang *et al.* (2014) obtained closed form solutions for uniform AFG Timoshenko beams whose bending stiffness and distributed mass density are assumed to obey a unified exponential law. Rajasekaran and Norouzzadeh Tochaei (2014) investigated the free vibration analysis of AFG Timoshenko beams carried out through the differential transformation element method and the quadrature element method. They introduce an element-based differential method that significantly improves the accuracy of results. They also introduced a lower order differential quadrature element based on differential quadrature element method. They showed the accuracy of both methods with several numerical examples.

Also, Sarkar and Ganguli (2014) founded closed form solutions for certain polynomial variations of the material mass density, elastic modulus and shear modulus, along the length of the beam. Gan *et al.* (2015) presented a finite element procedure for dynamic analysis of non-uniform AFG Timoshenko beams under multiple moving point loads. Sun *et al.* (2016) suggested a new initial value method to determine critical tip force and axial loading at buckling of a standing column with varying cross-section and variable material properties under self-weight and tip force. More recently, in 2017 Zhao *et al.* introduced a new approach based on Chebyshev polynomials theory and Tudjono *et al.* derived exact shape functions for both non-uniform (non-prismatic section) and inhomogeneous (functionally graded material) Timoshenko beam element formulation explicitly and Chen *et al.* (2017) developed the

*Corresponding author, Professor
E-mail: dbambill@criba.edu.ar
[a]Ph.D. Professor
E-mail: carossit@criba.edu.ar
[b]Ph.D. Student
E-mail: gonzalo.gilardi@uns.edu.ar



ISSN: 1225-4568 (Print), 1598-6217 (Online)

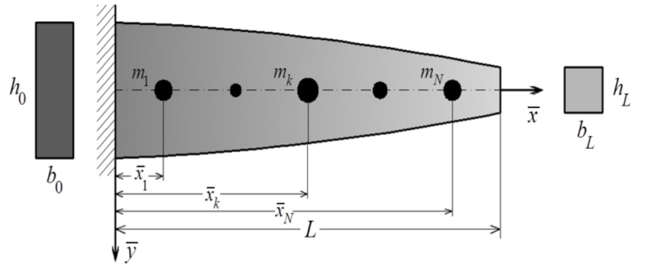


Fig. 1 AFG cantilever tapered beam with $N$ masses attached at arbitrary points

initial value method of a system of ordinary differential equations to determine the resonance frequencies of functionally graded nanocantilevers carrying a nanoparticle. In the present paper, we describe the determination of the natural frequencies of vibration of a Timoshenko cantilever beam with varying rectangular cross section and made of axially functionally graded material, carrying attached masses at arbitrary positions, having into account their rotatory inertia. In view of the mathematical difficulties of the problem since variable coefficients appear in the governing differential equations, approximated methods are required to solve general cases. The well-known Rayleigh-Ritz method (Ilanko and Monterrubio 2014) is employed and its suitability to apply to Timoshenko beams with properties varying according to its axial axis and attached masses is verified by comparison with particular cases of the posed model published in the literature.

## 2. Analytical approach

According to the Timoshenko beam theory, the determination of the beam frequencies involves the effects of rotational inertia, shear deformation and their combined effects.

For the normal modes of beam vibration (Fig. 1) it can be expressed

$$\begin{aligned} v(\bar{x},t) &= \bar{V}(\bar{x})\cos(\omega t) \\ \psi(\bar{x},t) &= \bar{\Psi}(\bar{x})\cos(\omega t) \end{aligned} \tag{1}$$

where $v$ is the transverse displacement of the mid-surface in the $y$-direction displacement, $\psi$ is the angle of rotation of the normal to the mid-surface of the beam, $t$ is the time and $\omega$ is the circular frequency in radians per second.

The non-dimensional coordinate is defined as

$$x = \bar{x}/L \tag{2}$$

where $L$ is the length of the beam and

$$\begin{aligned} V(x) &= \frac{\bar{V}(\bar{x})}{L}; \\ \Psi(x) &= \bar{\Psi}(\bar{x}) \end{aligned} \tag{3}$$

are the dimensionless expressions of the transverse displacement and bending angle, respectively.

If the cross section varies smoothly, the energy functional $J$ for the vibrating beam of length $L$ carrying attached $N$ masses $m_k$ at positions $x_k$ (see Fig. 1) is given by

$$\begin{aligned} J(V,\Psi) = {} & \frac{1}{2}\int_0^1 \left[ \frac{E(x)\,I(x)}{L^2}\left(\frac{d\Psi(x)}{dx}\right)^2 + \right. \\ & \left. + \frac{1}{2}\kappa\, G(x)\, A(x)\left(\frac{dV(x)}{dx} - \Psi(x)\right)^2 \right] L\,dx - \\ & - \frac{1}{2}\omega^2 \int_0^1 \rho(x)\left[ A(x)\left(V(x)\right)^2 L^2 + I(x)\left(\Psi(x)\right)^2 \right] L\,dx - \\ & - \frac{1}{2}\omega^2 \sum_{k=1}^{N} m_k\left(V(x_k)\right)^2 + m_k\, r_k^{\,2}\left(\Psi(x_k)\right)^2 \end{aligned} \tag{4}$$

where $A(x)$ is the varying cross section and $I(x)$ its second moment of area, the FGM density is $\rho(x)$, the Young´s modulo is $E(x)$ and the shear modulus is $G(x)$. $r_k$ defines the radius of gyration of the mass $m_k$ with respect to the neutral axis of the beam and $\kappa$ is the shear coefficient.

As the material and geometric characteristics of the beam may be general, one can define

$$\begin{aligned} E(x) &= E_0\, f_E(x) \\ G(x) &= G_0\, f_G(x) \\ \rho(x) &= \rho_0\, f_\rho(x) \\ A(x) &= A_0\, f_A(x) \\ I(x) &= I_0\, f_I(x) \\ b(x) &= b_0\, f_b(x) \\ h(x) &= h_0\, f_h(x) \end{aligned} \tag{5}$$

It was considered

$$G(x) = \frac{E(x)}{2(1+\nu)} \tag{6}$$

where $\nu$ is the Poisson ratio.

