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Static behavior of nonlocal Euler-Bernoulli beam model embedded in an elastic medium using mixed finite element formulation

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Abstract. The size-dependent behavior of single walled carbon nanotubes (SWCNT) embedded in the elastic medium and subjected to the initial axial force is investigated using the mixed finite element method. The SWCNT is assumed to be Euler-Bernoulli beam incorporating nonlocal theory developed by Eringen. The mixed finite element model shows its great advantage of dealing with nonlocal behavior of SWCNT subjected to a concentrated load owing to the existence of two coefficients α_1 and α_2. This is the first numerical approach to deal with a puzzling fact of nonlocal theory with concentrated load. Numerical examples are performed to show the accuracy and efficiency of the present method. In addition, parametric study is carefully carried out to point out the influences of nonlocal effect, the elastic medium, and the initial axial force on the behavior of the carbon nanotubes.

Keywords: nonlocal continuum theory; mixed finite element method; elastic medium; carbon nanotubes

1. Introduction

With the fast development of technology, a new research area is in growth for understanding of nano/micro structures, and the carbon nanotubes (CNT) and the graphene sheet (GS) are the most prevalent among nano/micro structures. Since its invention by Iijima (1991), the CNT has been attracted a vast quantity of research effort owing to its advance mechanical, chemical, physical, and electrical properties (Dai *et al*. 1996, Kim and Lieber 1999, Thostenson *et al*. 2001, Bachtold *et al*. 2001). Due to its superior feature, the application of CNT was expanded into many areas such as nano-electromechanical devices (Hierold *et al*. 2007), actuators (Baughman *et al*. 1999). Obviously, understanding and analyzing the behavior of CNT play an important role in research. Therefore, experiments as well as discrete atomistic methods such as molecular dynamics (MD) simulation (Frankland *et al*. 2002, Liew *et al*. 2004) have been utilized. However, these methods are either extremely difficult or highly expensive for computational cost. Because of these reasons, the size-dependent continuum model was developed as another alternative theoretical technique.

It is known that due to the absence of internal material length scale in the constitutive equation, the classical theory fails to predict the behavior at the nano-scale which was reported by Fleck *et al*. (1994), Stölken and Evans (1998) and Lam *et al*. (2003). In order to consider the size-dependent effect, numerous size-dependent continuum theories (e.g., nonlocal theory, strain gradient theory, surface theory, couple stress theory, and modified couple stress theory), which can capture the nano-scale effect, have been developed to study the behavior of nano/micro structures. In the local (classical) continuum theories based on the hyper-elastic constitutive relations, the stress state at a reference point is only a function of strain state at that point. The nonlocal continuum theory initially developed by Eringen (1972), and Eringen and Edelen (1972) indicate that the stress state at a reference point is a function of every strain state in the continuum body.

Although the nonlocal continuum theory was first introduced by Eringen (1972), the feasibility of the nonlocal theory for the analysis of nano/micro structures was initially presented in Peddieson *et al*. (2003). They concluded that the nonlocal effect could be useful and significant for nanostructures. In their work, the nonlocal effect was vanished for a cantilever beam subjected to any combination of concentrated loads. Lu *et al*. (2007) investigated the wave and vibration properties of single or multi-walled carbon nanotubes by using the nonlocal Euler-Bernoulli and Timoshenko beam models. They discussed some issues on deriving stress resultants, governing equations, and boundary conditions of the nonlocal beam model. Thai (2012) studied the static, buckling, vibration analysis of nonlocal beam using higher order shear deformation beam theory. Wang *et al*. (2008) expressed the analytical solutions for deflections, rotations and stress resultants for the nonlocal Timoshenko beam subjected to transverse distributed load. Pradhan and Murmu (2010) investigated the flapwise bending vibration of nonlocal Euler-Bernoulli cantilever beam. The governing equation was derived and solved by using differential quadrature method. Emam (2013) presented the analytical solution for

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buckling and post-buckling analysis of nonlocal nonlinear nanobeams with the classical, first order and higher-order shear deformation beam theories. Comprehensive research on the static, free vibration, and buckling were reported by Reddy and Pang (2008) based on Euler-Bernoulli and Timoshenko beam theories. The analytical solutions of the simply supported, cantilever, and clamped beams were obtained for bending, free vibration, and buckling problems. Reddy (2007) reformulated various beam theories including the Euler-Bernoulli, Timoshenko, Reddy, and Levinson beam theories incorporated with the nonlocal continuum theory for the bending, free vibration, and buckling of nonlocal beam. In particular, for bending analysis, the point load problem for the nonlocal theory was solved by using Navier solution which transforms every loading type into the distributed load by using Fourier series formulation. Reddy (2010) also developed his research for the nonlinear bending of classical and shear deformation theories of nonlocal beams and nonlocal plates. Aydogdu (2009) presented the Navier solution to deal with the point load problem for the simply supported beam. In his study, the length of beam was changed to investigate the length-scale effect. Moreover, in order to deal with the point load problem, Wang and Liew (2007) introduced Dirac delta function involved with point load in the governing equation of static analysis of nonlocal Euler-Bernoulli and Timoshenko beams. Owing to the advantage of Dirac delta function, the nonlocal effect would be presented for various nonlocal nanobeams subjected to the point load. Civalek and Demir (2011) also exploited this method to investigate the bending analysis of microtubules using nonlocal Euler-Bernoulli beam theory. The application of differential quadrature method was used to obtain the solution for different boundary conditions. On the other hand, finite element method was used to investigate the behavior of nonlocal Euler-Bernoulli beam and nonlocal Timoshenko beam by Phadikar and Pradhan (2010) and Pradhan (2012), respectively. A nonlocal beam subjected to the point load was not investigated in their study due to the fact that the nonlocal effect is excluded from the finite element model.

