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A magneto-thermo-viscoelastic problem with fractional order strain under GN-II model

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Abstract. In this work, we present a theoretical framework to study the thermovisco-elastic responses of homogeneous, isotropic and perfectly conducting medium subjected to inclined load. Based on recently developed generalized thermoelasticity theory with fractional order strain, the two-dimensional governing equations are obtained in the context of generalized magneto-thermo-viscoelasticity theory without energy dissipation. The Kelvin-Voigt model of linear viscoelasticity is employed to describe the viscoelastic nature of the material. The resulting formulation of the field equations is solved analytically in the Laplace and Fourier transform domain. On the application of inclined load at the surface of half-space, the analytical expressions for the normal displacement, strain, temperature, normal stress and tangential stress are derived in the joint-transformed domain. To restore the fields in physical domain, an appropriate numerical algorithm is used for the inversion of the Laplace and Fourier transforms. Finally, we have demonstrated the effect of magnetic field, viscosity, mechanical relaxation time, fractional order parameter and time on the physical fields in graphical form for copper material. Some special cases have also been deduced from the present investigation.

Keywords: fractional order strain; GN-II model; magnetic field; viscosity; Laplace and Fourier transforms; inclined load

1. Introduction

The analysis of heat conduction involving extremely short times or very high frequency has found numerous applications in many different areas of mathematical physics, applied sciences and engineering. In such situations, the classical Fourier's heat conduction theory becomes inaccurate and the non-Fourier effect becomes more reliable in describing the diffusion process and predicting the temperature distribution.

The Fourier's law is given as

$$\vec{q} = -k\nabla T. \tag{1}$$

Eq. (1) assumes that $\vec{q}$ and ∇T appear at the same time instant t and consequently implies that thermal signals propagate with an infinite speed. It shows that if the material is subjected to a thermal disturbance, the effects of the disturbance will be felt instantaneously at distances infinitely far from its source. This type of phenomenon is physically unrealistic.

The energy balance equation is given by

$$\nabla \cdot \vec{q} = \rho C_E \frac{\partial T}{\partial t}. \tag{2}$$

The classical heat conduction equation is given as

$$\nabla^2 T = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \tag{3}$$

where $\alpha = k / \rho C_E$, $\vec{q}$ is the heat flux vector, k is thermal conductivity, ρ is mass density, C_E is specific heat and ∇^2 is Laplace's differential operator. Fourier's law is quite accurate for most common engineering situations. However, for situations involving very short times, temperature near absolute zero or extreme thermal gradients, Fourier's law becomes invalid. The Eq. (3) is a parabolic-type partial differential equation that allows an infinite speed for thermal signals. The classical Fourier's law does not lead to the thermal wave behaviour, since the law permits the heat flux to respond immediately to changes in temperature gradient.

Despite of the Fourier theory, the non-Fourier or hyperbolic theory predicts that heat propagates with wave behaviour and finite speed. The non-Fourier heat flux equation has been developed by modifying the Fourier's Law that connects the heat flux to the temperature by adding an extra thermal inertia term.

The first model to remove the above mentioned paradox of classical Fourier's law was proposed by Cattaneo (1958) and Vernotte (1958) as

$$\vec{q} + \tau_0 \frac{\partial \vec{q}}{\partial t} = -k\nabla T, \tag{4}$$

where τ_0 is thermal relaxation time. The relaxation time τ_0 depends on the mechanism of heat transport and represents the time lag needed to establish steady state heat conduction in an element of volume when a temperature gradient is suddenly applied to that element. Here, the time derivative term makes the heat propagation speed finite. Eq. (4) tells us that the heat flux does not appear

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instantaneously but it grows gradually with a build-up time τ_0.

The modified Fourier equation coupled with the energy balance equation leads to a hyperbolic heat equation

$$\nabla^2 T = \frac{1}{\alpha}\left(\frac{\partial T}{\partial t} + \tau_0 \frac{\partial^2 T}{\partial t^2}\right). \tag{5}$$

Eq. (5) describes the heat propagation with a finite speed $v = \sqrt{\alpha / \tau_0}$ which diverges only for the unphysical assumption of $\tau_0 = 0$.

Based on this concept, Lord and Shulman (1967) derived the generalized thermoelasticity theory referred to as L-S theory or extended thermoelasticity theory, in which the Maxwell-Cattaneo's law replaces the Fourier's law of heat conduction by introducing a single parameter that acts as a relaxation time. The modified heat conduction equation in this theory is of the wave type and it ensures the finite speeds of propagation of heat and elastic waves. Green and Lindsay (1972) also developed another generalized thermoelasticity theory termed as G-L theory or the temperature-rate dependent theory, that includes the temperature-rate among constitutive variables and also predicts a finite speed for heat propagation. In addition, this theory modifies all the equations of the classical theory of thermoelasticity, not only the heat equation and contains two constants that act as relaxation times. Contrary to the L-S theory, the G-L theory does not violate Fourier's law of heat conduction when the solid has a center of symmetry. The L-S theory was further extended to homogeneous anisotropic heat conducting materials by Dhaliwal and Sherief (1980). As such, heat transport in solids is regarded as a wave phenomenon rather than a diffusion phenomenon. Extensive literature survey on the subject "generalized thermoelasticity" can be found in the review articles by Chandrasekharaiah (1986, 1998). Hetnarski and Ignaczak (1999) presented a survey article on the modeling of thermoelastic waves in a solid body in the context of five different theories of generalized thermoelasticity. Sharma *et al.* (2008) studied the dynamical behaviour in generalized thermoelastic diffusion medium under Green-Lindsay theory using Fourier transform method. Othman and Ahmed (2015) investigated the propagation of plane waves in generalized piezo-thermoelastic medium under the effect of rotation using normal mode analysis.

Green and Naghdi (1991, 1992, 1993) developed the non conventional heat conduction theory by using general entropy balance law rather than entropy inequality in an alternative way, predicting finite speed of the thermal disturbance. This theory is developed with a view to produce a rational continuum thermo-dynamical theory of solids which is capable of incorporating thermal pulse transmissions in a very logical manner. Green and Naghdi theory is further subdivided into three different theories referred to as GN-I, GN-II and GN-III theories. Under the assumption that when the respective theory is linearized, GN-I theory encompasses the classical heat conduction theory based on Fourier's constitutive prescription for the heat flux vector. In GN-II theory, the internal rate of production of entropy is assumed to be identical to zero i.e., there is no dissipation of thermal energy and this theory is referred as thermoelasticity without energy dissipation. GN-III theory includes the previous two models as special cases and admits dissipation of energy in general. Kumar *et al.* (2016) investigated the disturbances in a homogeneous transversely isotropic thermoelastic rotating medium with two temperature and in the presence of the combined effects of Hall currents and magnetic field under generalized thermoelasticity without energy dissipation.

