# Fracture analysis of functionally graded beams with considering material non-linearity

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**Abstract.** The present paper deals with a theoretical study of delamination fracture in the Crack Lap Shear (CLS) functionally graded beam configuration. The basic purpose is to analyze the fracture with taking into account the material non-linearity. The mechanical behavior of CLS was described by using a non-linear stress-strain relation. It was assumed that the material is functionally graded along the beam height. The fracture was analyzed by applying the *J*-integral approach. The curvature and neutral axis coordinate of CLS beam were derived in order to solve analytically the *J*-integral. The non-linear solution of *J*-integral obtained was verified by analyzing the strain energy release rate with considering material non-linearity. The effects of material gradient, crack location along the beam height and material non-linearity on fracture behavior were evaluated. The *J*-integral non-linear solution derived is very suitable for parametric studies of longitudinal fracture in the CLS beam. The results obtained can be used to optimize the functionally graded beam structure with respect to the fracture performance. The analytical approach developed in the present paper contributes for the understanding of delamination fracture in functionally graded beams exhibiting material non-linearity.

Keywords: functionally graded beams; fracture; material non-linearity; analytical modeling

#### 1. Introduction

Recently, the functionally graded materials have been widely used in aerospace technologies, automotive industry, electronics, optics, nuclear energy, etc. (Niino et al. 1987, Koizumi 1993, Markworth et al. 1995, Mortensen and Suresh 1995, Neubrand and Rödel 1997, Suresh and Mortensen 1998, Hirai and Chen 1999, Lu et al. 2009, Gasik 2010, Nemat-Allal et al. 2011, Parvanova et al. 2013, 2014, Bohidar et al. 2014). Due to the continuously varying properties with location within the material, the functionally graded materials have a number of advantages over the conventional structural materials. For instance, the functionally graded materials are able to withstand effects of harsh environment (for example, severe temperature gradients) while simultaneously maintaining structural integrity. With the development of fabrication technologies, various structural components (beams, plates, shells) may be produced by using functionally graded materials. Fracture is one of the fundamental problems that should be analyzed in the evaluation of functionally graded structures for durability. Therefore, fracture mechanics considerations play an important role in the design of these novel materials. This fact clearly indicates the need of development of fracture mechanics of these materials. A significant amount of attention has been given to fracture in functionally graded materials in recent years (Erdogan

Copyright © 2017 Techno-Press, Ltd. http://www.techno-press.com/journals/sem&subpage=7 1995, Pei and Asaro 1997, Paulino 2002, Tilbrook *et al.* 2005, Carpinteri and Pugno 2006, Upadhyay and Simha 2007, Zhang *et al.* 2013).

A strip of functionally graded material with a semiinfinite crack has been studied analytically by Pei and Asaro (1997). Bending moments and axial forces have been applied on the edge of strip. Methods of linear-elastic fracture mechanics have been used in the investigation. Analytical solutions have been derived for stress intensity factors.

Theoretical and computational studies have been reviewed of the fracture behavior of functionally graded linear-elastic materials by Tilbrook, Moon and Hoffman (2005). Analyses of stress intensity factors have been presented. Cracks of different orientation with respect to the material gradient direction have been considered.

Cracks in functionally graded materials have been analyzed also by Carpinteri and Pugno (2006).

Fracture has been investigated in functionally graded linear-elastic beams subjected to three-point bending by Upadhyay and Simha (2007). For this purpose, the compliance approach has been applied. An equivalent homogeneous beam of variable depth has been suggested for analysis of the stress intensity factor. It has been shown that equivalent beams are quite useful for engineering design considerations of cracked components.

The literature review indicates that fracture in functionally graded materials has been studied manly assuming linear-elastic behavior. However, in reality, the mechanical behavior may be non-linear. Obviously, there is a need of developing fracture analyses with taking into account the material non-linearity.

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Fig. 1 Geometry and loading of the CLS beam



Fig. 2 Non-linear stress-strain curve with strain energy density,  $u_0$ , and complementary strain energy density,  $u_0^*$ 

Therefore, the present paper reports a theoretical study of delamination fracture in the functionally graded CLS beam configuration assuming non-linear material behavior. The fracture was analyzed with the help of *J*-integral approach. The influence was evaluated of material gradient and crack location on the non-linear fracture. The *J*-integral solution derived is very suitable for parametric investigations. The analysis developed holds for non-linear elastic material behavior. However, the analysis is applicable also for elastic-plastic behavior, if the external load increases only, i.e., if the CLS beam undergoes active deformation (Lubliner 2006, Chakrabarty 2006).