Single-walled carbon nanotubes (SWCNT) can be modeled by the nonlocal beam model (Amara *et al*. 2010, Narendar and Gopalakrishnan 2009, Boumia *et al*. 2014, Wu and Lai 2015) or the nonlocal shell model (Hoseinzadeh and Khadem 2014, Khademolhosseini *et al*. 2010, Hu *et al*. 2008). Using the nonlocal beam model, Arash and Ansari (2010) studied the vibration characteristics of SWCNT subjected to an initial strain. The obtained results were compared with the MD simulation to assess the nonlocal parameter e_oa for clamped and cantilever SWCNT. Similarly, Duan *et al*. (2007) calibrated the parameter e_o by matching the closed-form solution for the free vibration of SWCNT and MD simulation. Their findings indicate that the parameter e_o varies from 0 to 19 depending on the length-to-diameter ratio, boundary conditions, and mode shapes of SWCNT. Recently, a lot of works have been done to investigate the behavior of SWCNT embedded in polymer or matrix as an elastic medium. Li *et al*. (2008) investigated the flexural wave behavior of SWCNT and double-walled carbon nanotubes (DWCNTs) embedded in an elastic medium by nonlocal theory and Timoshenko beam model. Ke *et al*. (2009) studied the non-linear free vibration of DWCNTs based on the nonlocal Timoshenko theory. The non-linearity was taken into consideration by von-Kármán non-linear strain tensor. Moreover, Mustapha and Zhong (2010) presented the free vibration analysis of the non-prismatic SWCNT with an axially initial force. The SWCNT is idealized as a nonlocal Rayleigh beam and the variable governing equation is solved with the Bubnov-Galerkin method. Aydogdu (2012) examined the axial free vibration characteristics of SWCNT embedded in an elastic medium. Narendar and Gopalakrishnan (2011) investigated the thermal buckling of SWCNT under the nonlocal Timoshenko framework. Numerous studies on vibration and buckling of CNT were perform by Murmu and his colleagues (Murmu and Pradhan 2009a, c, Pradhan and Murmu 2009, Murmu and Pradhan 2009b, Murmu and Adhikari 2011, Murmu *et al*. 2012). Recently, Wu and Lai (2015) developed the RMVT (Reissner mixed variational theorem) based nonlocal Timoshenko beam theory for the static behaviors of nanobeams and SWCNTs with four different boundary conditions, being embedded in an elastic medium. The results showed that the RMVT based nonlocal beam was superior to the PVD (principle of virtual displacement) based nonlocal one by comparing their solutions with the analytical ones available in the literature.

From the previously cited references, it is worth noting that in spite of the fact that extensive researches on nonlocal beam problems for SWCNT have been carried out, to the best of authors' knowledge, very little attention is given to the static analysis of the nonlocal nanobeams subjected to a point load. Navier solution of Reddy (2007) and Aydogdu (2009), and the Dirac delta function from Wang and Liew (2007), Civalek and Demir (2011) were derived to address the concentrated loading problem of nonlocal beam. However, such analytical solutions cannot generally handle arbitrary complicated geometries, material properties, loading types, and boundary conditions whereas finite element method by Phadikar and Pradhan (2010), Pradhan (2012) cannot take the point load problem of nonlocal beam into account.

Motivated by these reasons, this paper, which is an expansion of an earlier paper by Nguyen *et al.* (2015), presents the mixed finite element method for the static analysis of the nonlocal SWCNT embedded in an elastic medium under an initial axial force. The novelty and efficiency of this model are addressed and verified, especially for the concentrated load problem. The outline of this paper is as follows: The nonlocal elasticity theory is presented in Section 2. The mixed finite element model is formulated and derived in Section 3. The numerical results are carried out in Section 4. The accuracy and reliability of this study are presented and demonstrated by comparing the results with published works. Finally, concluding remarks are drawn.

2. Nonlocal continuum theory

It is known that in contrast to the constitutive equation in the classical elasticity, the nonlocal elasticity theory by Eringen (1983) states that the stress at a point $\mathbf{x}$ in an elastic continuum body depends not only on the strain at point $\mathbf{x}$ but also on those at all other points of the body. Therefore, the nonlocal stress tensor $\boldsymbol{\sigma}$ at point $\mathbf{x}$ is expressed as Reddy (2007) and Pradhan (2012)

$$\sigma_{ij} = \int_V \alpha\left(\left|\mathbf{x}' - \mathbf{x}\right|, \tau\right) t_{ij}(\mathbf{x}') d\mathbf{x}' \tag{1}$$

where $t_{ij}(\mathbf{x})$ are the components of the classical macroscopic stress tensor at point $\mathbf{x}$; the kernel function $\alpha\left(\left|\mathbf{x}'-\mathbf{x}\right|, \tau\right)$ is the nonlocal modulus or the attenuation function specifying the nonlocal effect at a reference point $\mathbf{x}$ produced by the local strain at the source $\mathbf{x}', \left|\mathbf{x}'-\mathbf{x}\right|$ being the distance (in Euclidean distance); τ is a material constant that depends on the internal and external characteristic lengths (such as the lattice spacing and wave length, respectively). The integration is taken for total volume V of elastic body. The macroscopic stress $\mathbf{t}$ at a point $\mathbf{x}$ in a Hookean solid is related to the strain $\boldsymbol{\varepsilon}$ at the point by the generalized Hookes law as follows

$$\mathbf{t}(\mathbf{x}) = \mathbf{C}(\mathbf{x}) : \boldsymbol{\varepsilon}(\mathbf{x}) \quad \text{or} \quad \mathrm{t_{ij}} = \mathrm{C_{ijkl}} \varepsilon_{\mathrm{kl}} \tag{2}$$

where $\mathbf{C}(\mathbf{x})$ is the fourth-order elasticity tensor and the colon denotes the double-dot product. The integral constitutive relation in Eq. (1) makes the elasticity problems difficult to solve, in addition to possible lack of determinism. Therefore, Eringen (1983) discussed in detail properties of the nonlocal kernel $\alpha\left(\left|\mathbf{x}'-\mathbf{x}\right|, \tau\right)$ and proved that when a kernel takes a Greens function of the linear differential operator

$$L_a \alpha\left(\left|\mathbf{x}'-\mathbf{x}\right|\right) = \delta\left(\left|\mathbf{x}'-\mathbf{x}\right|\right) \tag{3}$$

