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A novel simple two-unknown hyperbolic shear deformation theory for functionally graded beams

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Abstract. In this article, a novel simple higher-order shear deformation theory for bending and free vibration analysis of functionally graded (FG) beams is proposed. The beauty of this theory relies on its 2-unknowns displacement field as the Euler-Bernoulli beam theory, which is even less than the Timoshenko beam theory. A shear correction factor is, therefore, not needed. Equations of motion are obtained via Hamilton's principle. Analytical solutions for the bending and free vibration analysis are given for simply supported beams. Efficacy of the proposed model is shown through illustrative examples for bending and dynamic of FG beams. The numerical results obtained are compared with those of other higher-order shear deformation beam theory results. The results obtained are found to be accurate.

Keywords: a simple 2-unknown theory; bending; vibration; functionally graded beams

1. Introduction

It is well-known that the classical theory of bending of beam based on Euler-Bernoulli hypothesis neglects the influences of the shear deformation. The theory is applicable for thin beams and is not applicable for thick beams since it is based on the supposition that the sections normal to neutral axis before deformation remain so during deformation and after deformation, implying that the transverse shear strain is zero. Since the model disregards the transverse shear deformation, it underestimates deflections in case of deep beams where shear deformation impacts are considerable. Bresse (1859), Rayleigh (1877) and Timoshenko (1921) were the first researchers to introduce both the rotatory inertia and shear deformation effects in the beam theory. Timoshenko demonstrates that the influence of transverse shear is much greater than that of rotatory inertia on the behavior of transverse vibration of prismatic bars. This theory is also known as a first-order shear deformation theory (FSDT) of beams. However, this theory has the drawback of considering unrealistic constant transverse shear strain within the beam thickness. It also requires the employ of a shear correction factor (Adda Bedia *et al.* 2015, Meksi *et al.* 2015, Bellifa *et al.* 2016, Bouderba *et al.* 2016). The detailed investigations on employ of shear correction factors in Timoshenko beam theory are indicated by Cowper (1966), Jensen (1983), Hutchinson (2001). To remove the discrepancies in Euler-Bernoulli (EBT) and FSDT, higher order or refined shear deformation theories were proposed and are found in the open literature for bending and dynamic analysis of beam. Levinson (1981), Bickford (1982), Rehfield and Murty (1982), Krishna Murty (1984), Baluch *et al*. (1984), Bhimaraddi and Chandrashekhara (1993), Bousahla *et al*. (2014), Fekrar *et al*. (2014), Hamidi *et al*. (2015), Ait Atmane *et al*. (2015), Ait Yahia *et al*. (2015), Attia *et al*. (2015), Barati and Shahverdi (2016), Becheri *et al*. (2016), Beldjelili *et al*. (2016), Ahouel *et al* (2016), Belkorissat *et al*. (2016), Merdaci *et al*. (2016), Draiche *et al*. (2016), Klouche *et al*. (2017), Fahsi *et al*. (2017), Chikh *et al*. (2017), Meksi *et al*. (2017), Bellifa *et al*. (2017) developed nonlinear shear deformation models by considering a higher variation of axial displacement in terms of thickness coordinate. These models respect shear stress free boundary conditions on upper and lower surfaces of beam and hence obviate the need of shear correction coefficient. Irretier (1986) investigated the refined dynamical influences in linear, homogenous beam according to models, which exceed the limits of the EBT. These influences are rotary inertia, shear deformation, axial pre-stress, twist and coupling between bending and torsion. Stein (1989) proposed refined shear deformation model for deep beams including trigonometric function in terms of thickness coordinate in kinematic. However, with this model shear stress free boundary conditions are not verified at top and bottom surfaces of the beam. Ghugal and Dahake (2012) presented a flexural analysis of thick beam under parabolic load using refined shear deformation theory.