## 2. Fracture analysis by using non-linear stress-strain relation

The functionally graded beam configuration analyzed in the present article is shown in Fig. 1. There is a delamination crack of length *a* located arbitrarily along the beam height (it should be noted that the present study was motivated also by the fact that functionally graded materials can be built up layer by layer (Bohidar *et al.* 2014), which is a premise for appearance of delamination cracks between layers (Dolgov 2005, 2016)). The lower crack arm thickness is  $h_1$ . The loading consists of one longitudinal force, *F*, applied at the free end of lower crack arm as illustrated in Fig. 1. Therefore, the upper crack arm is stress free. The beam has a rectangular cross-section of width, *b*, and height, *2h*. The beam is clamped in the right-hand end.

It was assumed that the mechanical response of beam configuration considered can be described by using a non-linear stress-strain curve (Fig. 2).

The stress-strain equation was written as (Petrov 2014)

$$\sigma = D\varepsilon^{f} - L_{1}\varepsilon^{g_{1}}, \qquad (1)$$

where  $\sigma$  is the stress,  $\varepsilon$  is the strain, D,  $L_1$ , f and  $g_1$  are material properties. The present analysis was based on the small strain assumption (it should be noted that this assumption has been widely used in fracture analyses of functionally graded materials (Pei and Asaro 1997, Carpinteri and Pugno 2006, Upadhyay and Simha 2007)). It was also assumed that D is functionally graded along the beam cross-section height. The material property D was written as a linear function of the coordinate  $z_3$ 

$$D = D_0 + \frac{D_1 - D_0}{2h} (h + z_3), \qquad (2)$$

where  $D_0$  and  $D_1$  are the values of D in the upper and in lower edge of beam cross-section, respectively. The  $z_3$ -axis originates from the centre of beam cross-section and is directed downward.

The *J*-integral approach for functionally graded materials (Anlas, Santare and Lambros 2000), was applied here in order to perform a theoretical study of non-linear fracture in the beam configuration considered. The *J*-integral was written as

$$J = \int_{\Gamma} \left[ u_0 \cos \alpha - \left( p_x \frac{\partial u}{\partial x} + p_y \frac{\partial v}{\partial x} \right) \right] ds - \int_{A} \frac{\partial u_0}{\partial x} q dA, \quad (3)$$

where  $\Gamma$  is a contour of integration going from the lower crack face to the upper crack face in the counter clockwise direction,  $u_0$  is the strain energy density,  $\alpha$  is the angle between the outwards normal vector to the contour of integration and the crack direction,  $p_x$  and  $p_y$  are the components of stress vector, u and v are the components of displacement vector with respect to the crack tip coordinate system xy (x is directed along the crack), ds is a differential element along the contour, A is the area enclosed by that contour, q is a weight function with a value of unity at the crack tip, zero along the contour and arbitrary elsewhere. It should be specified that the partial derivative  $\partial u_0/\partial x$  exists only if the material property is an explicit function of x(Anlas, Santare and Lambros 2000).

The *J*-integral was solved by using integration contour,  $\Gamma$ , that consists of beam cross-sections ahead and behind the crack tip (Fig. 1). It is obvious that the *J*-integral has non-zero value only in segments  $A_1$  and B of the integration contour. Therefore, the *J*-integral solution was obtained by summation

$$J = J_{A_1} + J_B, \tag{4}$$

where  $J_{A_1}$  and  $J_B$  are the *J*-integral values in segments  $A_1$  and *B*, respectively.

First, the *J*-integral solution was obtained in segment  $A_1$  of the integration contour (this segment coincides with the lower crack arm cross-section behind the crack tip as illustrated in Fig. 1). The *J*-integral components in segment  $A_1$  were written as

$$p_{y} = 0, ds = dz_{1}, \cos \alpha = -1,$$
 (5)

where the  $z_1$ -coordinate varies in the interval  $[-h_1/2, h_1/2]$ . The curvature,  $\kappa_1$ , and neutral axis coordinate,  $z_{1n_1}$ , of lower crack arm, that are needed in order to determine the *J*-integral other components, were derived from the



Fig. 3 Distribution of stresses in the lower crack arm crosssection

equilibrium equations of lower crack arm cross-section.

It should be specified that the lower crack arm is loaded in eccentric tension, because the beam is functionally graded (the stresses distribution in lower crack arm is shown schematically in Fig. 3). Thus, the equilibrium equations of lower crack arm cross-section were written as

$$N = \int_{-\frac{h_1}{2}}^{\frac{h_1}{2}} \sigma(\varepsilon) b dz_1$$
(6)

$$M = \int_{-\frac{h_1}{2}}^{\frac{h_1}{2}} \sigma(\varepsilon) b z_1 d z_1$$
(7)

where, according to the Bernoulli's hypothesis for plane sections, the strain,  $\varepsilon$ , was expressed as

