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Geometrically nonlinear analysis of a laminated composite beam

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Abstract. The objective of this work is to analyze geometrically nonlinear static analysis a simply supported laminated composite beam subjected to a non-follower transversal point load at the midpoint of the beam. In the nonlinear model of the laminated beam, total Lagrangian finite element model of is used in conjunction with the Timoshenko beam theory. The considered non-linear problem is solved considering full geometric non-linearity by using incremental displacement-based finite element method in conjunction with Newton-Raphson iteration method. There is no restriction on the magnitudes of deflections and rotations in contradistinction to von-Karman strain displacement relations of the beam. In the numerical results, the effects of the fiber orientation angles and the stacking sequence of laminates on the nonlinear deflections and stresses of the composite laminated beam are examined and discussed. Convergence study is performed. Also, the difference between the geometrically linear and nonlinear analysis of laminated beam is investigated in detail.

Keywords: nonlinear analysis; composite laminated beams; Timoshenko beam theory; total lagragian; finite element method

1. Introduction

Laminated composite structures have been used many engineering applications, such as aircrafts, space vehicles, automotive industries, defence industries and civil engineering applications because these structures have higher strength-weight ratios, more lightweight and ductile properties than classical materials. With the great advances in technology, the using of the laminated composite structures is growing in applications.

In the literature, much more attention has been given to the linear analysis of laminated composite beam structures. However, nonlinear studies of Laminated composite beams are has not been investigated broadly. In the open literature, studies of the nonlinear behavior of laminated composite beams are as follows; Ghazavi and Gordaninejad (1989) studied geometrically nonliner static of laminated bimodular composite beams by using mixed finite element model. Singh *et al.* (1992) investigated nonlinear static responses of laminated composite beam based on higher shear deformation theory and von Karman's nonlinear type. Pai and Nayfeh (1992) presented three-dimensional nonlinear dynamics of anisotropic composite beams with von Karman nonlinear type. Di Sciuva and Icardi (1995) investigated large deflection of anisotropic laminated composite beams with Timoshenko beam theory and von Karman nonlinear strain-displacement relations by using Euler method. Donthireddy and Chandrashekhara (1997) investigated thermoelastic nonlinear static and dynamic analysis of laminated beams by using finite element method. Fraternali and Bilotti (1997) analyzed nonlinear stress of laminated composite curved beams. Ganapathi *et al.* (2009) studied nonlinear vibration analysis of laminated composite curved beams. Patel (1999) examined nonlinear post-buckling and vibration of laminated composite orthotropic beams/columns resting on elastic foundation with Von-Karman's strain-displacement relations. Oliveira and Creus (2003) investigated flexure and buckling behaviors of thin-walled composite beams with nonlinear viscoelastic model. Valido and Cardoso (2003) developed a finite element model for optimal desing of laminated composite thin-walled beams with geometrically nonlinear effects. Machado (2007) studied nonlinear buckling and vibration of thin-walled composite beams. Cardoso *et al.* (2009) investigated geometrically nonlinear behavior of the laminated composite thin-walled beam structures with finite element solution. Emam and Nayfeh (2009) investigated post-buckling of the laminated composite beams with different boundary conditions. Malekzadeh and Vosoughi (2009) studied large amplitude free vibration of laminated composite beams resting on elastic foundation by using differential quadrature method. Akgöz and Civalek (2011) and Civalek (2013) examined nonlinear vibration laminated plates resting on nonlinear-elastic foundation. Youzera *et al.* (2012) presented nonlinear dynamics of laminated composite beams with damping effect. Patel (2014) examined nonlinear static of laminated composite plates with the Green-Lagrange nonlinearity. Akbaş (2013b, 2014a, 2015a, b, c) investigated geometrically nonlinear of cracked and functionally graded beams. Stoykov and Margenov (2014) studied Nonlinear vibrations of 3D laminated composite Timoshenko beams. Cunedioğlu and Beylergil (2014) examined vibration of laminated composite beams under thermal loading. Mahi and Tounsi (2015) studied static and vibration of functionally graded,

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sandwich and laminated composite plates by using hyperbolic shear deformation theory. Draiche *et al.* (2016) investigated flexure analysis of laminated composite plate by using a refined theory with stretching effect. Chikh *et al.* (2017) investigated buckling of laminated plates under thermal loading with higher shear deformation theory. Li and Qiao (2015), Shen *et al.* (2016, 2017), Li and Yang (2016) investigated nonlinear postbuckling analysis of composite laminated beams. Kurtaran (2015), Mororó *et al.* (2015), Pagani and Carrera (2017) analyzed large deflections of laminated composite beams. Benselama *et al.* (2015), Liu and Shu (2015), Topal (2017) investigated buckling behavior of composite laminate beams. Latifi *et al.* (2016), Ebrahimi and Hosseini (2017) presented nonlinear dynamics of laminated composite structures. Also, there are many nonlinear, vibration, buckling studies of other type composite structures such as functionally graded materials, sandwich, nano composites etc. in the literature (Akbaş and Kocatürk 2012, Hebali *et al.* 2014, Zidi *et al.* 2014, Belabed *et al.* 2014, Meziane *et al.* 2014, Al-Basyouni *et al.* 2015, Yahia *et al.* 2015, Bourada *et al.* 2015, Akbaş 2013a, 2014b, 2015d, 2015e, 2017a, 2017b, 2017c, 2017d, 2017e, 2017b, 2018), Bouderba *et al.* (2013), 2016), Boukhari *et al.* (2016), Bellifa *et al.* (2016), Bennoun *et al.* (2016), Bounouara *et al.* (2016), Kocatürk and Akbaş (2010, 2011, 2012, 2013), Bousahla *et al.* (2014, 2016), Beldjelili *et al.* (2016), Kocatürk *et al.* (2011), Bellifa *et al.* (2017), El-Haina *et al.* (2017), Menasria *et al.* (2017), Bouafia *et al.* (2017), Abdelaziz *et al.* (2017), Youcef *et al.* (2018).

