Structural Engineering and Mechanics, *Vol. 64, No. 4 (2017) 495-504*
DOI: https://doi.org/10.12989/sem.2017.64.4.495

Extended implicit integration process by utilizing nonlinear dynamics in finite element

Saeed Mohammadzadeh[1a], Mehdi Ghassemieh[1b] and Yeonho Park[*2]

[1]*School of Civil Engineering, University of Tehran, Tehran, Iran*
[2]*Department of Civil Engineering, University of Texas, Arlington, USA*

(Received December 31, 2016, Revised June 3, 2017, Accepted August 4, 2017)

Abstract. This paper proposes a new direct numerical integration algorithm for solving equation of motion in structural dynamics problems with nonlinear stiffness. The new implicit method's degree of accuracy is higher than that of existing methods due to the higher order of the acceleration. Two parameters are defined, leading to a new family of unconditionally stable methods, which helps to take greater time steps in integration and eliminate concerns about the duration of solving. The method developed can be utilized for a number of solid plane finite elements, examples of which are given to compare the proposed method with existing ones. The results indicate the superiority of the proposed method.

Keywords: structural dynamics; direct time integration; numerical procedure; accuracy; unconditionally stable; nonlinear equation of motion

1. Introduction

The behavior of many dynamic systems undergoing time-dependent changes (transients) can be described by ordinary differential equations. To solve this kind of equation, three distinct methodologies are identified: modal analysis, frequency domain analysis, and direct numerical integration. Classical modal analysis and frequency domain analysis have severe limitations, as they are based on the principle of superposition. Thus, they are not directly applicable to nonlinear systems. Although there are some analytical or semi-analytical methods for solving nonlinear structures; they are limited to specific problems with specific conditions (Bayat and Pakar 2017). The direct numerical integration method is generally more time consuming, but it is a powerful approach which has become more attractive over the last two decades due to its increased ability to analyze realistic problems and achieve accurate responses (Park 1977, Felippa and Park 1979, Dokainish and Subbaraj 1989, Paz and Leigh 2003, Chopra 2007, Mohaamad Rezaiee-Pajand and JAVAD Alamatian 2008, Gao *et al.* 2012).

The fundamental idea of the direct time integration method is to approximate the solutions of the equation of motion with a set of algebraic equations which are evaluated in a step-by-step approach in time (Park 1977, Felippa and Park 1979, Dokainish and Subbaraj 1989, Humar 1990, Belytschko and Lu 1993, Chen *et al.* 2000, Paz and Leigh 2003, Chopra 2007, Liu *et al.* 2013).Within each time interval, a specific type of variation of the displacement, velocity, and acceleration is assumed. Several numerical integration algorithms are available for each type of variation assumed. This procedure is a form of finite difference solution for differential equations (Clough and Penzien 1983, Bathe 1996, Belytschko *et al.* 2000, Chen *et al.* 2000, Paz and Leigh 2003, Sha *et al.* 2003, Keierleber and Rosson 2005, Leontiev 2007, Hejranfar and Parseh 2016, Lindsay *et al.* 2016).

There are two basic categories of solving an equation of motion by using step-by-step integration methods: explicit method and implicit method (Pezeshk and Camp 1995, Paz and Leigh 2003, Alamatian 2013, Chang *et al.* 2015). The explicit method deals with the equation of motion in one time step to approximate the quantities of another time step (Houbolt 1950, Hughes 1987, Pezeshk and Camp 1995, Chang 2007, 2010). The implicit method deals with the equation of motion in the current time step to determine the quantities of the same time step (Tamma and Namburu 1988, Zhou and Tamma 2004, Bathe and Baig 2005, Bathe 2007, Razavi *et al.* 2007, Leontyev 2010, Gholampour *et al.* 2011, Bathe and Noh 2012, Gholampour and Ghassemieh 2013, Soares 2016).

The predictor-corrector integration method is a combination of the explicit and implicit integration methods, in which displacement and velocity are assumed to be functions of accelerations of several previous time steps (Howe 1991, Zhai 1996, Lourderaj *et al.* 2007, Rezaiee-Pajand and Alamatian 2008).

Implicit methods can be conditionally or unconditionally stable (Wilson *et al.* 1972, Krieg 1973, Hughes and Belytschko 1983, Dokainish and Subbaraj 1989, Subbaraj and Dokainish 1989, Pezeshk and Camp 1995, Bathe 1996). The Newmark family of methods stands

*Corresponding author, Faculty Research Associate
E-mail: ypark@uta.edu
[a]MSc.
E-mail: mohamadzade_71@ut.ac.ir
[b]Professor
E-mail: m.ghassemieh@ut.ac.ir

http://www.techno-press.com/journals/sem&subpage=7
ISSN: 1225-4568 (Print), 1598-6217 (Online)

in this category. In the Newmark integration method, depending on the values of the constants used, the acceleration varies linearly or remains constant within two instances of time (Newmark 1959, Wood *et al.* 1980). A very popular member of this family is the Trapezoidal Rule, in which the acceleration remains constant within two instances of time. Wilson-theta (Wilson 1962) is the other example of an implicit method.

In nonlinear systems, unconditional stable methods are preferred over any other method since there is a high chance of growing without bound. Stability and accuracy of the implicit methods have made them a popular choice for solving nonlinear dynamic problems. An unstable method makes the integration errors increase exponentially, and an arithmetic overflow can be expected even after just a few time steps (Goudreau and Taylor 1972, Bathe and Wilson 1973, Hilber 1977, Zhong and Zhu 1996, Chopra 2007, Kim and Kim 2015).

This paper deals with the accuracy and stability of direct time integration methods and proposes a method in which acceleration varies in a quadratic manner. By defining two parameters, a new family of unconditionally stable methods is created. The proposed method overcomes the stability problem, which is believed to be an imperfection of the classical methods. Although some classical methods are unconditionally stable, such as the trapezoidal rule, they demonstrate a lower order of accuracy than the proposed method. A proven way to deal with this fault is to use shorter time steps, but the duration of solving then becomes much greater, especially when the structure includes a number of degrees of freedom. Most problems which are related to finite element analysis are of this category. The simplicity of implementing the nonlinear solution systems of the finite element analysis makes the proposed method appealing.

2. Proposed algorithm

Consider the equation of motion in a single degree of freedom system with nonlinear stiffness, which is written in the following form

$$M\ddot{D}_{t+\Delta t} + C\dot{D}_{t+\Delta t} + K_t \Delta D_{t+\Delta t} = R_{t+\Delta t} - F_{s_t} \tag{1}$$

in which D is the displacement, $\dot{D}$ is the velocity, $\ddot{D}$ is the acceleration, M is the mass, C is the damping, K_t is the tangent stiffness, F_{s_t} is the internal force of the system at time t, and R is the exciting force. Notice that the incremental form of displacement is written between time t and t+Δt.