$$\mathcal{E} = \mathcal{K}_1(z_1 - z_{1n_1}). \tag{8}$$

It should be mentioned that the Bernoulli's hypothesis for plane sections has been frequently applied when analyzing fracture in functionally graded materials (Pei and Asaro 1997, Carpinteri and Pugno 2006, Upadhyay and Simha 2007). The axial force, N, and bending moment, M, in the lower crack arm were written as (Fig. 1)

$$N = F, \quad M = 0. \tag{9}$$

Eq. (2) was rewritten as

$$D = D_0^C + \frac{D_1 - D_0^C}{h_1} \left(\frac{h_1}{2} + z_1\right)$$
(10)

where

$$D_0^C = D_0 + \frac{D_1 - D_0}{2h} (2h - h_1)$$
(11)

is the value of D in the upper edge of lower crack arm. After substitution of Eqs. (1), (8) and (10) in Eqs, (6) and (7), we obtained

$$N = \frac{b\kappa_1^f}{f+1} \left[ \frac{1}{2} \left( D_0^C + D_1 \right) + \frac{1}{h_1} \left( D_1 - D_0^C \right) z_{1n_1} \right]$$
$$\left[ \left( \frac{h_1}{2} - z_{1n_1} \right)^{f+1} - \left( -\frac{h_1}{2} - z_{1n_1} \right)^{f+1} \right] +$$

$$+ \frac{b\kappa_{1}^{\prime f}}{h_{1}} (D_{1} - D_{0}^{c}) \frac{1}{f + 2} \begin{bmatrix} \left(\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+2} \\ - \left(-\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+2} \end{bmatrix}^{-1}$$
(12)  

$$- \frac{b\kappa_{1}^{\prime s_{1}} L_{1}}{g_{1} + 1} \begin{bmatrix} \left(\frac{h_{1}}{2} - z_{1n_{1}}\right)^{g_{1} + 1} - \left(-\frac{h_{1}}{2} - z_{1n_{1}}\right)^{g_{1} + 1} \end{bmatrix} \\ M = \kappa_{1}^{\prime f} b \begin{cases} D_{0}^{c} \begin{cases} \frac{1}{f + 2} \\ \frac{1}{f + 2} \\ - \left(-\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+2} \end{bmatrix}^{+} \\ + \frac{1}{f + 1} \begin{bmatrix} \left(\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+2} \\ \left(-\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+2} \\ \frac{1}{f + 2} \end{bmatrix} \end{bmatrix}^{+} \\ + \frac{1}{2} (D_{1} - D_{0}^{c}) \begin{cases} \frac{1}{f + 2} \begin{bmatrix} \left(\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+1} \\ \frac{1}{f + 2} \end{bmatrix} \end{cases}^{f+1} \\ - \left(-\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+1} - \left(-\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+2} \end{bmatrix}^{+} \\ + \frac{1}{f + 1} \begin{bmatrix} \left(\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+1} - \left(-\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+1} \end{bmatrix} \end{cases}^{f+1} \\ + \frac{1}{h_{1}} (D_{1} - D_{0}^{c}) \begin{cases} \frac{1}{f + 3} \begin{bmatrix} \left(\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+2} \\ - \left(-\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+2} \end{bmatrix}^{f+1} \\ - \left(-\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+2} \end{bmatrix}^{f+1} \\ + \frac{2z_{1n_{1}}}{f + 2} \begin{bmatrix} \left(\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+2} - \left(-\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+2} \end{bmatrix}^{f+2} \\ + \frac{2z_{1n_{1}}}{f + 1} \begin{bmatrix} \left(\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+1} - \left(-\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+2} \end{bmatrix}^{f+1} \\ + \frac{z_{1n_{1}}^{s_{1}}} \begin{bmatrix} \left(\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+1} - \left(-\frac{h_{1}}{2} - z_{1n_{1}}\right)^{g_{1}+2} \end{bmatrix}^{f+1} \\ + \frac{1}{g_{1} + 1} \begin{bmatrix} \left(\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+1} - \left(-\frac{h_{1}}{2} - z_{1n_{1}}\right)^{g_{1}+2} \end{bmatrix}^{f+1} \\ + \frac{1}{g_{1} + 1} \begin{bmatrix} \left(\frac{h_{1}}{2} - z_{1n_{1}}\right)^{g_{1}+1} \\ - \left(-\frac{h_{1}}{2} - z_{1n_{1}}\right)^{g_{1}+1} \\ - \left(-\frac{h_{1}}{2} - z_{1n_{1}}\right)^{g_{1}+1} \\ - \left(-\frac{h_{1}}{2} - z_{1n_{1}}\right)^{g_{1}+1} \end{bmatrix}^{f+1} \\ \end{bmatrix}^{f+1}$$

Obviously, at  $L_1=0$ , f=1 and  $D_1 = D_0^C = E$  (here *E* is the modulus of elasticity) the stress-strain relation (1) transforms into the Hooke's law. This means that at  $L_1=0$ , f=1 and  $D_1 = D_0^C = E$ , Eq. (13) has to transform in the formula for curvature of linear-elastic homogeneous beam. Indeed, by substitution of  $L_1=0$ , f=1 and  $D_1 = D_0^C = E$  in Eq. (13), we obtained

$$\kappa_1 = \frac{12M}{Ebh_1^3} \tag{14}$$

which is the known formula for curvature of homogeneous linear-elastic beam of width b and height  $h_1$ .