Obviously $f_A = f_b \times f_h$ , $f_I = f_b \times f_h^3$ , $f_G = f_E$ .

The subscript “0” refers to the cross section of the beam adopted as the reference section.

To apply the Rayleigh-Ritz method, it is necessary to approximate the spatial component of the solution

$$\begin{aligned} V(x) &\cong V_a(x) = \sum_{j=1}^{Nq} D_j\, q_j(x) \\ \Psi(x) &\cong \Psi_a(x) = \sum_{i=1}^{Np} C_i\, p_i(x) \end{aligned} \tag{7}$$

where $p_i$ and $q_j$ are coordinate functions that satisfy the essential boundary conditions, $C_i$ and $D_j$ are arbitrary constants, $Np$ and $Nq$ are the number of terms.

Following Rayleigh-Ritz procedure, the functional is minimized with respect to every arbitrary constant

$$\begin{aligned} \frac{\partial J\left[V_a(x),\Psi_a(x)\right]}{\partial C_i} &= 0 \quad ; \quad i = 1,2,\dots,Np. \\ \frac{\partial J\left[V_a(x),\Psi_a(x)\right]}{\partial D_j} &= 0 \quad ; \quad j = 1,2,\dots,Nq. \end{aligned} \tag{8}$$

Then a linear system of equations is formed

$$\mathbf{R}\begin{Bmatrix} C_i \\ D_j \end{Bmatrix} = \{0\} \tag{9}$$

which results in the following eigenvalue equation

$$\mathbf{A}=\mathbf{K} - \Omega^2\,\mathbf{M} \tag{10}$$

with $\Omega = \omega L^2 \sqrt{\dfrac{\rho_0 A_0}{E_0 I_0}}$ the frequency coefficient and where

$$\mathbf{K}=\begin{bmatrix} k_{ii} & k_{ij} \\ k_{ji} & k_{jj} \end{bmatrix} \tag{11}$$

$$\begin{aligned}
k_{ii} &= \int_0^1 \left( f_E\, f_I\, p_i'\, p_i' + \frac{s^2\kappa}{2(1+\nu)} f_E\, f_A\, p_i\, p_i \right) dx \\
k_{ij} &= -\int_0^1 \left( \frac{s^2\kappa}{2(1+\nu)} f_E\, f_A\, p_i\, q_j' \right) dx \\
k_{ji} &= -\int_0^1 \left( \frac{s^2\kappa}{2(1+\nu)} f_E\, f_A\, q_j'\, p_i \right) dx \\
k_{jj} &= \int_0^1 \left( \frac{s^2\kappa}{2(1+\nu)} f_E\, f_A\, q_j' q_j' \right) dx
\end{aligned} \tag{12}$$

$$\mathbf{M}=\begin{bmatrix} m_{ii} & m_{ij} \\ m_{ji} & m_{jj} \end{bmatrix} \tag{13}$$

$$\begin{aligned}
m_{ii} &= \frac{1}{s^2}\int_0^1 \left( f_\rho\, f_I\, p_i\, p_i \right) dx + \sum_{k=1}^{N} M_k\, c_k^2\, p_i(x_k)\, p_i(x_k) \\
m_{ij} &= 0 \\
m_{ji} &= 0 \\
m_{jj} &= \int_0^1 \left( f_\rho\, f_A\, q_j\, q_j \right) dx + \sum_{k=1}^{N} M_k\, q_j(x_k)\, q_j(x_k)
\end{aligned} \tag{14}$$

are the elements of matrices **K** and **M**, respectively; with

$$M_k = \frac{m_k}{\rho_0 A_0 L};\quad c_k = \frac{r_k}{L};\quad s = L\sqrt{\frac{A_0}{I_0}};\quad q_j' = \frac{dq_j}{dx};\quad p_j' = \frac{dp_j}{dx}. \tag{15}$$

Then, the eigenvalue problem can be expressed as

$$\left|\mathbf{K}\mathbf{M}^{-1} - \Omega^2\,\mathbf{I}\right| = \left|\mathbf{B}\text{-}\lambda\mathbf{I}\right| = 0 \tag{16}$$

where $\lambda = \Omega^2$ are the eigenvalues of matrix **B** and **I** the identity matrix.

For the cantilever beam, the following coordinate functions are chosen

$$\{p_i\}_{i=1}^{N_p} = \{x^i\}_{i=1}^{N_p} \tag{17}$$

$$\{q_j\}_{j=1}^{N_q} = \{x^j\}_{j=1}^{N_q} \tag{18}$$

which satisfy essential boundary conditions.

Table 1 Frequency coefficients for a tapered AFG Timoshenko beam ( $\Omega_i = \omega_i L^2\sqrt{(\rho_0 A_0)/(E_0 I_0)}$ )

| | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ |
|---|---|---|---|---|
| Shahba *et al.* | 3.9359 | 15.1577 | 31.2638 | 47.7164 |
| Rajasekaran *et al.* | 3.9358 | 15.1532 | 31.2236 | 47.5830 |
| Huang *et al.* | 3.93579 | 15.1533 | 31.2239 | 47.5857 |
| Zhao *et al.* | 3.93585 | 15.1540 | 31.2257 | 47.5871 |
| Present | 3.93579 | 15.1533 | 31.2239 | 47.5836 |

## 3. Numerical results

Since there were not found, in the technical literature, values of natural frequencies of vibration of AFG Timoshenko beams with attached masses in order to verify the accuracy of the proposed model, comparisons are made with particular cases available in the literature.

First, Table 1 compares values for a tapered Timoshenko beam made of axially functionally graded material studied by Shahba *et al.* (2011), Huang *et al.* (2013), Rajasekaran and Norouzzadeh Tochaei (2014) and Zhao *et al.* (2017). They obtained values for the first four natural frequency coefficients for a case that can be represented in the present model.