In modern technology, considerable interest has been evinced in the study of the interaction among the strain, temperature and electromagnetic field in an elastic solid due to its immense applications in various disciplines such as geophysics, magnetic structural elements, damping of acoustic waves in a magnetic field, emissions of electro-magnetic radiations from nuclear devices and electrical power engineering etc. The interplay of the Maxwell's electro-magnetic field with the motion of deformable solids is largely being undertaken by many authors owing to the possibility of its application to geophysical problems and certain topics in optics and acoustics.

The generation of magneto-thermoelastic waves by a thermal shock in a perfectly conducting half-space in contact with vacuum was investigated by Kaliski and Nowacki (1962). In this article, both the media were supposed to be permeated by a primary uniform magnetic field. Several other problems based on the magneto-thermoelasticity theory are established by Paria (1962), Willson (1963) and Nayfeh and Nemat-Nasser (1972). Youssef (2006) formulated a magneto-thermoelasticity theory with one relaxation time with variable material properties and studied one dimensional problem. Othman and Song (2006) investigated the effect of rotation on the reflection of magneto-thermoelastic waves under generalized thermo-elasticity theory without energy dissipation. Deswal and Kalkal (2006) analyzed the behaviour of plane harmonic waves in a magneto-thermo-viscoelastic medium with diffusion by using the methodology of normal mode analysis. Das and Kanoria (2012) studied the thermoelastic interactions in a magneto-thermoelastic half-space in the context of GN-II, GN-III and three-phase-lag models using Laplace and Fourier transform method. Deswal *et al.* (2013) illustrated a two-dimensional half-space problem subjected to thermo mechanical loading in the context of magneto-thermoelasticity theory with laser pulse heating.

With the rapid development of polymer science, plastic industry, application of biology and geology in engineering as well as the wide use of materials under high temperature, the theoretical study and applications in viscoelastic materials have become an important task for solid mechanics. Ilioushin and Pobedria (1970) established a mathematical model of thermo-viscoelasticity theory and obtained the solutions of some boundary value problems. However, the works of Tanner (1988) and Huilgol and Phan-Thien (1997) are devoted to find the solutions of boundary value problems for linear viscoelastic materials including temperature variations in both quasi-static and dynamic problems. El-Karamany and Ezzat (2004) has solved a one-dimensional thermal shock problem for a thermo-viscoelastic medium under different theories of thermoelasticity. Bakshi *et al.* (2008) studied a thermo-viscoelastic problem in an infinite isotropic medium in the presence of a point heat source by using joint Laplace-Fourier transform technique and eigen value approach. The problems related to the propagation of plane waves in an infinite thermo-viscoelastic medium of Kelvin-Voigt type are studied in the papers of Kumar and Partap (2011), Ezzat *et al.* (2013), Deswal and Yadav (2014) and Deswal and Kalkal (2015).

Fractional calculus is the branch of mathematics that concerns with integrals and derivatives of arbitrary order. During last few decades, fractional order differential equations have been successfully employed for modeling of many different processes and systems, specifically in the area of physics, chemistry, engineering, astrophysics, chemical mechanics, quantum mechanics, nuclear physics and quantum field theory etc. The historical development of the subject fractional calculus can be investigated in Ross (1977) and Miller and Ross (1993). Caputo and Mainardi (1971) and Caputo (1974) employed fractional order derivative for the description of viscoelastic materials and established the relationship between fractional derivative and the theory of linear viscoelasticity.

Recently, Magin and Royston (2010) applied fractional calculus to introduce a fractional order viscoelastic model with the idea that it is the order of derivative of strain that characterizes the material's behaviour. In this model, one-dimensional fractional order stress-strain relation can be expressed as

$$\sigma(t) = E^* \tau^\beta \frac{d^\beta \varepsilon(t)}{dt^\beta}, \tag{6}$$

where $\sigma(t)$ is stress and $\varepsilon(t)$ is strain as a function of time *t*, τ is mechanical relaxation time and β is the fractional parameter which takes values between 0 and 1. For $\beta = 0$, the stress-strain relation represents Hooke's Law with $E^* = E$ (Young's modulus) and for $\beta = 1$, it corresponds to Newtonian fluid with $E^* = \eta$ (coefficient of viscosity). Viscoelastic material occupies the intermediate range with a fractional order β between 0 and 1. In a similar manner, Meral *et al.* (2010) also developed a fractional order Voigt model and obtained the analytical solution of the surface waves on viscoelastic half-space for a finite circular disk located on the surface and oscillating normal to it. Some experimental results of the surface waves are also presented in order to compare different fractional order models.

Based on the fractional order strain model (Magin and Royston 2010), Youssef (2016) derived a new theory of thermoelasticity with fractional order strain which is considered as a new modification to Duhamel-Neumann's stress-strain relation. In this paper, the author postulated a new unified system of equations that govern seven different models of thermoelasticity in the context of one-temperature and two-temperature and one dimensional problem for an isotropic and homogeneous elastic half-space.

In the present paper, we consider a two-dimensional problem of generalized thermoelasticity theory without energy dissipation with fractional order strain. The transient responses of an isotropic, homogeneous and perfectly conducting magneto-thermo-viscoelastic half-space subjected to an inclined load are investigated. The solution of the problem is obtained in the Laplace-Fourier transform domain and a numerical inversion method of the Laplace-Fourier transforms is used to obtain the displacement, temperature, strain and stress fields in the physical domain. Finally, the effect of fractional order parameter, mechanical relaxation time, magnetic field, viscosity and time on the physical fields is illustrated in the figures. To the best of the authors' knowledge, the proposed mathematical model with fractional order strain is rather new and such a dynamical problem has not been considered previously.

2. Basic governing equations

In this section, we have formulated the basic governing equations for an isotropic, homogeneous magneto-thermo-viscoelastic solid of Kelvin-Voigt type under fractional order strain theory of generalized thermoelasticity without energy dissipation. In the absence of body forces, following Youssef (2016), the system of basic field equations and constitutive equations is obtained as under:

(i) the stress-strain-temperature relations for a viscoelastic material with fractional order strain are

$$\sigma_{ij} = 2\mu^*\left(1+\tau^\beta D_t^\beta\right)e_{ij} + \lambda^*\left(1+\tau^\beta D_t^\beta\right)e_{kk}\delta_{ij} - \beta_1\theta\delta_{ij}, \tag{7}$$

where $\lambda^* = \lambda_e\left(1+\alpha_0\frac{\partial}{\partial t}\right)$, $\quad \mu^* = \mu_e\left(1+\alpha_1\frac{\partial}{\partial t}\right)$,

$$\beta_1 = \beta_e\left(1+\beta_0\frac{\partial}{\partial t}\right), \qquad \beta_e = \left(3\lambda_e + 2\mu_e\right)\alpha_t,$$

$$\beta_0 = \left(3\lambda_e\alpha_0 + 2\mu_e\alpha_1\right)\alpha_t / \beta_e.$$

In this article we use the Caputo fractional derivative D_t^β of order β with respect to time *t*, which is defined as

$$D_t^\beta f(t) = \begin{cases} \dfrac{1}{\Gamma(1-\beta)}\displaystyle\int_0^t \frac{f'(\zeta)}{(t-\zeta)^\beta}d\zeta, & 0 \le \beta < 1, \\ \dfrac{df(t)}{dt}, & \beta = 1, \end{cases}$$