It is clear that at  $D_1 = D_0^C = 0$ ,  $g_1=1$  and  $-L_1=E$  the stress-strain relation Eq. (1) transforms also into the Hook's law. Indeed, at  $D_1 = D_0^C = 0$ ,  $g_1=1$  and  $-L_1=E$  Eq. (13) transforms also into Eq. (14).

Eq. (14) was obtained also by substitution of f=1,  $g_1=1$ ,  $D_1 = D_0^C = D$  and  $D-L_1=E$  in Eq. (13). This is a consequence from the fact that at f=1,  $g_1=1$ ,  $D_1 = D_0^C = D$  and  $D-L_1=E$ , the stress-strain relation Eq. (1) transforms again into the Hooke's law.

Eqs. (12) and (13) should be solved with respect to  $\kappa_1$  and  $z_{1n_1}$  by using the MatLab computer program.

The component  $p_x$  in Eq. (3) was written as

$$p_x = -\sigma = -D\varepsilon^f + L_1 \varepsilon^{g_1} \tag{15}$$

where  $\varepsilon$  was determined by Eq. (8).

The partial derivative that participates in the first integral in (3) was obtained by using the following formula from Mechanics of materials

$$\frac{\partial u}{\partial x} = \varepsilon,$$
 (16)

where  $\varepsilon$  was found from Eq. (8).

The strain energy density,  $u_0$ , is equal to the area *OPQ* enclosed by stress-strain curve (refer to Fig. 2)

$$u_0 = \int_0^\varepsilon \sigma d\varepsilon \, \cdot \tag{17}$$

After substitution of Eq. (1) in Eq. (17), we obtained

$$u_0 = D \frac{\varepsilon^{f+1}}{f+1} - L_1 \frac{\varepsilon^{g_1+1}}{g_1+1}$$
 (18)

Partial derivative,  $\partial u_0 / \partial x$ , in the second integral in Eq. (3) was written as

$$\frac{\partial u_0}{\partial x} = 0, \qquad (19)$$

since the strain energy density does not depend explicitly on x (the material property D is not a function of x, because the material is functionally graded along the beam height only (refer to Eq. (2)).

After substitution of Eqs. (5), (8), (10), (15), (16), (18) and (19) in Eq. (3), we obtained

$$J_{A_{1}} = \frac{\kappa_{1}^{f+1}D_{0}^{C}}{(f+1)(f+2)} \begin{bmatrix} \left(-\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+2} \\ -\left(\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+2} \end{bmatrix} + \frac{\kappa_{1}^{f+1}(D_{1} - D_{0}^{C})}{f+1} \begin{bmatrix} \frac{1}{2(f+2)} \begin{bmatrix} \left(-\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+2} \\ -\left(\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+2} \end{bmatrix} + \frac{\kappa_{1}^{f+1}(D_{1} - D_{0}^{C})}{f+1} \begin{bmatrix} \frac{1}{2(f+2)} \begin{bmatrix} \left(-\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+2} \\ -\left(\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+2} \end{bmatrix} + \frac{\kappa_{1}^{f+1}(D_{1} - D_{0}^{C})}{f+1} \begin{bmatrix} \frac{1}{2(f+2)} \begin{bmatrix} \left(-\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+2} \\ -\left(\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+2} \end{bmatrix} + \frac{\kappa_{1}^{f+1}(D_{1} - D_{0}^{C})}{f+1} \begin{bmatrix} \frac{1}{2(f+2)} \begin{bmatrix} \frac{1}{2(f+2)} \\ -\left(\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+2} \end{bmatrix} \end{bmatrix} + \frac{\kappa_{1}^{f+1}(D_{1} - D_{0}^{C})}{f+1} \begin{bmatrix} \frac{1}{2(f+2)} \\ -\left(\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+2} \end{bmatrix} + \frac{\kappa_{1}^{f+1}(D_{1} - D_{0}^{C})}{f+1} \end{bmatrix} + \frac{\kappa_{1}^{f+1}(D_{1} - D_{0}^{C})}{f+1} \begin{bmatrix} \frac{1}{2(f+2)} \\ -\left(\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+2} \end{bmatrix} + \frac{\kappa_{1}^{f+1}(D_{1} - D_{0}^{C})}{f+1} \end{bmatrix} + \frac{\kappa_{1}^{f+1}(D_{1} - D_{0}^{C})}{f+1} \begin{bmatrix} \frac{1}{2(f+2)} \\ -\left(\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+2} \end{bmatrix} + \frac{\kappa_{1}^{f+1}(D_{1} - D_{0}^{C})}{f+1} \end{bmatrix} + \frac{\kappa_{1}^{f+1}(D_{1} - D_{0}^{C})}{f+1} \begin{bmatrix} \frac{1}{2(f+2)} \\ -\left(\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+2} \end{bmatrix} \end{bmatrix} + \frac{\kappa_{1}^{f+1}(D_{1} - D_{0}^{C})}{f+1} \end{bmatrix} + \frac{\kappa_{1}^{f+1}(D_{1} - D_{0}^{C})}{f+1} \begin{bmatrix} \frac{1}{2(f+2)} \\ -\left(\frac{h_{1}}{2} - z_{1n_{1}}\right)^{f+2} \end{bmatrix} \end{bmatrix} + \frac{\kappa_{1}^{f+1}(D_{1} - D_{0}^{C})}{f+1} \end{bmatrix} + \frac{\kappa_{1}^{f+1}($$

