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Finite element vibration analysis of nanoshell based on new cylindrical shell element

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Abstract. In this paper, using modified couple stress theory in place of classical continuum theory, and using shell model in place of beam model, vibrational behavior of nanotubes is investigated via the finite element method. Accordingly classical continuum theory is unable to correctly compute stiffness and account for size effects in micro/nanostructures, higher order continuum theories such as modified couple stress theory have taken on great appeal. In the present work the mass-stiffness matrix for cylindrical shell element is developed, and by means of size-dependent finite element formulation is extended to more precisely account for nanotube vibration. In addition to modified couple stress cylindrical shell element, the classical cylindrical shell element can also be defined by setting length scale parameter to zero in the equations. The boundary condition were assumed simply supported at both ends and it is shown that the natural frequency of nano-scale shell using the modified coupled stress theory is larger than that using the classical shell theory and the results of Ansys. The results have indicated using the modified couple stress cylindrical shell element, the rigidity of the nano-shell is greater than that in the classical continuum theory, which results in increase in natural frequencies. Besides, in addition to reducing the number of elements required, the use of this type of element also increases convergence speed and accuracy.

Keywords: modified couple stress theory; FEM; cylindrical shell element; size dependent; Thin Shell Theory

1. Introduction

The development of the nanostructure in sciences such as communication, electronics, medicine, aerospace science, military science, robotics, chemistry, and optics has resulted in new achievements. In all these applications, according to experimental results, size effects play a significant role in proper study of the behavior of such structures. Therefore, since classical continuum theories, due to their lack of intrinsic length scales, are unable to correctly predict the behavior of these structures, use of higher order theories, such as nonlocal continuum theory, strain gradient theory and couple stress theory, which are able to account for size effects in computations, has become popular (Li *et al*. 2011a, b, c, Şimşek 2014, Zhang *et al*. 2014, Ghayesh *et al*. 2013, Sahmani and Ansari 2013, Li *et al*. 2015a, Fattahian and Tadi Beni 2017, Tadi Beni *et al*. 2015, Arash and Ansari 2010, Li 2014a, b, Pradhan and Phadikar 2009).

The classical couple stress theories was presented by Tiersten and Mindlin (1962), Toupin (1962), Koiter (1964) and Mindlin (1964). Using couple stress theory, Kang and Xi (2007) investigated the micro-beam resonant frequency and demonstrated the dependency of natural frequency on size effects. Also, using the couple stress theory, Zhou and Li (2001) investigated the mechanical and static behavior of a microbar subjected to torsional loading. Regarding the difficulties of determining length scale parameters, which classical couple stress theories consist of couple of them, Yang *et al*. (2002) proposed modified couple stress theories in which a new additional equilibrium equation and the equilibrium of the couples of moments, besides classical equilibrium equations of forces and their moments, exist and lead to one length scale parameter. This theory introduces a new equilibrium equation, i.e., the equilibrium of moments of couples, in addition to equations of equilibrium and momentum of forces. Many studies have employed modified couple stress theory. For instance, using Timoshenko beam model and modified couple stress theory, Ma *et al*. (2008) investigated the formulation of axial and transverse deformation of microstructures. In another study using modified couple stress theory, Reddy and Berry (2012) examined the bending of axisymmetric circular plates, demonstrating the possibility of use of the equations developed for extending the analytical response to free vibrations, bending, and buckling of linear cases. Afterwards, Mindlin (1964) introduced the general higher order stress theory by only considering the second order deformation gradient as supplementary deformation matrix, therefore the five linear elastic parameters are deduced from this additional part. Subsequently, Fleck and Hatchinson (1997) modified Mindlin's formulation and named it as strain gradient theory. In this new formulation, the stretch gradient tensor and rotation gradient tensor, are taken into account as the constitutive parts of second order

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deformation tensor. Afterwards, the modified strain gradient theory, which considered only the symmetric parts of strain gradient in the equations, was proposed by Lam *et al*. (2003). According to this theory, the five length scale parameters presented by Mindlin was reduced to three length scale parameters. These three parameters can be combined and reduced to only one single measurable parameter under the assumptions of modified couple stress theory.

In the recent years, the application of nonlocal continuum theory, the modified strain gradient theory and couple stress theory has attracted considerable attention in the investigation of the dynamic behavior of micro/nano electromechanical systems and nanotubes. Li *et al*. (2015a) demonstrated that the equivalent stiffness of a nanostructure predicted by the non-local theory may be larger or smaller than that by the classical theory, depending on the category of applied loads and showed that both the nano structural stiffness strengthening and reducing effects exist in nano mechanics and they are related to different surface properties, or the long range attractive and long range repulsive interactions on the surface of nanostructures cause the stiffness reducing and stiffness strengthening models, respectively. Ma *et al*. (2008) showed that the effects of Poisson's coefficient and size parameter on beam deflection and beam vibration are considerable. In another study, Tadi and Abadyan (2013) investigated the pull-in instability effect in a nano beam under torsion through strain gradient theory. Taking into consideration the Casmir forces, they compared their findings with couple stress theory. Considering the fact that the components of nano-devices could be modeled through the nano shell, it is essential to learn the correct model of the nano shell. Also, it is important to know that as dimensions are scaled down, many essential phenomena appear at the micro/nano-scale, which is not important at macro-scale.