The two constituent materials are Zirconia ($ZrO_2$) and Aluminum (Al).

$$ZrO_2\text{:}\quad E_0 = 200 \text{ GPa};\ \rho_0 = 5700\,\frac{kg}{m^3} \tag{19}$$

$$\text{Al:}\quad E_1 = 70 \text{ GPa};\ \rho_1 = 2702\,\frac{kg}{m^3} \tag{20}$$

and properties of AFG material, like mass density $\rho$, Young's modulus $E$, shear modulus $G$, continuously vary in the axial direction with a power law relation. Then, a generic material property $P(x)$ is assumed to vary along the beam axis $x$

$$P(x) = P_0\left[1 + \frac{(P_1 - P_0)}{P_0}x^n\right] \tag{21}$$

In order to compare results, it is taken $n$=2 in Eq. (21). The geometry of the varying section is defined adopting in Eqs. (5)

$$f_A = 1 - 0.1x\ ,\ f_I = (1 - 0.1x)^3 \tag{22}$$

They adopted

$$s = 10;\ \kappa = \frac{5}{6};\ \nu = 0.30 \tag{23}$$

Then, Table 2 compares values with the case studied by Torabi *et al.* (2013): A wedge cantilever homogeneous Timoshenko beam ($\nu$= 0.25, $\kappa$=2/3) with an attached tip mass ($M$=0.32) and whose cross-sectional properties are

Table 2 Frequency coefficients for a tapered homogeneous Timoshenko beam with a tip mass ( $\Omega_i = \omega_i L^2 \sqrt{(\rho_0 A_0)/(E_0 I_0)}$ )

| | $s$=10 | $s$=25 | Solution |
|---|---|---|---|
| $\Omega_1$ | 1.997 | 2.117 | Torabi *et al.* |
| | 1.9977 | 2.0957 | Present |
| $\Omega_2$ | 10.695 | 13.420 | Torabi *et al.* |
| | 10.6947 | 13.4311 | Present |
| $\Omega_3$ | 24.388 | 36.109 | Torabi *et al.* |
| | 24.3869 | 36.1016 | Present |
| $\Omega_4$ | 40.174 | 66.697 | Torabi *et al.* |
| | 40.1487 | 66.6219 | Present |
| $\Omega_5$ | 56.739 | 102.451 | Torabi *et al.* |
| | 56.7489 | 102.108 | Present |

defined by: $f_A = 1 - 0.4x$ , $f_I = (1-0.4x)^3$ . Two different slenderness ratios are considered and the first five non-dimensional natural frequencies are presented.

In all cases the calculations were done with $N_p$=$N_q$= 20 in Eqs. (17)-(18), and the cross section at the clamped edge ($x$=0) is taken as the reference cross section.

As it can be seen, the agreement is excellent in both cases. It is worth mentioning that the Rayleigh-Ritz method gives upper bounds of the wanted values.

Next, an example of the convenience of using Timoshenko theory in certain situations: in the paper that preceded and gave rise to the present (Rossit *et al.* 2017) are compared frequency values with a previous work (Chen and Liu, 2006) of a case of a Bernoulli-Euler beam with attached masses. Taking into account the geometry of the analyzed structure, the use of Timoshenko theory to describe the flexural behavior of the beam would have yielded more accurate results.

The physical properties and dimensions of the beam studied are: Young´s modulus $E$=2.051×10$^{11}$ N/m$^2$, mass density $\rho$=7850 kg/m$^3$, constant beam width $b$=0.1 m, beam length $L$= 1.60 m, lineal variable beam depth: 0.08 m at the free end and 0.40 m at the clamped end.

Each attached mass has a magnitude of one-fifth of the actual total mass of the beam: 60.288 kg.

In the present model, according to the definition used for the relative magnitude of the mass, it must be adopted

$$M_k = \frac{12}{100};\ c_k = 0;\ k = 1 \text{ to } 5.$$

$$\omega_i = \frac{\Omega_i}{L^2}\sqrt{\frac{EI_0}{\rho A_0}} \tag{24}$$

where, as indicated previously, $\omega_i$ is the natural frequency in radians per second.

Two particular situations were considered:

a) One point mass attached at the free end (Table 3) (Fig. 2).

Five equal point masses attached at coordinates: 0.125$L$, 0.3125$L$, 0.5$L$, 0.6875$L$ and 0.875$L$ respectively (Table 4) (Fig. 1).

Table 3 Frequency values in radians per second for a tapered homogeneous cantilever beam with one attached mass

| | Chen and Liu | Rossit *et al.* | Present |
|---|---|---|---|
| $\omega_1$ | 569.6279 | 569.3747 | 557.5622 |
| $\omega_2$ | 2508.895 | 2503.714 | 2297.209 |
| $\omega_3$ | 6743.232 | 6710.268 | 5548.167 |
| $\omega_4$ | 13408.53 | 13289.00 | 9823.906 |
| $\omega_5$ | 22570.17 | 22240.74 | 14743.28 |
| | Bernoulli-Euler | | Timoshenko |

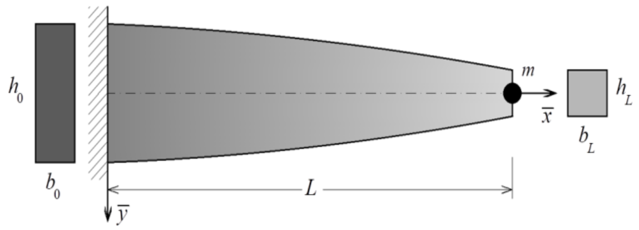


Fig. 2 AFG cantilever tapered beam with one mass $m$ attached at the free end

Table 4 Frequency values in radians per second for a tapered homogeneous cantilever beam with five attached masses

| | Chen and Liu | Rossit *et al.* | Present |
|---|---|---|---|
| $\omega_1$ | 613.220 | 613.194 | 594.523 |
| $\omega_2$ | 2525.54 | 2524.88 | 2259.80 |
| $\omega_3$ | 6366.50 | 6356.55 | 5056.30 |
| $\omega_4$ | 12184.0 | 12116.4 | 8716.58 |
| $\omega_5$ | 16089.9 | 15928.0 | 11551.5 |
| | Bernoulli-Euler | | Timoshenko |

When the free end of the beam has no attached mass, the summation in Eq. (18) starts with $j$=2 to partially satisfy the natural edge conditions and accelerate convergence of Rayleigh-Ritz method.