Lately, FG structures have attained a mentionable

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attention of the research community (Kar and Panda, 2015a,b.c, Alshorbagy *et al.* 2011, Ait Amar Meziane *et al.* 2014, Swaminathan and Naveenkumar 2014, Bakora and Tounsi 2015, Larbi Chaht *et al*. 2015, Akavci 2015, Liang *et al.* 2015, Arefi 2015a,b, Arefi and Allam 2015, Akbaş 2016, Celebi *et al.* 2016, Ghorbanpour Arani *et al.* 2016, Bousahla *et al.* 2016, Hadji *et al.* 2016, El-Haina *et al.* 2017, Benahmed *et al.* 2017). Benatta *et al*. (2009), Sallai *et al*. (2009) investigated the static response of simply supported FG hybrid beams subjected to uniformly distributed transverse loads by using a higher-order shear deformation theory. The finite element method and the third-order shear deformation theory (TSDT) are employed by Kadoli *et al*. (2008) to examine the bending of FG beams by considering different boundary conditions. Sankar (2001) proposed a beam model to study the static problem of a simply supported beam. Li (2008) discussed the bending and transverse vibrations problem of FG Timoshenko beams. Hebali *et al*. (2014) developed a novel quasi-three-dimensional hyperbolic shear deformation theory for the static and dynamic analysis of FG plate. Belabed *et al*. (2014) proposed an efficient and simple higher order shear and normal deformation theory for FG plates. Bourada *et al*. (2015) developed a new simple and refined trigonometric higher-order beam theory for static and vibration of FG beams with including the thickness stretching effect. Recently, a new class of plate theories with shear deformation effect is developed by both Tounsi *et al*. (2016) and Houari *et al*. (2016) by using only three unknowns in displacement field. Hassaine Daouadji and Adim (2017) investigated the mechanical behavior of FG sandwich plates using a quasi-3D higher order shear and normal deformation theory.

This article presents a new simple two -unknown hyperbolic shear deformation theory for FG beams. The effectiveness of the proposed theory is demonstrated through illustrative examples for static and free vibrations of FG beams of rectangular cross-section.

2. Mathematical formulation

Consider a simply supported FG beam with the length L and rectangular cross-section $b\times h$ with b being the width and h being the height. Unlike the previous mentioned theories, the number of unknown functions involved in the present theory is only two as in EBT. The beam is made of isotropic material with material properties varying smoothly in the thickness direction. The FG beam is isotropic with its material properties vary smoothly within the thickness of the beam.

The volume-fraction of ceramic V_c is defined by the following relation (Bessaim *et al.* 2013, Zidi *et al.* 2014, Bennai *et al.* 2015, Taibi *et al.* 2015, Benferhat *et al.* 2016, Bennoun *et al.* 2016, Besseghier *et al.* 2017)

$$V_c(z)=\left(\frac{z}{h}+\frac{1}{2}\right)^k \tag{1}$$

Where k is the gradient index, which takes the value greater or equal to zero. Material nonhomogeneous characteristics of a FG beam may be determined using the Voigt rule of mixture (Suresh and Mortensen 1998). Thus, using Eq. (1), the material nonhomogeneous properties (P) of FG beam P, as a function of thickness coordinate, become (Bouderba *et al.* 2013, Tounsi *et al.* 2013, Zemri *et al.* 2015, Meradjah *et al.* 2015, Mahi *et al.* 2015, Laoufi *et al.* 2016, Bouafia *et al.* 2017)

$$P(z)=\left(P_c-P_m\right)V_c(z)+P_m \tag{2}$$

where P is the effective material property of FG beam. P_m and P_c are the corresponding properties of the metal and ceramic, respectively. In the present study, we suppose that the elasticity modules E and the mass density ρ are defined by Eq. (2), while Poisson's ratio v, is assumed to be constant across the thickness (Bourada *et al.* 2015, Boukhari *et al.* 2016, Bounouara *et al.* 2016).