In the most of the nonlinear studies of laminated composite beams, the von-Karman strain displacement approximation is used. In the von-Karman strain, full geometric non-linearity cannot be considered because of neglect of some components of strain, satisfactory results can be obtained only for large displacements but moderate rotations. In the open literature, nonlinear studies of laminated composite beams with considering full geometric nonlinearity has not been investigated broadly.

In the present study, the geometrically nonlinear static analysis of a laminated Timoshenko beams is considered by using total Lagrangian finite element method in which full geometric nonlinearity can be considered as distinct from the studies by using von-Karman nonlinearity. The main purpose of this paper is to fill this gap for laminated composite beams. The distinctive feature of this study is geometrically nonlinear study of composite laminated beams with full geometric non-linearity. The effects of the fiber orientation angles and the stacking sequence of laminates on the nonlinear deflections and stresses of the composite laminated beam are examined and discussed. Convergence study is performed. Also, the difference between the geometrically linear and nonlinear analysis of the laminated beam is investigated in detail. The shortcomings of this study, the material nonlinearity and elasto-plastic behavior are not considered. It would be interesting to demonstrate the ability of the procedure through a wider campaign of investigations concerning elasto-plastic or material nonlinear analysis of laminated composite beams with geometrically nonlinearity.

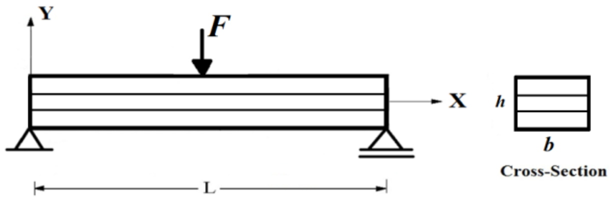


Fig. 1 A simply supported laminated beam subjected to a non-follower point load (F) at the midpoint of the beam and cross-section

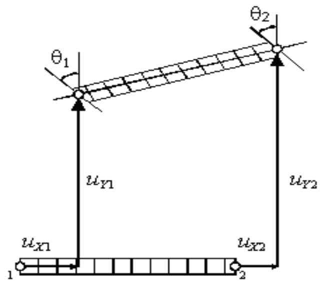


Fig. 2 Two-node C^0 beam element

2. Theory and formulation

A simply supported laminated composite beam with three layers of length L, width b and height h, as shown in Fig. 1. The beam is subjected to a non-follower transversal point load (F) at the midpoint of the beam as seen from Fig. 1. It is assumed that the layers are located as symmetry according to mid-plane axis. The height of each layer is equal to each other.

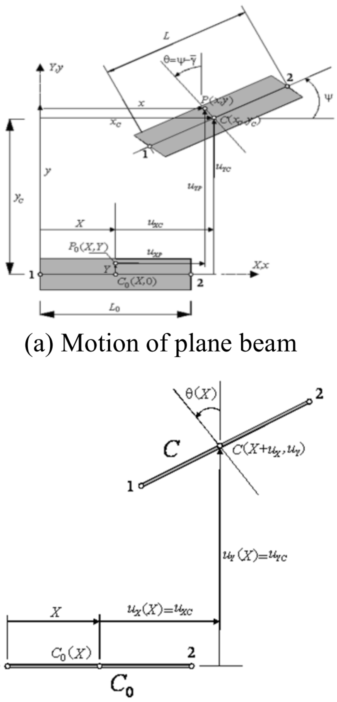
In the nonlinear kinematic model of the beam, the total Lagrangian approach is used within Timoshenko beam theory. The Lagrangian formulations of the problem are developed for laminated composite beam by using the formulations given by Felippa (2017) for isotropic and homogeneous beam material. The finite beam element of the problem is derived by using a two-node beam element shown in Fig. 2, of which each node has three degrees of freedom, i.e., two displacements u_{xi} and u_{yi} and one rotation θ_i about the Z axis.

In the deformation process, a generic point of the beam located at $P_0(X, Y)$ in the previous configuration C_0 moves to $P(x, y)$ in the current configuration C, as shown in Fig. 3. The projections of P_0 and P along the cross sections at C_0 and C upon the neutral axis are called $C_0(X, 0)$ and $C(x_c, y_c)$, respectively. It is assumed that the cross section of the beam remains unchanged, such that the shear distortion g<<1 and cosg can be replaced by 1 Felippa (2017).

The coordinates of the beam at the current C configuration are

$$\begin{aligned} x &= x_c - Y(sin\psi + sin\gamma\ cos\psi) \\ &= x_c - Y[\sin(\psi+\gamma) + (1-cos\psi)sin\psi] = x_c - Ysin\theta \end{aligned} \tag{1}$$

$$\begin{aligned} y &= y_c + Y(cos\psi - sin\gamma\ sin\psi) \\ &= y_c + Y[\cos(\psi+\gamma) + (1-cos\gamma)cos\psi] = y_c + Ycos\theta \end{aligned} \tag{2}$$