Considering a second ordered variation of acceleration given by Eq. (2)

$$\ddot{D}_{t+\Delta t} = \ddot{D}_t + \Delta t \cdot \dddot{D}_t + \frac{\Delta t^2}{2} \cdot \ddddot{D}_t \tag{2}$$

Integration from Eq. (2) leads to the formulas of velocity and displacement

$$\dot{D}_{t+\Delta t} = \dot{D}_t + \Delta t \cdot \ddot{D}_t + \frac{\Delta t^2}{2} \cdot \dddot{D}_t + \mu \cdot \Delta t^3 \cdot \ddddot{D}_t \tag{3}$$

$$D_{t+\Delta t} = D_t + \Delta t \cdot \dot{D}_t + \frac{\Delta t^2}{2} \cdot \ddot{D}_t + \frac{\Delta t^3}{6} \cdot \dddot{D}_t + \upsilon \cdot \Delta t^4 \cdot \ddddot{D}_t \tag{4}$$

In which υ and μ are constants designed to control the behavior of this method and to make it unconditionally stable. These constants are discussed in detail in the stability section.

The higher order derivatives for current time step are obtained as follows

$$\dddot{D}_t = \frac{1}{2\Delta t}\left(\ddot{D}_{t+\Delta t} - \ddot{D}_{t-\Delta t}\right) \tag{5}$$

$$\ddddot{D}_t = \frac{1}{\Delta t^2}\left(\ddot{D}_{t+\Delta t} - 2\ddot{D}_t + \ddot{D}_{t-\Delta t}\right) \tag{6}$$

By introducing Eqs. (5) and (6) into Eqs. (3) and (4), the standard form of new implicit method is generated as follows

$$\begin{aligned} D_{t+\Delta t} = D_t + \Delta t \cdot \dot{D}_t + \Bigg[\left(\upsilon - \frac{1}{12}\right) \cdot \ddot{D}_{t-\Delta t} + \\ \left(\frac{1}{2} - 2\upsilon\right) \cdot \ddot{D}_t + \left(\upsilon + \frac{1}{12}\right) \cdot \ddot{D}_{t+\Delta t}\Bigg] \cdot \Delta t^2 \end{aligned} \tag{7}$$

$$\begin{aligned} \dot{D}_{t+\Delta t} = \dot{D}_t + \Bigg[\left(\mu - \frac{1}{4}\right) \cdot \ddot{D}_{t-\Delta t} + \\ (1 - 2\mu) \cdot \ddot{D}_t + \left(\mu + \frac{1}{4}\right) \cdot \ddot{D}_{t+\Delta t}\Bigg] \cdot \Delta t \end{aligned} \tag{8}$$

Eqs. (7) and (8) can be used to approximate the displacement and velocity at time t+Δt, respectively. It can be proven that this strategy guarantees the second-order accuracy for any choices of υ and μ. For the special case of assuming μ=1/4 and υ=1/2, the proposed method leads to the linear acceleration method.

According to Eqs. (7) and (8), prior to starting the calculations, it is necessary to have the responses at the initial time step (t=0) and second time step (t=Δt). Note that D_0 and $\dot{D}_0$ are initial known quantities, and $\ddot{D}_0$ can be calculated by using Eq. (1) at initial time step (t=0). Any one-step method, such as the Trapezoidal Rule, can be utilized to get the response at the next time step (t=Δt).

Acceleration at the next time step ($\ddot{D}_{2\Delta t}$) is calculated by having the responses at the initial time step (t=0), as well as next time step (t=Δt), and introducing Eqs. (7) and (8) into Eq. (1). By substituting the calculated quantities in Eqs. (7) and (8), displacement and velocity at the next time step ($D_{2\Delta t}$, $\dot{D}_{2\Delta t}$) are calculated. To reach the answers of displacement, velocity, and acceleration at a specified time, this procedure must be repeated until the related time step. Section 4 of this article presents the simplified and computerized algorithm of this recursive procedure by defining some constants and matrices.

3. Stability

Evaluation of the stability is carried out, considering the equation of motion for a single degree of freedom, with free vibration at time step n+1 and calculating the amplification matrix [A]. The algorithm is stable if the eigenvalues of the amplification matrix are less than unit in modulus (Bathe and Wilson 1973, Chen *et al.* 2000, Chang 2002, Gholampour *et al.* 2011, Gholampour and Ghassemieh 2013).

Eq. (9) shows recursive matrix form of the proposed method in a free vibration.

$$\begin{Bmatrix} \ddot{D}_{t+\Delta t} \\ \ddot{D}_t \\ \dot{D}_{t+\Delta t} \\ D_{t+\Delta t} \end{Bmatrix} = [A] \begin{Bmatrix} \ddot{D}_t \\ \ddot{D}_{t-\Delta t} \\ \dot{D}_t \\ D_t \end{Bmatrix} \tag{9}$$

in which the amplification matrix and the constants of the matrix is obtained, as follows

$$[a] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$\begin{aligned}
a_{11} &= -(1/2-2\upsilon)\beta - (2-4\mu)\gamma \\
a_{12} &= -(\upsilon-1/12)\beta - (2\mu-1/2)\gamma \\
a_{13} &= \frac{1}{\Delta t}(-\beta-2\gamma); \quad a_{14} = \frac{1}{\Delta t^2}(-\beta) \\
a_{21} &= 1; \; a_{22} = a_{23} = a_{24} = 0 \\
a_{31} &= [1-2\mu-(1/2-2\upsilon)(\mu+1/4)\beta + \\
&\quad -(1-2\mu)(2\mu+1/2)\gamma]\cdot\Delta t \\
a_{32} &= [\mu-1/4-(\mu+1/4)(\upsilon-1/12)\beta + \\
&\quad -(\mu-1/4)(2\mu+1/2)\gamma]\cdot\Delta t \\
a_{33} &= 1-(\mu+1/4)\cdot(\beta+2\gamma); \quad a_{34} = -\frac{\beta}{\Delta t}(\mu+1/4) \\
a_{41} &= [1/2-2\upsilon-(1/2-2\upsilon)(\upsilon+1/12)\beta + \\
&\quad -(2-4\mu)(\upsilon+1/12)\gamma]\cdot\Delta t^2 \\
a_{42} &= [\upsilon-1/12-(\upsilon-1/12)(\upsilon+1/12)\beta + \\
&\quad -(2\mu-1/2)(\upsilon+1/12)\gamma]\cdot\Delta t^2 \\
a_{43} &= (1-(\upsilon+1/12)\cdot(\beta+2\gamma))\cdot\Delta t \\
a_{44} &= 1-(\upsilon+1/12)\cdot\beta
\end{aligned} \tag{10}$$

in which

$$\beta = \left(\frac{1}{\omega_n^2\Delta t^2} + \frac{2\xi(\mu+1/4)}{\omega_n\Delta t} + (\upsilon+1/12)\right)^{-1}; \gamma = \frac{\xi\beta}{\omega_n\Delta t} \tag{11}$$

In the above equations, ξ represents the damping ratio, and ω_n is the natural frequency of step n equivalence to tim t to t+Δt. Due to material nonlinearity, stiffness has different values in different steps, as does natural frequency.