(ii) the strain-displacement relations are

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{8}$$

(iii) the equations of motion for a perfectly conducting homogeneous elastic solid under uniform magnetic field are

$$\sigma_{ji,j} + F_i = \rho\ddot{u}_i, \tag{9}$$

where $\vec{F} = \mu_0 \vec{J} \times \vec{H}$,

(iv) heat conduction equation without heat source is

$$k^*\theta_{,ii} = \rho C_E \ddot{\theta} + \beta_1 T_0 \left(1+\tau^\beta D_t^\beta\right)\ddot{e}_{ij}\delta_{ij} . \tag{10}$$

The variations of electric and magnetic fields inside the medium are given by the Maxwell's equations as follows

$$\text{curl } \vec{h} = \vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}, \tag{11}$$

$$\text{curl } \vec{E} = -\mu_0 \frac{\partial \vec{h}}{\partial t}, \tag{12}$$

$$\vec{E} = -\mu_0 \left(\frac{\partial \vec{u}}{\partial t} \times \vec{H}\right), \tag{13}$$

$$\text{div}\,\vec{h} = 0, \tag{14}$$

$$\vec{h} = \text{curl}\left(\vec{u} \times \vec{H}\right), \tag{15}$$

where u_i are the components of displacement vector $\vec{u}$, $\theta = T - T_0$, T is the absolute temperature and T_0 is reference temperature, which are assumed to obey the inequality $|\theta/T_0| << 1$, σ_{ij} are the components of the stress tensor, e_{ij} are the components of strain tensor, δ_{ij} is the Kronecker delta function, $e_{kk} = e$ is the cubical dilatation, ρ is the density of the medium, $k^* = \dfrac{(\lambda_e + 2\mu_e)C_E}{4}$ is the material constant, α_t is the coefficient of linear thermal expansion, λ_e and μ_e are the Lame's constants, α_0 and α_1 are viscoelastic relaxation times, τ is the mechanical relaxation time, F_i are the components of Lorentz's body force vector $\vec{F}$, μ_0 is the magnetic permeability, ε_0 is the electric permittivity, $\vec{H}$ is the applied magnetic field, $\vec{J}$ is the current density vector, a comma followed by a suffix denotes material derivative and a superposed dot denotes the derivative with respect to time *t*.

3. Formulation and solution of problem

We consider a homogeneous, isotropic, thermally and electrically conducting generalized magneto-thermo-viscoelastic solid to study the dynamical interactions in the context of fractional order strain thermoelasticity theory without energy dissipation. We take the cartesian coordinate system (*x*, *y*, *z*) with *z*-axis pointing vertically into the medium and the half-space occupies the region $z \geq 0$. Now, we restrict our analysis to a two dimensional problem in *xz*-plane. Hence, all the considered functions will depend only on the space variables *x*, *z* and time *t*. The half-space is placed in a magnetic field with constant intensity $\vec{H} = (0,\, H_0,\, 0)$. The bounding surface of the half-space is assumed to be thermally insulated and subjected to a mechanical type inclined load Φ_0 with an inclination ϑ to *z*-axis. The geometry of the problem is given in Fig. 1.

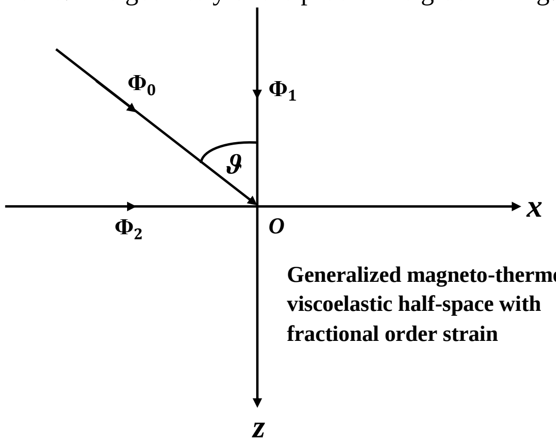

Fig. 1 Geometry of the problem

The components of displacement vector $\vec{u} = (u, v, w)$ assume the form

$$u = u(x,z,t), v = 0, w = w(x,z,t) . \tag{16}$$

The strain-displacement relation (8) gives

$$e = e_{kk} = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} . \tag{17}$$

The initial and regularity conditions for the thermoelastic half-space are given as

$$\begin{aligned} u(x,z,0) &= \dot{u}(x,z,0) = 0 , \\ w(x,z,0) &= \dot{w}(x,z,0) = 0 , \\ \theta(x,z,0) &= \dot{\theta}(x,z,0) = 0 , \end{aligned} \tag{18}$$

for $z \geq 0, -\infty \leq x \leq \infty,$ and

$$u(x,z,t) = w(x,z,t) = \theta(x,z,t) = 0 , \tag{19}$$

for $t > 0$ when $z \to \infty$.

The electric intensity $\vec{E}$ is normal to both the magnetic intensity and the displacement vectors and the induced magnetic field $\vec{h}$ is normal to the electric intensity $\vec{E}$. Hence, the Eqs. (12) and (13) give rise to the following expressions

$$\vec{E} = \mu_0 H_0 \left(\frac{\partial w}{\partial t}, 0, -\frac{\partial u}{\partial t}\right), \quad \vec{h} = \left(0, -H_0 e, 0\right), \tag{20}$$

Since the current density $\vec{J}$ is parallel to the electric intensity $\vec{E}$, it follows from the relations (11) and (20) that the electric current density $\vec{J}$ will have two component in *x* and *z*-direction, given by

$$J_x = H_0 \left(\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 w}{\partial z^2}\right) - \varepsilon_0 \mu_0 H_0 \frac{\partial^2 w}{\partial t^2}, \tag{21}$$

$$J_y = 0\,,$$

$$J_z = -H_0\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 w}{\partial x \partial z}\right)+\varepsilon_0\mu_0 H_0\frac{\partial^2 u}{\partial t^2}\,.$$

Inserting expressions (21) in the relation $\vec{F}=\mu_0\vec{J}\times\vec{H}$, we get the components of Lorentz's force as

$$F_x = \mu_0 H_0^2\left(\frac{\partial e}{\partial x}-\mu_0\varepsilon_0\frac{\partial^2 u}{\partial t^2}\right),$$

$$F_y = 0\,, \tag{22}$$

$$F_z = \mu_0 H_0^2\left(\frac{\partial e}{\partial z}-\mu_0\varepsilon_0\frac{\partial^2 w}{\partial t^2}\right).$$

To make the field equations simpler, we introduce the following dimensionless transformations

$$\left(x',z',u',w'\right)=c_0\eta_0\left(x,z,u,w\right),$$

$$\left(t',\tau',\alpha_0',\alpha_1',\beta_0'\right)=c_0^2\eta_0\left(t,\tau,\alpha_0,\alpha_1,\beta_0\right), \tag{23}$$

$$\sigma_{ij}'=\frac{\sigma_{ij}}{\beta_e T_0},\quad \theta'=\frac{\theta}{T_0},$$

where $c_0^2=\dfrac{\lambda_e+2\mu_e}{\rho}$, $\eta_0=\dfrac{\rho C_E\varpi}{k^*}$, $\varpi=\dfrac{\rho C_E c_0^3}{k^* h^*}$ and h^* is a fixed length.