The longitudinal dynamic behaviors of some common one-dimensional nanostructures were examined using the hardening nonlocal approach by Li *et al*. (2015b). The effects of a dimensionless nonlocal small scale parameter at molecular level unavailable in classical rods/tubes were investigated. The correlations between natural frequencies and the nonlocal nano scale parameter were obtained. Within the framework of hardening nonlocal stress theory, it concludes for the first time that the longitudinal free vibration frequencies of nano rods/nanotubes are higher than those based on the classical continuum mechanics but they are quite different from the softening nonlocal model. The strengthening effects on nonlocal stiffness of nano rods/nanotubes are observed and a comparative calculation for dimensional natural frequencies with respect to length of CNTs by different methodologies was provided to explain why the softening and hardening nonlocal models are both correct in nonlocal elasticity theory.

In order to correctly predict the behavior of micro/nanostructures, in addition to considering the length scale parameter, it is also necessary to use an appropriate geometric model to correctly model structures and elements. In recent years, in addition to laboratory experiments, methods such as molecular dynamic simulations and classical continuum theory have been used to simulate and study nano-structures and examine size effects. Besides, given the fact that methods such as MD are costly and include lengthy calculations, in order to examine the mechanical behavior in nanostructure, researchers have used non-classical continuum theories such as the nonlocal theory, strain gradient theory, and couple stress theory, which have ability to model size effects (Lim *et al*. 2012, Zeighampour and Tadi 2015, Kong *et al*. 2008, Abadyan *et al*. 2011, Noghrehabadi *et al*. 2011, Mohammadi and Tadi 2014, Tadi *et al*. 2014, Yang *et al*. 2008, Akgöz and Civalek 2013, Berrabah *et al*. 2013, Ji and Chen 2009, Kocaturk and Akbas 2013, Li 2013, Wang *et al*. 2013, Ebrahimi and Tadi 2016).

Besides the significance of consideration of size effect in micro/nano scales and the shear model in nano shells, it should be noted that, for correctly predicting structure behavior, it is essential to use an appropriate geometrical model in order to model structures and elements used in them. In addition, considering the extensive application of nanotubes in nano scale structures, it is necessary to use an appropriate geometric model (Kheibari and Tadi 2017, Mehralian and Tadi 2016). Many researchers have so far used beam model in order to model nanotubes (Chong and Lam 1999, Mohammadimehr and Alimirzaei 2016, Tadi 2016, Şimşek and Reddy 2013, Taghizadeh *et al* 2015, Wu *et al*. 2005). However, their topological structure which is in the shape of cylindrical shell is indicative of the superiority of the use of the shell model compared to other models. Zeighampour and Tadi (2014) investigated the dynamic behavior of double-walled carbon nanotubes using shell model and modified couple stress theory, demonstrating the effects of parameters such as size and fluid velocity on the results obtained by classical theory and modified couple stress theory. Using the three dimensional theory of elasticity and shell model, Alibeigloo and Shaban (2013) studied single walled carbon nanotube (SWCNT) vibration behavior and incorporated size effects into their calculations. Also, using the nonlocal theory, they demonstrated the effects of parameters such as the nonlocal parameter, thickness to radius ratio, and length to radius ratio on the results.

Since, due to the complexity of micro/nanostructures such as complicated loading or geometry, the use of analytical method is not always possible, it is especially important to use other current methods such as FEM which is one of the most common methods for investigating micro/nanostructures and which analyzes and simulates complex structures through a simple process. Chyuan (2008) used the finite element method to investigate the levitation of MEMS comb-drive. Metz *et al*. (2006) investigated the bending behavior of conducting polymer electromechanical actuators using the numerical method. Tajalli *et al*. (2009) investigated the dynamic pull-in of electrostatically actuated micro/nano plate by using the nine-node plate element and by calculating the nonlinear geometry and pressure of fluid.

It should be noted that all these studies have been conducted on the basis of the classical continuum theory. However, as mentioned previously, not only does classical continuum theory underestimate the stiffness of micro/nano structures, but it also excludes size effects in predicting the

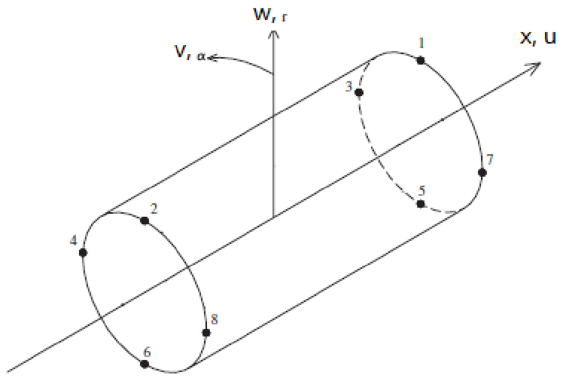

Fig. 1 8- node cylindrical shell element

behavior of micro/nanostructures. Hence, it is unable to correctly predict the behavior of micro/nanostructures. Since the study of nanoshells is conducted in nano dimensions, the mechanical properties of nanoshells could not be correctly predicted by the classical theory; therefore, taking into consideration the size effect, higher order continuum theories are used.

