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A unified formulation for modeling of inhomogeneous nonlocal beams

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Abstract. In this article, buckling and free vibration of functionally graded (FG) nanobeams resting on elastic foundation are investigated by developing various higher order beam theories which capture shear deformation influences through the thickness of the beam without the need for shear correction factors. The elastic foundation is modeled as linear Winkler springs as well as Pasternak shear layer. The material properties of FG nanobeam are supposed to change gradually along the thickness through the Mori–Tanaka model. The small scale effect is taken into consideration based on nonlocal elasticity theory of Eringen. From Hamilton's principle, the nonlocal governing equations of motion are derived and then solved applying analytical solution. To verify the validity of the developed theories, the results of the present work are compared with those available in literature. The effects of shear deformation, elastic foundation, gradient index, nonlocal parameter and slenderness ratio on the buckling and free vibration behavior of FG nanobeams are studied.

Keywords: buckling; dynamic analysis; vibration; nanostructures/nanotubes; functionally graded material; nonlocal elasticity theory

1. Introduction

In recent years, some different material compositions are used in structural elements such as beams and plates for the purpose of optimizing the mechanical responses of such structures. Due to the abrupt transitions in material properties, local stress concentrations are induced. To diminish these stresses the transition between different materials should be made gradually. Finally, this idea leads to the concept of functionally graded materials (FGMs) which are often made from a pair of ceramic-metal by supposing variable properties throughout the gradient directions. Hence, FGMs obtained broad potential applications for various systems and devices including aerospace, aircraft, automobile and defense structures and most recently the electronic devices.

With the growing application of FGMs in the structural components, several beam theories are developed to anticipate mechanical responses of these structures. The classical beam theory or Euler-Bernoulli beam theory (EBT) is known as the simplest theory which is only applicable for slender beams and should not be applied for thick beams and hence, the buckling loads and natural frequencies of thick beams are overestimated in which shear deformation effects are prominent. To overcome the imperfections of EBT, the first order shear deformation theory or Timoshenko beam theory (TBT) is suggested so that the shear deformation influence is considered in this theory. But, a disadvantage of this theory is to need a shear correction factor to properly demonstration of the deformation strain energy. To prevent using the shear correction factors, several higher-order shear deformation theories have been developed including the third-order shear deformation theory proposed by Reddy (2007), the generalized beam theory proposed by Aydogdu (2009), sinusoidal shear deformation theory of Touratier (1991) and exponential theory of Karma (2003). By verifying zero transverse shear stresses at the upper and lower surfaces of the beam, these theories have the potential to capture both the microstructural and shear deformation effects.

To develop higher order theories for mechanical analysis of FG structure, several works are performed. Kadoli *et al.* (2008) implemented a displacement field based on higher order shear deformation theory to study the static behavior of FG metal–ceramic beams under ambient temperature. Simsek (2010) analyzed fundamental frequency of FG beams having different boundary conditions within the framework of the classical, the first-order and different higher-order shear deformation beam theories. Mahi *et al.* (2010) presented exact solutions to study the free vibration of a beam made of symmetric functionally graded materials based on a unified higher order shear deformation theory. Thai and Vo (2012) studied bending and free vibration of functionally graded beams using various higher-order shear deformation beam theories. Neves *et al.* (2012) proposed a quasi-3D sinusoidal shear deformation theory for the static and free vibration of functionally graded plates. Larbi *et al.* (2013) presented an efficient shear deformation beam theory based on neutral surface position for bending and free vibration analysis of functionally graded beams. Vo *et al.* (2014) investigated static and vibration analysis of functionally graded beams using refined shear deformation theory. Nguyen *et al.* (2015) studied vibration and buckling of functionally

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graded sandwich beams by a new hyperbolic higher-order shear deformation theory. In another study, Vo *et al.* (2015) provided a finite element model for free vibration and buckling analyses of functionally graded (FG) sandwich beams by using a quasi-3D theory in which both shear deformation and thickness stretching effects are included. Also, Atmane *et al.* (2015) applied an efficient beam theory to study the effects of thickness stretching and porosity on mechanical responses of FGM beams resting on elastic foundation. Also, Mahi and Tounsi (2015) presented a new hyperbolic shear deformation theory applicable to bending and free vibration analysis of isotropic, functionally graded, sandwich and laminated composite plates. Recently, Kulkarni *et al.* (2015) proposed an analytical solution for bending and buckling analysis of functionally graded plates using inverse trigonometric shear deformation theory.

In addition, fast growing progress in the application of structural elements such as beams and plates with micro or nanolength scale in micro/nano electro-mechanical systems (MEMS/ NEMS) and nanosensors, due to their outstanding chemical, mechanical, and electrical properties, led to a provocation in modelling of micro/nano scale structures. Recently, nanotechnology is concerned with fabrication of functionally graded materials (FGMs) and engineering structures at nanoscale, which enables new generation of materials and devices with innovative properties. Small scale beams are the basic structures used in several applications such as nanoelectromechanical systems (NEMS), nano-probes, atomic force microscope (AFM), nanoactuators and nanosensors (Sedighi *et al.* 2016). For convince designing of nanostructure, the size and length-scale effects and the atomic forces should be included in mathematical formulation. It is clear that all of the above-mentioned studies on the mechanical behavior of FG structures using higher order theories are conducted with ignorance of small size influences. Recent experimental results indicate that when the size of the structures reduces to nano scale, the influences of small scale play a notable role in mechanical responses of such nanostructures. The defect of the classical continuum theory is that it does not take into account the size effects in micro/nano scale structures. So, Eringen's nonlocal elasticity theory is proposed to overcome this problem which includes small scale effects with good accuracy to model micro/nano scale devices and systems. Based on the nonlocal constitutive relations of Eringen, a number of studies have been carried out to predict the mechanical responses of nanobeams (Hosseini and Rahmani 2016). Eltaher *et al.* (2012) presented a finite element analysis for free vibration of FG nanobeams using nonlocal EBT. Based upon nonlocal Timoshenko and Euler beam theories, Simsek and Yurtcu (2013) investigated bending and buckling of FG nanobeam by analytical method. Also, Rahmani and Pedram (2014) Analysed the size effects on vibration of FG nanobeams based on nonlocal TBT. It is worth mentioning that on the basis of higher order beam theories, some studies are performed to investigate buckling and free vibration of FG nanobeams (Rahmani and Jandaghian 2015, Zemri *et al.* 2015, Rahmani *et al.* 2017, Ebrahimi and Barati 2016a, b, c, d, 2017a, b). Also, it can be useful for the readers to take a look at the other recent works dealing with mechanical answers of the small structures (Ebrahimi and Dabbagh 2017a, b, c, d, e, 2018, Ebrahimi and Heidari 2017, Ebrahimi and Daman 2017, Ebrahimi *et al.* 2016, 2017, Ebrahimi and Karimiasl 2017). Within recent years, another size-dependent theory has been developed for mechanical analysis of tiny structures, called nonlocal strain gradient elasticity. For more information about this theory, more curious readers are referred to other articles dealing with the mechanical characteristics of small size elements (Ebrahimi and Haghi 2017, Ebrahimi and Barati 2017c, d, e, 2018).