In terms of the non-dimensional quantities defined in Eq. (23), the governing Eqs. (7)-(10) in *xz*-plane reduce to (dropping the dashes for convenience)

$$\sigma_{zz}=\varepsilon_0^*\left(1+\alpha_0\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial x}+\left(1+\delta_0\frac{\partial}{\partial t}\right)\left(1+\tau^\beta D_t^\beta\right)\frac{\partial w}{\partial z} -\left(1+\beta_0\frac{\partial}{\partial t}\right)\theta\,, \tag{24}$$

$$\sigma_{zx}=\eta_1\left(1+\alpha_1\frac{\partial}{\partial t}\right)\left(1+\tau^\beta D_t^\beta\right)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right), \tag{25}$$

$$\varepsilon_2\frac{\partial^2 u}{\partial t^2}=\eta_1\left(1+\alpha_1\frac{\partial}{\partial t}\right)\left(1+\tau^\beta D_t^\beta\right)\nabla^2 u-\left(1+\beta_0\frac{\partial}{\partial t}\right)\frac{\partial\theta}{\partial x} +\left[\eta_2\left(1+\delta_1\frac{\partial}{\partial t}\right)\left(1+\tau^\beta D_t^\beta\right)+\varepsilon_1\right]\frac{\partial e}{\partial x}, \tag{26}$$

$$\varepsilon_2\frac{\partial^2 w}{\partial t^2}=\eta_1\left(1+\alpha_1\frac{\partial}{\partial t}\right)\left(1+\tau^\beta D_t^\beta\right)\nabla^2 w-\left(1+\beta_0\frac{\partial}{\partial t}\right)\frac{\partial\theta}{\partial z} +\left[\eta_2\left(1+\delta_1\frac{\partial}{\partial t}\right)\left(1+\tau^\beta D_t^\beta\right)+\varepsilon_1\right]\frac{\partial e}{\partial z}, \tag{27}$$

$$\nabla^2\theta=\varepsilon_3\frac{\partial^2\theta}{\partial t^2}+\varepsilon_4\left(1+\beta_0\frac{\partial}{\partial t}\right)\left(1+\tau^\beta D_t^\beta\right)\frac{\partial^2 e}{\partial t^2}, \tag{28}$$

where $\varepsilon_0^*=\dfrac{\lambda_e}{\lambda_e+2\mu_e}$, $\delta_0=\varepsilon_0^*\alpha_0+\left(1-\varepsilon_0^*\right)\alpha_1$, $\eta_1=\dfrac{1-\varepsilon_0^*}{2}$,

$\eta_2=\dfrac{1+\varepsilon_0^*}{2}$, $\delta_1=\dfrac{\varepsilon_0^*\alpha_0+\eta_1\alpha_1}{\eta_2}$, $\varepsilon_1=\dfrac{\mu_0 H_0^2}{\lambda_e+2\mu_e}$,

$\varepsilon_2=1+\dfrac{\varepsilon_0\mu_0^2 H_0^2}{\rho}$, $\varepsilon_3=C_E\left(\dfrac{\lambda_e+\mu_e}{k^*}\right)$, $\varepsilon_4=\dfrac{\beta_e^2 T_0}{k^*\rho}$.

To simplify the above system of equations, we introduce displacement potential functions through the relations $\vec{u}=\nabla\phi+\nabla\times\vec{\Psi}$, $\vec{\Psi}=\left(0,-\psi,0\right)$, which lead to (dimensionless form)

$$u=\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial z},\ w=\frac{\partial\phi}{\partial z}-\frac{\partial\psi}{\partial x}, \tag{29}$$

where $\nabla^2=\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial z^2}$ and the potential functions ϕ and $\vec{\Psi}$ represent the dilatational and rotational parts respectively of displacement vector $\vec{u}$.

It follows from Eqs. (17) and (29) that

$$e=\nabla^2\phi\,. \tag{30}$$

Now, we define the Laplace transform of the function $f\left(x,z,t\right)$ with respect to variable *t* as

$$L\left[f\left(x,z,t\right)\right]=\bar{f}\left(x,z,s\right)=\int_0^\infty f\left(x,z,t\right)e^{-st}dt\,, \tag{31}$$

and the Fourier transform of the function $\bar{f}\left(x,z,s\right)$ with respect to variable *x* as

$$F\left[\bar{f}\left(x,z,s\right)\right]=\hat{f}\left(\xi,z,s\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{f}\left(x,z,s\right)e^{-\iota\xi x}dx\,, \tag{32}$$

where s and ξ are the Laplace and Fourier transform parameters respectively.

The operational properties of the Fourier exponential transform are

$$F\left[\frac{\partial\bar{f}\left(x,z,s\right)}{\partial x}\right]=\iota\xi\hat{f}\left(\xi,z,s\right), \tag{33}$$

$$F\left[\frac{\partial^2\bar{f}\left(x,z,s\right)}{\partial x^2}\right]=-\xi^2\hat{f}\left(\xi,z,s\right), \tag{34}$$

and the Laplace transform rule for the fractional order derivative D_t^β under initial conditions is given by

$$L\left\{D_t^\beta f\left(t\right)\right\}=s^\beta L\left\{f\left(t\right)\right\}. \tag{35}$$

Introducing Laplace and Fourier transforms into Eqs. (26)-(28), after using Eqs. (29), (33)-(35) simultaneously, we get the following set of differential equations

$$\left(\frac{d^4}{dz^4}+L\frac{d^2}{dz^2}+M\right)\left(\hat{\phi},\hat{\theta}\right)=0\,, \tag{36}$$

and

$$\left(\frac{d^2}{dz^2}-\lambda_3^2\right)\hat{\psi}=0, \tag{37}$$

where $L=-\dfrac{2a_0\xi^2+\left(\varepsilon_2+\varepsilon_3 a_0+\varepsilon_4 a_4^2 a_3\right)s^2}{a_0}$,

$$M=\frac{a_0\xi^4+\varepsilon_2 s^2\xi^2+\varepsilon_3 s^2 a_0\xi^2+\varepsilon_4 a_2 a_4^2 s^2\xi^2+\varepsilon_2\varepsilon_3 s^4}{a_0},$$

$$\lambda_3^2=\xi^2+\frac{\varepsilon_2 s^2}{\eta_1 a_1 a_2}, \qquad a_0=\eta_1 a_1 a_2+\eta_2 a_3 a_2+\varepsilon_1,$$

$a_1=1+\alpha_1 s,\ a_2=1+\tau^\beta s^\beta,\quad a_3=1+\delta_1 s,\quad a_4=1+\beta_0 s.$