Considering the discussion made above, it could be argued that no study has to date been conducted on the cylindrical shell element formulation by using the modified couple stress theory; therefore in the present paper, using the finite element method and modified couple stress theory which are able to take size effects into account and correctly model micro/nanostructures, mass and stiffness matrix are developed using cylindrical shell element shape function, the application of this cylindrical shell element is outlined, and the results are compared with those obtained by classical continuum theory and Ansys. The correctness of the equations is demonstrated, and the results obtained by the modified couple stress theory in comparison with those obtained by the classical continuum theory are demonstrated. The results reveal using the modified couple stress cylindrical shell element, the rigidity of the nano-shell is greater than that in the classical continuum theory, which results in increase in natural frequencies and size-dependent finite element formulation with shell element is very appropriate for more precisely solving vibration of nanotubes using modified couple stress theory. It should be noted that above conclusion (stiffness strengthening effect on nanostructures) have been obtained with a lot of researchers in the last years (Liu *et al*. 2016, Li 2014, Li *et al*. 2011a, b, c) which show the results of this paper is correct, qualitatively.

2. Preliminaries

2.1 Element definition

A 8-node cylindrical shell element at length $\boldsymbol{L}$, radius $\boldsymbol{R}$, and thickness $\boldsymbol{h}$ is considered according to Fig. (1). Three independent coordinates are required to completely describe the position vector. The cylindrical coordinate system ($\boldsymbol{x}, \boldsymbol{\alpha}, \boldsymbol{r}$) is used as global coordinate system where $\boldsymbol{x}$, $\boldsymbol{r}$ and $\boldsymbol{\alpha}$ represent axial, radial and tangential axes (Fig. 1), and the element formulation is defined as follows

The following conditions must exist for the cylindrical shell element's shape function.

- The shape functions must be differentiable.
- One, and only one, shape function corresponding to each node must be 1 in that node and zero in other nodes.
- In accordance with the cylindrical shape of the shell element, all shape functions must by periodic in relation to coordinate α, with a period of 2π.

Based on the above conditions, the following shape functions for the cylindrical shell element are defined

$$
\begin{aligned}
N_1 &= \frac{1}{4}\left(\cos^2 \pi\gamma - \cos \pi\gamma\right)\left(1+\xi\right)\\
N_2 &= \frac{1}{4}\left(\cos^2 \pi\gamma - \cos \pi\gamma\right)\left(1-\xi\right)\\
N_3 &= \frac{1}{4}\left(\sin^2 \pi\gamma - \sin \pi\gamma\right)\left(1+\xi\right)\\
N_4 &= \frac{1}{4}\left(\sin^2 \pi\gamma - \sin \pi\gamma\right)\left(1-\xi\right)\\
N_5 &= \frac{1}{4}\left(\cos^2 \pi\gamma + \cos \pi\gamma\right)\left(1+\xi\right)\\
N_6 &= \frac{1}{4}\left(\cos^2 \pi\gamma + \cos \pi\gamma\right)\left(1-\xi\right)\\
N_7 &= \frac{1}{4}\left(\sin^2 \pi\gamma + \sin \pi\gamma\right)\left(1+\xi\right)\\
N_8 &= \frac{1}{4}\left(\sin^2 \pi\gamma + \sin \pi\gamma\right)\left(1-\xi\right)
\end{aligned}
\tag{1}
$$

and the local coordinate system (ξ,γ) is defined as

$$\xi = \frac{2x}{L}, \quad \gamma = \frac{\alpha}{\pi} - 1 \tag{2}$$

where

$$-\frac{L}{2} \le x \le \frac{L}{2}, \; 0 \le \alpha \le 2\pi \tag{3}$$

Therefore

$$-1 \le \xi, \gamma \le 1 \tag{4}$$

2.2 Strain-displacement relationships

Considering the shape functions, displacement of a point of the cylindrical shell element which can be represented by vector U with components u, v, and w along r, α and x is expressed as follows

$$\mathbf{U} = \{u \quad v \quad w\}^T \tag{5}$$

$$\mathbf{U}_{3\times1} = \mathbf{N}_{3\times24}\,\mathbf{q}_{24\times1} \tag{6}$$

where

$$\mathbf{q} = \{u_1 \quad v_1 \quad w_1 \cdots u_8 \quad v_8 \quad w_8\}^T \tag{7}$$

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}^u \\ \mathbf{N}^v \\ \mathbf{N}^w \end{bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & \cdots & N_8 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \cdots & 0 & N_8 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & \cdots & 0 & 0 & N_8 \end{bmatrix} \tag{8}$$

where N_i, as the shape functions, are defined according to Eq. (1).

Strain energy expression in the modified couple stress theory, contains a set of equilibrium equations similar to the classical equilibrium equations and a non-classical size parameter constant, too. Hence, the strain energy in area Λ (and element volume V), for the elastic and isotropic substance with infinitesimal deformation is obtained as Lam *et al*. (2003)

$$U=\frac{1}{2}\int_{\Omega}(\sigma_{ij}\varepsilon_{ij}+m_{ij}\chi_{ij})dV \tag{9}$$

Classical and non-classical components of the strain tensor for the cylindrical shell element are defined as

$$\varepsilon_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right) \tag{10}$$

$$\begin{aligned}\chi_{ij}&=\frac{1}{4}[e_{ipq}\eta_{jpq}+e_{jpq}\eta_{ipq}]\\ \Rightarrow \chi_{ij}&=\frac{1}{2}[\nabla\vec{\theta}+(\nabla\vec{\theta})^T]\end{aligned} \tag{11}$$

where u_i, e_{ipq} and η_{ipq} represent the components of displacement vector, permutation symbol, and deviator stretch gradient tensor, respectively. Also, σ_{ij} and m_{ij} respectively stand for the components of Cauchy tensor and higher order tensor components, which are defined using constitutive equations in modified couple stress theory in the elastic material as

$$\sigma_{ij}=C_{ijkl}\varepsilon_{kl} \tag{12}$$

$$m_{ij}=2l^2\mu\chi_{ij} \tag{13}$$

Finally μ and C_{ijkl} represent shear modulus, elastic constants parameters, respectively. Also, l is the additional independent length scale parameter associated with the symmetric rotation gradient.