2.1 Kinematics

The displacement field of the proposed two unknowns shear deformation theory is built upon the Euler-Bernoulli beam theory (EBT) including the hyperbolic function in terms of thickness coordinate to represent shear deformation and is assumed as follows (Mouffoki *et al.* 2017)

$$\begin{aligned} u(x,z,t)&=u_0(x,t)-z\frac{\partial w_0}{\partial x}-\beta\ f(z)\frac{\partial^3 w_0}{\partial x^3}\\ w(x,z,t)&=w_0(x,t)\end{aligned} \tag{3}$$

Where u_0 and w_0 are two unknown displacement functions of mid-axis of the beam. $f(z)$ is a shape function representing the variation of the transverse shear strains and shear stresses through the thickness of the beam and is given as (Soldatos 1992)

$$f(z)=h\sinh\left(\frac{z}{h}\right)-z\cosh\left(\frac{1}{2}\right) \tag{4}$$

The nonzero linear strains related to displacement field in Eq. (3) are

$$\varepsilon_x=\varepsilon_x^0+z\,k_x+\beta\,f(z)\eta_x,\quad \gamma_{xz}=\beta\,g(z)\gamma_{xz}^0 \tag{5}$$

Where

$$\varepsilon_x^0=\frac{\partial u_0}{\partial x},\quad k_x=-\frac{\partial^2 w_0}{\partial x^2},\quad \eta_x=-\frac{\partial^4 w_0}{\partial x^4},$$

$$\gamma_{xz}^0=-\frac{\partial^3 w_0}{\partial x^3} \tag{6}$$

And

$$g(z)=f'(z) \tag{7}$$

Where β is defined in Eq. (19).

2.2 Constitutive relations

By assuming that the material of FG beam obeys Hooke's law, the stresses in the beam become

$$\sigma_x=Q_{11}(z)\,\varepsilon_x\ \text{ and }\ \tau_{xz}=Q_{55}(z)\,\gamma_{xz} \tag{8}$$

Where (σ_x, τ_{xz}) and (ε_x, γ_{xz}) are the stress and strain components, respectively. The stiffness coefficients, Q_{ij}, can be expressed as

$$Q_{11}(z)=E(z)\,,\quad Q_{55}(z)=\frac{E(z)}{2(1+\nu)} \tag{9}$$

2.3 Governing equations

Hamilton's principle is used herein to derive the equations of motion. The principle can be stated in the following form

$$\delta\int_{t_1}^{t_2}(U+V-K)dt \tag{10}$$

Where t is the time; t_1 and t_2 are the initial and end time, respectively; U is the virtual variation of the strain energy; V is the virtual variation of the potential energy; and K is the virtual variation of the kinetic energy. The variation of the strain energy of the beam can be stated as

$$\begin{aligned}\delta U&=\int_0^L\int_{-h/2}^{h/2}(\sigma_x\delta\,\varepsilon_x+\tau_{xz}\delta\,\gamma_{xz})dxdz=\\&\int_0^L\left(N_x\delta\,\varepsilon_x^0+M_x\delta\,k_x+\beta(S_x\delta\,\eta_x+Q_{xz}\delta\,\gamma_{xz}^0)\right)dx\end{aligned} \tag{11}$$

In which the stress resultants N_x, M_x, S_x and Q_{xz} are defined by

$$(N_x,M_x,S_x)=\int_{-h/2}^{h/2}\sigma_x\,(1,z,\beta\,f(z))dz$$

$$Q_{xz}=\int_{-h/2}^{h/2}\tau_{xz}\,\beta\,g(z)dz \tag{12}$$

The variation of the potential energy by the applied transverse load q can be written as

$$\delta V=-\int_0^L q\delta w\,dx \tag{13}$$

The variation of kinetic energy is written as

$$\delta\,K\;=\int_0^L\int_{-h/2}^{h/2}[\dot u\delta\,\dot u+\dot w\delta\,\dot w]\rho(z)dxdz \tag{14}$$

$$=\left(\begin{aligned}&I_0(\dot u_0\delta\dot u_0+\dot w_0\delta\dot w_0)-I_1\left(\dot u_0\frac{\partial\delta\dot w_0}{\partial x}+\frac{\partial\dot w_0}{\partial x}\delta\,\dot u_0\right)-J_1\\&\beta\left(\dot u_0\frac{\partial^3\delta\dot w_0}{\partial x^3}+\frac{\partial^3\dot w_0}{\partial x^3}\delta\,\dot u_0\right)+I_2\left(\frac{\partial\dot w_0}{\partial x}\frac{\partial\delta\,\dot w_0}{\partial x}\right)\\&+K_2\beta^2\left(\frac{\partial^3\dot w_0}{\partial x^3}\frac{\partial^3\delta\,\dot w_0}{\partial x^3}\right)+J_2\beta\left(\frac{\partial\dot w_0}{\partial x}\frac{\partial^3\delta\,\dot w_0}{\partial x^3}+\frac{\partial^3\dot w_0}{\partial x^3}\frac{\partial\delta\,\dot w_0}{\partial x}\right)\end{aligned}\right)$$