Now, Eq. (36) can be factorized as

$$\left(\frac{d^2}{dz^2}-\lambda_1^2\right)\left(\frac{d^2}{dz^2}-\lambda_2^2\right)\left(\hat{\phi},\hat{\theta}\right)=0, \tag{38}$$

where

$$\lambda_1^2=\frac{-L+\sqrt{L^2-4M}}{2},$$

and

$$\lambda_2^2=\frac{-L-\sqrt{L^2-4M}}{2}.$$

Following the regularity conditions given in Eq. (19), the solution of Eqs. (37) and (38) can be expressed as

$$\hat{\phi}=A_1 e^{-\lambda_1 z}+A_2 e^{-\lambda_2 z}, \tag{39}$$

$$\hat{\theta}=B_1 e^{-\lambda_1 z}+B_2 e^{-\lambda_2 z}, \tag{40}$$

$$\hat{\psi}=A_3 e^{-\lambda_3 z}, \tag{41}$$

where A_i, $B_i\,(i=1,2)$ and A_3 are the unknown coefficients dependent on s and ξ such that $B_i=b_i A_i$ with $b_i=\varepsilon_4 a_2 a_4\left(\dfrac{\lambda_i^2-\xi^2}{\lambda_i^2-\xi^2-\varepsilon_3 s^2}\right)\,(i=1,2)$.

4. Application

We have considered an isotropic, homogeneous magneto-thermo-viscoelastic half-space $z\geq 0$ with fractional order strain. The bounding surface $z=0$ is assumed to be thermally insulated and is subjected to a mechanical type inclined load Φ_0 with an inclination ϑ to *z*-axis as given in Fig. 1. Hence, the normal line load Φ_1 and tangential line load Φ_2 are expressed as $\Phi_1=\Phi_0\cos\vartheta$ and $\Phi_2=\Phi_0\sin\vartheta$ respectively. In order to solve the problem, we assume that the initial conditions of the problem are taken to be homogeneous, while the boundary conditions at the surface *z* = 0 are expressed as

$$\sigma_{zz}(x,z,t)=-\Phi_1\psi_1(x)\delta(t), \tag{42}$$

$$\sigma_{zx}(x,z,t)=-\Phi_2\psi_2(x)\delta(t), \tag{43}$$

$$\frac{\partial\theta(x,z,t)}{\partial t}=0, \tag{44}$$

where $\psi_1(x)$ and $\psi_2(x)$ are normal and horizontal load functions respectively and $\delta(t)$ is the Dirac delta function.

Now, we further assume that the load is linearly distributed over a strip of width 2*d* (as shown in Fig. 2), which is expressed in terms of load functions $\psi_1(x)$ and $\psi_2(x)$ by relation

$$\{\psi_1(x),\psi_2(x)\}=\begin{cases}1-\dfrac{|x|}{d} & \text{if } |x|\leq d\\ 0 & \text{if } |x|>d\end{cases}. \tag{45}$$

The Fourier transform of expression (45) is given by

$$\{\hat{\psi}_1(\xi),\hat{\psi}_2(\xi)\}=\frac{2(1-\cos(\xi d))}{\xi^2 d}. \tag{46}$$

Introducing Eqs. (24), (25) and (29) along with the relations $\Phi_i'=\dfrac{\Phi_i}{\beta_e T_0}(i=1,2)$ into the Eqs. (42)-(44) (dropping the primes) and applying Laplace-Fourier transforms along with the use of Eqs. (39)-(41) and (46) in the resulting expressions, we obtain

$$P_1A_1+P_2A_2+P_3A_3=P, \tag{47}$$

$$Q_1A_1+Q_2A_2+Q_3A_3=Q, \tag{48}$$

$$R_1A_1+R_2A_2=0, \tag{49}$$

where $P_i=a_2a_5\lambda_i^2-\varepsilon_0^* a_6\xi^2-a_4 b_i$, $Q_i=2\iota\xi\eta_1 a_1 a_2\lambda_i$, $R_i=-\lambda_i b_i\ (i=1,2)$, $P_3=\iota\xi\lambda_3\left(\varepsilon_0^* a_6-a_2a_5\right)$, $Q_3=\eta_1 a_1 a_2\left(\lambda_3^2+\xi^2\right)$,

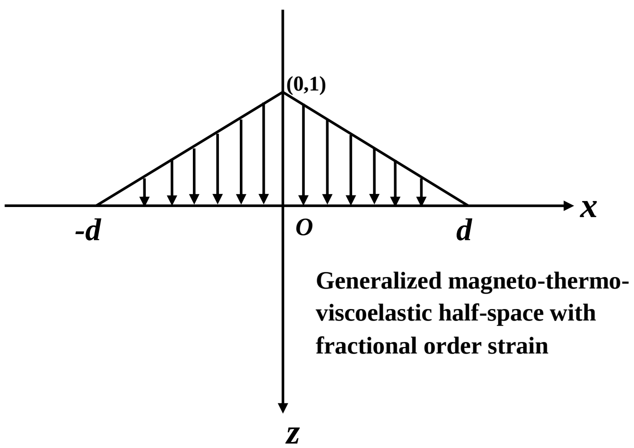

Fig. 2 Linearly distributed load

$$P=-\frac{2\Phi_0\cos\vartheta\left(1-\cos\left(\xi d\right)\right)}{\xi^2 d},$$

$$Q=-\frac{2\Phi_0\sin\vartheta\left(1-\cos\left(\xi d\right)\right)}{\xi^2 d},$$

$a_5=1+\delta_0 s,\quad a_6=1+\alpha_0 s.$

Solution of Eqs. (47)-(49) is given by

$$A_1=\frac{\Delta_1}{\Delta},\quad A_2=\frac{\Delta_2}{\Delta},\quad A_3=\frac{\Delta_3}{\Delta},\tag{50}$$

where $\Delta=R_1\left(P_2Q_3-Q_2P_3\right)-R_2\left(P_1Q_3-Q_1P_3\right)$,

$\Delta_1=-R_2\left(PQ_3-QP_3\right),\qquad \Delta_2=R_1\left(PQ_3-QP_3\right),$

$\Delta_3=R_1\left(QP_2-PQ_2\right)-R_2\left(P_1Q-Q_1P\right).$

By virtue of expressions in Eq. (50), Eqs. (39)-(41) can be rewritten as

$$\hat{\phi}=\frac{1}{\Delta}\left(\Delta_1e^{-\lambda_1z}+\Delta_2e^{-\lambda_2z}\right),\tag{51}$$

$$\hat{\theta}=\frac{1}{\Delta}\left(b_1\Delta_1e^{-\lambda_1z}+b_2\Delta_2e^{-\lambda_2z}\right),\tag{52}$$

$$\hat{\psi}=\frac{\Delta_3}{\Delta}e^{-\lambda_3z}.\tag{53}$$

Substitution of Eqs. (51)-(53) into Eqs. (24)-(25) and (29)-(30) after applying Laplace and Fourier transforms, leads to the following expressions of physical fields in the transform domain