The nonlocal constitutive relation in Eq. (1) is reduced to the following differential equation

$$L_a \sigma_{ij} = t_{ij} \tag{4}$$

Thus, Eringen (1983) proposed a nonlocal model with the linear differential operator La by matching the dispersion curves with lattice models as follows

$$L_a = 1 - \left(e_o a\right)^2 \nabla^2 \tag{5}$$

where ∇^2 is the Laplace operator; e_o is a parameter to adjust the model to match the reliable results by experiments or other models; a is an internal characteristics length (e.g., granular distance, lattice parameter); $e_o a$ denotes the nonlocal parameter which reveals the small scale effect on the response of structure for nanosize. Eringen (1983) proposed the $e_o = \sqrt{(\pi^2 - 4)} / 2\pi \cong 0.39$ and $e_o = 1/\sqrt{12} \cong 0.288$ was given by Wang (2005). For a beam type structure, the nonlocal behavior can be neglected in the thickness direction. Thus, the constitutive relation for the nonlocal elasticity can be represented by following form

$$\sigma_{xx} - \left(e_o a\right)^2 \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E \varepsilon_{xx} \tag{6}$$

where σ_{xx} and ε_{xx} are the axial normal stress and the axial strain, respectively, and E is the elasticity modulus. When the nonlocal parameter is taken as $e_0 a$, the constitutive relation of the local theory is obtained. Integrating Eq. (6) over the area of cross-section, we can obtain the moment-curvature relation as follows

$$M - \left(e_o a\right)^2 \frac{d^2 M}{dx^2} = \int_A z E \varepsilon_{xx} dA \tag{7}$$

where M_x and A are the bending moment and the area of the cross-section, respectively.

It is worth noting that the bending moment in Eq. (7) is defined from the nonlocal constitutive law. It is so-called as the nonlocal bending moment which is associated to the nonlocal theory.

3. Beam formulation

Let consider a SWCNT assumed to be modeled as an Euler-Bernoulli beam embedded in an elastic medium and subjected to an initial axial force in Fig. 1. The kinematic relations according to the Euler-Bernoulli beam theory is given as

$$u\left(x, z\right) = -z \frac{dw_o(x)}{dx} \tag{8}$$

$$w\left(x, z\right) = w_o\left(x\right) \tag{9}$$

where u and w are the axial and transverse displacements, respectively, at any general point in the deformed state; w_o is the transverse displacement calculated at the mid-plane. The definition of the axial strain is written by

$$\varepsilon_{xx} = \frac{du}{dx} = -z \frac{d^2 w_o}{dx^2} \tag{10}$$

Substituting Eq. (10) into Eq. (7), the moment-curvature relation of the nonlocal Euler-Bernoulli beam theory can be obtained as follows

$$M - \left(e_o a\right)^2 \frac{d^2 M}{dx^2} = -EI \frac{d^2 w_o}{dx^2} \tag{11}$$

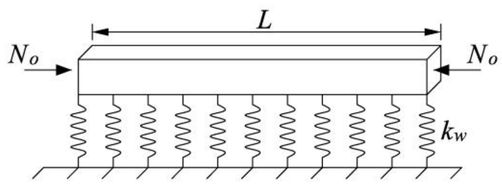

Fig. 1 Schematic of SWCNT embedded in an elastic medium and subjected to an initial axial force

where I is the moment inertia of the cross-section.

Now consider a SWCNT subjected to the initial axial force N_o, transverse distributed load q and embedded in an elastic medium of elastic modulus k_w, using the principle of virtual work, the following equilibrium equation is obtained as given by Murmu and Pradhan (2009a), Pradhan and Reddy (2011)

$$\frac{d^2M^E}{dx^2} = N_o \frac{d^2w_o}{dx^2} + k_w w_o - q \tag{12}$$

where the notation M^E stands for the equilibrium bending moment which is not dependent on the nonlocal or local constitutive laws.

According to Reddy (2005), the weak-form finite element method requires the Hermite cubic interpolation for w_0. In order to reduce the differentiability requirements on w_0 and include the bending moment as a nodal degree of freedom, two governing equations for the mixed finite element method are derived from Eqs. (11) and (12) as follows

$$\frac{d^2M^E}{dx^2} - N_o \frac{d^2w_o}{dx^2} - k_w w_o + q = 0 \tag{13}$$

$$M - (e_o a)^2 \frac{d^2M}{dx^2} = -EI \frac{d^2w_o}{dx^2} \tag{14}$$

It is important to note that M denotes the nonlocal bending moment which depends on the nonlocal theory. In order to develop the mixed finite element model, the element $\Omega_e = (x_e, x_{e+1})$ which only includes two nodes per element is divided from the domain. The weak forms of two above equations are

$$\int_{x_e}^{x_{e+1}} v_1 \left(N_o \frac{d^2w_o}{dx^2} + k_w w_o - \frac{d^2M^E}{dx^2} - q \right) = 0 \tag{15}$$