(a) Motion of plane beam

(b) Reduction to one-dimensional element

Fig. 3 Lagrangian kinematics of the C^0 beam element with X-aligned reference configuration Felippa (2017)

where, $x_c = X + u_{XC}$ and $y_c = u_{XC}$. Consequently, $x = X + u_{XC} - Y\sin\theta$ and $y = u_{YC} + Y\cos\theta$. From now on, we shall call u_{XC} and u_{YC} simply u_X and u_Y, respectively. Thus, the Lagrangian representation of the coordinates of the generic point at C is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} X + u_X - Y sin\theta \\ u_Y + Y cos\theta \end{bmatrix} \tag{3}$$

in which u_X, u_Y and θ are functions of X only. This concludes the reduction to a one-dimensional model, as sketched in Fig. 3(b). For a two-node C_0 element, it is natural to express the displacements and rotation as linear functions of the node degrees

$$\mathbf{w} = \begin{bmatrix} u_X(X) \\ u_Y(X) \\ \theta(X) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1-\xi & 0 & 0 & 1+\xi & 0 & 0 \\ 0 & 1-\xi & 0 & 0 & 1+\xi & 0 \\ 0 & 0 & 1-\xi & 0 & 0 & 1+\xi \end{bmatrix} \begin{bmatrix} u_{X1} \\ u_{Y1} \\ \theta_1 \\ u_{X2} \\ u_{Y2} \\ \theta_2 \end{bmatrix} = \mathbf{Nu} \tag{4}$$

in which $\xi = (2X/L_0) - 1$ is the isoparametric coordinate that varies from $\xi = -1$ at node 1 to $\xi = 1$ at node 2.

The Green-Lagrange strains are given as follows Felippa (2017)

$$[\boldsymbol{e}] = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} e_{XX} \\ 2e_{XY} \end{bmatrix} = \begin{bmatrix} (1+u'_X)cos\theta + u'_Y\, sin\theta - Y\theta' - 1 \\ 2e_{XY} \end{bmatrix} = \begin{bmatrix} e - Y\kappa \\ \gamma \end{bmatrix} \tag{5}$$

$$e=(1+u'_X)\cos\theta+u'_Y\sin\theta-1 \tag{6a}$$

$$\gamma=-(1+u'_X)\sin\theta+u'_Y\sin\theta-1,\ \kappa=\theta' \tag{6b}$$

where e is the axial strain, γ is the shear strain, and κ is curvature of the beam, $u'_X = \mathrm{d}u_X/\mathrm{d}X$, $u'_Y = \mathrm{d}u_Y/\mathrm{d}X$, $\theta' = \mathrm{d}\theta/\mathrm{d}X$. The equivalent Young's modulus of kth layer in the x direction (E_x^k) is used the following formulation (Vinson and Sierakowski 2002)

$$\frac{1}{E_x^k} = \frac{cos^4(\theta_k)}{E_{11}} + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_{11}}\right) cos^2(\theta_k)\, sin^2(\theta_k) + \frac{sin^4(\theta_k)}{E_{22}} \tag{7}$$

where, E_{11} and E_{22} indicate the Young's modulus in the longitudinal and transverse directions, respectively, G_{12} and ν_{12} are shear modulus and Poisson ratio, respectively. $m=cos\,\theta$ and $n= sin\,\theta$, θ indicates the fiber orientation angle. By assuming that the material of the laminated composite beam obeys Hooke's law, the axial force N, shear force V and bending moment M are given as follows

$$N = A_{11}\, e + B_{11}\, \kappa \tag{8a}$$

$$V = A_{55}\, \gamma \tag{8b}$$

$$M = B_{11}\, e + D_{11}\, \kappa \tag{8c}$$

where A_{11}, B_{11}, D_{11} and A_{55} are the extensional, coupling, bending, and transverse shear rigidities respectively, and their expressions are defined as

$$A_{11} = \sum_{k=1}^{n} bE_x^k(z_{k+1} - z_k\,) \tag{9a}$$

$$B_{11} = \frac{1}{2}\sum_{k=1}^{n} bE_x^k(z_{k+1}^2 - z_k^2) \tag{9b}$$

$$D_{11} = \frac{1}{3}\sum_{k=1}^{n} bE_x^k(z_{k+1}^3 - z_k^3) \tag{9c}$$

Expression of the transverse shear rigidity A_{55} given as follows (Vinson and Sierakowski 2002)

$$A_{55} = \frac{5}{4}\sum_{k=1}^{n} bQ_{55}^k(z_{k+1} - z_k\ - \frac{4}{3h^2}(z_{k+1}^3 - z_k^3)) \tag{10}$$

where Q_{55}^k is given below

$$Q_{55}^k = G_{13}cos^2(\theta_k) + G_{23}sin^2(\theta_k) \tag{11}$$

For the solution of the geometrically nonlinear problem in the total Lagrangian coordinates, a small-step incremental approach based on Newton-Raphson iteration method is used. In the Newton-Raphson solution for the problem, the applied load is divided by a suitable number of increments according to its value. After completing an iteration process, the previous accumulated load is increased by a load increment.

The solution for the n+1 st load increment and ith iteration is performed using the following relation

$$d\mathbf{u}_n^i = (\mathbf{K}_T^i)^{-1}\mathbf{R}_{n+1}^i \tag{12}$$

where $\mathbf{K}_T^i$ is the tangent stiffness matrix of the system at the i th iteration, $d\mathbf{u}_n^i$ is the displacement increment vector at the i th iteration and n+1 st load increment, $(R_{n+1}^i)_S$ is the residual vector of the system at the i th iteration and n+1 st load increment. This iteration procedure is continued until the difference between two successive solution vectors is less than a preset tolerance in the Euclidean norm, given by

$$\sqrt{\frac{\left[(d\boldsymbol{u}_n^{i+1}-d\boldsymbol{u}_n^i)^T(d\boldsymbol{u}_n^{i+1}-d\boldsymbol{u}_n^i)\right]^2}{\left[(d\boldsymbol{u}_n^{i+1})^T(d\boldsymbol{u}_n^{i+1})\right]^2}} \le \xi_{tol} \tag{13}$$