Eigenvalues of the amplification matrix can be calculated. For various values of ϑ and μ and for the most critical case being ξ=0, the stability condition is in the form

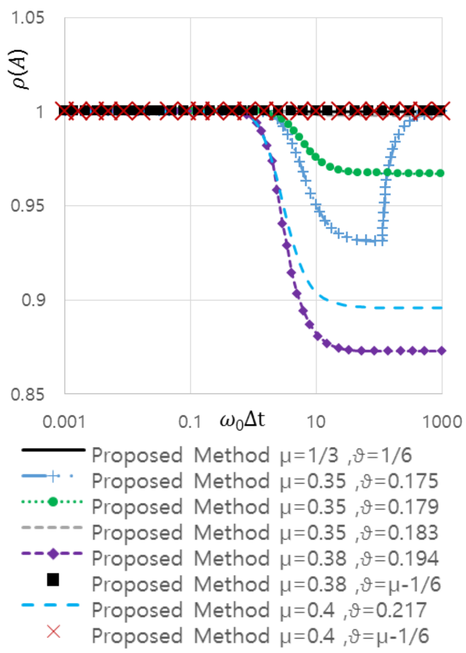

Fig. 1 Spectral radius as a function of natural frequency multiplied by time step

of the following equation

$$\mu \geq 1/3; \quad \mu/2 \leq \upsilon \leq \mu - 1/6 \tag{12}$$

Fig. 1 shows the spectral radius as the maximum eigenvalue of the amplification matrix in modulus (A), as a function of natural frequency of the system multiplied by the time step size. According to this figure, the mentioned condition in Eq. (12) is a border between conditional and unconditional stability. In other words, if values of parameters μ and υ meet the condition of Eq. (12), the spectral radius will remain less than unit. There are no time step size limitations in Eq. (12); therefore, the proposed method becomes unconditionally stable. Another interesting fact about this figure is that if μ≥1/3 then the spectral radius value remains unit, provided that υ=μ−1/6.

4. Accuracy

Accuracy of the time step marching methods is measured by three significant factors: order of accuracy, amplitude decay, and period elongation. The proposed method is second-order accurate. Percentage amplitude decay and percentage period elongation can be evaluated by following the approach presented in several studies (Wilson *et al.* 1972, Bathe 1996, Bathe and Baig 2005). The evaluation is performed by solving a SDOF system without damping, and with unit initial displacement and zero initial velocity.

Fig. 2 presents the percentage amplitude decay for various methods, as a function of $\Delta t/T$, where Δt is the time step size and T is the natural period of the system. The proposed method is seen to have a reasonable amount of amplitude decay when parameters have been adopted in the range given in Eq. (12). In some nonlinear cases, this

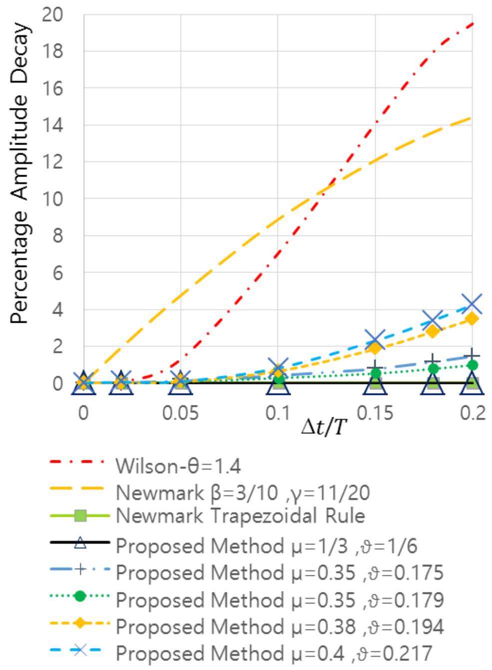

Fig. 2 Percentage amplitude decay for various methods

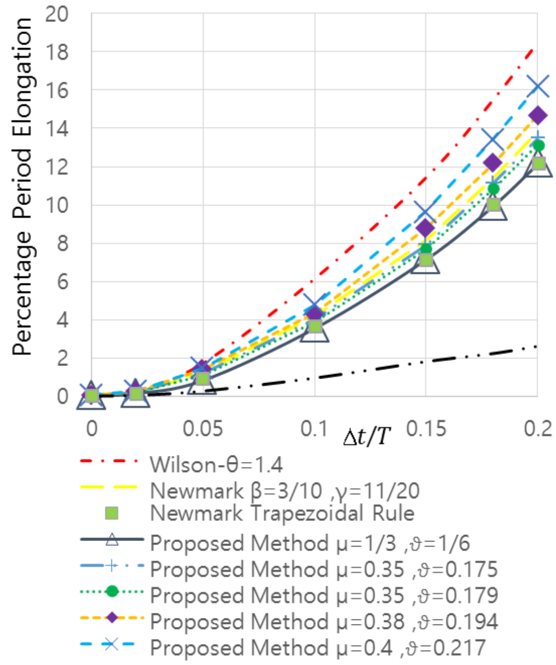

Fig. 3 Percentage period elongation for various methods

amount of amplitude decay would help to prevent the answer from growing without bound. On the other hand, there is a special choice of parameters which produces zero amplitude decay. This means that the amount of amplitude decay in the proposed method can be controlled through parameters μ and υ. It is worthy of paying attention that choosing $\mu>1/3$ and $\upsilon=\mu-1/6$ will produce no amplitude decay in the response.

Fig. 3 present the percentage period elongation for various methods. According to this figure, the proposed method produces the same amount of period elongation that is produced by the Newmark Trapezoidal Rule. Other choices of constant parameters given in Eq. (12), in the unconditional stability range, result in slightly higher percentage period elongation; as they demonstrate for Newmark family of methods. Please be noticed that the choices of parameters being $\mu=0.38$ or $\mu=0.35$ with $\upsilon=\mu-1/6$ produce an amount of period elongation close to Newmark trapezoidal rule; thus, in order to have a clear figure, their related curves have been removed from the Fig. 3.

To summarize, according to Figs. 2 and 3, the amplitude decay and period elongation errors are the same for the proposed method and the Newmark Trapezoidal Rule. In pseudo-dynamic testing, or in some nonlinear cases damping out high-frequency responses, is particularly important because the higher modes of a system are more sensitive to experimental errors than the lower ones (Shing and Mahin 1985). However when introducing numerical damping (or amplitude decay) in the Newmark family of methods $\gamma\neq1/2$, the method drops from second-order to first-order accurate, a significant reduction in the accuracy of the method. The proposed method, regardless of the values chosen for constant parameters, is second-order accurate, which is a huge advantage.

5. Steps of the proposed algorithm

Steps of the proposed algorithm for the nonlinear structural dynamic problems are as follows:

1. Initial Calculations

A. Form elastic stiffness matrix $[K]$, mass matrix $[M]$, and damping matrix $[C]$.

B. Select time step duration Δt and specify parameters υ and μ as follows:

$$\mu\geq1/3;\ \mu/2\leq\upsilon\leq\mu-1/6$$

C. Calculate acceleration ($\ddot{D}_1$), velocity ($\dot{D}_1$), and displacement (D_1) vectors at the end of the first time step, using a one-step method.