Where dot-superscript convention indicates the differentiation with respect to the time variable t; $\rho(z)$ is the mass density; and (I_0, I_1, J_1, I_2, J_2, K_2) are mass inertias defined as

$$\begin{aligned}&(I_0,I_1,J_1,I_2,J_2,K_2)=\\&\sum_{n=1}^{3}\int_{h_{n-1}}^{h_n}(1,z,f,z^2,z\,f,f^2)\rho(z)dz\end{aligned} \tag{15}$$

Using the expressions for δU, δV, and δK from Eqs. (11), (13), and (14) into Eq. (10) and integrating by parts, and collecting the coefficients of δu_0 and δw_0, the following equations of motion of the beam are obtained

$$\begin{aligned}&\delta\,u_0:\;\frac{\partial N_x}{\partial x}=I_0\ddot u_0-I_1\frac{\partial\ddot w_0}{\partial x}-\beta\,J_1\frac{\partial^3\ddot w_0}{\partial x^3}\\&\delta\,w_0:\;\frac{\partial^2 M_x}{\partial x^2}+\beta\left(\frac{\partial^4 S_x}{\partial x^4}-\frac{\partial^3 Q_{xz}}{\partial x^3}\right)+q\\&=I_0\ddot w_0+I_1\left(\frac{\partial\ddot u_0}{\partial x}\right)-I_2\left(\frac{\partial^2\ddot w_0}{\partial x^2}\right)\\&+\beta\,J_1\left(\frac{\partial^3\ddot u_0}{\partial x^3}\right)-2\beta\,J_2\left(\frac{\partial^4\ddot w_0}{\partial x^4}\right)-\beta^2\,K_2\left(\frac{\partial^6\ddot w_0}{\partial x^6}\right)\end{aligned} \tag{16}$$

2.4 Governing equations in terms of displacements

By substituting Eq. (5) into Eq. (8) and the subsequent results into Eq. (12), the stress resultants can be written as below

$$N_x=A_{11}\varepsilon_x^0+B_{11}k_x+\beta\,B_{11}^s\eta_x \tag{17a}$$

$$M_x=B_{11}\varepsilon_x^0+D_{11}k_x+\beta\,D_{11}^s\eta_x \tag{17b}$$

$$S_x=\beta\,B_{11}^s\varepsilon_x^0+\beta\,D_{11}^s k_x+\beta^2\,H_{11}^s\eta_x \tag{17c}$$

$$Q_{xz}=\beta^2\,A_{55}^s\gamma_{xz}^0 \tag{17d}$$

Where

$$\begin{aligned}&\{A_{11},B_{11},D_{11},B_{11}^s,D_{11}^s,H_{11}^s\}=\\&\int_{-h/2}^{h/2}Q_{11}(z)\left(1,z,z^2,\,f(z),z\,f(z),\,f^2(z)\right)dz\end{aligned} \tag{18a}$$

$$A_{55}^s=\int_{-h/2}^{h/2}Q_{55}(z)[g(z)]^2\,dz, \tag{18b}$$

The expression of shape parameter 'β' is evaluated in the post-processing phase and is found to be as follows

$$\beta=-\frac{(D_{11}^s A_{11}-B_{11}B_{11}^s)}{(A_{11}A_{44}^s+A_{11}\lambda^2 H_{11}^s-B_{11}^s\lambda^2)}, \tag{19a}$$