$$\int_{x_e}^{x_{e+1}} v_2 \left(-\frac{d^2w_o}{dx^2} - \frac{M}{EI} + \frac{(e_o a)^2}{EI} \frac{d^2M}{dx^2} \right) = 0 \tag{16}$$

where v_1 and v_2 are weight functions which have the interpolation of the virtual deflection δw_o and the virtual equilibrium bending moment δM^E, respectively. We integrate by part Eqs. (15) and (16) and obtain the following equations

$$\int_{x_e}^{x_{e+1}} \left(-N_o \frac{dv_1}{dx}\frac{dw_o}{dx} + k_w v_1 w_o + \frac{dv_1}{dx}\frac{dM^E}{dx} - v_1 q \right) dx - v_1(x_e) Q_1^e - v_1(x_{e+1}) Q_2^e = 0 \tag{17}$$

$$\int_{x_e}^{x_{e+1}} v_2 \left(\frac{dv_2}{dx}\frac{dw_o}{dx} - \frac{v_2 M}{EI} - \frac{(e_o a)^2}{EI}\frac{dv_2}{dx}\frac{dM}{dx} \right) - v_2(x_e)\theta_1 + v_2(x_{e+1})\theta_2 + \frac{(e_o a)^2}{EI}\left[v_2 \frac{dM}{dx} \right]_{x_e}^{x_{e+1}} = 0 \tag{18}$$

where

$$Q_1^e = -\left(\frac{dM^E}{dx} - N_o \frac{dw_o}{dx} \right)_{x=x_e} ; Q_2^e = \left(\frac{dM^E}{dx} - N_o \frac{dw_o}{dx} \right)_{x=x_{e+1}}$$
$$\theta_1 = \left(-\frac{dw_o}{dx} \right)_{x=x_e} ; \theta_2 = \left(-\frac{dw_o}{dx} \right)_{x=x_{e+1}} \tag{19}$$

In which the last term in Eq. (18) denotes the nonlocal boundary conditions. It is worthy to note that the mixed finite element formulation for the nonlocal theory leads to nonlocal boundary conditions which are associated with the first derivative of the nonlocal bending moment dM/dx and the equilibrium bending moment δM^E as shown in Eq. (19). For local theory ($e_o a = 0$) we do not have this term in mixed finite element formulation. However, it is clear from Eqs. (17) and (18) that there are three variables including w_o, M and M^E in mixed finite element model but we only have two equations, thus some assumptions need to be supposed. Now it is assumed that the nonlocal bending moment is equivalent to the equilibrium bending moment except for the natural and essential boundary conditions where the equilibrium bending moment is modified by the coefficient α_1 and α_2, respectively

$$\begin{aligned} &M = M^E && \text{for entire domain} \\ &M = \alpha_1 M^E && \text{for essential boundary conditions} \\ &M = \alpha_2 M^E && \text{for natural boundary conditions} \end{aligned} \tag{20}$$

Herein, the essential boundary conditions are known as the specified geometric constraints and natural boundary conditions are known as the specified concentrated applied load. In this study, it is supposed that the modified coefficients α_1 and α_2 are taken in the range [0, 1].

From Eqs. (17) and (18), we suggest that both deflection and bending moment could be interpolated by linear Lagrange interpolation function

$$w_0 = \sum_{i=1}^{2} w_i \phi_i, \quad M = M^E = \sum_{i=1}^{2} w_i \psi_i \tag{21}$$

where ϕ_i and ψ_i are linear Lagrange interpolation functions for w_o and M, M^E, respectively. It is well-known that the regular finite element method for Euler-Bernoulli beam requires Hermite cubic interpolation function with C^1 continuity to properly represent the displacement field. However, for the mixed finite element method, Lagrange linear interpolation functions (C^0 continuity) is sufficient to obtain accurate solution.

Substituting Eq. (21) into Eqs. (17) and (18), we obtain the mixed finite element model as follows

$$\begin{bmatrix} K_0^e & K^e \\ (K^e)^T & -G^e \end{bmatrix} \begin{Bmatrix} w^e \\ M^e \end{Bmatrix} = \begin{Bmatrix} F^e \\ \bar{\theta}^e \end{Bmatrix} + \begin{Bmatrix} Q^e \\ \aleph^e \end{Bmatrix} \tag{22}$$

where $\aleph^e = \left\{ -\frac{(e_o a)^2}{EI} Q^e, -\alpha_1 \frac{(e_o a)^2}{EI} Q^e, -\alpha_2 \frac{(e_o a)^2}{EI} Q^e \right\}$ are used for the arbitrary nodes, the essential boundary conditions, and the natural boundary conditions, respectively, and

$$K_{ij}^e = \int_{x_e}^{x_{e+1}} \frac{d\phi_i}{dx}\frac{d\psi_j}{dx} dx; \qquad G_{ij}^e = \frac{1}{EI}\int_{x_e}^{x_{e+1}} \left[(e_o a)^2 + \psi_i \psi_j \right] dx$$
$$(K_0)_{ij}^e = \int_{x_e}^{x_{e+1}} \left(-N_o \frac{d\phi_i}{dx}\frac{d\phi_j}{dx} + k_w \phi_i \phi_j \right) dx; \quad F_{ij}^e = \int_{x_e}^{x_{e+1}} q\phi_i dx; \bar{\theta}_i^e = (-1)^{i+1}\theta_i^e \tag{23}$$

From the studies of Phadikar and Pradhan (2010) and Pradhan (2012) using the regular finite element method, one can see that the nonlocal parameter is directly associated with the transverse distributed load in force vector. As a result, the regular finite element model can only capture the nonlocal effect for the nonlocal beam subjected to distributed load. For a nonlocal beam subjected to the point load, it is easy to see that the regular finite element method cannot take the influence of the nonlocality into consideration. In the other words, the regular finite element solutions for the nonlocal beam subjected to point load would be identical with the local theory solution in this case. As aforementioned discussion, in order to overcome the problem of applied point load, some techniques are exploited such as Navier solution from Reddy (2007), Aydogdu (2009) and Dirac delta function from Wang and Liew (2007), Civalek and Demir (2011). On the contrary, for the mixed finite element model, the nonlocal parameter e_oa exists inside the stiffness matrix as given in Eq. (23), regardless of force vector. Therefore, the mixed finite element model has the capacity to capture the nonlocal effect for arbitrary loading types. The efficiency and accuracy of the present method will be addressed in next Section.