A series of successive iterations at the n+1 st incremental step gives

$$\boldsymbol{u}_{n+1}^{i+1}\ \boldsymbol{u}_{n+1}^i + d\boldsymbol{u}_{n+1}^i = \boldsymbol{u}_n + \Delta\boldsymbol{u}_n^i \tag{14}$$

where

$$\Delta\boldsymbol{u}_n^i = \sum_{k=1}^i d\boldsymbol{u}_n^k \tag{15}$$

The residual vector $\mathbf{R}_{n+1}^i$ for the structural system is given as follows

$$\mathbf{R}_{n+1}^i = \mathbf{f} - \mathbf{p} \tag{16}$$

Where **f** is the vector of total external forces and **p** is the vector of total internal forces, as given in the appendix. The element tangent stiffness matrix for the total Lagrangian Timoshenko beam element as given (Felippa 2017) is

$$\mathbf{K}_T = \mathbf{K}_M + \mathbf{K}_G \tag{17}$$

where $\mathbf{K}_G$ is the geometric stiffness matrix, and $\mathbf{K}_M$ is the material stiffness matrix given as follows

$$\boldsymbol{K}_M = \int_{L_o} B_m^T S\, B_m\, dX \tag{18}$$

The explicit expressions of the terms in Eq. (17) are given in the appendix. After integration of Eq. (18), the matrix $\mathbf{K}_M$ can be expressed as follows

$$\mathbf{K}_M = \mathbf{K}_M^a + \mathbf{K}_M^c + \mathbf{K}_M^b + \mathbf{K}_M^s \tag{19}$$

where $\mathbf{K}_M^a$ is the axial stiffness matrix, $\mathbf{K}_M^c$ the coupling stiffness matrix, $\mathbf{K}_M^b$ the bending stiffness matrix, and $\mathbf{K}_M^s$ the shearing stiffness matrix, of which the explicit expressions are given in the Appendix.

3. Numerical results

In the numerical examples, geometrically nonlinear deflections, namely large deflections and stresses of the simply supported laminated beam are calculated and presented for different fiber orientation angles and the stacking sequence of laminates under non-follower transversal point load (F) at the midpoint of the beam (Fig. 1). Also, geometrically linear and nonlinear results are presented and discussed for laminated composite beams. Using the conventional assembly procedure for the finite elements, the tangent stiffness matrix and the residual vector are obtained from the element stiffness matrices and residual vectors in the total Lagrangian sense for finite element model of the laminated Timoshenko beams. After that, the solution process outlined in the preceding section is used to obtain the solution for the problem of concern. In obtaining the numerical results, graphs and solution of the nonlinear finite element model, MATLAB program is used. Numerical calculations of the integrals seen in the rigidity matrices will be performed by using five-point Gauss rule.

In the numerical examples, the material properties of the layers are used in Loja *et al.* (2001): E_1 = 129.207 *GPa*, E_2 = E_3= 9.42512 *GPa*, G_{12} = 5.15658 *GPa*, G_{13}=4.3053 *GPa*, G_{23}=2.5414 *GPa*, ν_{12}=ν_{13}=0.3, ν_{23}= 0.218837. The geometry properties of the beam are considered as follows: b=0.3 m, h=0.3 m and L=3 m. It is mentioned before that the thickness of layers is equal to each other.

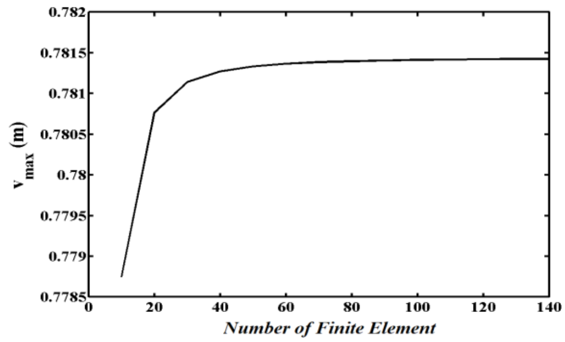
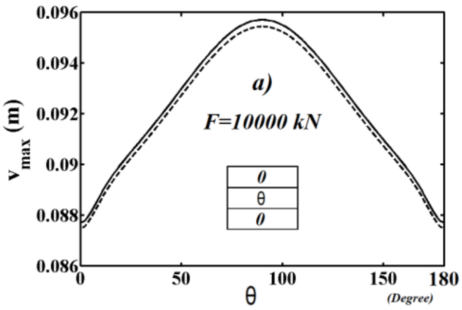


Fig. 4 Convergence study for nonlinear vertical displacements at the midpoint of the beam

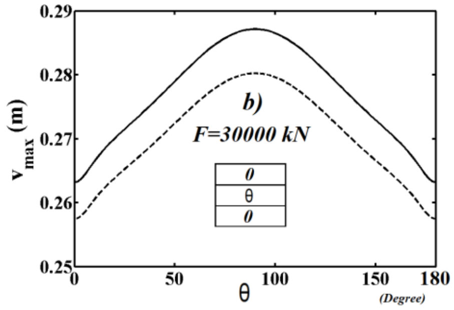
In order to obtain the optimum number of the finite element for the numerical calculations, the convergence study is performed in Fig. 4. In Fig. 4, nonlinear maximum vertical displacements (at the midpoint of the beam) of the laminated composite beam are calculated for different numbers of finite elements for the point load F=100000 kN and the stacking sequence of laminates [0,90,0]. It is seen from Fig. 4 that the nonlinear maximum displacements converge perfectly after the finite element n=100. So, the number of finite elements is taken as 100 in the numerical calculations.