Next, some cases are evaluated to show the influence of the application of Timoshenko theory on an AFG vibrating beam with attached masses. The effect of the slenderness of the beam: $L/h_0 = s/\sqrt{12}$ on such influence is evidenced.

Among the infinity of cases that could be evaluated due to the quantity and variability of the parameters involved in the description of the behavior of the posed model, just the case of a beam whose height varies quadratically is considered: $h(x) = h_0\left(1-0.5x^2\right)$ .In all cases, there are taken into account the quantity and positions of masses of Tables 3 and 4.

Properties of AFG materials, like mass density $\rho$, Young's modulus $E$, shear modulus $G$, continuously vary in the axial direction according to Eq. (21).

Three cases are considered in Eq. (21), $n$=0 (homogeneous material), $n$=1 (AFG material with linear variation) and $n$=2 (AFG material with quadratic variation).

Table 5 Frequency coefficients for a homogeneous tapered beam with a mass at the free end

| $L/h_0$ | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|---|---|---|---|---|---|
| 5 | 2.61342 | 12.4385 | 25.7434 | 42.1832 | 66.2152 |
| 10 | 2.65308 | 13.1585 | 27.9641 | 50.8473 | 87.2670 |
| 50 | 2.66619 | 13.4136 | 28.8753 | 55.3339 | 100.861 |
| 100 | 2.66661 | 13.4218 | 28.9059 | 55.4978 | 101.411 |
| 500 | 2.66674 | 13.4244 | 28.9157 | 55.5506 | 101.589 |
| 1000 | 2.66674 | 13.4245 | 28.9160 | 55.5522 | 101.595 |
| B-E | 2.66674 | 13.4245 | 28.9161 | 55.5528 | 101.597 |

Table 6 Frequency coefficients for a tapered beam with AFG material (Alum-St) varying linearly and a mass at the free end

| $L/h_0$ | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|---|---|---|---|---|---|
| 5 | 3.37453 | 15.5940 | 30.9115 | 54.5280 | 88.4008 |
| 10 | 3.42831 | 16.2669 | 33.4780 | 66.0183 | 116.311 |
| 50 | 3.44611 | 16.4974 | 34.5370 | 71.8186 | 133.950 |
| 100 | 3.44667 | 16.5047 | 34.5726 | 72.0287 | 134.657 |
| 500 | 3.44685 | 16.5071 | 34.5840 | 72.0964 | 134.886 |
| 1000 | 3.44686 | 16.5072 | 34.5844 | 72.0985 | 134.893 |
| B-E | 3.44686 | 16.5072 | 34.5845 | 72.0992 | 134.896 |

Table 7 Frequency coefficients for a tapered beam with AFG material (Alum-St) varying quadratically and a mass at the free end

| $L/h_0$ | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|---|---|---|---|---|---|
| 5 | 3.57172 | 16.9323 | 33.1276 | 59.9104 | 97.620 |
| 10 | 3.62692 | 17.5915 | 35.9643 | 72.8383 | 128.876 |
| 50 | 3.64517 | 17.8147 | 37.1361 | 79.3762 | 148.734 |
| 100 | 3.64574 | 17.8218 | 37.1755 | 79.6132 | 149.531 |
| 500 | 3.64592 | 17.8240 | 37.1881 | 79.6896 | 149.789 |
| 1000 | 3.64593 | 17.8241 | 37.1885 | 79.6919 | 149.797 |
| B-E | 3.64593 | 17.8241 | 37.1886 | 79.6927 | 149.800 |

Table 8 Frequency coefficients for a homogeneous tapered beam with 5 masses attached

| $L/h_0$ | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|---|---|---|---|---|---|
| 5 | 2.54668 | 11.4422 | 25.0435 | 40.2884 | 55.9815 |
| 10 | 2.59584 | 12.3499 | 28.1789 | 47.0494 | 70.4052 |
| 50 | 2.61217 | 12.6836 | 29.4089 | 49.8045 | 76.3278 |
| 100 | 2.61269 | 12.6944 | 29.4493 | 49.8951 | 76.5124 |
| 500 | 2.61285 | 12.6979 | 29.4623 | 49.9241 | 76.5714 |
| 1000 | 2.61286 | 12.6980 | 29.4627 | 49.9250 | 76.5733 |
| B-E | 2.61286 | 12.6980 | 29.4628 | 49.9253 | 76.5739 |

Table 9 Frequency coefficients for a tapered beam with AFG material (Alum-St) varying linearly and 5 masses attached

| $L/h_0$ | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|---|---|---|---|---|---|
| 5 | 3.34776 | 14.4042 | 31.0773 | 49.7108 | 68.3974 |
| 10 | 3.41463 | 15.4125 | 34.4194 | 57.6441 | 89.0157 |
| 50 | 3.43689 | 15.7767 | 35.6846 | 60.7844 | 97.5336 |
| 100 | 3.43759 | 15.7884 | 35.7258 | 60.8869 | 97.7912 |
| 500 | 3.43782 | 15.7922 | 35.7390 | 60.9198 | 97.8734 |
| 1000 | 3.43783 | 15.7923 | 35.7394 | 60.9208 | 97.8759 |
| B-E | 3.43783 | 15.7923 | 35.7395 | 60.9211 | 97.8768 |