In this article, various higher-order shear deformation beam theories such as parabolic, sinusoidal, hyperbolic, exponential as well as inverse trigonometric shear deformation theory for buckling and free vibration of size-dependent FG nanobeams resting on elastic foundation are developed. These theories provide a constant transverse displacement and higher-order variation of axial displacement through the depth of the nanobeam so that there is no need for any shear correction factors. Material properties of FG nanobeam are assumed to change continuously along the thickness according to Mori-Tanaka model. From Hamilton's principle the governing equations of motion are derived and Navier type solution method is used to solve the equations. Numerical and illustrative results are presented to show the effects of the shear deformation, gradient index, nonlocality and elastic foundation parameters on the buckling and free vibration of FG nanobeams.

2. Theory and formulation

2.1 Mori-Tanaka FGM beam model

In this study, Mori-Tanaka homogenization technique is employed to model the effective material properties of the FG nanobeam. According to Mori-Tanaka homogenization technique the local effective material properties of the FG nanobeam such as effective local bulk modulus K_e and shear modulus μ_e can be calculated

$$\frac{K_e - K_m}{K_c - K_m} = \frac{V_c}{1 + V_m (K_c - K_m)/(K_m + 4\mu_m / 3)} \tag{1}$$

$$\frac{\mu_e - \mu_m}{\mu_c - \mu_m} = \frac{V_c}{1 + V_m (\mu_c - \mu_m)/[(\mu_m + \mu_m (9K_m + 8\mu_m)/(6(K_m + 2\mu_m))]} \tag{2}$$

Therefore from Eq. (4), the effective Young's modulus (E), Poisson's ratio (v) based on Mori-Tanaka scheme can be expressed by

$$E(z) = \frac{9K_e \mu_e}{3K_e + \mu_e} \tag{3}$$

$$\nu(z) = \frac{3K_e - 2\mu_e}{6K_e + 2\mu_e} \tag{4}$$

The shear modulus G(z) of FG nanobeam with respect to Mori-Tanaka homogenization is defined as

$$G(z) = \frac{E(z)}{2(1 + \nu(z))} \tag{5}$$

The material composition of FG nanobeam at the upper surface (z=+h/2) is supposed to be the pure ceramic and it changes continuously to the opposite side surface(z=-h/2) which is pure metal.

2.2 Kinematic relations

Based on the various shear deformation beam theories, the displacement field at any point of the beam can be written as

$$u_x\left(x,z\right)=u\left(x\right)-z\frac{\partial w_b}{\partial x}-f(z)\frac{\partial w_s}{\partial x} \tag{6}$$

$$u_z(x,z)=w_b(x)+w_s(x) \tag{7}$$

where u is longitudinal displacement and w_b, w_s are the bending and shear components of transverse displacement of a point on the midplane of the beam. $f(z)$ is the shape function determining the distribution of the transverse shear strain and shear stress through the thickness of the beam. Nonzero strains of the present beam model are expressed as follows

$$\varepsilon_{xx}=\frac{\partial u}{\partial x}-z\frac{\partial^2 w_b}{\partial x^2}-f(z)\frac{\partial^2 w_s}{\partial x^2} \tag{8}$$

$$\gamma_{xz}=g\frac{\partial w_s}{\partial x} \tag{9}$$

where $g(z)=1-d_f/d_z$. By using the Hamilton's principle, in which the motion of an elastic structure in the time interval t1 < t < t2 is so that the integral with respect to time of the total potential energy is extremum

$$\int_0^t \delta(U+V-K)\,dt=0 \tag{10}$$

Here U is strain energy, V is work done by external forces and K is kinetic energy. The virtual strain energy can be calculated as

$$\delta U=\int_v \sigma_{ij}\delta\varepsilon_{ij}\,dV=\int_v(\sigma_{xx}\delta\varepsilon_{xx}+\sigma_{xz}\delta\gamma_{xz})dV \tag{11}$$

Substituting Eqs. (8)-(9) into Eq. (11) yields

$$\delta U=\int_0^L\left(N\frac{d\delta u}{dx}-M_b\frac{d^2\delta w_b}{dx^2}-M_s\frac{d^2\delta w_s}{dx^2}+Q\frac{d\delta w_s}{dx}\right)dx \tag{12}$$

In which the variables introduced in arriving at the last expression are defined as follows

$$N=\int_A\sigma_{xx}\,dA,\ M_b=\int_A z\sigma_{xx}\,dA,\ M_s=\int_A f\sigma_{xx}dA,\ Q=\int_A g\sigma_{xz}\,dA \tag{13}$$