In Figs. 5 and 6, the effects of the fiber orientation angles on the maximum vertical displacements (at the midpoint of the beam) are displayed for the stacking sequences [0/θ/0] and [θ/0/θ], respectively, for different values of transversal point load (F) in both linear and nonlinear analysis.

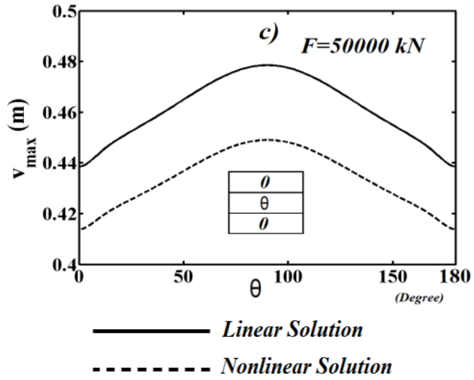
It is observed from Figs. 5 and 6 that increasing the fiber orientation angles to 0° from 90°, the deflections increase significantly. At the fiber orientation angle θ=90°, the



(a) for F=10000 kN

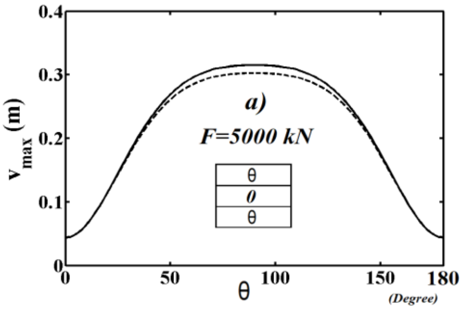


(b) for F=30000 kN

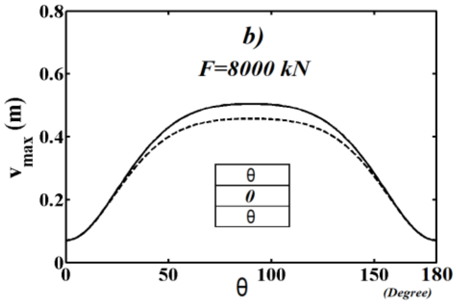


(c) for F=50000 kN

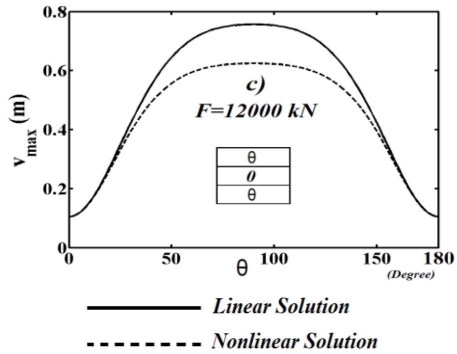
Fig. 5 The relationship between fiber orientation angles and maximum deflections for linear and nonlinear solution for the stacking sequence [0/θ/0]



(a) for F=5000 kN



(b) for F=8000 kN



(c) for F=12000 kN

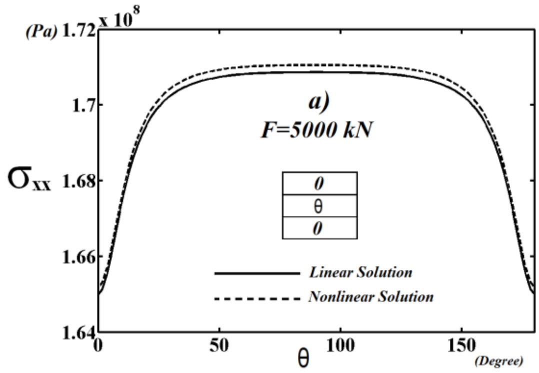
Fig. 6 The relationship between fiber orientation angles and maximum deflections for linear and nonlinear solution for the stacking sequence [θ/0/θ]

deflections are the greatest value for each layer arrangements. The equivalent Young's modulus and bending rigidity increase according to the Eq. (7). As a result, the strength of the beam increases. Another result of Figs. 5 and 6 that the increase in load causes increase in difference between the displacement values of the linear and the nonlinear solutions. Increase in load is more effective in the vertical displacements and rotations of the linear solution. Also, the difference between laminated beam in the linear case is bigger than in the nonlinear's. In the stacking sequences [θ/0/θ], the difference between the displacement values of the linear and the nonlinear solutions increase considerably with increasing the fiber orientation angles to 0° from 90° as seen from Fig. 6. However, this difference does not change in the stacking sequences [0/θ/0] with increasing θ as seen from Fig. 5. In the higher value of fiber orientation angles and load, the difference between linear and nonlinear solutions is quite big. It shows that the stacking sequences and fiber orientation angles play very important role on the nonlinear mechanical behaviour of the laminated beams.

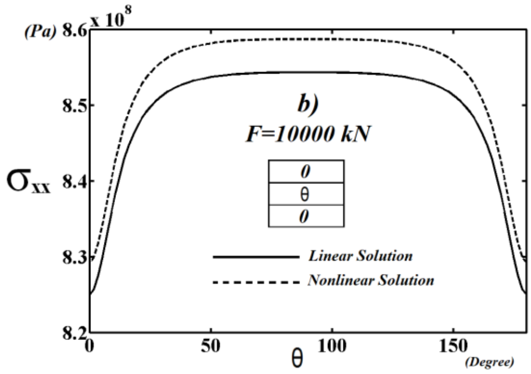
Figs. 7 and 8 show that effect of the fiber orientation angles on the Cauchy normal stresses (σ_{xx}) at the midpoint of the beam (X=L/2 and Y=0.5h) for the stacking sequences [0/θ/0] and [θ/0/θ], respectively for different values of transversal point load (F) in both linear and nonlinear analysis. It is seen from Figs. 7 and 8 that the normal stresses in the stacking sequence [0/θ/0] increase with increasing the fiber orientation angles.