Table 10 Frequency coefficients for a tapered beam with AFG material (Alum-St) varying quadratically and 5 masses attached

| $L/h_0$ | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|---|---|---|---|---|---|
| 5 | 3.53034 | 15.4729 | 33.2349 | 53.0152 | 72.9484 |
| 10 | 3.59840 | 16.5480 | 36.7570 | 61.5285 | 94.8568 |
| 50 | 3.62101 | 16.9345 | 38.0812 | 64.8727 | 103.644 |
| 100 | 3.62173 | 16.9470 | 38.1242 | 64.9813 | 103.905 |
| 500 | 3.62196 | 16.9510 | 38.1380 | 65.0161 | 103.988 |
| 1000 | 3.62196 | 16.9511 | 38.1384 | 65.0172 | 103.990 |
| B-E | 3.62196 | 16.9511 | 38.1386 | 65.0176 | 103.991 |

In the calculations, the AFG material made of steel and aluminum oxide $Al_2O_3$ (alumina) proposed by Su *et al.* (2013) is used. Their Young's modulus and density are

$$E_{St} = 210GPa\,;\ \ \rho_{St} = 7800kg/m^3\,;\ \ E_{Alum} = 390GPa\,; \\ \rho_{Alum} = 3960kg/m^3\,;\ \ \nu_{St} = \nu_{Alum} = 0.30 \tag{25}$$

Note that the alumina, more rigid, is lighter than steel. It is adopted $\kappa = 5/6$ and the masses are all the same

$$M_k = 0.20,\ \ c_k = \frac{r_k}{L} = 0.10 \tag{26}$$

Tables 5-7 show the values of the frequency coefficient $\Omega_i = \omega_i L^2 \sqrt{(\rho_{st} A_0)/(E_{st} I_0)}$ for the three compositions of the cantilever beam considered when a mass is attached at the free end.

Different values of the slenderness of the beam are taken into account. The value for the Bernoulli-Euler beam is indicated at the last row of every table for comparison

Tables 8 to 10 show the results when five masses are attached to the beam.

As can be seen and as might be expected, as the beam's slenderness increases the frequency values tend to the values obtained using the simpler theory of Bernoulli-Euler.

Figs. 3 to 7 show the differences of the values of the first five frequencies of the different beam cases analyzed with respect to the respective value of the Bernoulli-Euler beam. The graphs indicate the decrease in absolute value $|\Delta\%|$, according to the slenderness of the beams for the analyzed cases:

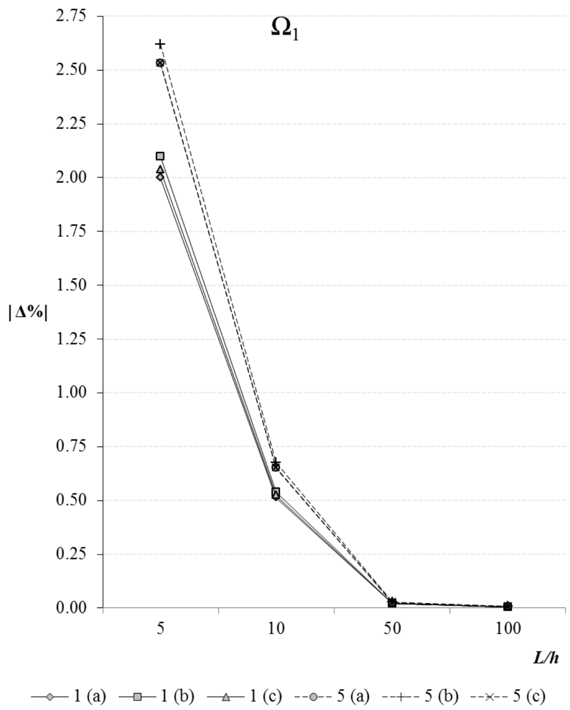


Fig. 3 Variations of the first frequency by the use of theory of Timoshenko

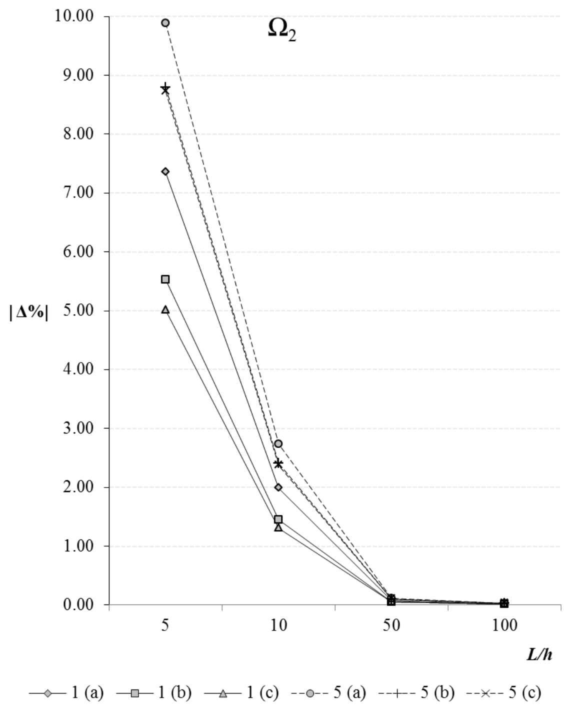


Fig. 4 Variations of the second frequency by the use of theory of Timoshenko

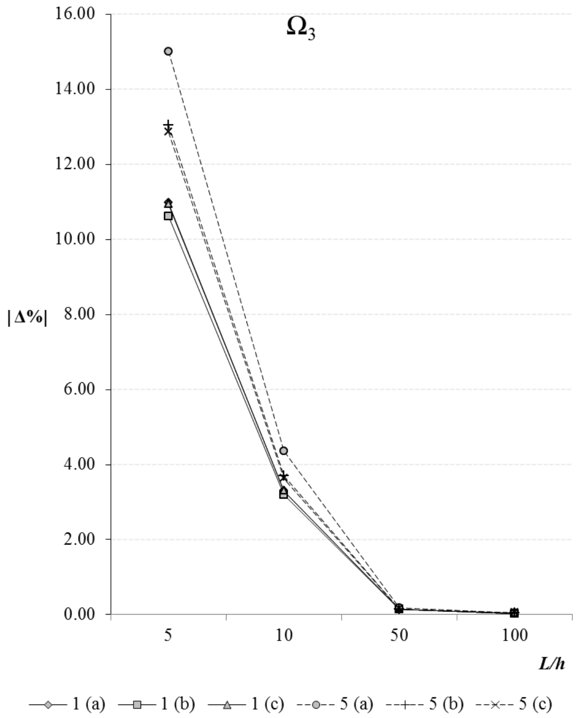


Fig. 5 Variations of the third frequency by the use of theory of Timoshenko

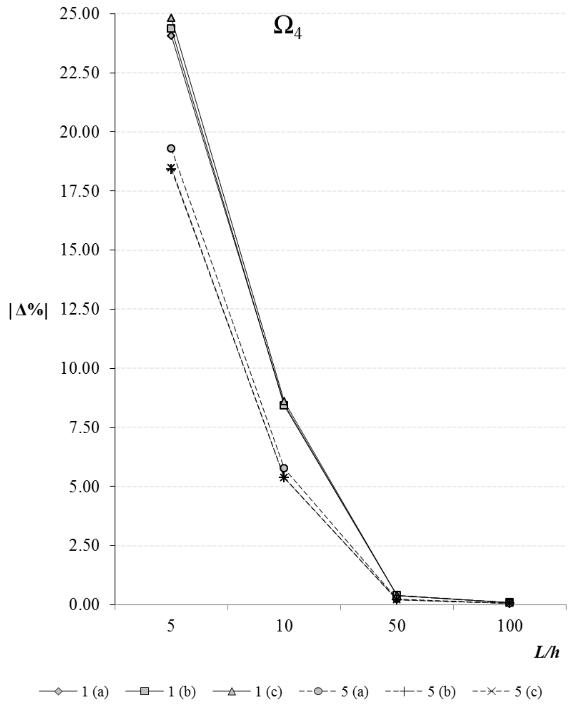


Fig. 6 Variations of the fourth frequency by the use of theory of Timoshenko

1(a): Homogeneous beam with one mass attached.
1(b): AFG (linear) beam with one mass attached.
1(c): AFG (quadratic) beam with one mass attached.
5(a): Homogeneous beam with five masses attached
5(b): AFG (linear) beam with five masses attached.
5(c): AFG (quadratic) beam with five masses attached.

It is plotted up to the ratio $L/h = 100$ to improve the representation of trends, since for the slenderest beams the differences are very small.

It is observed that for the first three frequencies, the influence of considering the theory of Timoshenko is greater in the beams with five masses for all the material compositions. However, for the fourth and fifth frequencies, the influence is greatest on beams carrying just one mass.

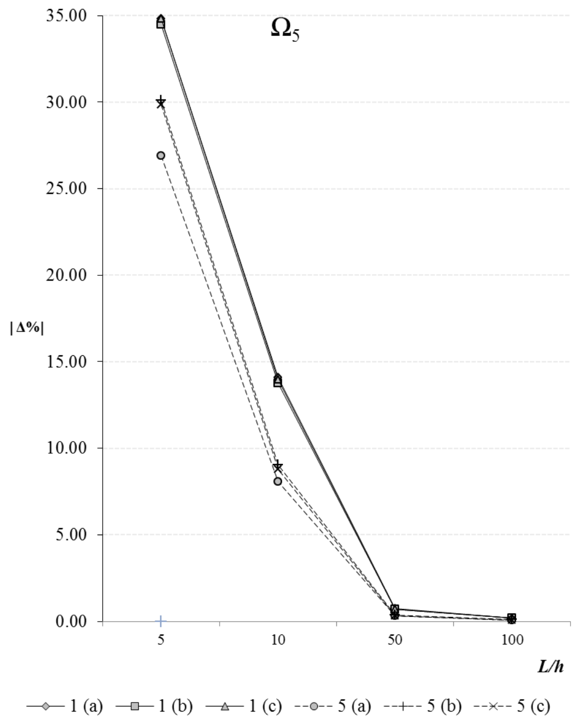


Fig. 7 Variations of the fifth frequency by the use of theory of Timoshenko

Table 11 Convergence study (case of the first row in Table 10)

| $N_{p,q}$ | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|---|---|---|---|---|---|
| 5 | 3.53362 | 15.5960 | 34.2454 | 62.0190 | 97.4123 |
| 10 | 3.53129 | 15.5170 | 33.5176 | 53.8933 | 74.6629 |
| 15 | 3.53053 | 15.4876 | 33.3527 | 53.3795 | 73.5660 |
| 20 | 3.53034 | 15.4729 | 33.2349 | 53.0152 | 72.9484 |
| 25 | 3.53022 | 15.4690 | 33.1829 | 52.6410 | 72.1216 |
| 30 | 3.53014 | 15.4651 | 33.1366 | 52.5256 | 71.8635 |
| 35 | 3.53010 | 15.4601 | 33.0987 | 52.4371 | 71.6952 |

Table 12 Frequency coefficients for a homogeneous tapered beam with a mass at the free end

| $L/h_0$ | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|---|---|---|---|---|---|
| 5 | 1.27563 | 2.14165 | 8.58443 | 21.5106 | 39.2133 |
| 10 | 1.27843 | 2.15684 | 8.87121 | 23.0863 | 43.8814 |
| 50 | 1.27932 | 2.16182 | 8.96939 | 23.6722 | 45.8118 |
| 100 | 1.27935 | 2.16197 | 8.97252 | 23.6913 | 45.8767 |
| 500 | 1.27936 | 2.16202 | 8.97352 | 23.6974 | 45.8975 |
| 1000 | 1.27936 | 2.16202 | 8.97355 | 23.6976 | 45.8981 |
| B-E | 1.27936 | 2.16202 | 8.97354 | 23.6976 | 45.8983 |