$$+ \frac{1}{h_{1}(f+3)} \Biggl[ \Biggl( -\frac{h_{1}}{2} - z_{1n_{1}} \Biggr)^{f+3} - \Biggl( \frac{h_{1}}{2} - z_{1n_{1}} \Biggr)^{f+3} \Biggr] + \\ + \frac{z_{1n_{1}}}{h_{1}(f+2)} \Biggl[ \Biggl( -\frac{h_{1}}{2} - z_{1n_{1}} \Biggr)^{f+2} - \Biggl( \frac{h_{1}}{2} - z_{1n_{1}} \Biggr)^{f+2} \Biggr] \Biggr\} + \\ + \kappa_{1}^{f+1} \Biggl\{ \frac{D_{0}^{C}}{f+2} \Biggl[ \Biggl( \frac{h_{1}}{2} - z_{1n_{1}} \Biggr)^{f+2} \\ - \Biggl( -\frac{h_{1}}{2} - z_{1n_{1}} \Biggr)^{f+2} \Biggr] + \\ + \frac{D_{1} - D_{0}^{C}}{2(f+2)} \Biggl[ \Biggl( \frac{h_{1}}{2} - z_{1n_{1}} \Biggr)^{f+2} - \Biggl( -\frac{h_{1}}{2} - z_{1n_{1}} \Biggr)^{f+2} \Biggr] + \\ + \frac{D_{1} - D_{0}^{C}}{h_{1}(f+3)} \Biggl[ \Biggl( \frac{h_{1}}{2} - z_{1n_{1}} \Biggr)^{f+3} - \Biggl( -\frac{h_{1}}{2} - z_{1n_{1}} \Biggr)^{f+3} \Biggr] + \\ + \frac{z_{1n_{1}} \Bigl( D_{1} - D_{0}^{C} \Biggr] \Biggl[ \Biggl( \frac{h_{1}}{2} - z_{1n_{1}} \Biggr)^{f+2} \\ - \Biggl( -\frac{h_{1}}{2} - z_{1n_{1}} \Biggr)^{f+2} \Biggr] \Biggr\} + \\ + \frac{z_{1n_{1}} \Bigl( D_{1} - D_{0}^{C} \Biggr] \Biggl[ \Biggl( \frac{h_{1}}{2} - z_{1n_{1}} \Biggr)^{f+2} \\ - \Biggl( -\frac{h_{1}}{2} - z_{1n_{1}} \Biggr)^{f+2} \Biggr] \Biggr\} + \\ + \frac{L_{1} g_{1} \kappa_{1}^{g_{1}+1}}{(g_{1} + 1)(g_{1} + 2)} \Biggl[ \Biggl( \frac{h_{1}}{2} - z_{1n_{1}} \Biggr)^{g_{1}+2} \\ - \Biggl( -\frac{h_{1}}{2} - z_{1n_{1}} \Biggr)^{g_{1}+2} \Biggr] .$$

The segment, B, of integration contour coincides with the cross-section of un-cracked beam portion (Fig. 1), which is loaded non-centrically by a tensile force, F. Thus, the axial load, N, and the bending moment, M, in the uncracked beam portion were written as

$$N = F , \qquad (21)$$

$$M = F\left(h - \frac{h_1}{2}\right),\tag{22}$$

The normal stresses in the cross-section of un-cracked beam portion induced by N and M are distributed non-linearly (refer to Eq. (1)) as shown in Fig. 4.