$$\hat{u}=\frac{1}{\Delta}\left[\iota\xi\left(\Delta_1e^{-\lambda_1z}+\Delta_2e^{-\lambda_2z}\right)-\lambda_3\Delta_3e^{-\lambda_3z}\right],\tag{54}$$

$$\hat{w}=-\frac{1}{\Delta}\left[\left(\lambda_1^2-\xi^2\right)\Delta_1e^{-\lambda_1z}+\left(\lambda_2^2-\xi^2\right)\Delta_2e^{-\lambda_2z}+\iota\xi\Delta_3e^{-\lambda_3z}\right],\tag{55}$$

$$\hat{e}=\frac{1}{\Delta}\left[\lambda_1\Delta_1e^{-\lambda_1z}+\lambda_2\Delta_2e^{-\lambda_2z}\right],\tag{56}$$

$$\hat{\sigma}_{zz}=\frac{1}{\Delta}\left[C_1\Delta_1e^{-\lambda_1z}+C_2\Delta_2e^{-\lambda_2z}-C_3\Delta_3e^{-\lambda_3z}\right],\tag{57}$$

$$\hat{\sigma}=\frac{1}{\Delta}\left[D_1\Delta_1e^{-\lambda_1z}+D_2\Delta_2e^{-\lambda_2z}-D_3\Delta_3e^{-\lambda_3z}\right],\tag{58}$$

where $C_i=a_2a_5\lambda_i^2-\varepsilon_0^*a_6\xi^2-a_4b_i$,

$D_i=2\iota\xi\eta_1a_1a_2\lambda_i\left(i=1,2\right),\quad C_3=\iota\xi\left(\varepsilon_0^*a_6-a_2a_5\right)\lambda_3,$

$D_3=\eta_1a_1a_2\left(\lambda_3^2+\xi^2\right).$

5. Special cases

Case I: Without viscous and magnetic effects

If we assume that the thermal and mechanical fields are independent of the viscosity, then the results obtained from the present analysis remain for magneto-thermo-elastic medium. This can be done by setting $\alpha_0=\alpha_1=0$, which leads to the following relations $\lambda^*=\lambda_e$, $\mu^*=\mu_e$, $\beta_0=0$ and $\beta_1=\beta_e$. In addition, if we assume that the material properties are independent of the magnetic field (i.e., $H_0=0$), then the present problem reduces to the fractional order strain theory of generalized thermoelasticity without energy dissipation. Furthermore, for one dimensional case, the relevant problem coincides with Youssef (2016) with appropriate change in the boundary conditions.

Case II: Without fractional order strain

To discuss the wave phenomena for generalized theory of magneto-thermo-viscoelasticity without energy dissipation, we substitute $\tau=0$ with $\beta\neq 0$ in the basic governing equations, which leads to $a_2=1$. The corresponding expressions for the physical variables considered in the problem for this case can be procured from Eqs. (52) and (54)-(58).

6. Inversion of integral transforms

It is difficult to find the analytical inversion of Laplace and Fourier transforms for displacement, strain, temperature and stress fields in the space-time domain. But this can be conveniently managed through numerical evaluations of the inversion integrals. The fields in the Laplace-Fourier transform domain are the functions of the form $\hat{f}\left(\xi,z,s\right)$. First, we invert the Fourier's transform, which gives us the Laplace transform expression $\bar{f}\left(x,z,s\right)$ for the function $f\left(x,z,t\right)$ as

$$F^{-1}\left[\hat{f}\left(\xi,z,s\right)\right]=\bar{f}\left(x,z,s\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}\left(\xi,z,s\right)e^{\iota\xi x}d\xi.\tag{59}$$

The inversion formula of the Laplace transform for the function $\bar{f}\left(x,z,s\right)$ is defined as

$$L^{-1}\left[\bar{f}\left(x,z,s\right)\right]=f\left(x,z,t\right)=\frac{1}{2\pi\iota}\int_{c-\iota\infty}^{c+\iota\infty}e^{st}\bar{f}\left(x,z,s\right)ds,\tag{60}$$

where c is an arbitrary real number larger than the real parts of all the singularities of $\bar{f}\left(x,z,s\right)$.

Taking $s=c+\iota w$, the preceding integral takes the form

$$f\left(x,z,t\right)=\frac{e^{ct}}{2\pi}\int_{-\infty}^{\infty}\bar{f}\left(x,z,c+\iota w\right)e^{\iota wt}dw.\tag{61}$$

Expanding the function $h\left(x,z,t\right)=e^{-ct}f\left(x,z,t\right)$ in a Fourier series in the interval $\left[0,2t_1\right]$, we obtain the approximate formula (Honig and Hirdes 1984)

$$f\left(x,z,t\right)=f_\infty\left(x,z,t\right)+E_D,\tag{62}$$

where

$$f_\infty(x,z,t) = \frac{1}{2}c_0 + \sum_{k=1}^{\infty} c_k, \ 0 \le t \le 2t_1 \tag{63}$$

and

$$c_k = \frac{e^{ct}}{t_1}\left[e^{\frac{\iota k\pi t}{t_1}} \bar{f}\left(x, z, c + \frac{\iota k\pi t}{t_1} \right) \right]. \tag{64}$$

The discretization error E_D can be made arbitrarily small by choosing c large enough (Honig and Hirdes 1984). Since the infinite series in Eq. (63) can be summed upto a finite number of terms N, the approximate value of $f(x,z,t)$ becomes

$$f_N(x,z,t) = \frac{1}{2}c_0 + \sum_{k=1}^{N} c_k, \ 0 \le t \le 2t_1 . \tag{65}$$

Using the preceding formula to evaluate $f(x,z,t)$, we introduce a truncation error E_{t_1} that must be added to the discretization error to produce the total approximation error.

Two methods are used to reduce the total error. First, the 'Korrektur' method is applied to reduce the discretization error. Next, the ε -algorithm is used to accelerate convergence (Honig and Hirdes 1984).

The Korrektur method uses the following formula to evaluate the function $f(x,z,t)$

$$f(x,z,t) = f_\infty(x,z,t) - e^{-2ct_1} f_\infty(x,z,2t_1+t) + E'_D, \tag{66}$$

where the discretization error $|E'_D| \le |E_D|$. Thus, the approximate value of $f(x,z,t)$ becomes

$$f_{NK}(x,z,t) = f_N(x,z,t) - e^{-2ct_1} f_{N'}(x,z,2t_1+t), \tag{67}$$

where N' is an integer such that $N' < N$.

We shall now describe the ε -algorithm that is used to accelerate the convergence of the series in Eq. (65). Let $N = 2q+1,$ where q is a natural number and $s_m = \sum_{k=1}^{m} c_k$ is the sequence of the partial sum of the series in Eq. (65).