4. Numerical results

In this section, various numerical examples are performed to investigate the nonlocal behavior of SWCNT in conjunction with the elastic medium effect and the initial axial force. For this purpose, the deflection of midpoint is calculated for clamped-free (CF), simply supported (SS), clamped-simply supported (CS), and clamped-clamped (CC) boundary conditions. In addition, the parametric studies are carried out by investigating the total strain energy as

$$U = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} \tag{24}$$

where $\mathbf{u}$ and $\mathbf{K}$ are displacement vector and stiffness matrix, respectively. The following normalized definitions are used in the examples

$$N_o^* = \frac{N_o}{P_{cr}}\times 100; \quad P_{cr} = \frac{\pi^2 EI}{(KL)^2} \tag{25a}$$

$$k_w^* = \frac{k_w L^4}{EI} \tag{25b}$$

$$w^* = \frac{100EI}{qL^4} w_{\max} \tag{25c}$$

$$w^* = \frac{100EI}{PL^3} w_{\max} \tag{25d}$$

where K is effective length factor.

4.1 Verification

In order to show the accuracy of present methodology, the normalized deflections of SWCNT embedded in an elastic medium and subjected to an initial axial force, separately, are presented in Table 1 and Table 2 for e_oa=0 nm, respectively. It is obvious to see that the present results are in good agreement with those of Wang *et al*. (1998) and Chen (1987). The results are not affected by the nonlocal coefficients α_1 and α_2 since e_oa=0 nm.

The nonlocal static behavior of SWCNT without elastic medium and initial force effects is investigated. The comparison of normalized deflections of simply supported (SS) SWCNT subjected to uniform load and a point load at midpoint are tabulated in Table 3 and Table 4, respectively. It notes that the nonlocal coefficients α_1 and α_2 have no effects on the solution of SS boundary condition with

Table 1 Comparison of the normalized deflections for SWCNT embedded in an elastic medium under uniform load

BCs	k_w^*	Wang *et al.* (1998)	This study
SS	0	1.3033	1.3021
	10	1.1814	1.1809
	100	0.6403	0.6413
CC	0	0.2616	0.2604
	10	0.2565	0.2553
	100	0.2174	0.2167

Table 2 Comparison of the normalized deflections for SS SWCNT subjected to a compressive axial force and uniform load

N_0^*	Chen (1987)	This study
0	1.3021	1.3021
10	1.4473	1.4469
20	1.6287	1.6279
30	1.8621	1.8605
40	2.1732	2.1703
50	2.6089	2.6035

Table 3 Comparison of the normalized deflections for SS SWCNT subjected to uniform load (L/h=10)

$(e_oa)^2$	Reddy and Pang (2008)	Wang *et al.* (2008)	Pradhan (2012)	Şimşek and Yurtcu (2013)	This study
0	1.3021	1.3021	1.3021	1.3020	1.3021
1	1.4271	1.4271	1.4271	1.4270	1.4271
2	1.5521	1.5521	1.5521	1.5520	1.5521
3	1.6771	1.6771	-	1.6770	1.6771
4	1.8021	1.8021	-	1.8020	1.8021

Table 4 Comparison of the normalized deflections for SS SWCNT subjected to a point load P at midpoint (L/h=10)

Methods	$(e_oa)^2$				
	0	1	2	3	4
Wang and Liew (2007)	2.0833	2.3333	2.5833	2.8333	3.0833
This study	2.0833	2.3333	2.5833	2.8333	3.0833

Table 5 Comparison of the normalized deflections for SWCNT with different BCs and loading

BCs- Loading	x/L	Local solution	Wang and Liew (2007)	This study (α_1, α_2)				
				(1, 1)	(1, 0)	(0, 1)	(0, 0)	(0.5, 0.5)
CF-q	0.5	4.4270	3.9270	3.9270	3.9270	5.9272	5.9272	4.9271
	0.75	8.3495	7.2247	7.2247	7.2247	10.2245	10.2245	8.7248
	1	12.5001	10.5000	10.5000	10.5000	14.4998	14.4998	12.5001
CF-P	0.5	4.1667	4.1667	4.1667	4.1667	6.1669	6.1669	5.1668
	0.75	7.2916	6.2915	7.2916	6.2915	10.2919	9.2918	8.2917
	1	10.4165	8.4168	10.4165	8.4168	14.4169	12.4167	11.4166
CC-P	0.25	0.2604	0.3854	0.2604	0.3854	0.6354	0.7604	0.5104
	0.5	0.5209	1.0208	0.5209	1.0208	1.0208	1.5208	1.0208
	0.75	0.2604	0.3854	0.2604	0.3854	0.6354	0.7604	0.5104