The first variation of the work done by applied forces can be written in the form

$$\begin{aligned}\delta V=&\int_0^L(\overline{N}(\frac{dw_b}{dx}\frac{\partial d\,w_b}{dx}+\frac{dw_s}{dx}\frac{d\delta\,w_s}{dx})-q\,\delta(w_b+w_s)-k_w\delta(w_b+w_s)\\&+k_p(\frac{d^2\delta w_b}{dx^2}+\frac{d^2\delta w_s}{dx^2}))dx\end{aligned} \tag{14}$$

where $\overline{N}$ and q are the applied axial compressive and transverse load, respectively and k_w and k_p are linear and shear coefficient of elastic foundation. The variation of the kinetic energy can be expressed as

$$\begin{aligned}\delta K=&\int_0^L(I_0[\frac{du}{dt}\frac{d\delta u}{dt}+(\frac{dw_b}{dt}+\frac{dw_s}{dt})(\frac{d\delta\,w_b}{dt}+\frac{d\delta\,w_s}{dt})]-I_1(\frac{du}{dt}\frac{d^2\delta w_b}{dxdt}+\frac{d^2w_b}{dxdt}\frac{d\delta u}{dt})\\&+I_2(\frac{d^2w_b}{dxdt}\frac{d^2\delta w_b}{dxdt})-J_1(\frac{du}{dt}\frac{d^2\delta w_s}{dxdt}+\frac{d^2w_s}{dxdt}\frac{d\delta u}{dt})+K_2(\frac{d^2w_s}{dxdt}\frac{d^2\delta w_s}{dxdt})+J_2(\frac{d^2w_b}{dxdt}\frac{d^2\delta w_s}{dxdt}\\&+\frac{d^2w_s}{dxdt}\frac{d^2\delta w_b}{dxdt}))dx\end{aligned} \tag{15}$$

where

$(I_0,I_1,J_1,I_2,J_2,K_2)=\int_A\rho(z)(1,z,f,z^2,zf,f^2)dA$. By substituting Eqs. (12), (14) and (15) into Eq. (10) and setting the coefficients of δu, δw and $\delta\varphi$ to zero, the following Euler-Lagrange equation can be obtained

$$\frac{\partial N}{\partial x}=I_0\frac{d^2u}{dt^2}-I_1\frac{d^3w_b}{dxdt^2}-J_1\frac{d^3w_s}{dxdt^2} \tag{16}$$

$$\frac{d^2M_b}{dx^2}+q=\overline{N}\frac{d^2w_b}{dx^2}+I_0(\frac{d^2w_b}{dt^2}+\frac{d^2w_s}{dt^2})+I_1\frac{d^3u}{dxdt^2}-I_2\frac{d^4w_b}{dx^2dt^2}-J_2\frac{d^4w_s}{dx^2dt^2}+k_ww_b-k_p\frac{d^2w_b}{dx^2} \tag{17}$$

$$\begin{aligned}\frac{d^2M_s}{dx^2}+\frac{dQ}{dx}+q=&\overline{N}\frac{d^2w_s}{dx^2}+I_0(\frac{d^2w_b}{dt^2}+\frac{d^2w_s}{dt^2})+J_1\frac{d^3u}{dxdt^2}-J_2\frac{d^4w_b}{dx^2dt^2}-K_2\frac{d^4w_s}{dx^2dt^2}\\&+k_ww_s-k_p\frac{d^2w_s}{dx^2}\end{aligned} \tag{18}$$

2.3 The nonlocal elasticity model for FG nanobeam

According to Eringen nonlocal elasticity model (Eringen and Edelen 1972), the stress state at a point inside a body is regarded to be function of strains of all points in the neighbor regions. For homogeneous elastic solids the nonlocal stress-tensor components σ_{ij} at each point x in solid can be defined as

$$\sigma_{ij}(x)=\int_\Omega\alpha(|x'-x|,\tau)\,t_{ij}(x')\,d\Omega(x') \tag{19}$$

where $t_{ij}(x')$ are the components available in local stress tensor at point x which are associated to the strain tensor components ε_{kl} as

$$t_{ij}=C_{ijkl}\varepsilon_{kl} \tag{20}$$

The concept of Eq. (19) is that the nonlocal stress at any point is weighting average of local stress of all points in the near region that point, the size that is related to the nonlocal kernel $\alpha(|x'-x|,\tau)$. Also $|x'-x|$ is Euclidean distance and τ is a constant as follows

$$\tau=\frac{e_0a}{l} \tag{21}$$

which indicates the relation of a characteristic internal length, (for instance lattice parameter, C-C bond length and granular distance) and a characteristic external length, l (for instance crack length and wavelength) using a constant, e_0, dependent on each material. The value of e_0 is experimentally estimated by comparing the scattering curves of plane waves and atomistic dynamics. According to (Eringen and Edelen 1972, Eringen 1983) for a class of physically admissible kernel $\alpha(|x'-x|,\tau)$ it is possible to represent the integral constitutive relations given by Eq. (19) in an equivalent differential form as

$$(1-(e_0a)\nabla^2)\sigma_{kl}=t_{kl} \tag{22}$$

where ∇_2 is the Laplacian operator. Thus, the scale length e_0a considers the influences of small scale on the response of nano-structures. The magnitude of the small-scale parameter relies on several parameters including mode shapes, boundary conditions, chirality and the essence of motion. So, for a material in the one-dimension case, the constitutive relations of nonlocal theory can be expressed as

$$\sigma_{xx}-(e_0a)^2\frac{\partial^2\sigma_{xx}}{\partial x^2}=E\,\varepsilon_{xx} \tag{23}$$

$$\sigma_{xz}-(e_0a)^2\frac{\partial^2\sigma_{xz}}{\partial x^2}=G\,\gamma_{xz} \tag{24}$$

where σ and ε are the nonlocal stress and strain, respectively and E is the Young's modulus. For a nonlocal FG beam, Eqs. (23) and (24) can be written as

$$\sigma_{xx}-\mu\frac{\partial^2\sigma_{xx}}{\partial x^2}=E(z)\varepsilon_{xx} \tag{25}$$

$$\sigma_{xz}-\mu\frac{\partial^2\sigma_{xz}}{\partial x^2}=G(z)\gamma_{xz} \tag{26}$$

where $\mu=(e_0a)^2$. Integrating Eqs. (25) and (26) over the beam's cross-section area, we obtain the force-strain and the moment-strain of the nonlocal refined beam theory can be obtained as follows