We define the ε -sequence by

$$\varepsilon_{0,m} = 0 , \ \ \varepsilon_{1,m} = s_m , \tag{68}$$

and

$$\varepsilon_{p+1,m} = \varepsilon_{p-1,m+1} + \frac{1}{\varepsilon_{p,m+1} - \varepsilon_{p,m}}, \, p = 1,2,3,... \tag{69}$$

It can be shown that [Honig and Hirdes (1984)] the sequence $\varepsilon_{1,1}$, $\varepsilon_{3,1}$, $\varepsilon_{5,1}$, …, $\varepsilon_{N,1}$, … converges to $f(x,z,t) + E_D - \frac{c_0}{2}$ faster than the sequence of partial sums s_m , $m = 1,2,3,...$

The actual procedure used to invert the Laplace transform consists of using Eq. (67) together with the ε-algorithm. The values of c and t_1 are chosen according to the criteria outlined by Honig and Hirdes (1984).

Simultaneous computations of the inversion of the Fourier transform are performed by evaluating the infinite integral (59) numerically by seven-point Gaussian quadrature formula for several prescribed values of the variables x and z (Rakshit and Mukhopadhyay 2007).

7. Numerical discussion

To validate the results of the present study, numerical results for homogeneous, isotropic, perfectly conducting thermo-viscoelastic medium under a uniform magnetic field are obtained within the framework of GN-II generalized thermoelasticity theory with fractional order strain. A linearly distributed inclined load is applied on the bounding surface of half-space with insulated boundary as described in Figures 1 and 2.

In the present work, copper material is considered for the purpose of numerical illustrations. The values of the material properties (i.e. Lame's constants, coefficients of heat conduction and thermal expansion, magnetic field parameters and viscosity constants) of copper are assumed to be

$\lambda_e = 7.76 \times 10^{10} kgm^{-1}s^{-2}$, $\mu_e = 3.86 \times 10^{10} kgm^{-1}s^{-2}$,
$T_0 = 293K$, $C_E = 383.1 Jkg^{-1}K^{-1}$, $\alpha_t = 1.78 \times 10^{-5} K^{-1}$,
$\rho = 8954 kgm^{-3}$, $\mu_0 = 4\pi \times 10^{-7} Hm^{-1}$, $\alpha_0 = 0.06s$,
$\varepsilon_0 = (10^{-9}/36\pi) Fm^{-1}$, $H_0 = (10^7/4\pi) Am^{-1}$, $\alpha_1 = 0.09s.$

The other numerical constants related to fractional order strain model and mechanical load are assumed as:

$$\tau = 0.02, \quad \vartheta = 45°, \quad \Phi_0 = 10, \quad d = 2, \quad h^* = 10.$$

Considering the above physical data, we have evaluated the numerical values of the field quantities with the help of a computer program and the results are displayed graphically.

The effect of fractional order parameter β and mechanical relaxation time τ on the thermoelastic responses (i.e., normal displacement w, strain e, temperature θ , normal stress σ_{zz} and tangential stress σ_{zx}) is analyzed in Figs. 3-7 at x=1.0 and t=0.02. In these figures, the case $\tau = 0, \beta \ne 0$ corresponds to generalized thermoelasticity theory without energy dissipation in the absence of fractional order derivative. The effects of magnetic field and viscosity on the physical fields under generalized thermoelasticity theory of fractional order strain without energy dissipation are presented in Figs. 8-12 at $\beta = 0.1,$ $x = 1.0$ and $t = 0.02.$ In these figures, we use the following abbreviations:

(i) GTFOSMV-Generalized thermoelasticity theory of fractional order strain with magnetic field and viscosity, (ii) GTFOSV-Generalized thermoelasticity theory of fractional order strain with viscosity and (iii) GTFOSM-Generalized thermoelasticity theory of fractional order strain with magnetic field. The wave phenomena of the studied physical fields in a thermoelastic half space depending upon various values of time t (0.01, 0.015, 0.02) are illustrated in Figs. 13-17 at $\beta = 0.1$ and x=1.0.

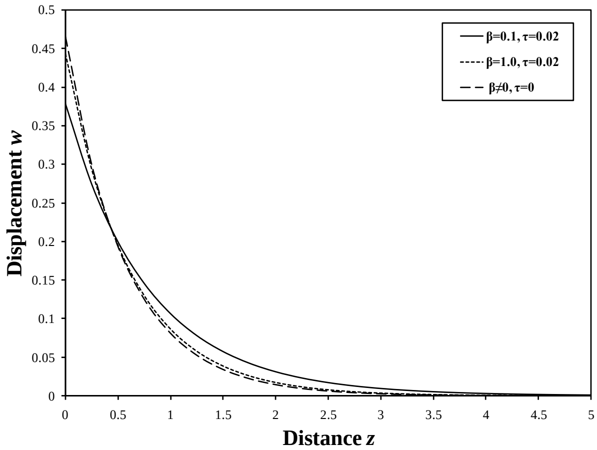

Fig. 3 Dependence of displacement field w on β and τ at t=0.02

Fig. 3 shows the variations of normal displacement field w versus z to study the effects of fractional parameter β and mechanical relaxation time τ. It is indicated that the response of the displacement field is similar for all the values of fractional parameter β and mechanical relaxation time τ. Figure also reveals that the increase in the fractional parameter β is accompanied by the enlargement in the field values in the range $0 \le z < 0.5$ and reduction in the range $0.5 \le z \le 4.5$. Moreover, the effect of β and τ is noticed to be quite prominent on this field.

The response of strain field e for three different cases $\beta = 0.1$, $\tau = 0.02$; $\beta = 1.0$, $\tau = 0.02$ and $\beta \neq 0$, $\tau = 0$ is predicted in Figure 4. As can be seen, the strain field has maximum amplitude on the point of application of source which is a physically plausible situation and as we move away from the source, it shows a decreasing trend for all the three cases. However, the fractional parameter β and mechanical relaxation time τ have both increasing and decreasing effects on the amplitude of this field.