Whereas, the normal stresses in the stacking sequence [θ/0/θ] decrease with increasing the fiber orientation angles. Also, difference between the stresses values of the linear and the nonlinear solutions increase significantly with increasing the load. It shows that nonlinear theory must be considered in the higher load values and large displacements problems. Otherwise, linear theory fails to satisfy large displacement problems.

In Figs. 9 and 10 display Cauchy normal stresses distributions along the height at the midpoint of the

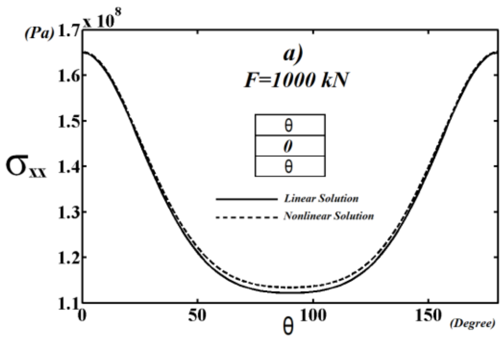


(a) for F=10000 kN

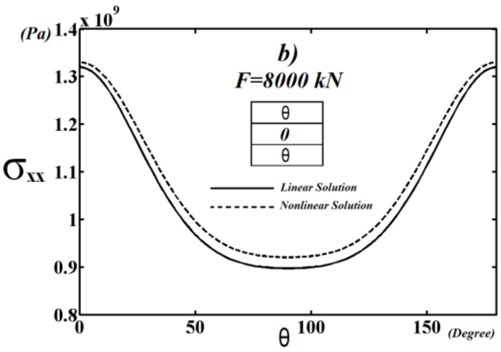


(b) for F=30000 kN

Fig. 7 The relationship between fiber orientation angles and Cauchy normal stresses (σ_{xx}) at the midpoint of the beam (X=L/2 and Y=0.5h) for linear and nonlinear solution for the stacking sequence [0/θ/0]

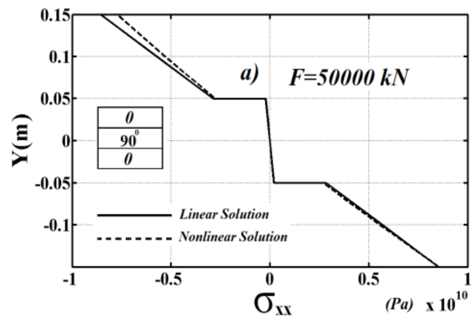


(a) for F=10000 kN

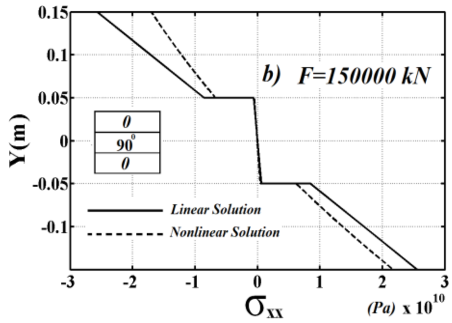


(b) for F=30000 kN

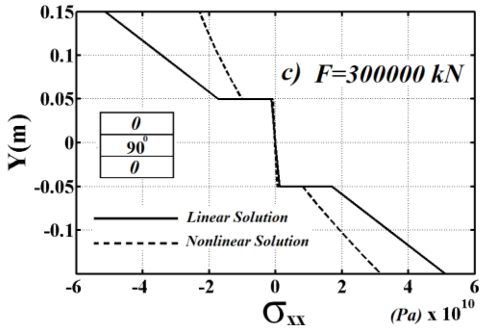
Fig. 8 The relationship between fiber orientation angles and Cauchy normal stresses (σ_{xx}) at the midpoint of the beam (X=L/2 and Y=0.5h) for linear and nonlinear solution for the stacking sequence [θ/0/θ]



(a) for F=50000 kN



(b) for F=150000 kN



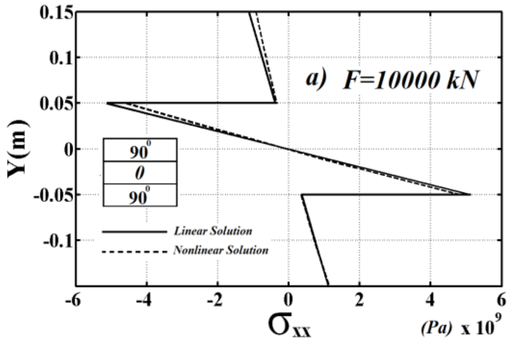
(c) for F=300000 kN

Fig. 9 Normal stress distributions along the height at the midpoint of the beam for linear and nonlinear solution for the stacking sequence [0/90/0]

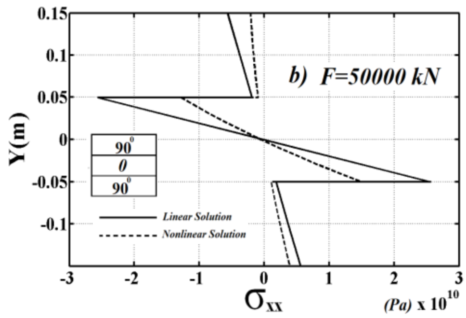
laminated beam for the stacking sequences [0/90/0] and [90/0/90], respectively for different values of transversal point load (F) in both linear and nonlinear cases. It is seen from Figs. 9 and 10 that with change in the stacking sequence of laminas, Cauchy normal stresses change seriously. The stress values in the linear case are bigger than the nonlinear case's. In higher load values, the difference between stress distribution in the linear and the nonlinear cases increases considerably. In order to obtain more realistic results and real stress values for laminated composite structures, the nonlinear effects must be considered, especially for higher load values and large deflection problems. As seen from the stress distribution graphs, the stacking sequence of laminates is very effective in the stress distribution.