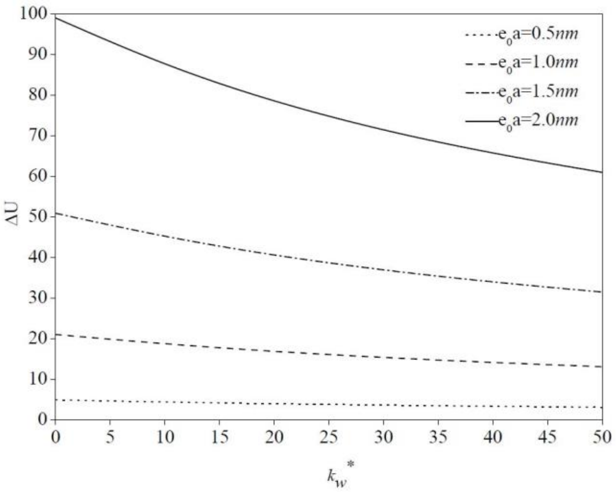

Fig. 2 The effect of elastic medium on nonlocal behavior for SS SWCNT subjected to uniform load ($N_0^* = 0$)

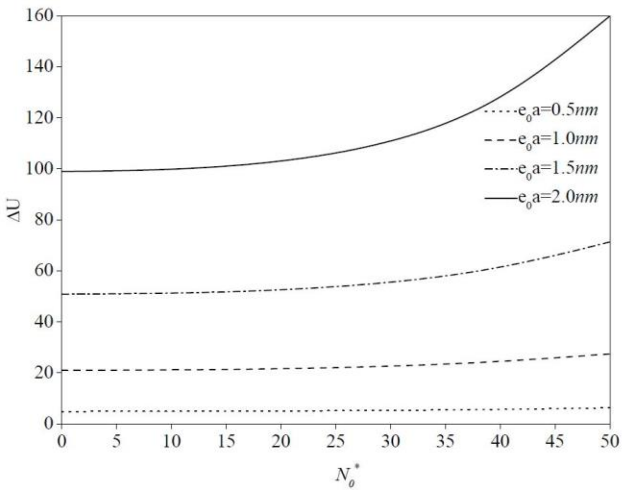

Fig. 3 The effect of initial force on nonlocal behavior for SS SWCNT subjected to uniform load ($k_w^* = 0$)

uniform load case due to the absence of the equilibrium bending moment M^E and the natural boundary conditions. The obtained results also agree very well with those from the literature in Table 3. In Table 4, the obtained results with α_2=0 coincide with the analytical solutions by Wang and Liew (2007). Therefore, it implies that for SS beams, the nonlocal coefficient α_2=0 gives the appropriate solution for the mixed finite element method with the nonlocal theory.

The normalized deflection of SWCNT with respect to various values of coefficients α_1 and α_2 for CF and CC boundary conditions are tabulated in Table 5. The nonlocal parameter and the length of SWCNT are set as e_0a=2 nm and L=10 nm, respectively. In general, the mixed finite element results with α_1=1, α_2=0 exactly match the analytical solutions from Wang and Liew (2007) for CF and CC boundary conditions under uniform load case and a point load case. Consequently, it indicates that the coefficients α_1=1, α_2=0 provide the proper solution for the present method with the typical boundary conditions. To this end, the coefficients α_1=1, α_2=0 are utilized in the following examples.

4.2 Parametric study

The influences of nonlocality on the static behavior of SWCNT are investigated in conjunction with the elastic medium effect and the initial axial load. For this purpose, the effect of nonlocal parameter e_0a on the variation of the total strain energy is defined as

$$\Delta U = \frac{U_1 - U_0}{U_0} \times 100 \tag{26}$$

where U_1 is the total strain energy with e_oa=0.5, 1, 1.5 and 2 nm, respectively, and U_0 is the total strain energy with e_oa=0 nm.

One can see that ΔU shows the variation of the total strain energy of nonlocal SWCNT with respect to the local total strain energy. The effect of elastic medium and initial axial force on the variation of total strain energy is plotted in Fig. 2 and Fig. 3, respectively, for SS SWCNT subjected to uniform load. It can be seen that the total strain energy is regardless of the elastic medium and the initial axial load proximately increases to 4.8, 21.0, 50.9, 98.9% with e_oa=0.5, 1, 1.5, 2 nm, respectively. On the other words, the nonlocal effect reduces the rigidity of SWCNT. With the appearance of the elastic medium effect, the variation of total strain energy gradually decreases whereas the variation of total strain energy increases with the increase of initial axial force as shown in Fig. 3. It means that the elastic medium lessens the effect of nonlocality on total strain energy. Meanwhile, the initial axial force intensifies the effect of nonlocality.

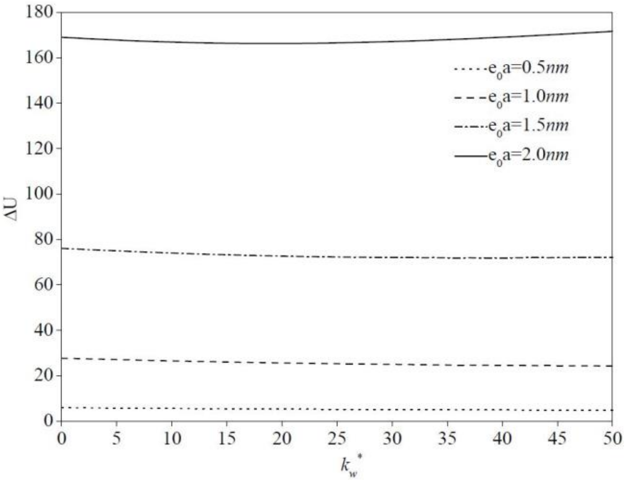

Fig. 4 The effect of elastic medium on nonlocal behavior for SS SWCNT subjected to a point load at midpoint ($N_0^* = 0$)