D. Calculate the following integration constants obtained from Eqs. (7) and (8):

$$a_1=(\upsilon-1/12)\cdot\Delta t^2\,;\ a_2=(1/2-2\upsilon)\cdot\Delta t^2$$
$$a_3=(\upsilon+1/12)\cdot\Delta t^2\,;\ a_4=(\mu-1/4)\cdot\Delta t$$
$$a_5=(1-2\mu)\cdot\Delta t\qquad;\ a_6=(\mu+1/4)\cdot\Delta t$$

2. For each time step iteration:

A. Calculate tangent stiffness matrix $[K_t]$ according to stress level in the material behavior and related constitutive matrix.

B. Calculate the following matrices obtained by inputting Eqs. (7) and (8) into Eq. (1):

$$b_1=[C]+[K_t]\cdot\Delta t$$
$$b_2=[C]\cdot\Delta t\cdot(1-2\mu)+[K_t]\cdot\Delta t^2\cdot(1/2-2\upsilon)$$
$$b_3=[C]\cdot\Delta t\cdot(\mu-1/4)+[K_t]\cdot\Delta t^2\cdot(\upsilon-1/12)$$
$$b_4=[M]+[C]\cdot\Delta t\cdot(\mu+1/4)+[K_t]\cdot\Delta t^2\cdot(\upsilon+1/12)$$

C. The acceleration vector at the end of the time step can be calculated as follows:

$$\ddot{D}_{t+\Delta t}=(b_4)^{-1}\cdot\left(R_{t+\Delta t}-F_{s_t}-b_1\cdot\dot{D}_t-b_2\cdot\ddot{D}_t-b_3\cdot\ddot{D}_{t-\Delta t}\right)$$

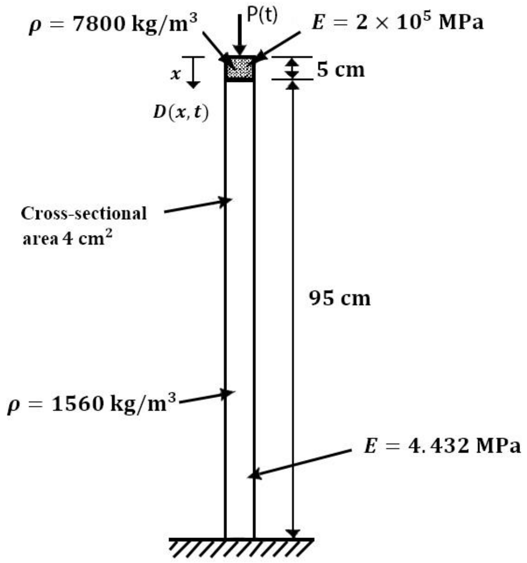

Fig. 4 Bar under dynamic axial force

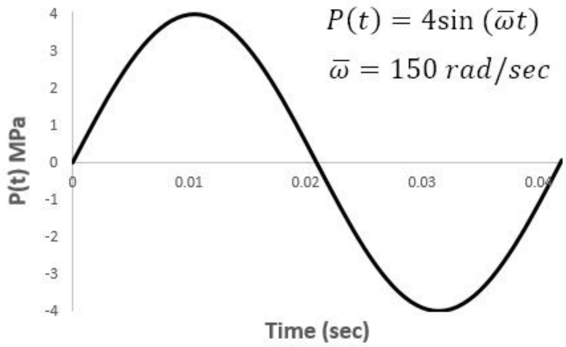

Fig. 5 Exciting force equation acting on the bar

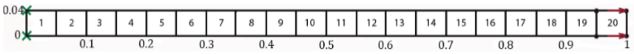

Fig. 6 Finite element model of the bar structure

$$D_{t+\Delta t} = D_t + \Delta t \cdot \dot{D}_t + \Delta t^2 \cdot (1/2 - 2\upsilon) \cdot \ddot{D}_t + $$
$$+ \Delta t^2 \cdot (\upsilon - 1/12) \cdot \ddot{D}_{t-\Delta t} + \Delta t^2 \cdot (\upsilon + 1/12) \cdot \ddot{D}_{t+\Delta t}$$
$$\dot{D}_{t+\Delta t} = \dot{D}_t + \Delta t^2 \cdot (1 - 2\mu) \cdot \ddot{D}_t + $$
$$+ \Delta t \cdot (\mu - 1/4) \cdot \ddot{D}_{t-\Delta t} + \Delta t^2 \cdot (\mu + 1/4) \cdot \ddot{D}_{t+\Delta t}$$

3. In the equilibrium path iteration i, Newton-Raphson, Modified Newton-Raphson, or any other equilibrium path tracker algorithm can be utilized in order to find ΔD, as follows:

$$\Delta R^i_{t+\Delta t} = P_{t+\Delta t} - M \cdot \ddot{D}_{t+\Delta t} - C \cdot \dot{D}_{t+\Delta t} - F_{st}^{\,i}$$
$$K_t \cdot \Delta D^i = \Delta R^i_{t+\Delta t}$$

4. In the material point iteration, based on the above displacement increment, values of strain increment $\Delta\varepsilon$ and stress increment $\Delta\sigma$ can be obtained by using the radial return algorithm or other algorithms. Thus, according to the $[B]$ matrix, which is the strain displacement transformation matrix, the following internal force is obtained:

$$F_{st}^{\,i+1} = \int [B]^T \cdot \sigma \cdot d\Omega$$

5. Verify whether the tolerance condition for equilibrium path tracking algorithm is met. If it is, proceed to the next time step; if it is not, depending on the chosen equilibrium path tracking algorithm, any of the steps above could be the next step.

Note that in linear cases, steps 2.A and 2.B can be performed in the initial calculations, so steps 3 and 4 must be eliminated. In that case, internal force can be calculated as follows:

$$F_{s\,t+\Delta t} = K_t \cdot D_{t+\Delta t}$$

6. Numerical examples

To provide a quantitative assessment of the proposed method, the following two examples are benchmark problems chosen from (Bathe 1996), and the rest of the examples were generated by the authors.

It is noteworthy that the analyses for all of the examples were conducted through a code written using MATLAB software. Nonlinear examples used the isotropic hardening material model, along with the von Mises yield function for the analysis (Chen and Han 2007) and the Modified Newton-Raphson algorithm, to track the equilibrium path (Crisfield 1979).

6.1 Example 1 -Bar problem

The bar shown in Fig. 4 was initially at rest and was subjected to a dynamic concentrated end load, as shown in Fig. 5. The response of the bar at time 0.01 *sec.* was sought. It is noteworthy that the bar consisted of two materials, which gave it a stiff and flexible section, and the load was applied to the stiff section of the bar. Both stiff and flexible sections are represented using the consistent mass matrix.

In the analysis of this problem, the bar has been assumed to have linear elastic behavior, and according to (Bathe 1996), the static correction rendered the superposition method highly improved in accuracy. Consequently, in this example, the mode superposition method is considered as the reference solution.

Solving this problem requires a sufficient number of elements; therefore, as shown in Fig. 6, the problem is modeled with twenty quadrilateral four-noded isoparametric elements.

Fig. 7 shows the bar's displacement responses, using the mode superposition, Trapezoidal Rule with two different time increments (Bathe 1996), and the proposed method. The results of the reference solution and Trapezoidal Rule, using Δt=0.0004 sec. time increment and the proposed method using Δt=0.002 sec. time increment, are very similar. A close look at Fig. 7 reveals perceptible differences between the results around the middle (coordinate 0.4 m to 0.6 m from the bottom). In this region, the proposed method, with a time increment of 5 times greater than the one used in the Trapezoidal Rule, still provides acceptable accuracy, even with the use of a quadrilateral element.