$$N-\mu\frac{d^2N}{dx^2}=A\frac{du}{dx}-B\frac{d^2w_b}{dx^2}-B_s\frac{d^2w_s}{dx^2} \tag{27}$$

$$M_b-\mu\frac{d^2M_b}{dx^2}=B\frac{du}{dx}-D\frac{d^2w_b}{dx^2}-D_s\frac{d^2w_s}{dx^2} \tag{28}$$

$$M_s-\mu\frac{d^2M_s}{dx^2}=B_s\frac{du}{dx}-D_s\frac{d^2w_b}{dx^2}-H_s\frac{d^2w_s}{dx^2} \tag{29}$$

$$Q-\mu\frac{d^2Q}{dx^2}=A_s\frac{dw_s}{dx} \tag{30}$$

In which the cross-sectional rigidities are defined as follows

$$(A,B,B_s,D,D_s,H_s)=\int_A E(z)(1,z,f,z^2,zf,f^2)\,dA \tag{31}$$

$$A_s=\int_A g^2G(z)dA \tag{32}$$

The nonlocal governing equations of higher order shear deformable FG nanobeams in terms of the displacement can be derived by substituting for N, M_b, M_s and Q from Eqs. (27)-(30), respectively, into Eqs. (16)-(18) as follows

$$\begin{aligned}&A\frac{d^2u}{dx^2}-B\frac{\partial^3w_b}{\partial x^3}-B_s\frac{\partial^3w_s}{\partial x^3}-I_0\frac{d^2u}{dt^2}+I_1\frac{d^3w_b}{dxdt^2}+J_1\frac{d^3w_s}{dxdt^2}\\&+\mu\Big(I_0\frac{d^4u}{dx^2dt^2}-I_1\frac{d^5w_b}{dx^3dt^2}-J_1\frac{d^5w_s}{dx^3dt^2}\Big)=0\end{aligned} \tag{33}$$

$$\begin{aligned}&B\frac{d^3u}{dx^3}-D\frac{d^4w_b}{dx^4}-D_s\frac{d^4w_s}{dx^4}+q-\overline{N}\frac{d^2w_b}{dx^2}-I_0\Big(\frac{d^2w_b}{dt^2}+\frac{d^2w_s}{dt^2}\Big)-I_1\frac{d^3u}{dxdt^2}+I_2\frac{d^4w_b}{dx^2dt^2}\\&+J_2\frac{d^4w_s}{dx^2dt^2}-k_ww_b+k_p\frac{d^2w_b}{dx^2}+\mu\Big(-\frac{d^2q}{dx^2}+\overline{N}\frac{d^4w_b}{dx^4}+I_0\Big(\frac{d^4w_b}{dx^2dt^2}+\frac{d^4w_s}{dx^2dt^2}\Big)+I_1\frac{d^5u}{dx^3dt^2}\\&-I_2\frac{d^6w_b}{dx^4dt^2}-J_2\frac{d^6w_s}{dx^4dt^2}+k_w\frac{d^2w_b}{dx^2}-k_p\frac{d^4w_b}{dx^4}\Big)=0\end{aligned} \tag{34}$$

$$\begin{aligned}&B_s\frac{d^3u}{dx^3}-D_s\frac{d^4w_b}{dx^4}-H_s\frac{d^4w_s}{dx^4}+A_s\frac{d^2w_s}{dx^2}+q-\overline{N}\frac{d^2w_s}{dx^2}-I_0\Big(\frac{d^2w_b}{dt^2}+\frac{d^2w_s}{dt^2}\Big)-J_1\frac{d^3u}{dxdt^2}\\&+J_2\frac{d^4w_b}{dx^2dt^2}+K_2\frac{d^4w_s}{dx^2dt^2}-k_ww_s+k_p\frac{d^2w_s}{dx^2}+\mu\Big(-\frac{d^2q}{dx^2}+\overline{N}\frac{d^4w_s}{dx^4}+I_0\Big(\frac{d^4w_b}{dx^2dt^2}+\frac{d^4w_s}{dx^2dt^2}\Big)\\&+J_1\frac{d^5u}{dx^3dt^2}-J_2\frac{d^6w_b}{dx^4dt^2}-K_2\frac{d^6w_s}{dx^4dt^2}+k_w\frac{d^2w_s}{dx^2}-k_p\frac{d^4w_s}{dx^4}\Big)=0\end{aligned} \tag{35}$$

3. Solution procedures

Here, on the basis the Navier method, an analytical solution of the governing equations for buckling and free vibration of a simply supported FG nanobeam is presented. To satisfy governing equations of motion and the simply supported boundary condition, the displacement variables are adopted to be of the form

$$u(x,t)=\sum_{n=1}^{\infty}U_n\cos(\alpha x)e^{i\omega_nt} \tag{36}$$

$$w_b(x,t)=\sum_{n=1}^{\infty}W_{bn}\sin(\alpha x)e^{i\omega_nt} \tag{37}$$

$$w_s(x,t)=\sum_{n=1}^{\infty}W_{sn}\sin(\alpha x)e^{i\omega_nt} \tag{38}$$

where $\alpha=\dfrac{n\pi}{L}$ and (U_n, W_{bn}, W_{sn}) are the unknown Fourier coefficients to be determined for each n value. Substituting Eqs. (36)-(38) into Eqs. (33)-(35) respectively, leads to

$$\left\{\begin{pmatrix}k_{11}&k_{12}&k_{13}\\k_{21}&k_{22}&k_{23}\\k_{31}&k_{32}&k_{33}\end{pmatrix}-\bar{\omega}_n^{\,2}\begin{pmatrix}m_{11}&m_{12}&m_{13}\\m_{21}&m_{22}&m_{23}\\m_{31}&m_{32}&m_{33}\end{pmatrix}\right\}\begin{Bmatrix}U_n\\W_{bn}\\W_{sn}\end{Bmatrix}=0 \tag{39}$$