To calculate equations of deviator stretch tensor and symmetric rotation gradients the definitions of classic and higher order strains in orthogonal coordinate system are used as follows (Eringen 1980, Zhao and Pedroso 2008)

$$\begin{aligned}\varepsilon_{(j)}^{(i)}=\varepsilon_j^i\sqrt{\frac{g_{\underline{ii}}}{g_{\underline{jj}}}}=\frac{1}{2}\sqrt{\frac{g_{\underline{ii}}}{g_{\underline{jj}}}}\Bigg[&\left(\left(\frac{u^{(i)}}{\sqrt{g_{\underline{ii}}}}\right)_{,j}+\Gamma_{mj}^{i}\frac{u^{(m)}}{\sqrt{g_{\underline{mm}}}}\right)\\ &+g_{nj}g^{im}\left(\left(\frac{u^{(n)}}{\sqrt{g_{\underline{nn}}}}\right)_{,m}+\Gamma_{qm}^{n}\frac{u^{(q)}}{\sqrt{g_{\underline{qq}}}}\right)\Bigg]\end{aligned} \tag{14}$$

$$\eta_{(i)(j)}^{(k)}=\eta_{ij}^k\sqrt{\frac{g_{\underline{kk}}}{g_{\underline{ii}}g_{\underline{jj}}}}=\frac{1}{2}\sqrt{\frac{g_{\underline{kk}}}{g_{\underline{ii}}g_{\underline{jj}}}}\left(u^k\big|_{ij}+u^k\big|_{ji}\right) \tag{15}$$

$$\begin{aligned}u^k\big|_{lm}=&\left(\frac{u^{(k)}}{\sqrt{g_{\underline{kk}}}}\right)_{,lm}+\Gamma_{ql}^{k}\left(\frac{u^{(q)}}{\sqrt{g_{\underline{qq}}}}\right)_{,m}\\ &+\Gamma_{qm}^{k}\left(\frac{u^{(q)}}{\sqrt{g_{\underline{qq}}}}\right)_{,l}-\Gamma_{ml}^{q}\left(\frac{u^{(k)}}{\sqrt{g_{\underline{kk}}}}\right)_{,q}\\ &+\left(\left(\Gamma_{lp}^{k}\right)_{,m}+\Gamma_{qm}^{k}\Gamma_{pl}^{q}-\Gamma_{pq}^{k}\Gamma_{ml}^{q}\right)\left(\frac{u^{(p)}}{\sqrt{g_{\underline{pp}}}}\right)\end{aligned} \tag{16}$$

where u_i, ε_{ij} and η_{kij} are the physical components of displacement vector u_i, displacement gradient ε_{ij} and higher order displacement gradient η_{kij}, and g_{ii} and Γ_{ijk} represent the components of metric tensor and Christoffel symbols of the second kind. The underscores are placed under the indices to indicate lack of addition on them. In the cylindrical coordinate system, the components of metric tensor and Christoffel symbol are expressed as

$$\begin{aligned}&g_{xx}=1,\ g_{\theta\theta}=\left(R\left(1+\frac{z}{R}\right)\right)^2,\ g_{zz}=1,\ g_{kl}=0\ (k\neq l)\\ &\Gamma_{z\theta}^{\theta}=\Gamma_{\theta z}^{\theta}=\frac{1}{R\left(1+\frac{z}{R}\right)},\ \Gamma_{\theta\theta}^{z}=-R\left(1+\frac{z}{R}\right)\end{aligned} \tag{17}$$

Classical strain components are obtained by substituting Eq. (17) into Eq. (14) as follows

$$\begin{aligned}\varepsilon_{zz}&=\frac{\partial w}{\partial z}\\ \varepsilon_{xx}&=\frac{\partial u}{\partial x}\\ \varepsilon_{\theta\theta}&=\frac{1}{R(1+z/R)}\left[\frac{\partial v}{\partial\theta}+w\right]\\ \varepsilon_{\theta z}=\varepsilon_{z\theta}&=\frac{1}{2}\left[\frac{1}{R(1+z/R)}\frac{\partial w}{\partial\theta}+\frac{\partial v}{\partial z}-\frac{v}{R(1+z/R)}\right],\\ \varepsilon_{zx}=\varepsilon_{xz}&=\frac{1}{2}\left[\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right],\\ \varepsilon_{x\theta}=\varepsilon_{\theta x}&=\frac{1}{2}\left[\frac{\partial v}{\partial x}+\frac{1}{R(1+z/R)}\frac{\partial u}{\partial\theta}\right]\end{aligned} \tag{18}$$