For an isotropic beam

$$\beta=-\frac{D_{11}^s}{(A_{44}^s+\lambda^2 H_{11}^s)}, \tag{19b}$$

By substituting Eq. (17) into Eq. (16), the governing equations can be written in terms of generalized displacements (u_0 and w_0) as

$$\begin{aligned}&A_{11}\frac{\partial^2 u_0}{\partial x^2}-B_{11}\frac{\partial^3 w_0}{\partial x^3}-\beta\,B_{11}^s\frac{\partial^5 w_0}{\partial x^5}\\&=I_0\ddot u_0-I_1\frac{\partial\ddot w_0}{\partial x}-\beta\,J_1\frac{\partial^3\ddot w_0}{\partial x^3},\end{aligned} \tag{20a}$$

$$
\begin{aligned}
& B_{11}\frac{\partial^3 u_0}{\partial x^3} - D_{11}\frac{\partial^4 w_0}{\partial x^4} + \beta\left(B_{11}^s\frac{\partial^5 u_0}{\partial x^5} - 2D_{11}^s\frac{\partial^6 w_0}{\partial x^6}\right) \\
& -\beta^2\left(H_{11}^s\frac{\partial^8 w_0}{\partial x^8} - A_{44}^s\frac{\partial^6 w_0}{\partial x^6}\right) + q \\
& = I_0\ddot{w}_0 + I_1\left(\frac{\partial \ddot{u}_0}{\partial x}\right) - I_2\left(\frac{\partial^2 \ddot{w}_0}{\partial x^2}\right) + \beta J_1\left(\frac{\partial^3 \ddot{u}_0}{\partial x^3}\right) \\
& -2\beta J_2\left(\frac{\partial^4 \ddot{w}_0}{\partial x^4}\right) - \beta^2 K_2\left(\frac{\partial^6 \ddot{w}_0}{\partial x^6}\right)
\end{aligned} \tag{20b}
$$

3. Analytical solution

The above governing equations are analytically solved for bending problems of a simply supported beam. Based on Navier solution procedure, the displacements are assumed as follows

$$
\begin{Bmatrix} u_0(x) \\ w_0(x) \end{Bmatrix} = \sum_{m=1}^{\infty} \begin{Bmatrix} U_m \cos(\lambda x) e^{i\omega t} \\ W_m \sin(\lambda x) e^{i\omega t} \end{Bmatrix} \tag{21}
$$

Where $\lambda = m\pi/a$, (U_m, W_m) are arbitrary parameters to be determined, ω is the eigenfrequency associated with m-th eigenmode, The transverse load q is also expanded in Fourier sine series as

$$
q(x) = \sum_{m=1}^{\infty} Q_m \sin(\lambda x) \tag{22}
$$

Where Q_m is the load amplitude calculated from

$$
Q_m = \frac{2}{L}\int_0^L q(x)\sin(\lambda x)\,dx \tag{23}
$$

The coefficients Q_m are given below for some typical loads. For the case of a sinusoidally distributed load, we have

$$
m = 1 \quad \text{and} \quad Q_1 = q \tag{24}
$$

and for the case of uniform distributed load, we have

$$
q_{mn} = \frac{4q_0}{m\pi}, \left(m = 1,3,5,\ldots\ldots\right) \tag{25}
$$

Substituting the expansions of u_0, w_0, and q from Eqs. (21) and (22) into the equations of motion, Eq. (20), the analytical solutions can be obtained from the following equations

$$
\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix}\right)\begin{Bmatrix} U_m \\ W_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ Q_m \end{Bmatrix} \tag{26}
$$

Where

$$
\begin{aligned}
a_{11} &= -A_{11}\lambda^2 \\
a_{12} &= B_{11}\lambda^3 - \beta B_{11}^s\lambda^5 \\
a_{22} &= -D_{11}\lambda^4 - 2\beta D_{11}^s\lambda^6 - \beta^2\left(H_{11}^s\lambda^8 + A_{55}^s\lambda^6\right) \\
m_{11} &= -I_0 \\
m_{12} &= I_1\lambda + \beta J_1\lambda^3 \\
m_{33} &= -I_0 - I_2\lambda^2 - 2\beta J_2\lambda^4 + \beta^2 K_2\lambda^6
\end{aligned} \tag{27}
$$

4. Numerical results and discussion

In this work, bending and free vibration analysis of the simply supported FG beams is studied using the present 2-