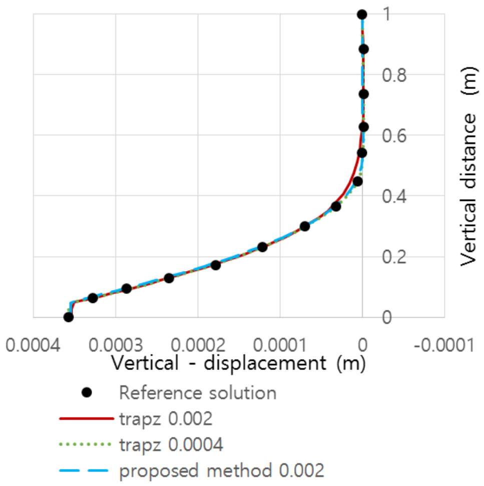

Fig. 7 Responses of the axially loaded bar

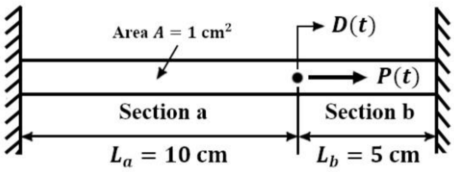

Fig. 8 Clamped uniaxial bar

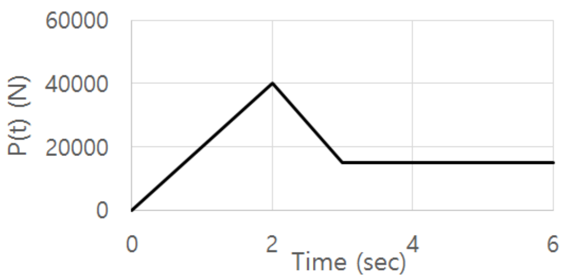

Fig. 9 Load acting on the clamped uniaxial bar

6.2 Example 2 - Clamped uniaxial bar problem

This example assessed the behavior of the proposed method in tracking the equilibrium path in inelastic regions of the analysis. According to (Bathe 1996), the load is applied slowly so that the entry to the inelastic part of the analysis is slow, which results in assessing merely the behavior of the method in inelastic regions.

Consider the clamped uniaxial bar shown in Fig. 8. This bar was subjected to an axial dynamic load P (t), which is shown in Fig. 9. The response of D (t) (displacement of the point shown in Fig. 9) as a function of P (t) was sought.

Fig. 10 shows that the material used in the uniaxial bar behaved in a bilinear manner, in both tension and compression.

As is shown in Fig. 11, in order to get satisfactory responses, the bar structure was modeled using thirty quadrilateral four-noded isoparametric elements.

The solution of this example was achieved by employing three different numerical procedures, namely Trapezoidal Rule, Wilson-theta, and the proposed method. From the results presented in Fig. 12, it is seen that, as

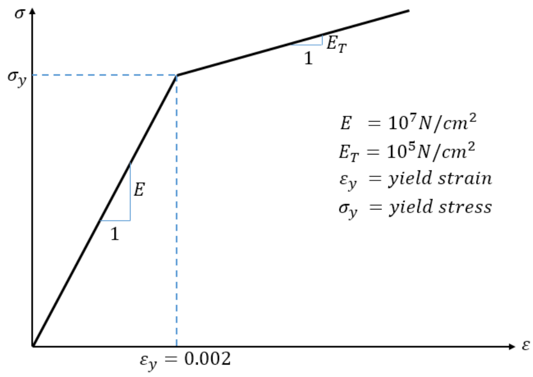

Fig. 10 Stress-strain relation

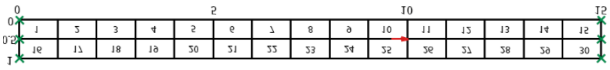

Fig. 11 Finite element model of the clamped bar structure

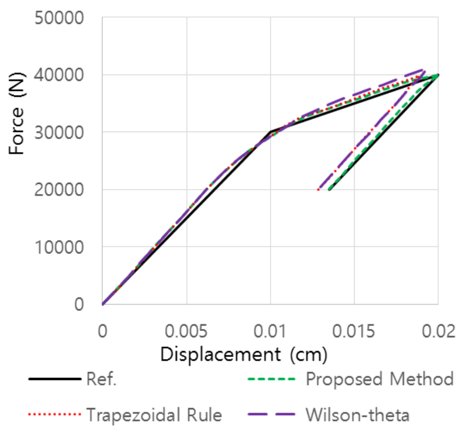

Fig. 12 Force-displacement response of the clamped beam

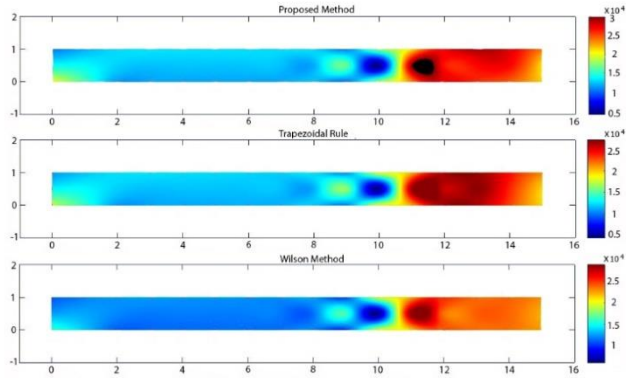

Fig. 13 Stress contour in clamped bar at 6^{th} second

expected, the responses were very similar in the elastic region. However, in the inelastic region, there were deviations from the referenced solution for the Trapezoidal Rule and Wilson-theta method, while the proposed method kept tracking the reference solution.

To analyze further, the von Mises stress contour of the clamped bar at the 6^{th} second of the analysis is presented in Fig. 13. From the stress color bar provided in the figure, it can be observed that in the loading region, the proposed method resulted in having different stress distribution than

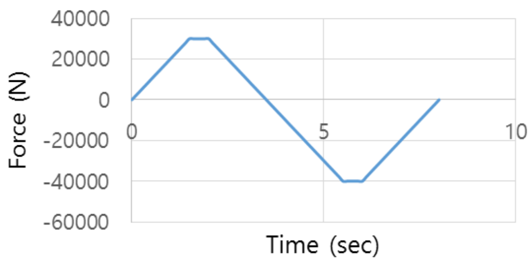

Fig. 14 Dynamic force pattern applied to the clamped bar

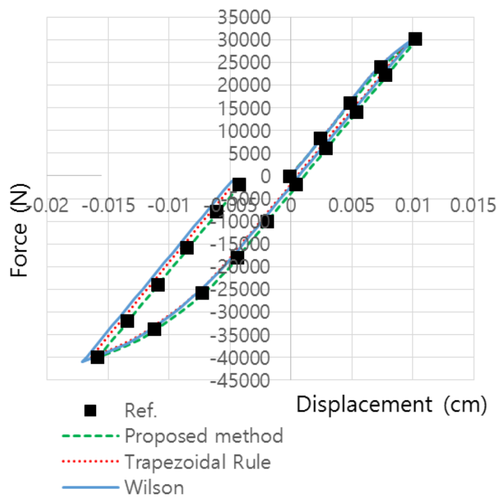

Fig. 15 Responses of the clamped beam under second load shape

Table 1 Solution time of each method for second load shape

Methods	Proposed Method	Trap. Rule	Wilson Method
Solution Time (sec.)	93.4	104.7	267.3

the other two methods.