where

$$\begin{aligned}&k_{1,1}=-A\alpha^2\ ,\ \ k_{1,2}=B\alpha^3,\ k_{1,3}=B_s\alpha^3,\ k_{2,3}=-D_s\alpha^4\ ,\\&k_{2,2}=\overline{N}\alpha^2(1+\mu\alpha^2)-k_p\alpha^2(1+\mu\alpha^2)-k_w(1+\mu\alpha^2)-D\alpha^4\ ,\\&k_{3,3}=\overline{N}\alpha^2(1+\mu\alpha^2)-k_p\alpha^2(1+\mu\alpha^2)-k_w(1+\mu\alpha^2)-A_s\alpha^2-H_s\alpha^4,\\&m_{1,1}=I_0(1+\mu\alpha^2),\ \ m_{1,2}=-I_1\alpha-\mu I_1\alpha^3,\ m_{1,3}=-J_1\alpha-\mu J_1\alpha^3,\\&m_{2,2}=I_0(1+\mu\alpha^2)+I_2\alpha^2+\mu I_2\alpha^4,\ \ m_{2,3}=I_0(1+\mu\alpha^2)+J_2\alpha^2+\mu J_2\alpha^4;\\&m_{3,3}=I_0(1+\mu\alpha^2)+K_2\alpha^2+\mu K_2\alpha^4,\end{aligned} \tag{40}$$

4. Numerical results and discussions

Through this section, the influences of FG composition, shear deformation, nonlocality and slenderness ratio on the natural frequencies and buckling loads of the FG nanobeam as shown in Fig. 1 will be figured out. The FG nanobeam is a combination of Steel and Alumina (Al_2O_3) where their

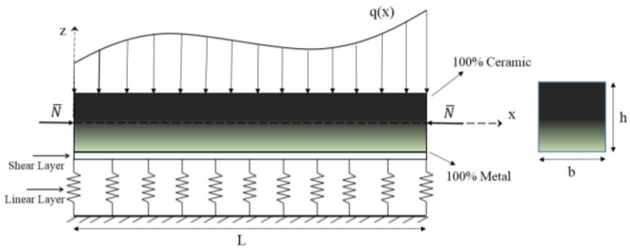

Fig. 1 Geometry and coordinates of functionally graded nanobeam resting on elastic foundation

Table 1 Shape functions

Beam theory	$f(z)$
Parabolic (Reddy 2007)	$\frac{4z^3}{3h^2}$
Sinusoidal (Touratier 1991)	$z-\frac{h}{\pi}\sin(\frac{\pi z}{h})$
Exponential (Karma *et al.* 2003)	$z-ze^{-2(\frac{z}{h})^2}$
Hyperbolic(1) (Soldatos 1992)	$z-h\sinh(\frac{z}{h})+z\cosh(\frac{1}{2})$
Hyperbolic(2) (Kettaf *et al.* 2013)	$z-z[1+\frac{3\pi}{2}\mathrm{sech}^2(\frac{1}{2})]+\frac{3\pi}{2}h\tanh(\frac{z}{h})$
Cotangential inverse trigonometric (Kulkarni *et al.* 2015)	$z-\cot^{-1}(\frac{rh}{z})+\frac{4r}{h(4r^2+1)};\ r=0.46$

Table 2 Material properties of FGM constituents

Properties	Steel	Alumina (Al_2O_3)
E	210 (GPa)	390 (GPa)
ρ	7800 (kg/m^3)	3960 (kg/m^3)
ν	0.3	0.24

properties are given in Table 1. The following dimensions for the beam geometry is considered: L (length) = 10000 nm, b (width) = 1000 nm (Eltaher *et al*. 2012, Rahmani and Pedram 2014).

Also, for better presentation of the results the following dimensionless quantities are adopted

$$\hat{\omega}=\omega L^2\sqrt{\frac{\rho_c A}{E_c I}},\ N_{cr}=\bar{N}\frac{L^2}{E_c I},\ K_w=k_w\frac{L^4}{E_c I},\ K_p=k_p\frac{L^2}{E_c I} \tag{41}$$

where I=bh^3/12 is the moment inertia of the beam's cross section. For the verification purpose, in Table 3 the non-dimensional frequency of simply supported FG nanobeam using third order or parabolic shear deformation theory with various nonlocal parameters and gradient indexes are compared with the results presented by Ebrahimi and Salari (2015). It can be observed from Table 3 that the result of nonlocal third higher beam theory are smaller than those of nonlocal Euler beam theory. This is attributed to the fact that Euler-Bernoulli beam model is unable to capture the influence of shear deformation.

The variations of the dimensionless frequencies and buckling loads of FG nanobeam resting on elastic foundation for various beam theories are presented in Tables 4-6. It is observed that all of the presented beam theories provide approximately a same result for buckling and free vibration of FG nanobeams and only some negligible differences exist. According to these tables, it must be noted that as the gradient index increases the dimensionless frequencies and buckling load reduce. The reason is that increasing the gradient index results in reduction of the rigidity of the beam. Similar to gradient index, nonlocal parameter has a decreasing influence on stiffens of the beam and hence the dimensionless frequencies and buckling reduce. Therefore, nonlocality and gradient index have a notable effect on the mechanical responses of size-dependent FG nanobeams. Also, it is found that when the Winkler and Pasternak parameters increase the non-dimensional frequency and buckling load increase due to the stiffening effect of foundation parameters on the FG nanobeam structure.