Table 1 Comparison of non-dimensional deflections and stresses of FG beams under uniform

k	Method	L/h=5				L/h=20			
		$\bar{w}$	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{xz}$	$\bar{w}$	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{xz}$
0	Li *et al*. (2010)	3.1657	0.9402	3.8020	0.7500	2.8962	0.2306	15.0130	0.7500
	Ould Larbi *et al*. (2013)	3.1651	0.9406	3.8043	0.7489	2.8962	0.2305	15.0136	0.7625
	Present	3.1654	0.9397	3.8017	0.7312	2.8962	0.2306	15.0129	0.7429
0.5	Li *et al*. (2010)	4.8292	1.6603	4.9925	0.7676	4.4645	0.4087	19.7005	0.7676
	Ould Larbi *et al*. (2013)	4.8282	1.6608	4.9956	0.7660	4.4644	0.4087	19.7013	0.7795
	Present	4.8285	1.6595	4.9920	0.7484	4.4644	0.4087	19.7003	0.7599
1	Li *et al*. (2010)	6.2599	2.3045	5.8837	0.7500	5.8049	0.5686	23.2054	0.7500
	Ould Larbi *et al*. (2013)	6.2590	2.3052	5.8875	0.7489	5.8049	0.5685	23.2063	0.7625
	Present	6.2594	2.3036	5.8831	0.7312	5.8049	0.5685	23.2052	0.7429
2	Li *et al*. (2010)	8.0602	3.1134	6.8812	0.6787	7.4415	0.7691	27.0989	0.6787
	Ould Larbi *et al*. (2013)	8.0683	3.1146	6.8878	0.6870	7.4421	0.7691	27.1005	0.7005
	Present	8.0675	3.1127	6.8819	0.6685	7.4420	0.7691	27.0989	0.6802
5	Li *et al*. (2010)	9.7802	3.7089	8.1030	0.5790	8.8151	0.9133	31.8112	0.5790
	Ould Larbi *et al*. (2013)	9.8345	3.7128	8.1187	0.6084	8.8186	0.9134	31.8151	0.6218
	Present	9.8271	3.7097	8.1095	0.5883	8.8181	0.9134	31.8127	0.5998
10	Li *et al*. (2010)	10.8979	3.8860	9.7063	0.6436	9.6879	0.9536	38.1372	0.6436
	Ould Larbi *et al*. (2013)	10.9413	3.8898	9.7203	0.6640	9.6907	0.9537	38.1408	0.6788
	Present	10.9375	3.8859	9.7111	0.6445	9.6905	0.9536	38.1383	0.6572

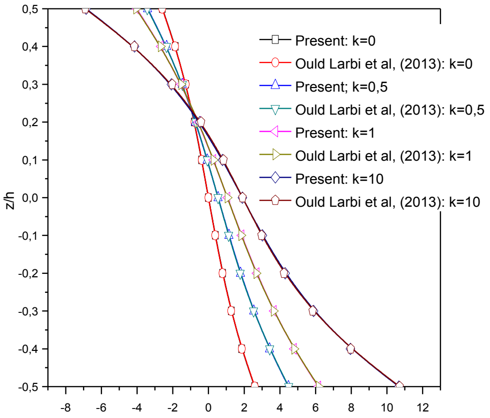

Fig. 1 The variation of the axial displacement $\bar{u}$ through the thickness of an FG beam (L=2h)

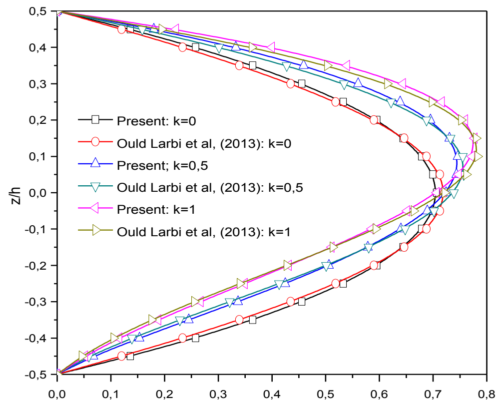

Fig. 3 The variation of the shear stress $\bar{\tau}_{xz}$ through the thickness of an FG beam (L=2h)