Higher order strain components are obtained by substituting Eq. (16) and Eq. (17) into Eq. (15) as follows

$$\eta_{xxx}=\frac{\partial^2u}{\partial x^2}$$

$$\eta_{xx\theta}=\frac{\partial^2v}{\partial x^2}$$

$$\eta_{x\theta\theta}=\eta_{\theta x\theta}=\frac{1}{R(1+z/R)}\left[\frac{\partial^2v}{\partial x\partial\theta}+\frac{\partial w}{\partial x}\right]$$

$$\eta_{\theta xx}=\eta_{x\theta x}=\frac{1}{R(1+z/R)}\frac{\partial^2u}{\partial x\partial\theta}$$

$$\eta_{\theta\theta x}=\frac{1}{R(1+z/R)}\left[\frac{1}{R(1+z/R)}\frac{\partial^2u}{\partial\theta^2}+\frac{\partial u}{\partial z}\right]$$

$$\eta_{\theta\theta\theta}=\frac{1}{\left(R(1+z/R)\right)^2}\left[\frac{\partial^2v}{\partial\theta^2}+2\frac{\partial w}{\partial\theta}+R(1+z/R)\frac{\partial v}{\partial z}-v\right]$$

$$\eta_{\theta\theta z}=\frac{1}{\left(R(1+z/R)\right)^2}\left[\frac{\partial^2w}{\partial\theta^2}-2\frac{\partial v}{\partial\theta}+R(1+z/R)\frac{\partial w}{\partial z}-w\right]$$

$$\eta_{z\theta\theta}=\eta_{\theta z\theta}=\frac{1}{R\left(1+z/R\right)}\begin{bmatrix}\dfrac{\partial^2 v}{\partial z\,\partial\theta}-\dfrac{1}{R\left(1+z/R\right)}\dfrac{\partial v}{\partial\theta}\\ +\dfrac{\partial w}{\partial z}-\dfrac{w}{R\left(1+z/R\right)}\end{bmatrix}$$

$$\eta_{zz\theta}=\frac{\partial^2 v}{\partial z^2}$$

$$\eta_{z\theta z}=\eta_{\theta zz}=\frac{1}{R\left(1+z/R\right)}\begin{bmatrix}\dfrac{\partial^2 w}{\partial z\,\partial\theta}-\dfrac{1}{R\left(1+z/R\right)}\dfrac{\partial w}{\partial\theta}\\ -\dfrac{\partial v}{\partial z}-\dfrac{v}{R\left(1+z/R\right)}\end{bmatrix}$$

$$\eta_{xxz}=\frac{\partial^2 w}{\partial x^2} \tag{19}$$

$$\eta_{zxx}=\eta_{xzx}=\frac{\partial^2 u}{\partial x\partial z}$$

$$\eta_{zzz}=\frac{\partial^2 w}{\partial z^2}$$

$$\eta_{zzx}=\frac{\partial^2 u}{\partial z^2}$$

$$\eta_{zxz}=\eta_{xzz}=\frac{\partial^2 w}{\partial x\partial z}$$

$$\eta_{x\theta z}=\eta_{\theta xz}=\frac{1}{R\left(1+z/R\right)}\left[\frac{\partial^2 w}{\partial x\,\partial\theta}-\frac{\partial v}{\partial x}\right]$$

In an elastic isotropic thin shell, the stress-strain relations for the plane stress condition (σ_{zz}=0) are expressed as

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{\theta\theta}\\ \sigma_{x\theta}\end{Bmatrix}=\begin{pmatrix}\dfrac{E}{1-\nu^2} & \dfrac{\nu E}{1-\nu^2} & 0\\ \dfrac{\nu E}{1-\nu^2} & \dfrac{E}{1-\nu^2} & 0\\ 0 & 0 & \mu\end{pmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{\theta\theta}\\ \varepsilon_{x\theta}\end{Bmatrix} \tag{20}$$

And, the components of strain tensor are determined as follows

$$\varepsilon_{xx}=\frac{\partial u}{\partial x}-z\frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_{\theta\theta}=\frac{1}{R}\frac{\partial v}{\partial\theta}+\frac{w}{R}-z\frac{\partial^2 w}{R^2\partial\theta^2} \tag{21}$$

$$\varepsilon_{x\theta}=\frac{1}{R}\frac{\partial u}{\partial\theta}+\frac{\partial v}{\partial x}-2z\frac{\partial^2 w}{R\partial x\partial\theta}$$

which can be expressed in the matrix form as

$$\boldsymbol{\varepsilon}=\mathbf{LU} \tag{22}$$

where

$$\mathbf{L}=\begin{pmatrix}\dfrac{\partial}{\partial x} & 0 & -z\,\dfrac{\partial^2}{\partial x^2}\\ 0 & \dfrac{1}{R}\dfrac{\partial}{\partial\theta} & \dfrac{1}{R}-z\,\dfrac{\partial^2}{R^2\partial\theta^2}\\ \dfrac{1}{R}\dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial x} & -2z\,\dfrac{\partial^2}{R\partial x\,\partial\theta}\end{pmatrix} \tag{23}$$

Therefore, using Eq. (6), Eq. (22) can be expressed as

$$\boldsymbol{\varepsilon}=\mathbf{LU}=\mathbf{LN\,q}=\mathbf{B\,q} \tag{24}$$

Therefore,

$$\mathbf{B}_{6\times24}=\mathbf{L}_{6\times3}\,\mathbf{N}_{3\times24} \tag{25}$$