In terms of the composition, it can be said that the influence of using Timoshenko theory in general is similar for the two cases of AFG material studied and for the homogeneous beam. Only in the case of the beam with 5 masses and for the fifth frequency the influence of the theory of Timoshenko is greater for the two cases with AFG

Table 13 Frequency coefficients for a tapered beam with AFG material (Alum-St) varying linearly and a mass at the free end

| $L/h_0$ | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|---|---|---|---|---|---|
| 5 | 1.36309 | 2.39179 | 11.2089 | 28.1562 | 51.5514 |
| 10 | 1.36512 | 2.41198 | 11.5495 | 29.9232 | 56.7953 |
| 50 | 1.36576 | 2.41860 | 11.6656 | 30.5684 | 58.8905 |
| 100 | 1.36578 | 2.41880 | 11.6692 | 30.5893 | 58.9602 |
| 500 | 1.36579 | 2.41887 | 11.6704 | 30.5960 | 58.9826 |
| 1000 | 1.36579 | 2.41887 | 11.6705 | 30.5963 | 58.9833 |
| B-E | 1.36579 | 2.41887 | 11.6704 | 30.5962 | 58.9833 |

Table 14 Frequency coefficients for a tapered beam with AFG material (Alum-St) varying quadratically and a mass at the free end

| $L/h_0$ | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|---|---|---|---|---|---|
| 5 | 1.40096 | 2.48318 | 12.3504 | 31.2174 | 57.0748 |
| 10 | 1.40280 | 2.50489 | 12.7168 | 33.1502 | 62.8409 |
| 50 | 1.40338 | 2.51200 | 12.8413 | 33.8538 | 65.1354 |
| 100 | 1.40340 | 2.51222 | 12.8453 | 33.8766 | 65.2117 |
| 500 | 1.40341 | 2.51229 | 12.8465 | 33.8839 | 65.2361 |
| 1000 | 1.40341 | 2.51230 | 12.8466 | 33.8841 | 65.2369 |
| B-E | 1.40341 | 2.51229 | 12.8466 | 33.8841 | 65.2369 |

materials than for the homogeneous beam.

## 4. Convergence of the procedure

### *4.1 Selection of number of terms*

The situation of the first row of Table 10 will be used to evaluate the convergence of the approach because it is the most complex of the analyzed ones: greater shear effect, height of the cross section varying quadratically and five attached masses. Table 11 indicates the results for different numbers of terms in the summations of Eqs. (17)-(18).

The values in Table 11 prove the convergence of the method (it is recalled that Rayleigh-Ritz gives upper bounds of the wanted values). The computation time is significantly increased for $N_p$,$N_q$>20. Therefore, considering that the results are sufficiently indicative from an engineering viewpoint and evaluating the time of data processing, was taken $Np = Nq = 20$ for all cases.

### *4.2 Stability of the solution in a case of large taper ratio*

For large taper ratios, the model moves further away from the classical theory of strength of materials, so it is important to verify whether the proposed solution tend steadily to the values obtained using the simpler theory of Bernoulli-Euler as the beam's slenderness increases.

Table 15 Frequency coefficients for a homogeneous tapered beam with 5 masses attached

| $L/h_0$ | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|---|---|---|---|---|---|
| 5 | 1.97783 | 5.36874 | 10.3193 | 19.4102 | 27.3144 |
| 10 | 1.99942 | 5.48667 | 10.7394 | 21.2467 | 29.5836 |
| 50 | 2.00644 | 5.48576 | 10.8857 | 21.9391 | 30.4221 |
| 100 | 2.00666 | 5.48667 | 10.8904 | 21.9617 | 30.4493 |
| 500 | 2.00673 | 5.48696 | 10.8919 | 21.9689 | 30.4580 |
| 1000 | 2.00673 | 5.48697 | 10.8919 | 21.9691 | 30.4583 |
| B-E | 2.00674 | 5.48697 | 10.8919 | 10.8919 | 30.4584 |

Table 16 Frequency coefficients for a tapered beam with AFG material (Alum-St) varying linearly and 5 masses attached

| $L/h_0$ | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|---|---|---|---|---|---|
| 5 | 2.45983 | 6.23187 | 12.3419 | 23.6946 | 31.9500 |
| 10 | 2.48622 | 6.31773 | 12.8358 | 25.8668 | 34.3836 |
| 50 | 2.49480 | 6.34590 | 13.0068 | 26.6812 | 35.2872 |
| 100 | 2.49507 | 6.34678 | 13.0123 | 26.7077 | 35.3165 |
| 500 | 2.49515 | 6.34707 | 13.0140 | 26.7162 | 35.3259 |
| 1000 | 2.49515 | 6.34708 | 13.0141 | 26.7165 | 35.3262 |
| B-E | 2.49516 | 6.34708 | 13.0141 | 26.7165 | 35.3263 |

Table 17 Frequency coefficients for a tapered beam with AFG material (Alum-St) varying quadratically and 5 masses attached

| $L/h_0$ | $\Omega_1$ | $\Omega_2$ | $\Omega_3$ | $\Omega_4$ | $\Omega_5$ |
|---|---|---|---|---|---|
| 5 | 2.60538 | 6.58682 | 13.1136 | 25.2506 | 34.0783 |
| 10 | 2.63303 | 6.67367 | 13.6388 | 27.5666 | 36.6714 |
| 50 | 2.64202 | 6.70212 | 13.8203 | 28.4326 | 37.6315 |
| 100 | 2.64231 | 6.70301 | 13.8261 | 28.4607 | 37.6626 |
| 500 | 2.64240 | 6.70330 | 13.8280 | 28.4698 | 37.6726 |
| 1000 | 2.64240 | 6.70331 | 13.8280 | 28.4700 | 37.6729 |
| B-E | 2.64240 | 6.70331 | 13.8280 | 28.4701 | 37.6730 |

The case $h(x) = h_0\left(1-0.95x\right)$ is evaluated for the same materials composition, quantity and positions of masses of Tables 5 to 10.

As can be seen in Tables 12 to 17 the trend of the model is suitable.