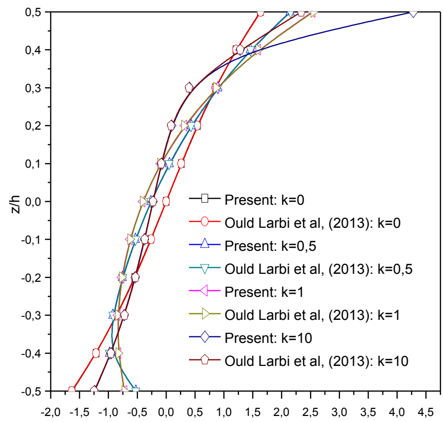

Fig. 2 The variation of the axial stress $\bar{\sigma}_x$ through the thickness of an FG beam (L=2h)

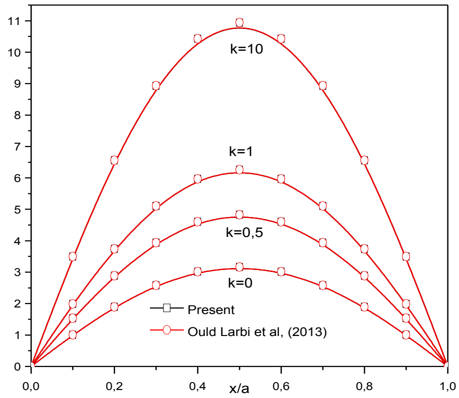

Fig. 4 The variation of the transverse displacement $\bar{w}$ versus non-dimensional length of an FG beam (L=5h)

unknowns hyperbolic shear deformation theory.

The FG beam is considered to be made of aluminum and alumina with the following material properties

- Ceramic (P_c: Alumina, Al_2O_3): E_c=380 GPa, v=0.3, ρ_c=3960 kg/m^3.
- Metal (P_m: Aluminum, Al): E_m=70 GPa, v=0.3, ρ_m=2707 kg/m^3.

For convenience, the following non-dimensional parameters are employed

$$\bar{w}=100\frac{E_m h^3}{q_0 L^4}w\left(\frac{L}{2}\right),\quad \bar{u}=100\frac{E_m h^3}{q_0 L^4}u\left(0,\frac{-h}{2}\right),$$

$$\bar{\sigma}_x=\frac{h}{q_0 L}\sigma_x\left(\frac{L}{2},\frac{h}{2}\right),\quad \bar{\tau}_{xz}=\frac{h}{q_0 L}\tau_{xz}(0,0),$$

$$\bar{\omega}=\frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}}$$

4.1 Results of bending analysis

Table 1 provides non-dimensional displacements and stresses of FG beams subjected to uniform load q_0 for different values of gradient index (k) and span-to-depth ratio (L/h). The computed results are compared with the analytical solutions reported by Li *et al*. (2010) and the high shear deformation theory of Ould Larbi *et al*. (2013). It can be seen that our results are in an excellent agreement to those reported by Li *et al*. (2010), Ould Larbi *et al*. (2013).

In Figs. 1-3, the variation of the axial displacement $\bar{u}$, normal stresses $\bar{\sigma}_x$, and transverse shear stress $\bar{\tau}_{xz}$ within the depth of the FG beam under uniform load is presented. A comparison with higher shear deformation beam theory developed by Ould Larbi *et al*. (2013) is also demonstrated in these figures for different values of the gradient k. It is deduced that there is an excellent agreement between the proposed two-unknown hyperbolic shear deformation theory and the theory of Ould Larbi *et al*. (2013) which involves three unknowns functions. It can be observed from Fig. 1 that increasing the gradient index k leads to an increase of the axial displacement $\bar{u}$ and especially at the upper and lower surfaces of the beam. In Fig. 2, the longitudinal stress $\bar{\sigma}_x$ is tensile state at the upper surface and compressive state at the lower surface. The fully ceramic beam k=0 yields the maximum compressive stresses at the lower surface and the minimum tensile stresses at the upper surface of the beam. In Fig. 3 we plotted the through-the-thickness variations of the transverse shear stress $\bar{\tau}_{xz}$. The through-the-thickness variations of $\bar{\tau}_{xz}$ for FG beams are not parabolic as in the case of fully metal or ceramic beams. Fig. 4 shows the