Figs. 2 and 3 demonstrate the influence of shear deformation and slenderness ratio on the variation of the

Table 3 Comparison of the non-dimensional frequency of a Mori-Tanaka based FG nanobeam without elastic foundation (L/h=20)

μ	Gradient index							
	0		0.2		1		5	
	EBT (Ebrahimi and Salari 2015)	Present RBT	EBT (Ebrahimi and Salari 2015)	Present RBT	EBT (Ebrahimi and Salari 2015)	Present RBT	EBT (Ebrahimi and Salari 2015)	Present RBT
0	9.8594	9.82957	8.5788	8.55411	6.9131	6.89566	5.8869	5.86719
1	9.4062	9.37769	8.1844	8.16086	6.5953	6.57866	5.6163	5.59746
2	9.0102	8.98289	7.8399	7.81730	6.3176	6.30170	5.3798	5.36182
3	8.6603	8.63410	7.5354	7.51376	6.0723	6.05701	5.1709	5.15362
4	8.3483	8.32302	7.2639	7.24305	5.8536	5.83878	4.9846	4.96794

Table 4 The variation of the first three non-dimensional frequencies of FG nanobeam for various beam theories (K_w=K_p=0, L/h=20)

μ	Beam theory	Mode 1			Mode 2			Mode 3		
		p=0.2	p=1	p=5	p=0.2	p=1	p=5	p=0.2	p=1	p=5
0	Parabolic	8.55411	6.89566	5.86719	33.8128	27.2447	23.1588	74.6504	60.1073	51.0165
	Sinusoidal	8.55415	6.89568	5.86719	33.8134	27.2451	23.1588	74.6535	60.1092	51.0166
	Exponential	8.55426	6.89577	5.86725	33.8152	27.2465	23.1598	74.6621	60.1158	51.0217
	Hyperbolic (1)	8.55411	6.89566	5.86719	33.8128	27.2448	23.1588	74.6504	60.1074	51.0168
	Hyperbolic (2)	8.55413	6.89567	5.86718	33.8131	27.2449	23.1587	74.652	60.1081	51.0161
	Cotangential	8.55459	6.89603	5.86748	33.8202	27.2505	23.1634	74.6861	60.1351	51.0389
1	Parabolic	8.16086	6.57866	5.59746	28.6304	23.069	19.6093	54.3251	43.7417	37.1261
	Sinusoidal	8.1609	6.57868	5.59746	28.6309	23.0693	19.6093	54.3274	43.7431	37.1262
	Exponential	8.16101	6.57876	5.59752	28.6325	23.0705	19.6101	54.3337	43.7479	37.1299
	Hyperbolic (1)	8.16086	6.57866	5.59747	28.6304	23.069	19.6093	54.3251	43.7418	37.1263
	Hyperbolic (2)	8.16088	6.57866	5.59745	28.6307	23.0692	19.6092	54.3263	43.7423	37.1258
	Cotangential	8.16132	6.57901	5.59774	28.6367	23.0739	19.6132	54.3511	43.762	37.1424
2	Parabolic	7.8173	6.3017	5.36182	25.2759	20.3661	17.3118	44.8003	36.0725	30.6168
	Sinusoidal	7.81733	6.30172	5.36181	25.2764	20.3664	17.3118	44.8022	36.0736	30.6169
	Exponential	7.81744	6.3018	5.36187	25.2777	20.3674	17.3125	44.8073	36.0776	30.6199
	Hyperbolic (1)	7.8173	6.3017	5.36182	25.2759	20.3661	17.3118	44.8003	36.0725	30.6169
	Hyperbolic (2)	7.81731	6.30171	5.36181	25.2762	20.3662	17.3117	44.8013	36.073	30.6166
	Cotangential	7.81773	6.30204	5.36208	25.2815	20.3704	17.3152	44.8217	36.0892	30.6302

Table 5 The variation of the first three non-dimensional frequencies of FG nanobeam for various beam theories (K_w=25, K_p=5, L/h=20)

μ	Beam theory	Mode 1			Mode 2			Mode 3		
		p=0.2	p=1	p=5	p=0.2	p=1	p=5	p=0.2	p=1	p=5
0	Parabolic	11.6455	9.81165	8.61547	36.3538	29.6845	25.4800	76.9517	62.3212	53.1149
	Sinusoidal	11.6430	9.80882	8.61203	36.3490	29.6792	25.4734	76.9452	62.3131	53.1036
	Exponential	11.6403	9.80590	8.60846	36.3453	29.6748	25.4676	76.944	62.3096	53.0970
	Hyperbolic (1)	11.1894	9.30439	8.06717	36.8022	30.0105	25.7578	78.9815	64.1312	54.9342
	Hyperbolic (2)	11.6442	9.81018	8.61365	36.3513	29.6817	25.4764	76.9483	62.3168	53.1084
	Cotangential	11.7067	9.87498	8.68707	36.5196	29.8511	25.6633	77.2972	62.6618	53.4818
1	Parabolic	11.3577	9.58911	8.43119	31.585	25.898	22.2945	57.4346	46.7233	39.9438
	Sinusoidal	11.3550	9.58606	8.42748	31.5791	25.8914	22.2864	57.4234	46.7106	39.9279
	Exponential	11.3521	9.5829	8.42362	31.5738	25.8855	22.2789	57.4158	46.701	39.9151
	Hyperbolic (1)	10.8487	9.03185	7.83601	31.8485	26.0237	22.3599	59.1176	48.1061	41.2610
	Hyperbolic (2)	11.3563	9.58752	8.42923	31.582	25.8946	22.2901	57.4288	46.7165	39.9350
	Cotangential	11.4225	9.65626	8.50721	31.7795	26.0941	22.5109	57.8868	47.1739	40.4338
2	Parabolic	11.1113	9.39882	8.27367	28.573	23.5148	20.2938	48.5103	39.619	33.9594
	Sinusoidal	11.1084	9.39555	8.26970	28.5658	23.507	20.2843	48.4954	39.6027	33.9397
	Exponential	11.1053	9.39215	8.26556	28.5593	23.4997	20.2753	48.4834	39.5888	33.9223
	Hyperbolic (1)	10.5525	8.79502	7.63545	28.6552	23.4554	20.1731	49.7924	40.5855	34.8457
	Hyperbolic (2)	11.1098	9.39712	8.27157	28.5693	23.5107	20.2887	48.5026	39.6104	33.9487
	Cotangential	11.1796	9.46973	8.35403	28.7928	23.7372	20.5397	49.0492	40.1584	34.5478

Table 6 The variation of the non-dimensional buckling load of FG nanobeam for various beam theories (L/h=20)