4. Conclusions

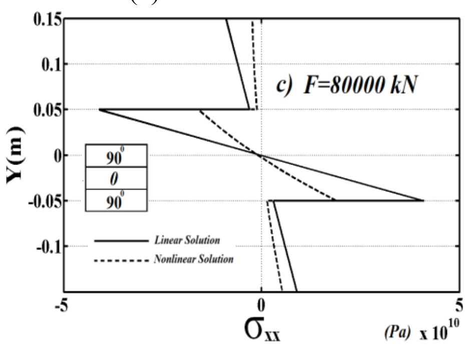
Geometrically nonlinear static analysis of a simply



(a) for F=10000 kN



(b) for F=50000 kN



(c) for F=80000 kN

Fig. 10 Normal stress distributions along the height at the midpoint of the beam for linear and nonlinear solution for the stacking sequence [90/0/90]

supported laminated composite beam is investigated by using total Lagrangian finite element model with the Timoshenko beam theory.

The considered non-linear problem is solved by using incremental displacement-based finite element method in conjunction with Newton-Raphson iteration method. Convergence study is performed. The fibber orientation angles and the stacking sequence of laminates on nonlinear displacements of the laminated beam are studied and discussed in both linear and nonlinear cases.

It is observed from the investigations that the fiber orientation angles and the stacking sequences of laminates have a great influence on the geometrically non-linear static response of the laminated beams. There is a significant difference between the geometrically linear and non-linear analysis for the laminated beams in higher values of loads. Also, with change the fiber orientation angles and the stacking sequences of laminates, difference between the geometrically linear and non-linear results change seriously. In order to obtain more realistic results and real stress and displacement values for laminated composite structures, the nonlinear effects must be considered in large deflection problems and higher values of loads.

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Appendix

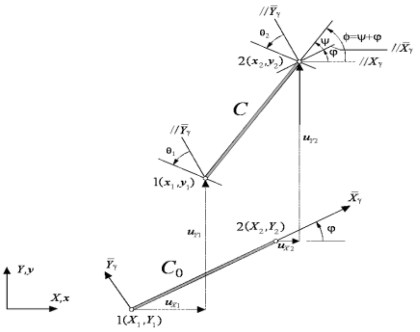

Fig. A1 Plane beam element with arbitrarily oriented reference configuration Felippa (2017)

In this Appendix, the entries of the following matrices are given: axial stiffness matrix $\mathbf{K}_M^a$, coupling stiffness matrix $\mathbf{K}_M^c$, bending stiffness matrix $\mathbf{K}_M^b$, and shearing stiffness matrix $\mathbf{K}_M^s$ are developed for laminated composite beam by using the formulations given by Felippa (2018) for isotropic and homogeneous beam material.

$$\boldsymbol{K}_M^a = \frac{\mathrm{A}_{11}}{L_0}\begin{bmatrix} c_m^2 & c_m s_m & -c_m\gamma_m L_0/2 & -c_m^2 & -c_m s_m & -c_m\gamma_m L_0/2 \\ c_m s_m & s_m^2 & -s_m\gamma_m L_0/2 & -c_m s_m & -s_m^2 & -s_m\gamma_m L_0/2 \\ -c_m\gamma_m L_0/2 & -s_m\gamma_m L_0/2 & \gamma_m^2 L_0^2/4 & c_m\gamma_m L_0/2 & -s_m\gamma_m L_0/2 & \gamma_m^2 L_0^2/4 \\ -c_m^2 & -c_m s_m & c_m\gamma_m L_0/2 & c_m^2 & c_m s_m & c_m\gamma_m L_0/2 \\ -c_m s_m & -s_m^2 & s_m\gamma_m L_0/2 & c_m s_m & s_m^2 & s_m\gamma_m L_0/2 \\ c_m\gamma_m L_0/2 & -s_m\gamma_m L_0/2 & \gamma_m^2 L_0^2/4 & c_m\gamma_m L_0/2 & s_m\gamma_m L_0/2 & \gamma_m^2 L_0^2/4 \end{bmatrix} \quad (A1)$$

$$\boldsymbol{K}_M^c = \frac{\mathrm{B}_{11}}{L_0}\begin{bmatrix} 0 & 0 & -c_m & 0 & 0 & c_m \\ 0 & 0 & -s_m & 0 & 0 & s_m \\ -c_m & -s_m & \gamma_m L_0 & c_m & s_m & 0 \\ 0 & 0 & c_m & 0 & 0 & -c_m \\ 0 & 0 & s_m & 0 & 0 & -s_m \\ c_m & s_m & 0 & -c_m & -s_m & -\gamma_m L_0 \end{bmatrix} \quad (A2)$$

$$\boldsymbol{K}_M^b = \frac{\mathrm{D}_{11}}{L_0}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \quad (A3)$$