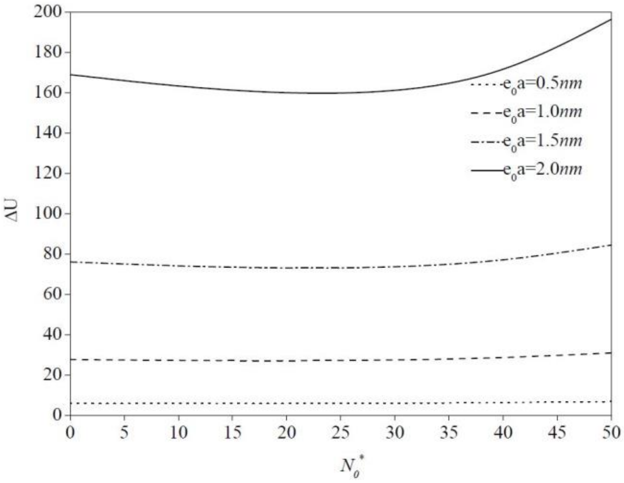

Fig. 5 The effect of initial force on nonlocal behavior for SS SWCNT subjected to a point load at midpoint ($k_w^* = 0$)

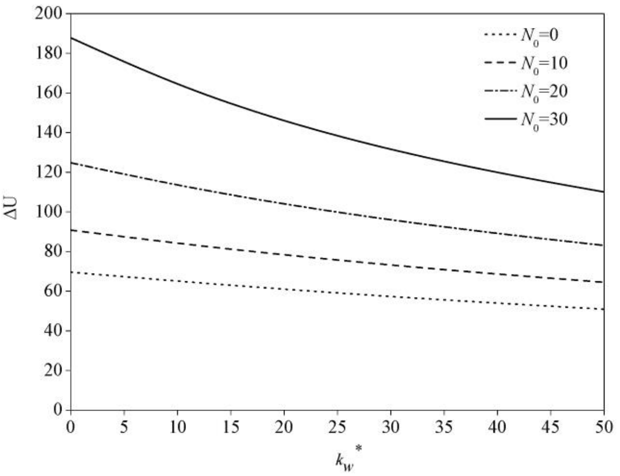

Fig. 6 The effect of elastic medium on nonlocal behavior for CS SWCNT subjected to uniform load (e_oa=2 nm)

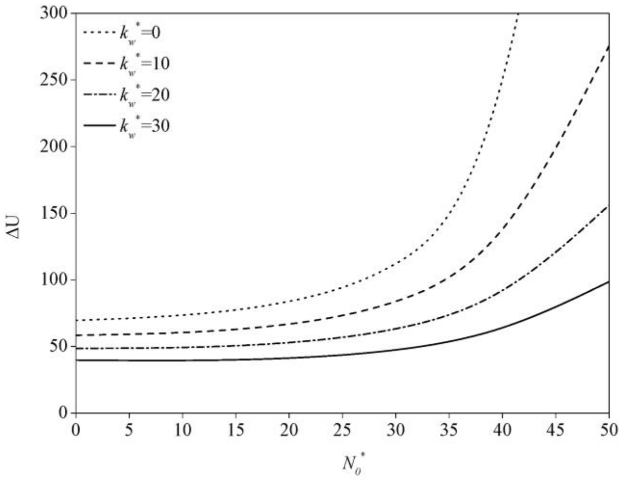

Fig. 7 The effect of initial force on nonlocal behavior for CS SWCNT subjected to uniform load (e_oa=2 nm)

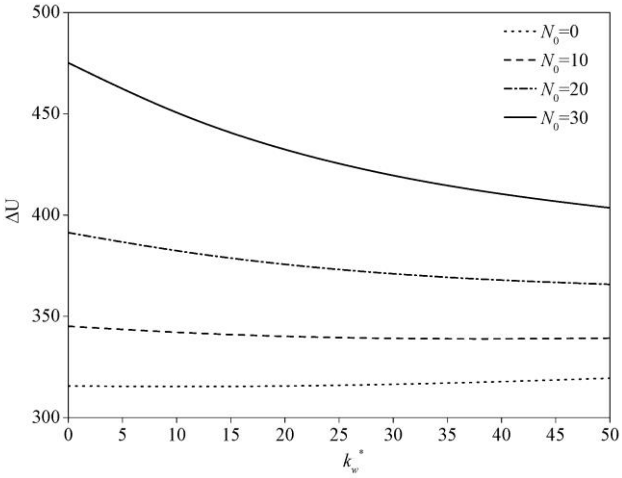

Fig. 8 The effect of elastic medium on nonlocal behavior for CS SWCNT subjected to a point load at midpoint (e_oa=2 nm)

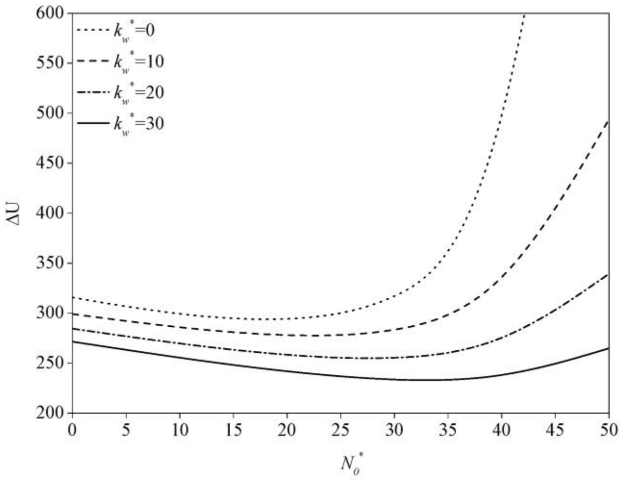

Fig. 9 The effect of initial force on nonlocal behavior for CS SWCNT subjected to a point load at midpoint (e_oa=2 nm)