μ	Beam theory	K_w=K_p=0				K_w=25, K_p=5			
		P=0	P=0.2	P=1	P=5	P=0	P=0.2	P=1	P=5
0	Parabolic	9.8658	8.67766	7.2035	6.35864	17.3988	16.2107	14.7365	13.8917
	Sinusoidal	9.86442	8.67651	7.20246	6.35751	17.3974	16.2095	14.7355	13.8905
	Exponential	9.86267	8.67505	7.20115	6.35607	17.3957	16.2081	14.7342	13.8891
	Hyperbolic (1)	9.86916	8.68045	7.20603	6.36146	17.4022	16.2135	14.7391	13.8945
	Hyperbolic (2)	9.86511	8.67709	7.20298	6.35807	17.3981	16.2101	14.736	13.8911
	Cotangential	9.81071	8.63035	7.16272	6.31750	17.3437	16.1634	14.6958	13.8505
1	Parabolic	8.97955	7.89815	6.55641	5.78744	16.5126	15.4312	14.0894	13.3205
	Sinusoidal	8.97829	7.89710	6.55546	5.78641	16.5113	15.4301	14.0885	13.3194
	Exponential	8.97670	7.89577	6.55427	5.78510	16.5097	15.4288	14.0873	13.3181
	Hyperbolic (1)	8.98261	7.90068	6.55871	5.79001	16.5156	15.4337	14.0917	13.3230
	Hyperbolic (2)	8.97892	7.89763	6.55593	5.78692	16.5120	15.4307	14.0890	13.3200
	Cotangential	8.92941	7.85509	6.51929	5.74999	16.4624	15.3881	14.0523	13.2830
2	Parabolic	8.23940	7.24714	6.01599	5.31041	15.7724	14.7802	13.549	12.8434
	Sinusoidal	8.23825	7.24617	6.01512	5.30946	15.7713	14.7792	13.5482	12.8425
	Exponential	8.23679	7.24496	6.01402	5.30826	15.7698	14.7780	13.5471	12.8413
	Hyperbolic (1)	8.24221	7.24946	6.01810	5.31276	15.7752	14.7825	13.5511	12.8458
	Hyperbolic (2)	8.23883	7.24666	6.01555	5.30993	15.7719	14.7797	13.5486	12.8430
	Cotangential	8.19339	7.20762	5.98193	5.27605	15.7264	14.7407	13.5150	12.8091

dimensionless frequency and buckling load of S-S FG nanobeam, respectively at gradient index p=0.2, nonlocal parameter μ=2 and foundation parameters K_w=25, K_p=5.

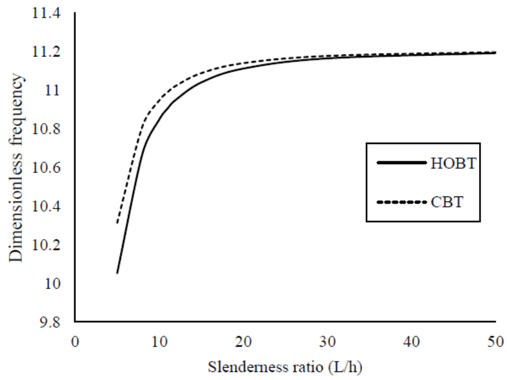

Fig. 2 Variation of dimensionless frequency of FG nanobeam with respect to slenderness ratio for both classical and higher order beam theories (K_w=25, K_p=5, p=0.2, μ=2)

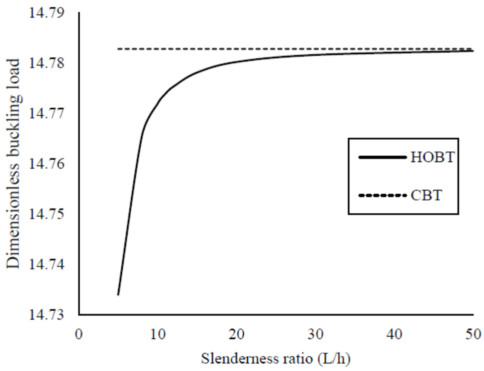

Fig. 3 Variation of dimensionless buckling load of FG nanobeam with respect to slenderness ratio for both classical and higher order beam theories (K_w=25, K_p=5, p=0.2, μ=2)

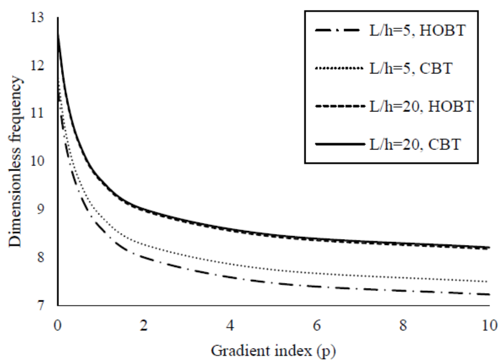

Fig. 4 Variation of dimensionless frequency of FG nanobeam with respect to gradient index for both classical and higher order beam theories (K_w=25, K_p=5, μ=1)

Due to the fact the classical beam theory (CBT) disregards the effects of shear deformation, it overestimates the buckling and frequency results. Also, it must be mentioned that the differences between the classical and higher order beam theories is less considerable for larger values of slenderness ratio. Moreover, it is seen that buckling responses of FG nanobeams based on CBT is independent of slenderness ratio.

The effect of slenderness ratio on the variation of the

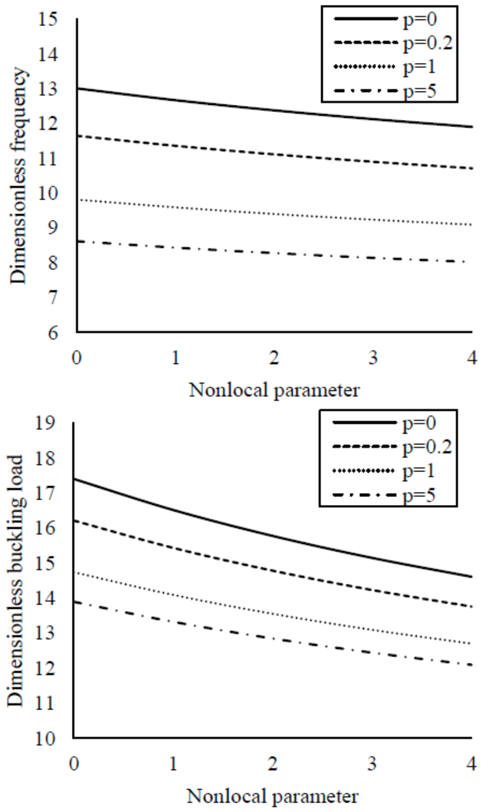

Fig. 5 Variation of dimensionless frequency and buckling load of FG nanobeam with respect to nonlocal parameter (L/h=20, K_w=25, K_p=5)