According to Eq. (11) and Eq. (19), the components of higher order strain tensor are determined as

$$\chi_{rr}=0$$

$$\chi_{xx}=\frac{1}{R}\frac{\partial^2 w}{\partial x\,\partial\theta}$$

$$\chi_{\theta\theta}=-\frac{1}{R}\frac{\partial^2 w}{\partial x\,\partial\theta}$$

$$\chi_{x\theta}=\chi_{\theta x}=\frac{1}{2}\left(\frac{1}{R^2}\frac{\partial^2 w}{\partial\theta^2}-\frac{\partial^2 w}{\partial x^2}\right) \tag{26}$$

$$\chi_{rx}=\chi_{xr}=\frac{1}{4}\left(\frac{\partial^2 v}{\partial x^2}-\frac{1}{R}\frac{\partial^2 u}{\partial x\,\partial\theta}\right)$$

$$\chi_{r\theta}=\chi_{\theta r}=\frac{1}{4}\left(\frac{1}{R}\frac{\partial^2 v}{\partial x\,\partial\theta}-\frac{1}{R^2}\frac{\partial^2 u}{\partial\theta^2}\right)$$

This can be expressed in matrix form as

$$\boldsymbol{\chi}=\bar{\mathbf{L}}\,\mathbf{U} \tag{27}$$

where

$$\bar{\mathbf{L}}=\begin{bmatrix}0 & 0 & \dfrac{1}{R}\dfrac{\partial^2}{\partial x\,\partial\theta}\\ 0 & 0 & -\dfrac{1}{R}\dfrac{\partial^2}{\partial x\,\partial\theta}\\ 0 & 0 & 0\\ 0 & 0 & \dfrac{1}{2}(\dfrac{\partial^2}{R^2\partial\theta^2}-\dfrac{\partial^2}{\partial x^2})\\ -\dfrac{1}{4}\dfrac{\partial^2}{R\partial x\,\partial\theta} & \dfrac{1}{4}\dfrac{\partial^2}{\partial x^2} & 0\\ -\dfrac{1}{4}\dfrac{\partial^2}{R^2\partial\theta^2} & \dfrac{1}{4}\dfrac{\partial^2}{R\partial x\,\partial\theta} & 0\end{bmatrix} \tag{28}$$

Hence, using Eq. (6), Eq. (27) can be expressed as

$$\boldsymbol{\chi}=\bar{\mathbf{L}}\mathbf{U}=\bar{\mathbf{L}}\mathbf{N}\,\mathbf{q}=\bar{\mathbf{B}}\,\mathbf{q} \tag{29}$$

Therefore

$$\bar{\mathbf{B}}_{6\times48}=\bar{\mathbf{L}}_{6\times3}\,\mathbf{N}_{3\times48} \tag{30}$$

2.3 Stress-strain relationships & Element stiffness and mass matrix

The components of Cauchy stress tensor and the symmetric part of the higher order stress tensor are determined as follows

$$\boldsymbol{\sigma}=\mathbf{C}\,\boldsymbol{\varepsilon}=\mathbf{C}\,\mathbf{Bq} \tag{31}$$

$$\mathbf{m}=\mathbf{D}\boldsymbol{\chi}=\mathbf{D}\,\bar{\mathbf{B}}\mathbf{q} \tag{32}$$

where

$$\mathbf{C} = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 & 0 & 0 & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{33}$$

$$\mathbf{D} = 2l^2\mu \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where E, v and μ represent elasticity modulus, Poisson's coefficient, and shear modulus, respectively, and l are the additional independent material length scale parameter related to the symmetric rotation gradient.

According to the modified couple stress theory which was first developed by Yang *et al*, strain energy includes the following two parts (Yang *et al* 2002):

- A classical part, (1/2) $\sigma_{ij}\,\varepsilon_{ij}$
- A non-classical part, (1/2) $m_{ij}\,\chi_{ij}$

and is defined as follows

$$U = \frac{1}{2}\int_{\Omega}\left(\boldsymbol{\sigma}:\boldsymbol{\varepsilon}+\mathbf{m}:\boldsymbol{\chi}\right)\,dV \tag{34}$$

in which $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{m}$ and $\boldsymbol{\chi}$ are the components of the Cauchy stress tensor, strain tensors, the symmetric part of the higher order stress tensor and the symmetric part of rotation gradient tensor.

Using the principle of virtual displacements, the following result is achieved

$$\begin{aligned}\int_V\left(\delta\boldsymbol{\varepsilon}^T\boldsymbol{\sigma}+\delta\boldsymbol{\chi}^T\mathbf{m}\right)dV &= \int_V \delta\mathbf{U}^T\mathbf{f}^B dV \\ &+\int_S \delta\mathbf{U}^{S^T}\mathbf{f}^S dS+\sum_i \delta\mathbf{U}^{i^T}\mathbf{R}_c^i\end{aligned} \tag{35}$$

where f^B, f^S and R^c are body forces per unit volume, surface tractions over a very small area and concentrated loads.

According to Eq. (6), Eq. (24) and Eq. (29) can be concluding

$$\delta\mathbf{U} = \mathbf{N}\;\delta\mathbf{q} \tag{36a}$$

$$\delta\boldsymbol{\varepsilon} = \mathbf{B}\;\delta\mathbf{q} \tag{36b}$$

$$\delta\boldsymbol{\chi} = \bar{\mathbf{B}}\;\delta\mathbf{q} \tag{36c}$$

Therefore, by substituting Eq. (36) into Eq. (35), and using Eqs. (31)- (32), the following result is achieved