In order to highlight the number of errors created by each method, the dynamic load shown in Fig. 14 was applied to the bar. Since this loading was more periodic, it revealed that negligible errors of the first inelastic zone experience perpetuated as the method experienced more inelastic zones, until the errors became tangible.

As shown in Fig. 15, the responses confirmed that the errors increased in the elastic-inelastic zone of the analysis cycle. This means that the Trapezoidal Rule and the Wilson method continued deviating from the reference solution as the analysis progressed, until the errors became tangible. For the proposed method, as presented in Fig. 15, the deviation was less than for the other two methods, resulting in more reliable responses.

For the dynamic analysis, the time increment of 0.1 sec. was used for all of the methods except for the reference response, in which a time step of 0.01 sec. was used in the Newmark Trapezoidal Rule. The time increment of 0.1 sec. made other numerical methods, other than unconditional stable ones, grow without bound. The duration of solving the second shape of loading is given in Table 1.

Meeting the tolerance condition is a major difficulty in nonlinear problems; therefore, it takes longer to solve the problems. The higher order of accuracy in the proposed method helps to meet the tolerance conditions more quickly than the other methods do, which is why the proposed method has the lowest solution time.

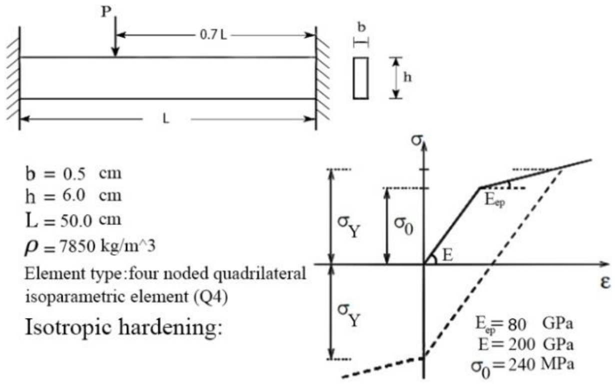

Fig. 16 Clamped beam and its properties

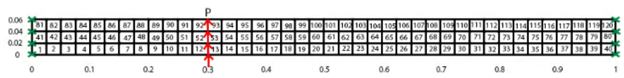

Fig. 17 Finite element mesh of the clamped beam with exciting force

Fig. 18 Exciting force applied to each one of the loaded nodes

6.3 Example 3 - Clamped beam problem

A clamped beam was considered in this example to provide better insight into the effectiveness of the proposed method in solving nonlinear problems, using numerical damping. The beam was subjected to an exciting harmonic force (Fig. 16).

Fig. 17 presents the finite element mesh of the modeled beam, consisting of one hundred and twenty quadrilateral four-noded isoparametric elements. The exciting forces shown in Fig. 17 are of the form shown in Fig. 18.

Analysis of this problem was conducted by using different methods, and it was found that only unconditional stable methods with a time step of lower than 0.2 sec. offered satisfactory responses. Analysis of this problem with a time step of longer than 0.2 sec. would need numerical damping. As is mentioned in section 4, numerical damping in the Newmark method results in lowering the order of accuracy, which makes the responses grow without bound.

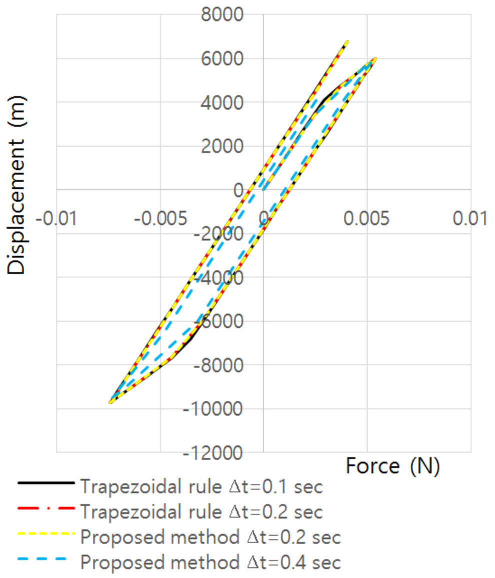

Fig. 19 Responses of the clamped beam

Table 2 CPU solution time of different methods

Methods	Trap. Rule Δt=0.1	Trap. Rule Δt=0.2	Prop. Method Δt=0.2.	Prop. Method Δt=0.4
Solution Time (sec.)	75.2	41.5	35.7	24.7

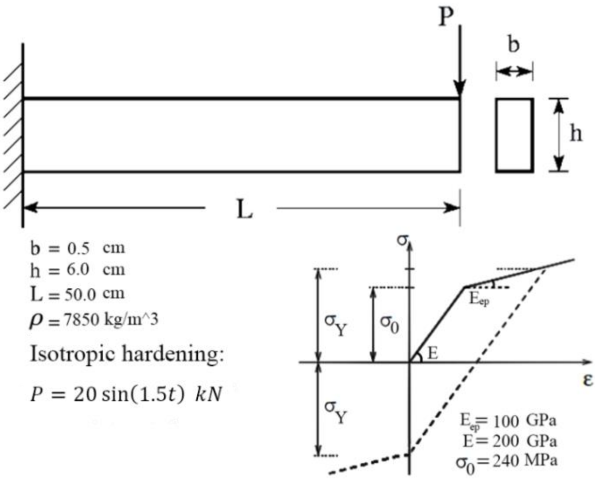

Fig. 20 Cantilever beam and its properties

However, Fig. 19 shows that the proposed method, with a time step of 0.4 sec., continues to give acceptable responses. The fact that all of the other methods grow without bound or give extraneous responses using a time step longer than 0.2 sec., while the proposed method continues giving acceptable responses with a time step of 0.4 sec., makes the proposed method a reliable method in dealing with problems with high nonlinearity.

Table 2 presents the CPU solution time of each method. According to this table, the proposed method, as mentioned in example 2, due to having higher order of accuracy, had lower solution time. Moreover, if the error created by this method using time step of 0.4 sec is acceptable, the proposed method is deemed to be the most satisfactory.