Fig. 4 Distribution of stresses in the cross-section of uncracked beam portion

The equations for equilibrium of un-cracked beam cross-section ahead of the crack tip were written as

$$N = \int_{-h}^{h} \sigma(\varepsilon) b dz_3, \qquad (23)$$

$$M = \int_{-h}^{h} \sigma(\varepsilon) b z_3 d z_3, \qquad (24)$$

where

+

$$\mathcal{E} = \kappa_3 (z_3 - z_{3n_3}) \tag{25}$$

After substitution of Eqs. (1), (2) and (25) in Eqs. (23) and (24), we obtained

$$N = \frac{b\kappa_{3}^{f}}{f+1} \left[ \frac{1}{2} (D_{0} + D_{1}) + \frac{1}{2h} (D_{1} - D_{0}) z_{3n_{3}} \right] \\ \left[ (h - z_{3n_{3}})^{f+1} - (-h - z_{3n_{3}})^{f+1} \right] + \\ + \frac{b\kappa_{3}^{f}}{2h} (D_{1} - D_{0}) \frac{1}{f+2} \left[ (h - z_{3n_{3}})^{f+2} - (-h - z_{3n_{3}})^{g+1} \right] - \\ - \frac{b\kappa_{3}^{g}}{g_{1} + 1} L_{1} \left[ (h - z_{3n_{3}})^{g_{1} + 1} - (-h - z_{3n_{3}})^{g_{1} + 1} \right] \\ M = \kappa_{3}^{f} b \left\{ D_{0} \left\{ \frac{1}{f+2} \left[ (h - z_{3n_{3}})^{f+2} - (-h - z_{3n_{3}})^{f+2} \right] + \\ + \frac{1}{f+1} \left[ (h - z_{3n_{3}})^{f+1} z_{3n_{3}} - (-h - z_{3n_{3}})^{f+1} z_{3n_{3}} \right] \right\} + \\ + \frac{1}{2} (D_{1} - D_{0}) \left\{ \frac{1}{f+2} \left[ (h - z_{3n_{3}})^{f+2} - (-h - z_{3n_{3}})^{f+2} \right] + \\ + \frac{2s_{3n_{3}}}{f+1} \left[ (h - z_{3n_{3}})^{f+1} - (-h - z_{3n_{3}})^{f+1} \right] \right\} + \\ + \frac{2z_{3n_{3}}}{f+1} \left[ (h - z_{3n_{3}})^{f+1} - (-h - z_{3n_{3}})^{f+3} \right] + \\ + \frac{2z_{3n_{3}}}{f+2} \left[ (h - z_{3n_{3}})^{f+2} - (-h - z_{3n_{3}})^{f+2} \right] + \\ + \frac{2z_{3n_{3}}}{f+1} \left[ (h - z_{3n_{3}})^{f+1} - (-h - z_{3n_{3}})^{f+2} \right] + \\ + \frac{2s_{3n_{3}}}{f+1} \left[ (h - z_{3n_{3}})^{f+1} - (-h - z_{3n_{3}})^{f+1} \right] \right\} - \\ - \kappa_{3}^{g_{1}} b L_{1} \left\{ \frac{1}{g_{1}+2} \left[ (h - z_{3n_{3}})^{g_{1}+1} z_{3n_{3}} - (-h - z_{3n_{3}})^{g_{1}+1} z_{3n_{3}} \right] \right\}.$$

It should be noted that at  $L_1=0$ , f=1 and  $D_1=D_0=E$ , Eq. (27) transforms in

$$\kappa_3 = \frac{3M}{2Ebh^3} \tag{28}$$

which is the known formula for curvature of homogeneous

linear-elastic beam of width b and height 2h.

Eq. (28) was obtained also by substitution of  $D_1=D_0=0$ ,  $g_1=1$  and  $-L_1=E$  in Eq. (27).

Also, at f=1,  $g_1=1$ ,  $D_1=D_0=D$  and  $D-L_1=E$ , Eq. (27) transforms again in Eq. (28).

Eqs. (26) and (27) should be solved as an algebraic system with unknowns  $\kappa_3$  and  $z_{3n_3}$  by using the MatLab program.

The *J*-integral components in segment B of the integration contour (Fig. 1) were written as

$$p_{x} = \sigma = D\varepsilon^{f} - L_{1}\varepsilon^{g_{1}}, \quad p_{y} = 0, \quad (29)$$

$$ds = -dz_3, \ \cos\alpha = 1, \tag{30}$$

$$\frac{\partial u}{\partial x} = \varepsilon = \kappa_3 (z_3 - z_{3n_3}). \tag{31}$$

The strain energy density was determined by substitution of Eq. (25) in Eq. (18)

$$u_0 = D \frac{\left[\kappa_3 \left(z_3 - z_{3n_3}\right)\right]^{f+1}}{f+1} - L_1 \frac{\left[\kappa_3 \left(z_3 - z_{3n_3}\right)\right] \varepsilon^{g_1 + 1}}{g_1 + 1}$$
(32)