$$\begin{aligned}&\int_V\left(\delta\boldsymbol{\varepsilon}^T\boldsymbol{\sigma}+\delta\boldsymbol{\chi}^T\mathbf{m}\right)dV = \\ &\int_V\left(\left(\mathbf{B}\delta\mathbf{q}\right)^T\mathbf{CB}+\left(\bar{\mathbf{B}}\delta\mathbf{q}\right)^T\mathbf{D}\bar{\mathbf{B}}\right)dV = \\ &\delta\mathbf{q}^T\left(\int_V\left(\mathbf{B}^T\mathbf{CB}+\bar{\mathbf{B}}^T\mathbf{D}\bar{\mathbf{B}}\right)dV\right)\mathbf{q}\end{aligned} \tag{37}$$

$$\begin{aligned}&\int_V \delta\mathbf{U}^T\mathbf{f}^B dV+\int_S \delta\mathbf{U}^{S^T}\mathbf{f}^S dS+\sum_i \delta\mathbf{U}^{i^T}\mathbf{R}_c^i \\ &= \delta\mathbf{q}^T\left(\mathbf{R}_c^i+\int_V \mathbf{N}^T\mathbf{f}^B dV+\int_S \mathbf{N}^T\mathbf{f}^S dS\right)\end{aligned} \tag{38}$$

Therefore, assuming $\mathbf{f}^S$=0, Eq. (35) is expressed as

$$\begin{aligned}&\left(\int_V\left(\mathbf{B}^T\mathbf{CB}+\bar{\mathbf{B}}^T\mathbf{D}\bar{\mathbf{B}}\right)dV\right)\mathbf{q} \\ &=\left(\mathbf{R}_c^i+\int_V \mathbf{N}^T\mathbf{f}^B dV\right)\end{aligned} \tag{39}$$

And the stiffness and force matrix is as follows

$$\mathbf{K} = \int_V\left(\mathbf{B}^T\mathbf{CB}+\bar{\mathbf{B}}^T\mathbf{D}\bar{\mathbf{B}}\right)dV \tag{40}$$

$$\mathbf{F} = \mathbf{R}_c^i+\int_V \mathbf{N}^T\mathbf{f}^B dV \tag{41}$$

The infinitesimal volume dv in cylindrical coefficient systems as global coordinate system is defined as

$$dV = rhdxd\theta \tag{42}$$

By using Jacobian transfer to transfer the integral in Eq. (35) from $(\boldsymbol{x},\boldsymbol{\theta})$ space to $(\boldsymbol{\xi},\boldsymbol{\gamma})$ space, the following equation is made

$$dV = \left|\det\left(\mathbf{J}\right)\right| rhdxd\theta = \frac{L\pi}{2} rhdxd\theta \tag{43}$$

By substitution Eq. (43) into Eq. (40) yields the following equation

$$\mathbf{K} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left(\mathbf{B}^T\mathbf{CB}+\bar{\mathbf{B}}^T\mathbf{D}\bar{\mathbf{B}}\right)\frac{L\pi}{2}Rhd\xi d\gamma \tag{44}$$

The mass matrix which expresses the features of an element's mass is developed as follows

$$\mathbf{M} = \int_V \mathbf{N}^T\rho\mathbf{N}\,dV \tag{45}$$

In which integral transfer from global coordinate to local coordinate is expressed as follows

$$\mathbf{M} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\mathbf{N}^T\rho\mathbf{N}\frac{L\pi}{2}Rhd\xi d\gamma \tag{46}$$

3. Results and discussion

The size-dependent vibration of nano-shell is examined using cylindrical shell element based on modified couple stress theory for the simply supported boundary conditions. The problem of vibration of single walled carbon nanotube (SWCNT) is solved at the following geometrical and mechanical properties

$$\begin{aligned}&R = 12\text{nm}\;,\; h = 0.34\text{nm}\;,\; E = 10^{12}\text{ Pa} \\ &\upsilon = 0.25\;,\; \rho_t = 2300\text{ Kg/m}^3\;,\; l = h\end{aligned} \tag{47}$$

A dimensionless natural frequency was defined as $\Omega = \omega R\sqrt{\frac{\rho}{E}}$ and the effects of material length scale and shell length are obtained by using modified couple stress theory and the classical theory. Considering the equation of motion Eq. (48) and Eq. (49), the natural frequency

Fig. 2The effect of *L/R* on dimensionless natural frequency

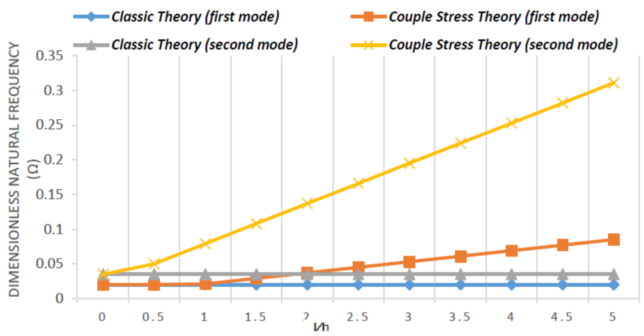

Fig. 3The effect of l/h on dimensionless natural frequency

Table 1 Dimensionless natural frequency of SWCNTs with different theories

Natural frequency	Classical theory (Alibeigloo 2013)	Couple stress theory (Zeighampour 2014)	Couple stress theory (*l*=*h*) (present study)	Ansys (320 elements)
First mode	0.197	0.198	0.198	0.196
Second mode	0.256	0.279	0.271	0.258
Third mode	0.277	0.392	0.364	0.292

Table 2 Natural frequency

Mode number	Classic theory	Couple stress theory (15 cylindrical shell element)	Ansys (320 element)
1	312	297	294
2	1339	1323	1310
3	1548	1492	1474