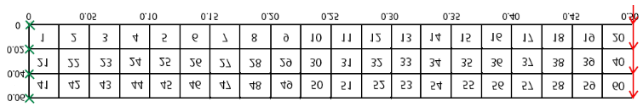

Fig. 21 Finite element mesh of the cantilever beam

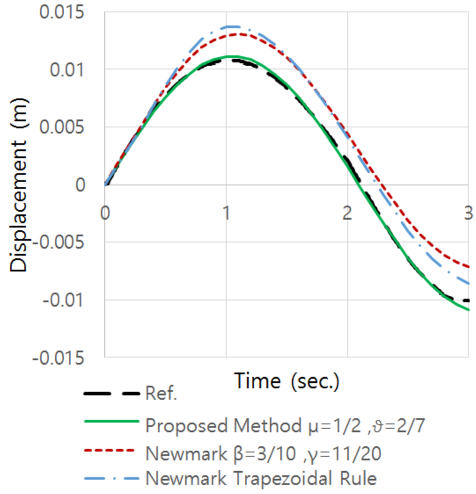

Fig. 22 Displacement of the free end of the beam with respect to time

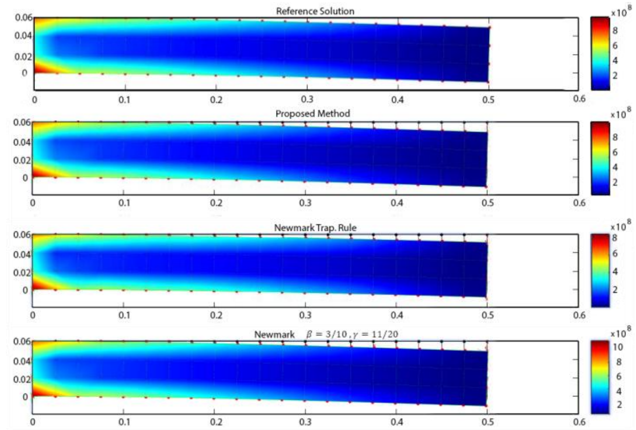

Fig. 23 Von Mises stress distributions at time 3 *sec.*

6.4 Example 4 - Cantilever beam problem

To further investigate the strength of the proposed method in nonlinear problems, the free end of a cantilever beam was subjected to an exciting harmonic force (Fig. 20). Numerical damping was used to solve this example. All of the methods were implemented with a time step of 0.1 sec. except for the reference solution, (Newmark method), in which a time step of 0.01 sec. was used.

Fig. 21 presents the finite element mesh of the modeled beam, consisting of sixty quadrilateral four-noded isoparametric elements. The load was applied at the tip of the free end.

Fig. 22 illustrates the displacement at the free end of the beam. Compared to the reference solution, the displacement response of the proposed method, due to having numerical damping and higher order of accuracy, was highly accurate. According to the figure, the Newmark method with μ=1/2

& υ=2/7 yielded a better response than the Newmark Trapezoidal Rule. This was because of the numerical damping, but the numerical damping in the Newmark method lowered the order of accuracy of this method.

Fig. 23 presents the von Mises stress distributions of the methods at time 3 sec. to provide better insight into the accuracy of the responses.

According to the figure and considering the color bar of each method, it is easy to see that the stress distribution in the proposed method is the most accurate. The accuracy of the Newmark methods is low because they either lack enough order of accuracy or don’t possess numerical damping.

7. Conclusions

In this research, a new direct numerical integration algorithm is proposed, with quadratic variation of acceleration. Whether the problem is linear or nonlinear, the proposed method has a greater degree of accuracy in its responses than other unconditionally stable methods. Moreover, the duration of the solving of linear systems can be controlled by stretching the time step increment. Even though the stretching of the time increment is limited, as it lowers the accuracy, the proposed method, with a remarkably greater time increment, indicates the same accuracy as the Trapezoidal Rule. In nonlinear systems, the proposed method, with the same time step increment, yields more accurate responses than the other methods, which continue digressing from the reference solution as the analysis proceeds. The proposed method also proved to have satisfactory responses when applied to nonlinear systems, using an extended time step. Additionally, keeping the order of accuracy while numerical damping is inserted into the analysis renders the proposed method a reliable method for solving such nonlinear problems. Because of the lower number of loops needed to meet the tolerance condition in the Modified Newton-Raphson loop of analysis, the proposed method has the lowest CPU solution time. Finally the simplicity of employing this method to nonlinear systems provides an additional motive to offering the proposed method.

References

Alamatian, J. (2013), “New implicit higher order time integration for dynamic analysis”, *Struct. Eng. Mech.*, **48**(5), 711-736.

Bathe, K.J. (2007), “Conserving energy and momentum in nonlinear dynamics: a simple implicit time integration scheme”, *Comput. Struct.*, **85**(7), 437-445.

Bathe, K.J. and Baig, M.M.I. (2005), “On a composite implicit time integration procedure for nonlinear dynamics”, *Comput. Struct.*, **83**(31), 2513-2524.

Bathe, K.J. and Noh, G. (2012), “Insight into an implicit time integration scheme for structural dynamics”, *Comput. Struct.*, **98**, 1-6.

Bathe, K. (1996), *Finite Element Procedures*, Prentice-Hall, Englewood Cliffs, NJ.

Bathe, K.J. and Wilson, E.L. (1973), “Stability and accuracy analysis of direct time integration methods”, *Earthq. Eng. Struct. Dyn.*, **1**, 283-291.

Bayat, M. and Pakar, I. (2017), “Accurate semi-analytical solution for nonlinear vibration of conservative mechanical problems”, *Struct. Eng. Mech.*, **61**(5), 657-661.

Belytschko, T. and Lu, Y. (1993), “Explicit multi-time step integration for first and second order finite element semi-discretizations”, *Comput. Meth. Appl. Mech. Eng.*, **3-4**(108), 353-383.

Belytschko, T., Liu, W.K. and Moran, B. (2000), *Nonlinear Finite Elements for Continua and Structures*, 3rd Edition, Wiley, Chichester, U.K.

Chang, S.Y. (2002), “Integrated equations of motion for direct integration methods”, *Struct. Eng. Mech.*, **13**(5), 569-589.

Chang, S.Y. (2007), “Improved explicit method for structural dynamics”, *J. Eng. Mech.*, **133**(7), 748-760.

Chang, S.Y. (2010), “A new family of explicit methods for linear structural dynamics”, *Comput. Struct.*, **88**(11), 755-772.

Chang, S.Y., Wu, T.H. and Tran, N.C. (2015), “A family of dissipative structure-dependent integration methods”, *Struct. Eng. Mech.*, **55**(4), 815-837.

Chen, S., Hansen, J.M. and Tortorelli, D.A. (2000), “Unconditionally energy stable implicit time integration: application to multibody system analysis and design”, *Int. J. Numer. Meth. Eng.*, **48**(6), 791-822.

Chen, W.F. and Han, D.J. (2007), *Plasticity for Structural Engineers*, J. Ross Publishing.

Chopra, A. (2007), *Dynamics of Structures: Theory and Applications to Earthquake Engineering*, 3rd Ed. Edition, Prentice-Hall, Upper Saddle River, NJ.

Clough, R.W. and Penzien, J. (1983), *Dynamics of Structures*, McGraw Hill, New York.

Crisfield, M. (1979), “A faster modified Newton-Raphson iteration”, *Comput. Meth. Appl. Mech. Eng.*, **20**(3), 267-278.

Dokainish, M. and Subbaraj, K. (1989), “A survey of direct time-integration methods in computational structural dynamics-I. Explicit methods”, *Comput. Struct.*, **32**(6), 1371-1386.

Felippa, C.A. and Park, K.C. (1979), “Direct time integration methods in nonlinear structural dynamics”, *Comput. Meth. Appl. Mech. Eng.*, **17-18**(2), 277-313.