It is required to clarify that in the cases in which no natural condition is satisfied at the free edge (Tables 12 to 14) it has been necessary to take $Np = Nq = 25$ in Eqs. (17) and (18) to obtain stability in the trend.

Values of Tables 15 to 17 are obtained taken $Np = Nq = 20$ in Eqs. (17) and (18).

## 5. Conclusions

The study of the use of FG materials in resistant structures has had an important development due to the ability of these materials to resist great efforts and to reduce the weight of structures.

For this purpose, it is important to have tools that allow carrying out studies of the performance of structures in different work situations. Among them, the behavior of structural elements that support motors or machines attached to them is usual in many applications and their operation can cause severe efforts in the structure.

In that case, the knowledge of the dynamic parameters of the beam-mass system is fundamental for what is necessary to have a model that correctly describes its behavior. In the case of short beams or higher frequencies, Timoshenko theory should be used to describe the flexural behavior of the beam.

The well-known Rayleigh-Ritz method provides an accurate and convenient procedure to tackle the problem of a vibrating Timoshenko AFG beam carrying attached masses, a problem of which the authors have not found data in the literature.

As a general conclusion, one may say that the influence of the use of Timoshenko theory on the frequencies of cantilever AFG beams supporting masses follows the same trends as for homogeneous beams.

## Acknowledgments

The present study has been sponsored by the Universidad Nacional del Sur (UNS), the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET) and the Comisión de Investigaciones Científicas-Buenos Aires Province (CIC). It was developed in the Department of Enginering. The authors are indebted to the anonymous Reviewers for their valuable comments and suggestions.

## References

Chen, D.W. and Liu, T.L. (2006), "Free and forced vibrations of a tapered cantilever beam carrying multiple point masses", *Struct. Eng. Mech.*, **23**(2), 209-216.

Chen, D.W., Sun, D.L. and Li, X.F. (2017), "Surface effects on resonance frequencies of axially functionally graded Timoshenko nanocantilevers with attached nanoparticle", *Compos. Struct.*, **173**(1), 116-126.

Elishakoff, I., Kaplunov, J. and Nolde, E. (2015), "Celebrating the centenary of Timoshenko's study of effects of shear deformation and rotary inertia", *Appl. Mech. Rev.*, **67**(6), 060802.

Gan, B.S., Trinh, T.H., Le, T.H. and Nguyen, D.K. (2015), "Dynamic response of non-uniform Timoshenko beams made of axially FGM subjected to multiple moving point loads", *Struct. Eng. Mech.*, **53**(5), 981-995.

He, P., Liu, Z. and Li, C. (2013), "An improved beam element for beams with variable axial parameters", *Shock Vibr.*, **20**(4), 601-617.

Huang, Y., Yang, L. and Luo, Q. (2013), "Free vibration of axially functionally graded Timoshenko beams with non-uniform cross-section", *Compos.: Part B*, **45**(1), 1493-1498.

Ilanko, S. and Monterrubio, L.E. (2014), *The Rayleigh-Ritz Method for Structural Analysis*, John Wiley & Sons Inc., London, U.K.

Nikolić, A. (2017), "Free vibration analysis of a non-uniform axially functionally graded cantilever beam with a tip body", *Arch. Appl. Mech.*, **87**(7), 1227-1241.

Rajasekaran, S. and Norouzzadeh Tochaei, E. (2014), "Free vibration analysis of axially functionally graded tapered Timoshenko beams using differential transformation element method and differential quadrature element method of lowest-order", *Meccan.*, **49**(4), 995-1009.

Rossit, C.A., Bambill, D.V. and Gilardi, G.J. (2017), "Free vibrations of AFG cantilever tapered beams carrying attached masses", *Struct. Eng. Mech.*, **61**(5), 685-691.

Sarkar, K. and Ganguli, R. (2014), "Closed-form solutions for axially functionally graded Timoshenko beams having uniform cross-section and fixed-fixed boundary condition", *Compos.: Part B*, **58**, 361-370.

Shahba, A., Attarnejad, R., Marvi, M.T. and Hajilar, S. (2011), "Free vibration and stability analysis of axially functionally graded tapered Timoshenko beams with classical and non-classical boundary conditions", *Compos.: Part B*, **42**(4), 801-808.

Su, H., Banerjee, J.R. and Cheung, C.W. (2013), "Dynamic stiffness formulation and free vibration analysis of functionally graded beams", *Compos. Struct.*, **106**, 854-862.

Sun, D.L., Li, X.F. and Wang, C.Y. (2016), "Buckling of standing tapered Timoshenko columns with varying flexural rigidity under combined loadings", *Int. J. Struct. Stab. Dyn.*, **16**(6), 1550017.

Tang, A.Y., Wu, J.X., Li, X.F. and Lee, K.Y. (2014), "Exact frequency equations of free vibration of exponentially non-uniform functionally graded Timoshenko beams", *Int. J. Mech. Sci.*, **89**, 1-11.

Timoshenko, S.P. (1921), "On the correction for shear of the differential equation for transverse vibration of prismatic bar", *Philosoph. Mag.*, **41**(245), 744-746.

Timoshenko, S.P. (1922), "On the transverse vibrations of bars of uniform cross section", *Philosoph. Mag.*, **43**(253), 125-131.

Torabi, K., Afshari, H. and Heidari-Rarani M. (2013), "Free vibration analysis of a non-uniform cantilever Timoshenko beam with multiple concentrated masses using DQEM", *Eng. Sol. Mech.*, **1**(1), 9-20.

Tudjono, S., Han, A., Nguyen, D., Kiryu, S. and Gan, B. (2017), "Exact shape functions for Timoshenko beam element", *IOSR J. Comput. Eng.*, **19**(3), 12-20.

Zhao, Y., Huang, Y. and Guo, M. (2017), "A novel approach for free vibration of axially functionally graded beams with non-uniform cross-section based on Chebyshev polynomials theory", *Compos. Struct.*, **168**, 277-284.

*CC*