Table 3 Comparison of resonant frequency (THz)

L/D	Molecular dynamics method (Ansari *et al.* 2012)	Couple stress theory (FEM)
4.67	0.2476	0.2685
7.55	0.1079	0.1146
10.07	0.0562	0.0581
13.69	0.0312	0.0336

equation was obtained according to Eq. (50) as follow

$$\mathbf{M}\ddot{\mathbf{Q}}+\mathbf{K}\mathbf{Q}=0 \tag{48}$$

$$\mathbf{Q}=\mathbf{A}_{\mathbf{i}}\sin(\omega_i) \tag{49}$$

$$\mathbf{M}\omega_i^2-\mathbf{K}=0 \tag{50}$$

where M, K, Q and A_i respectively are, the system mass, the stiffness matrices, the global degree of freedom vector and the amplitude vector. The effect of length to radius ratio on dimensionless natural frequency of the SWCNT for the first two frequency modes shown for different theories in Fig. 2.

As it is clear from Fig. 2, the increase of length-radius ratio results in the decrease of SWCNT natural frequency, which is due to reduction in SWCNT rigidity at higher length-radius ratios. It is shown that in the second mode the increase in shell length has a weaker effect on the decrease of the natural frequency than in the first mode. In addition, in shorter lengths, there is smaller difference between the values of the natural frequency. It is concluded from Fig. 2 that the variation of natural frequency in the second mode between the couple stress theory and the classical theory is more than that in the first mode.

The effect of material length scale parameter to thickness ratio on dimensionless natural frequency of the SWCNT for the first two frequency modes shown for modified coupled stress theory and the classical shell theory in Fig. 3. It is shown that by increasing the size parameter, the SWCNT's natural frequency will increase too, which is due to the increase in SWCNT rigidity. The forecasted values of natural frequency from couple stress theory is greater than that of classical continuum theory, which is due to the presence of one size parameter in couple stress theory.

In order to validate the results using this modified couple stress cylindrical shell element, the natural frequencies of SWCNT using the modified couple stress cylindrical shell element were calculated and compared with those obtained by other references for simply supported boundary conditions in Table 1. The material properties of the cylindrical shell were considered to be E=1.06 GPa and ν=0.3, and cylinder dimensions were considered to be R=2.32 nm, L/R=10, h/R=0.02, $l=h$. It is shown that the resonant frequencies calculated have good agreement with other references.

Table 1 illustrate the natural frequencies obtained from couple stress theory using the modified couple stress cylindrical shell element are greater than that of classical theory and Ansys, which is due to the presence of one size parameter in couple stress theory.

In another case for modal analysis, a nano cylindrical shell at radius r=0.05 (mm), length L=0.6 (mm), elasticity modulus E=2 $\times 10^{11}$ (MPa), Poisson's coefficient υ=0.3 and density ρ=7800 (Kg/m^3) were considered.

Table 2 shows the natural frequencies obtained from different theories for clamped- free boundary conditions.

The resonant frequencies of SWCNTS using this cylindrical shell element were compared with the results of molecular dynamics method that reported by Ansari *et al.* (2012) at thickness=0.34e^{-9} and diameter=0.678 nm with Clamped-Free boundary conditions. According to the results it is obvious that in addition to reducing the number of elements required, the use of this type of element also increases convergence speed and accuracy.

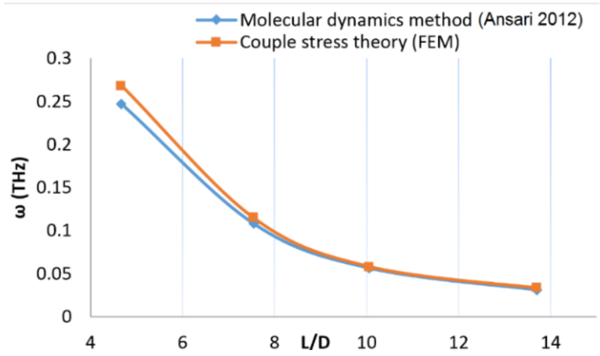

Fig. 4 The effects of length-diameter ratio on natural frequency

Fig. 4 presents the natural frequencies of SWCNTS using this cylindrical shell element obtained from couple stress theory are greater than that of molecular dynamics method, which is due to the presence of one size parameter in couple stress theory. It is shown that increase of length-diameter ratio results in the decrease of SWCNT natural frequency, which is due to reduction in SWCNT rigidity at higher length-diameter ratios.

4. Conclusions

In this study, by considering the size effect using modified couple stress theory and shell model, a modified couple stress cylindrical shell element was developed. The mass-stiffness matrix for cylindrical shell element is developed, and by means of size-dependent finite element formulation is extended to more precisely account for nanotube vibration. In addition to modified couple stress cylindrical shell element, the classical cylindrical shell element can also be defined by setting length scale parameter to zero in the equations. It is shown that increase of length-radius ratio results in the decrease of SWCNT natural frequency, which is due to reduction in SWCNT rigidity at higher length-radius ratios and by increasing the size parameter, the SWCNT's natural frequency will increase too which is due to the increase in SWCNT rigidity. Finally, in order to validate the results, the natural frequencies of SWCNT using the modified couple stress cylindrical shell element were calculated and compared with those obtained by other references, and an appropriate consistency was demonstrated and it is shown that the use of this type of element increases convergence speed and accuracy.

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