Gao, Q., Wu, F., Zhang, H., Zhong, W., Howson, W. and Williams, F. (2012), “A fast precise integration method for structural dynamics problems”, *Struct. Eng. Mech.*, **43**(1), 1-13.

Gholampour, A.A. and Ghassemieh, M. (2013), “Nonlinear structural dynamics analysis using weighted residual integration”, *Mech. Adv. Mater. Struct.*, **20**, 199-216.

Gholampour, A.A., Ghassemieh, M. and Razavi, H. (2011), “A time stepping method in analysis of nonlinear structural dynamics”, *Appl. Comput. Mech.*, **5**, 143-150.

Goudreau, G.L. and Taylor, R.L. (1972), “Evaluation of numerical integration methods in elastodynamics”, *Comput. Meth. Appl. Mech. Eng.*, **2**, 69-97.

Hejranfar, K. and Parseh, K. (2016), “Numerical simulation of structural dynamics using a high-order compact finite-difference scheme”, *Appl. Math. Model.*, **40**(3), 2431-2453.

Hilber, H.M. (1977), *Analysis and Design of Numerical Integration Methods in Structural Dynamics*, Earthquake Engineering Research Center, University of California, Berkeley.

Houbolt, J.C. (1950), “A recurrence matric solution for the dynamic response of aircraft in gusts”, *NACA TN*, **2060**.

Howe, R. (1991), “A new family of real-time redictor-corrector integration algorithms”, *Simul.*, **57**(3), 177-186.

Hughes, T. (1987), *The Finite Element Methods*, Eaglewood Cliffs, New Jersy, NJ.

Hughes, T. and Belytschko, T. (1983), “A precis of developments in computational methods for transient analysis”, *J. Appl. Mech.*, **50**(4b), 1033-1041.

Humar, J.L. (1990), *Dynamics of Structures*, Prentice-Hall, Englewood Cliffs, NJ.

Keierleber, C. and Rosson, B. (2005), "Higher-order implicit dynamic time integration method", *J. Struct. Eng.*, **131**(8), 1267-1276.

Kim, J. and Kim, D. (2015), "A quadratic temporal finite element method for linear elastic structural dynamics", *Math. Comput. Simul.*, **117**, 68-88.

Krieg, R. (1973), "Unconditional stability in numerical time integration methods", *J. Appl. Mech.*, **40**(2), 417-421.

Leontiev, V. (2007), "Extension of LMS formulations for L stable optimal integration methods with U0-V0 overshoot properties in structural dynamics: the level symmetric (LS) integration methods", *Int. J. Numer. Meth. Eng.*, **71**(13), 1598-1632.

Leontyev, V. (2010), "Direct time integration algorithm with controllable numerical dissipation for structural dynamics: two-step Lambda method", *Appl. Numer. Math.*, **60**(3), 277-292.

Lindsay, P., Parks, M. and Prakash, A. (2016), "Enabling fast, stable and accurate peridynamic computations using multi-time-step integration", *Comput. Meth. Appl. Mech. Eng.*, **306**, 382-405.

Liu, Q., Ma, X., Bai, Z. and Zhuansun, X. (2013), "A Split-Step-Scheme-Based Precise Integration Time Domain Method for Solving Wave Equation", *COMPEL: Int. J. Comput. Math. Elec. Electron. Eng.*, **33**(1/2), 9-9.

Lourderaj, U., Song, K., Windus, T. L., Zhuang, Y. and Hase, W.L. (2007), "Direct dynamics simulations using Hessian-based predictor-corrector integration algorithms", *J. Chem. Phys.*, **126**(4), 044105.

Newmark, N.M. (1959), "A method of computation for structural dynamics", *J. Eng. Mech. Div.*, **85**(3), 67-94.

Park, K.C. (1977), "Practical aspects of numerical time integration", *Comput. Struct.*, **7**(3), 343-353.

Paz, M. and Leigh, W. (2003), *Structural Dynamics: Theory and Computation*, Springer, Netherlands.

Pezeshk, S. and Camp, C. (1995), "An explicit time-integration method for damped structural systems", *Struct. Eng. Mech.*, **3**(2), 145-162.

Pezeshk, S. and Camp, C.V. (1995), "An explicit time integration technique for dynamic analysis", *Int. J. Numer. Meth. Eng.*, **38**(13), 2265-2281.

Razavi, S., Abolmaali, A. and Ghassemieh, M. (2007), "A weighted residual parabolic acceleration time integration method for problems in structural dynamics", *Comput. Meth. Appl. Math. Comput. Meth. Appl. Math.*, **7**(3), 227-238.

Rezaiee-Pajand, M. and Alamatian, J. (2008), "Nonlinear dynamic analysis by dynamic relaxation method", *Struct. Eng. Mech.*, **28**(5), 549-570.

Rezaiee-Pajand, M. and Alamatian, J. (2008), "Numerical time integration for dynamic analysis using a new higher order predictor-corrector method", *Eng. Comput.*, **25**(6), 541-568.

Sha, D., Zhou, X. and Tamma, K. (2003), "Time discretized operators. Part 2: towards the theoretical design of a new generation of a generalized family of unconditionally stable implicit and explicit representations of arbitrary order for computational dynamics", *Comput. Meth. Appl. Mech. Eng.*, **192**(3), 291-329.

Shing, P.S.B. and Mahin, S.A. (1985), "Computational aspects of a seismic performance test method using online computer control", *Earthq. Eng. Struct. Dyn.*, **13**(4), 507-526.

Soares, D. (2016), "An implicit family of time marching procedures with adaptive dissipation control", *Appl. Math. Model.*, **40**(4), 3325-3341.

Subbaraj, K. and Dokainish, M. (1989), "A survey of direct time-integration methods in computational structural dynamics—II. Implicit methods", *Comput. Struct.*, **32**(6), 1387-1401.

Tamma, K.K. and Namburu, R.R. (1988), "A new finite element based Lax-Wendroff/Taylor-Galerkin methodology for computational dynamics", *Comput. Meth. Appl. Mech. Eng.*, **71**(2), 137-150.

Wilson, E.L. (1962), *Dynamic Response by Step-By-Step Matrix Analysis*, Labortorio Nacional de Engenharia Civil, Lisbon, Portugal, Lisbon, Portugal.

Wilson, E.L., Farhoomand, I. and Bathe, K.J. (1972), "Nonlinear dynamic analysis of complex structures", *Int. J. Earthq. Eng. Struct. Dyn.*, **3**(1), 241-252.

Wood, W., Bossak, M. and Zienkiewicz, O. (1980), "An alpha modification of Newmark's method", *Int. J. Numer. Meth. Eng.*, **15**(10), 1562-1566.

Zhai, W.M. (1996), "Two simple fast integration methods for large scale dynamic problems in engineering", *International Int. J. Numer. Meth. Eng.*, **39**(24), 4199-4214.

Zhong, W. and Zhu, J. (1996), "On a new fourth order self-adaptive time integration algorithm", *Struct. Eng. Mech.*, **4**(6), 589-600.

Zhou, X. and Tamma, K. (2004), "A new unified theory underlying time dependent linear first order systems: a prelude to algorithms by design", *Int. J. Numer. Meth. Eng.*, **60**(10), 1699-1740.

PL