After substitution of Eqs. (19), (29), (30), (31) and (32) in Eq. (3), we derived:

$$J_{B} = -\frac{\kappa_{3}^{f+1}D_{0}}{(f+1)(f+2)} \begin{bmatrix} (-h-z_{3n_{3}})^{f+2} \\ -(h-z_{3n_{3}})^{f+2} \end{bmatrix} - \frac{\kappa_{3}^{f+1}(D_{1}-D_{0})}{f+1} \left\{ \frac{1}{2(f+2)} \begin{bmatrix} (-h-z_{3n_{3}})^{f+2} \\ -(h-z_{3n_{3}})^{f+2} \end{bmatrix} + \frac{1}{2h(f+3)} \begin{bmatrix} (-h-z_{3n_{3}})^{f+3} - (h-z_{3n_{3}})^{f+3} \end{bmatrix} + \frac{2n(f+2)}{2h(f+2)} \begin{bmatrix} (-h-z_{3n_{3}})^{f+2} - (h-z_{3n_{3}})^{f+2} \end{bmatrix} \right\} - \frac{2n(f+2)}{f+2} \begin{bmatrix} (h-z_{3n_{3}})^{f+2} - (h-z_{3n_{3}})^{f+2} \end{bmatrix} \right\} - \frac{2n(f+2)}{2h(f+2)} \begin{bmatrix} (h-z_{3n_{3}})^{f+2} - (-h-z_{3n_{3}})^{f+2} \end{bmatrix} + \frac{2n(f+2)}{2h(f+2)} \begin{bmatrix} (h-z_{3n_{3}})^{f+2} - (-h-z_{3n_{3}})^{f+2} \end{bmatrix} + \frac{2n(f+2)}{2h(f+2)} \begin{bmatrix} (h-z_{3n_{3}})^{f+3} - (-h-z_{3n_{3}})^{f+2} \end{bmatrix} + \frac{2n(f+2)}{2h(f+2)} \begin{bmatrix} (h-z_{3n_{3}})^{f+3} - (-h-z_{3n_{3}})^{f+3} \end{bmatrix} + \frac{2n(f+2)}{2h(f+2)} \begin{bmatrix} (h-z_{3n_{3}})^{f+2} - (-h-z_{3n_{3}})^{f+2} \end{bmatrix} \end{bmatrix}$$

where  $\kappa_3$  and  $z_{3n_3}$  were determined from Eqs. (26) and (27).

After substitution of Eqs. (20) and (33) in Eq. (4), the *J*-integral final non-linear solution was written as

$$J = \frac{\kappa_1^{f+1} D_0^C}{(f+1)(f+2)} \begin{bmatrix} \left(-\frac{h_1}{2} - z_{1n_1}\right)^{f+2} \\ -\left(\frac{h_1}{2} - z_{1n_1}\right)^{f+2} \end{bmatrix} +$$



It should be mentioned that by substitution of  $L_1=0, f=1, D_1=D_0^C=D_0=E$  and  $h_1=h_2=h$  in Eq. (34), we found

$$J = \frac{F^2}{16Eb^2h} \tag{35}$$

Eq. (35) coincides with the formula for strain energy release rate in homogeneous linear-elastic CLS configuration, when the crack is located in the beam midplane (Hutchinson and Suo 1992, Szekrenyes 2012). It should be noted that at  $D_1=D_0=0$ ,  $g_1=1$ ,  $-L_1=E$  and  $h_1=h_2=h$  Eq. (34) transforms also in Eq. (35). Besides, at f = 1,  $g_1=1$ ,  $D_1 = D_0^C = D_0 = D$ ,  $D-L_1=E$  and  $h_1=h_2=h$  Eq. (34) transforms again in Eq. (35).

An analysis was developed of the strain energy release rate, G, in the functionally graded CLS beam (Fig. 1) with considering material non-linearity in order to verify the *J*integral solution Eq. (34). For this purpose, an elementary increase of the crack area,  $dA_a$ , was given (the external loading was kept constant). The strain energy release rate, associated with  $dA_a$ , was written as

$$G = \frac{dW_{ext} - dU}{dA_a} \tag{36}$$

where  $dW_{ext}$  and dU are the changes of external work and strain energy, respectively. The change of external work was expressed as

$$dW_{ext} = dU^* + dU \tag{37}$$

where  $dU^*$  is the change of complementary strain energy. By substitution of Eq. (37) in Eq. (36), we derived

$$G = \frac{dU^*}{dA_a} \tag{38}$$

where

$$dA_a = bda \tag{39}$$

Here, da is an elementary crack length increase.