$$\boldsymbol{K}_M^a = \frac{\mathrm{A}_{55}}{L_0}\begin{bmatrix} s_m^2 & -c_m s_m & -s_m\alpha_1 L_0/2 & -s_m^2 & c_m s_m & -s_m\alpha_1 L_0/2 \\ -c_m s_m & c_m^2 & c_m\alpha_1 L_0/2 & c_m s_m & -c_m^2 & c_m\alpha_1 L_0/2 \\ -s_m\alpha_1 L_0/2 & c_m\alpha_1 L_0/2 & \alpha_1^2 L_0^2/4 & s_m\alpha_1 L_0/2 & -c_m\alpha_m L_0/2 & \alpha_1^2 L_0^2/4 \\ -s_m^2 & c_m s_m & s_m\alpha_1 L_0/2 & s_m^2 & -c_m s_m & s_m\alpha_1 L_0/2 \\ c_m s_m & -c_m^2 & -c_m\alpha_1 L_0/2 & -c_m s_m & c_m^2 & -c_m\alpha_1 L_0/2 \\ -s_m\alpha_1 L_0/2 & c_m\alpha_1 L_0/2 & \alpha_1^2 L_0^2/4 & s_m\alpha_1 L_0/2 & -c_m\alpha_1 L_0/2 & \alpha_1^2 L_0^2/4 \end{bmatrix} \quad (A4)$$

where m denotes the midpoint of the beam, $x = 0$, and

$q_m = (q_1 + q_2)/2$, $w_m = q_m + j$, $c_m = \cos w_m$, $s_m = \sin w_m$,

$e_m = L\cos(q_m - y)/L_0 - 1$, $a_1 = 1 + e_m$ and $g_m = L\sin(y - q_m)/L_0$

(See Fig. A1 for symbols). The initial axis of the beam considered is taken as horizontal, therefore j=0. The matrix $\mathbf{S}$ is defined as follows

$$\boldsymbol{S} = \begin{bmatrix} \mathrm{A}_{11} & 0 & -\mathrm{B}_{11} \\ 0 & \mathrm{A}_{55} & 0 \\ -\mathrm{B}_{11} & 0 & \mathrm{D}_{11} \end{bmatrix} \quad (A5)$$

The matrix $\mathbf{B}_m$ is given as follows

$$\boldsymbol{B}_m = \boldsymbol{B}_m|_{\xi=0} = \frac{1}{L_0}\begin{bmatrix} -c_m & -s_m & -\frac{1}{2}L_0\gamma_m & c_m & s_m & -\frac{1}{2}L_0\gamma_m \\ s_m & -c_m & \frac{1}{2}L_0(1+e_m) & s_m & -c_m & \frac{1}{2}L_0(1+e_m) \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \quad (A6)$$

The geometric stiffness matrix K_G is given as follows

$$\boldsymbol{K}_G = \frac{\mathrm{N_m}}{2}\begin{bmatrix} 0 & 0 & s_m & 0 & 0 & s_m \\ 0 & 0 & -c_m & 0 & 0 & -c_m \\ s_m & -c_m & -\frac{1}{2}L_0(1+e_m) & -s_m & c_m & -\frac{1}{2}L_0(1+e_m) \\ 0 & 0 & -s_m & 0 & 0 & -s_m \\ 0 & 0 & c_m & 0 & 0 & c_m \\ s_m & -c_m & -\frac{1}{2}L_0(1+e_m) & -s_m & c_m & -\frac{1}{2}L_0(1+e_m) \end{bmatrix} + \frac{\mathrm{N_m}}{2}\begin{bmatrix} 0 & 0 & c_m & 0 & 0 & c_m \\ 0 & 0 & s_m & 0 & 0 & s_m \\ c_m & s_m & -\frac{1}{2}L_0\gamma_m & -c_m & -s_m & -\frac{1}{2}L_0\gamma_m \\ 0 & 0 & -c_m & 0 & 0 & -c_m \\ 0 & 0 & -s_m & 0 & 0 & -s_m \\ c_m & s_m & -\frac{1}{2}L_0\gamma_m & -c_m & -s_m & -\frac{1}{2}L_0\gamma_m \end{bmatrix} \quad (A7)$$

in which N_m and V_m are the axial and shear forces evaluated at the midpoint. The internal nodal force vector is Felippa (2017)

$$\boldsymbol{p} = L_0\boldsymbol{B}_m^T\boldsymbol{z} = \begin{bmatrix} -c_m & -s_m & -\frac{1}{2}L_0\gamma_m & c_m & s_m & \frac{1}{2}L_0\gamma_m \\ s_m & -c_m & -\frac{1}{2}L_0(1+e_m) & s_m & -c_m & -\frac{1}{2}L_0(1+e_m) \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} N \\ V \\ M \end{bmatrix} \quad (A8)$$

where $z^T=[N\ V\ M]$. The external nodal force vector is

$$\boldsymbol{z}^T = \mathrm{b}\int_{\mathrm{h}}\int_{L_0}\begin{bmatrix} 1-\xi_1 & 0 & 0 \\ 0 & 1-\xi_1 & 0 \\ 0 & 0 & 1-\xi_1 \\ 1-\xi_2 & 0 & 0 \\ 0 & 1-\xi_2 & 0 \\ 0 & 0 & 1-\xi_2 \end{bmatrix}\begin{bmatrix} f_X \\ f_Y \\ 0 \end{bmatrix} dX\,dY + b\int_{L_0}\int_{\mathrm{h}}\int_{L_0}\begin{bmatrix} 1-\xi_1 & 0 & 0 \\ 0 & 1-\xi_1 & 0 \\ 0 & 0 & 1-\xi_1 \\ 1-\xi_2 & 0 & 0 \\ 0 & 1-\xi_2 & 0 \\ 0 & 0 & 1-\xi_2 \end{bmatrix}\begin{bmatrix} t_X \\ t_Y \\ mt_Y \end{bmatrix} dX \quad (A9)$$

where f_X, f_Y are the body forces, t_X, t_Y, m_Z are the surface loads in the X, Y directions and about the Z axis.