Table 2 Comparison of non-dimensional fundamental frequencies of FG beam

L/h	Theory	*P* 0	0.5	1	2	5	10
5	Ould Larbi *et al.* (2013)	5.1529	4.4108	3.9905	3.6263	3.4001	3.2812
	Present	5.1527	4.4107	3.9904	3.6265	3.4004	3.2817
20	Ould Larbi *et al.* (2013)	5.4603	4.6511	4.2051	3.8361	3.6484	3.5389
	Present	5.4603	4.6511	4.2051	3.8361	3.6485	3.5390

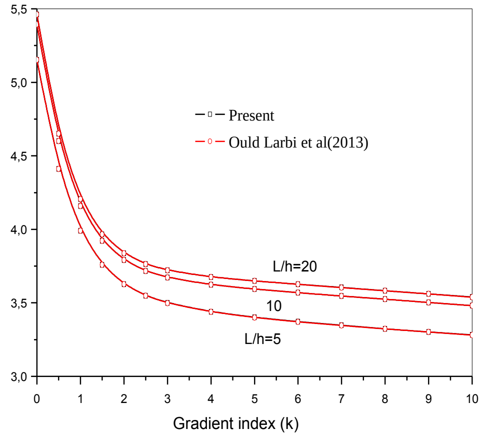

Fig. 5 Variation of the non-dimensional fundamental frequency $\overline{w}$ of an FG beam with the gradient index k and span-to-depth ratio L/h

distribution of the non-dimensional deflection $\overline{w}$ versus non-dimensional length for different gradient index. It can be seen that the results predicted by the proposed theory are almost identical to those given by Ould Larbi *et al.* (2013). The results show also that the increase of the gradient index k leads to an increase of deflection $\bar{\tau}_{xz}$.

Also, one can note that, the proposed theory involves only two unknowns variables against the three unknowns variables in case of Timoshenko beam theory. Furthermore, it can be indicated that, the Timoshenko theory requires the use of a shear correction factor. In contrast, the proposed theory does not require a shear correction factor.

4.2 Results of free vibration analysis

In this part of study, the non-dimensional fundamental frequencies $\overline{\varpi}$ predicted by the proposed theory are compared with those reported by Ould Larbi *et al*. (2013) of FG beams for different values of gradient index k and span-to-depth ratio L/h and the results are listed in Table 2. It can be observed that the proposed theory with only two unknown's variables gives almost identical results to those of Ould Larbi *et al*. (2013) three unknown's variables.

Fig. 5 presents the variation of non-dimensional fundamental natural frequency $\overline{\varpi}$ versus the gradient index k for different values of span-to-depth ratio L/h and the results are compared to those computed using the theory developed by Ould Larbi *et al*. (2013). The examination of this figure demonstrates an excellent agreement between the proposed theory and that of Ould Larbi *et al*. (2013). It can be seen that the increase of the gradient index lead to a reduction of the frequency. The highest frequency is found for the fully ceramic beams (k=0). However, the lowest frequency values are obtained for fully metal beams ($k\to\infty$). This is due to the fact that an increase in the value of the gradient index results in a reduction in the value of elasticity modulus.

5. Conclusions

In this article, a simple two-unknown shear deformation theory for bending and free vibrations of a FG beam of rectangular cross-section is presented. Some of the important aspects of the beam theory presented herein can be summarized as follows:

- The governing differential equation of the theory involves only two unknown variables as the Euler-Bernoulli beam theories which are even less than the Timoshenko beam theory and other HSDTs.
- The displacement field of the proposed beam theory gives rise to a realistic parabolic distribution of transverse shear stress across the beam cross-section. Furthermore, proposed theory does not require a shear correction factor.
- Efficacy of the developed theory is shown within illustrative examples for bending and dynamic of rectangular cross-section FG beams. The obtained numerical results are compared with those of other higher-order shear deformation beam theory results. The obtained are found to be accurate.

In conclusion, the beam theory proposed herein is a simple and an accurate theory for bending and free vibrations analysis of FG beams of rectangular cross-section.

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