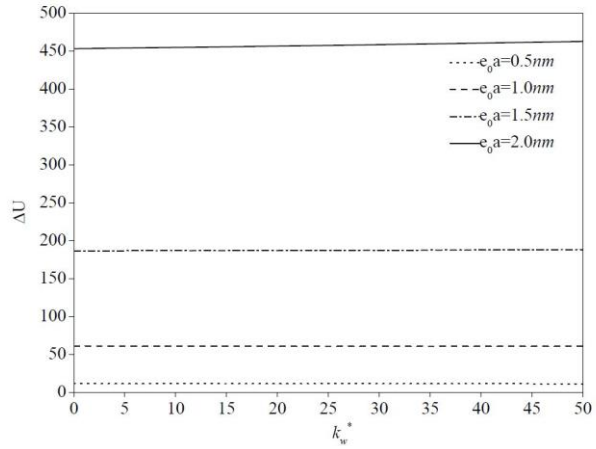

Fig. 10 The effect of elastic medium on nonlocal behavior for CC SWCNT subjected to a point load at midpoint ($N_0^*=0$)

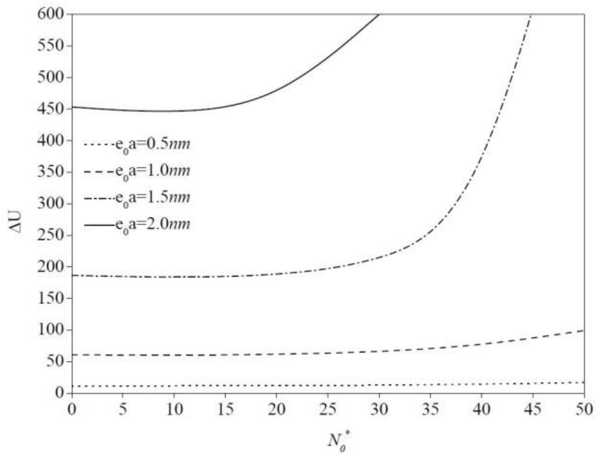

Fig. 11 The effect of initial force on nonlocal behavior for CC SWCNT subjected to a point load at midpoint ($k_w^*=0$)

Fig. 4 and Fig. 5 show the behavior of simply supported SWCNT under a point load at midpoint. In contrast to the phenomenon of uniform load case, the variation of total strain energy almost does not change with respect to the elastic medium modulus in Fig. 4. It means that there is no interactive effect between nonlocal effect and elastic medium. On the other hand, the variation of total strain energy slightly decreases with respect to the first increase of initial axial force, then the variation increases when the normalized initial axial force N_o^* is roughly beyond 30 for e_oa=2.0 nm. For e_oa=0.5, 1, 1.5 nm, the influence of the initial force on variation is nearly negligible. Moreover, it is interesting to observe that the increase of total strain energy with $N_o^*=k_w^*=0$ for a point load case is larger than uniform load case's. It implies that the point load case is more sensitive to nonlocal effect.

Fig. 6 to Fig. 9 show the influences of elastic medium and initial force on ΔU for the clamped-simply (CS) supported SWCNT under uniform load and a point load cases, respectively, when e_oa is set as 2 nm. Similarly, the consistent variations of total strain energy can be pointed out in Fig. 6 to Fig. 9. It is easy to see that the variation of total strain energy decreases with respect to the elastic medium modulus in Fig. 6 for uniform load case. Whereas the initial axial force plays an inverse proportion behavior in Fig. 7. In compliance with SS SWCNT under a point load, the elastic medium has no effect on the variation of total strain energy for CS SWCNT under a point load case ($N_o^*=0$) which is expressed by dot line in Fig. 8. However, it is valuable notice that for a point load case, the increases of total strain energy become much more significant than those of uniform load case. For instant, the total strain energy for a point load case raises to 315% ($N_o^*=k_w^*=0$), approximately, in comparison with 70% for uniform load case. In Fig. 9, owing to the interaction between the elastic medium effect and the initial axial force, ΔU initially decreases with respect to N_o^* and gradually increases as N_o^* is approximately larger than 30.

Finally, the variations of total strain energy of CC SWCNT subjected to a point load are plotted in Fig. 10 and Fig. 11 corresponding to the effect of the elastic medium and the initial axial force, respectively. As would be expected, the elastic medium do not influence on the change of total strain energy. On the contrary, the initial axial force induces the remarkable increasing effect of nonlocality with e_oa=1.5 nm due to the buckling behavior. Indeed, the critical buckling load of clamped boundary condition greatly reduces with nonlocal effect according to Reddy and Pang (2008).

5. Conclusions

The mixed finite element method is presented in this paper to study the behavior of SWCNT embedded in an elastic medium incorporated with the initial axial force based on the nonlocal continuum theory. The Euler-Bernoulli beam theory was used in the analysis. The numerical examples indicate that the coefficients α_1=1, α_2=0 for the nonlocal boundary conditions give the accurate solution in comparison with the valid literature. Moreover, the total strain energy is evaluated to investigate the effect of the elastic medium and the initial axial force on the nonlocal behavior. From results of the parametric study, the prominent observations are drawn as follows:

- In general, the nonlocal effect reduces the total rigidity of SWCNT due to the increase of the total strain energy.
- For uniform load case, the elastic medium reduces the effect of nonlocality while the initial axial force intensifies the nonlocal behavior.
- For a point load case, there is no interactive influence between elastic medium and nonlocal effect. Additionally, the total strain energy of a point load case is more delicate than uniform load case's under the nonlocal effect.

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