The complementary strain energy density,  $u_0^*$ , was integrated in the beam volume to obtain the complementary strain energy,  $U^*$ 

$$U^* = \int_{-\frac{h_1}{2}}^{\frac{h_1}{2}} u_0^* badz_1 + \int_{-h}^{h} u_0^* b(l-a)dz_3$$
(40)

The complementary strain energy density is equal to the area OQR that supplements the area OPQ to a rectangle (Fig. 2). Thus, the complementary strain energy density was found as

$$u_0^* = \sigma \varepsilon - u_0 \tag{41}$$

By substitution of Eqs. (1) and (18) in Eq. (41), we obtained

$$u_0^* = D \frac{\varepsilon^{f+1} f}{f+1} - L_1 \frac{\varepsilon^{g_1+1} g_1}{g_1+1}$$
(42)

where  $\varepsilon$  in the lower crack arm and in un-cracked beam



Fig. 5 The *J*-integral value in non-dimensional form as a function of  $D_1/D_0$  ratio for  $h_1/2h=0.25$ , 0.50 and 0.75

portion was calculated by Eqs. (8) and (25), respectively. After combining of Eqs. (2), (8), (10), (38), (39) and (42), and performing the necessary mathematical operations, we derived the equation of strain energy release rate that is exact match of Eq. (34). This fact verifies the *J*-integral non-linear solution Eq. (34).

### 3. Effects of material gradient and crack location on the non-linear fracture

A parametric study was performed in order to evaluate the effects of material gradient and crack location along the beam height on the non-linear fracture behavior of functionally graded CLS configuration shown in Fig. 1. For this purpose, the J-integral value was calculated by using Eq. (34) and the results obtained were presented in nondimensional form by formula  $J_N = J/(D_0 b)$ . In these calculations, it was assumed that b=0.02 m, h=0.004 m and F=500 N. Also, the crack location along the beam height was characterized by  $h_1/2h$  ratio (Fig. 1) in the parametric study. The J-integral values generated by the calculations were plotted in non-dimensional form against  $D_1/D_0$  ratio at  $L_1/D_0 = 0.2$  for  $h_1/2h = 0.25$ , 0.50 and 0.75 in Fig. 5. The curves in Fig. 5 indicate that the J-integral value decreases with increasing  $D_1/D_0$  ratio (this finding was attributed to increase of the functionally graded CLS beam stiffness). Also, it can be observed in Fig. 5 that increase of  $h_1/2h$  ratio leads to decease of the J-integral value (this is due to increase of the lower crack arm stiffness and to decrease of the bending moment in un-cracked beam portion).

The influence of material non-linearity on the fracture behavior was analyzed too. For this purpose, the *J*-integral value, calculated by Eq. (34), was plotted in nondimensional form against the external load magnitude, *F*, at  $D_1/D_0=1.5$ ,  $L_1/D_0=0.2$  and  $h_1/2h=0.25$  as shown in Fig. 6. Also, calculations were performed of the *J*-integral at  $D_1/D_0=1.5$  and  $h_1/2h=0.25$  assuming linear-elastic material behavior of the functionally graded CLS beam (the *J*-integral linear-elastic solution was derived by substitution of f=1 and  $L_1=0$  in Eq. (34)) and the values obtained were plotted in non-dimensional form against the external load in Fig. 6 for comparison with the non-linear solution. One can



Fig. 6 The *J*-integral value in non-dimensional form plotted against the external load, F (curve 1-linear-elastic material behaviour, curve 2-non-linear material behaviour)

observe in Fig. 6 that the material non-linear behavior leads to increase of the *J*-integral. Therefore, the material non-linearity has to be taken into account in fracture mechanics based safety design of functionally graded structural members.

### 4. Conclusions

Delamination fracture behavior of the CLS functionally graded beam configuration was studied theoretically with taking into account the material non-linearity.

The mechanical response of CLS beam was modeled analytically with the help of non-linear stress-strain relation. It was assumed that the material is functionally graded transversally to the beam (linear variation of one of the material properties in the stress-strain relation was assumed along the beam height). Fracture was analyzed with the help of J-integral approach. A non-linear solution of the Jintegral was derived for a delamination crack located arbitrary along the beam height. The solution was verified by analyzing the strain energy release rate with considering the material non-linearity. The influence of material gradient and the crack location along the beam height on the non-linear fracture behavior was investigated. It was found that the J-integral value decreases with increasing lower crack arm thickness. This finding was attributed to increase of the lower crack arm stiffness and to decrease of the bending moment in un-cracked beam portion. Analysis revealed also that the J-integral value deceases with increasing  $D_1/D_0$  ratio (refer to Eq. (2)). The influence of material non-linearity on the fracture behavior was analyzed too. It results obtained indicate that the non-linear material behavior leads to increase of the J-integral value (therefore, the material non-linearity has to be taken into account in fracture mechanics based safety design of functionally graded structural members). The analytical approach developed in the present paper is very suitable for parametric investigations. The results obtained can be used for optimization of functionally graded beam structures with respect to their fracture performance. The analysis developed contributes for the understanding of delamination fracture behavior of functionally graded beams with

material non-linearity.

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