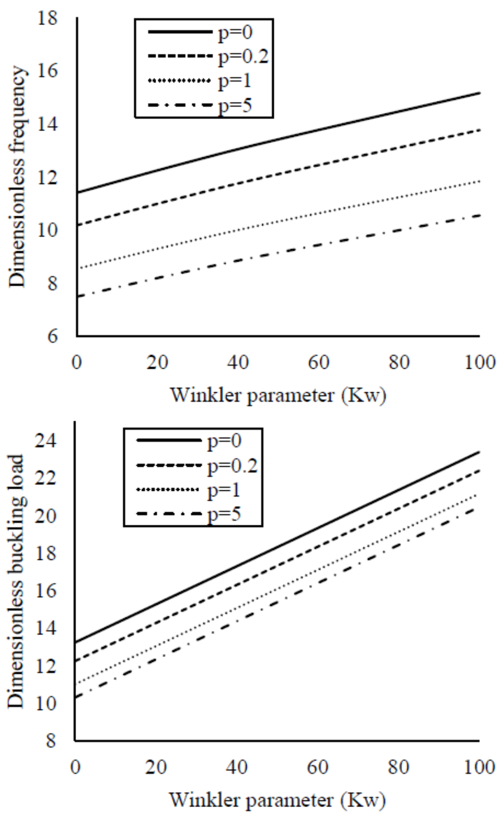

Fig. 6 Variation of dimensionless frequency and buckling load of FG nanobeam with respect to Winkler parameter (L/h=20, K_p=5, K_p=5, μ=2)

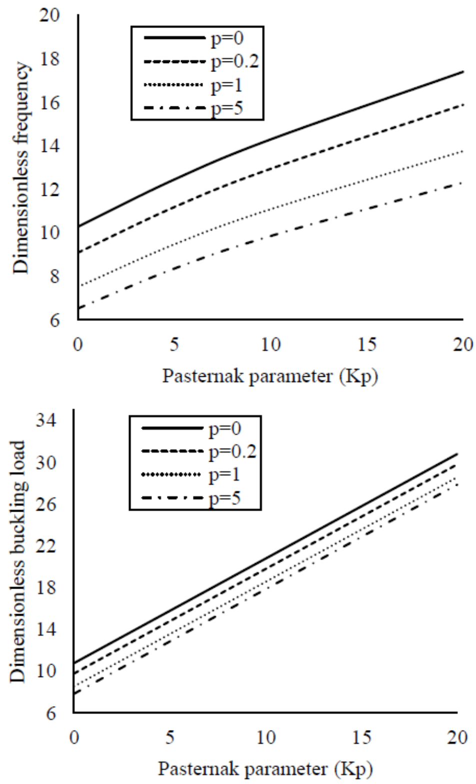

Fig. 7 Variation of dimensionless frequency and buckling load of FG nanobeam with respect to Pasternak parameter (L/h=20, K_w=25, μ=2)

first dimensionless frequency of S-S FG nanobeam resting on elastic foundation for both classical and higher order beam theory at K_w=25, K_p=5 and μ=1 is plotted in Fig. 4. It is observable that the dimensionless frequency reduces with high rate for lower values of gradient index than that of larger gradient index. Also, it is seen that increasing slenderness ratio results in rise of natural frequency. In addition, the difference in predicted frequencies based on CBT and HOBTs are significant only for smaller values of slenderness ratio. So, for larger slenderness ratios or thin beams the shear deformation effect is less prominent.

The softening impact of nonlocal parameter on the dimensionless frequency and buckling load of S-S FG nanobeams for various gradient index at L/h=20 and K_w=25, K_p=5 is shown in Fig. 5. Therefore, as the nonlocal parameter growths, the dimensionless frequency and buckling load reduce for all gradient indexes. So, as a consequence nonlocal parameter has a significant influence on the beam structure as well as mechanical responses of size-dependent nanobeams.

The variation of the dimensionless frequency and buckling load of S-S FG nanobeam with respect to Winkler and Pasternak parameter for different gradient indexes is presented in Figs. 6 and 7, respectively at L/h=20. In this figure, it is seen that with increase of the Winkler and Pasternak parameter both dimensionless frequency and buckling load increase for all values of gradient index. According to these figures it is found that the influence of the Pasternak parameter (K_p) on the non-dimensional buckling load is more significant than that of the Winkler parameter (K_w). So, it is very important to consider the shear layer of an elastic foundation in the analysis of FG nanostructures.

5. Conclusions

In the present work, free vibration and buckling analysis of size-dependent FG nanobeams embedded in two-parameter elastic foundation is performed based on various higher order shear deformation beam theories in conjunction with Navier analytical method. To define material properties of FG nanobeam Mori-Tanaka model is considered. The nonlocal governing differential equations in elastic medium are derived by implementing Hamilton's principle and using nonlocal constitutive equations of Eringen. The effects of shear deformation, small scale parameter, gradient index, foundation parameters and slenderness ratio on mechanical behavior of FG nanobeams are investigated. As a general consequence, all of the higher order theories present accurate and same results for the vibrational and stability analysis of FG nanobeams. It is observed that shear deformation influence on the responses of FG nanobeams is more significant for lower values of slenderness ratio. Also, it is indicated that with an increase of Winkler or Pasternak parameter, the beam becomes more rigid and the dimensionless frequency and buckling load of FG nanobeams increase. Moreover, it is found that nonlocality and gradient index has a notable decreasing effect on the natural frequency and buckling load of FG nanobeams.

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