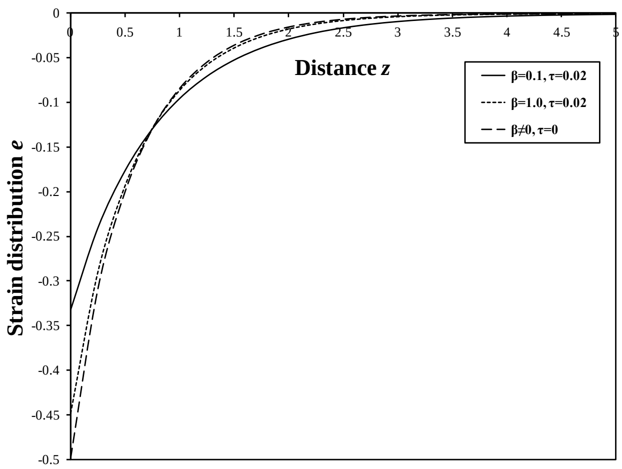

Fig. 4 Dependence of strain distribution e on β and τ at t=0.02

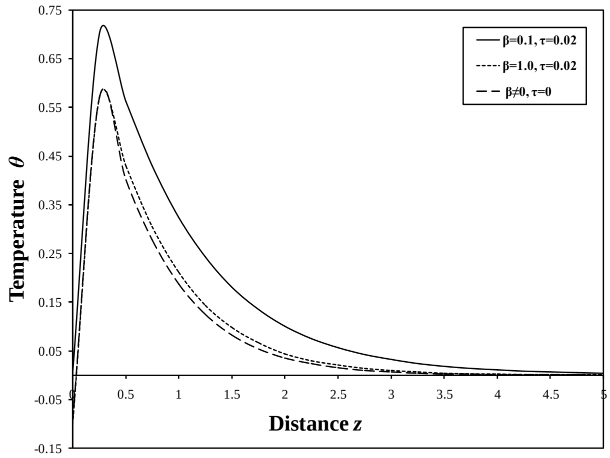

Fig. 5 Dependence of temperature θ on β and τ at t=0.02

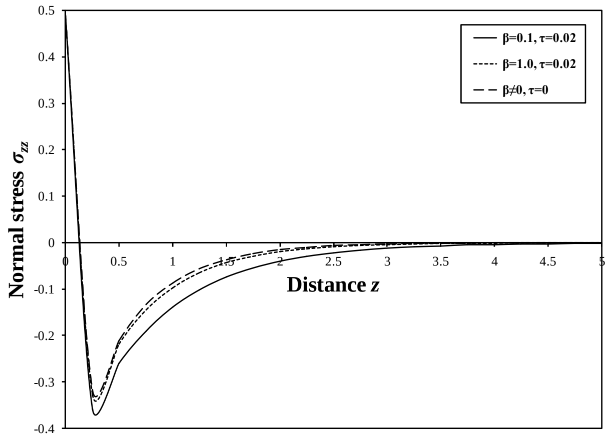

Fig. 6 Dependence of normal stress σ_{zz} on β and τ at t=0.02

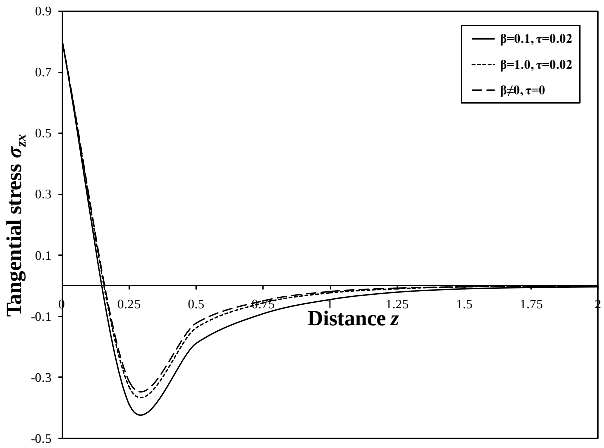

Fig. 7 Dependence of tangential stress σ_{zx} on β and τ at t=0.02

Fig. 5 illustrates the variations of temperature field θ with respect to distance z corresponding to three different cases of the values of β and τ. For all the cases, the temperature profile is experiencing an increasing pattern in the range $0 \le z < 0.5$ and decreasing pattern in the range

$0.5 \le z \le 5$ and thereafter it diminishes to zero. Hence, all the variations are restricted to a limited region which confirms the wave nature of heat propagation, whereas, this type of wave phenomenon is absent in classical theories of thermoelasticity. Additionally, increase in the value of β reduces the values of temperature field.

The normal stress σ_{zz} and tangential stress σ_{zx} are plotted with respect to distance z in Figs. 6 and 7 respectively. It is evident from these figures that the stress fields are tensile near the application of source with maximum intensity at z=0, which is followed by the mechanical boundary conditions applied. However, these stress fields become compressive after some distance from the source with a sufficient peak value and then continuously increase in the compressive region to reach to the steady state and finally diminish to zero. The reason for this is that due to the application of mechanical load, the region near the source experiences a reaction force with compressive nature. It is also worthy to mention here that these fields are significantly affected by the fractional parameter β and mechanical relaxation time τ.

Fig. 8 is drawn to represent the response of normal displacement w versus distance z for three different cases GTFOSMV, GTFOSV and GTFOSM. It is noted that all the curves of displacement field show similar pattern and all restrained in a finite region. There are significant differences between the variations predicted by different cases. Absence of magnetic field decreases the values of this field, whereas viscosity has reverse effect.

Figure 9 is illustrated to show the effect of magnetic field and viscosity on the profile of the strain field e versus distance z. It is interesting to note that the curves corresponding to GTFOSMV and GTFOSV show similar trends but for the case GTFOSM, it behaves distinctly, which clearly shows the dominance of viscosity on the strain field. It should be mentioned that the magnetic field acts to decrease the strain field. This is mainly due to the fact that the magnetic field corresponds to a term signifying a positive force, which tends to accelerate the solid particles.

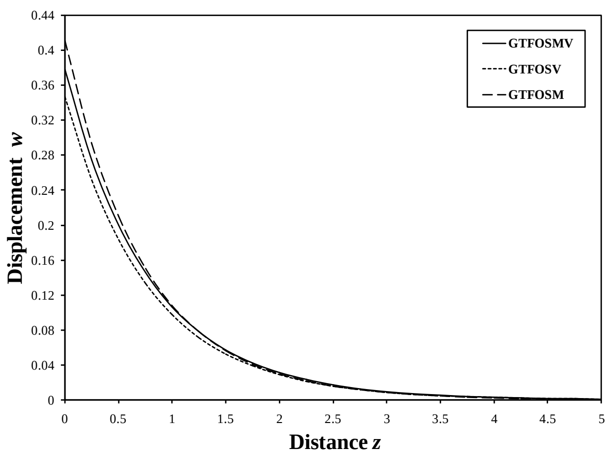

Fig. 8 Effect of magnetic field and viscosity on displacement field w at β=0.1 and t=0.02

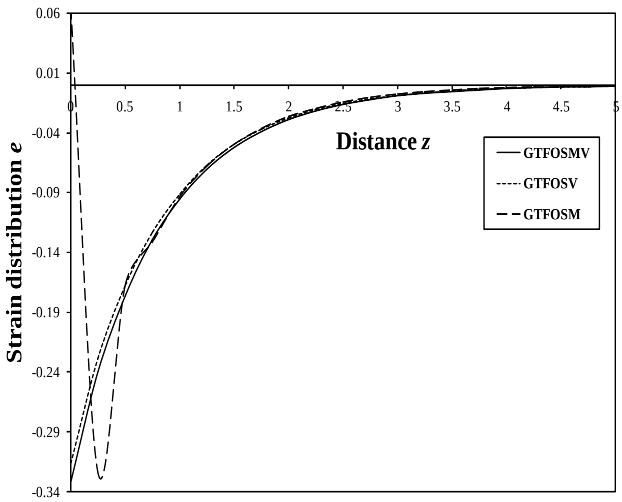

Fig. 9 Effect of magnetic field and viscosity on strain distribution e at $\beta = 0.1$ and t=0.02

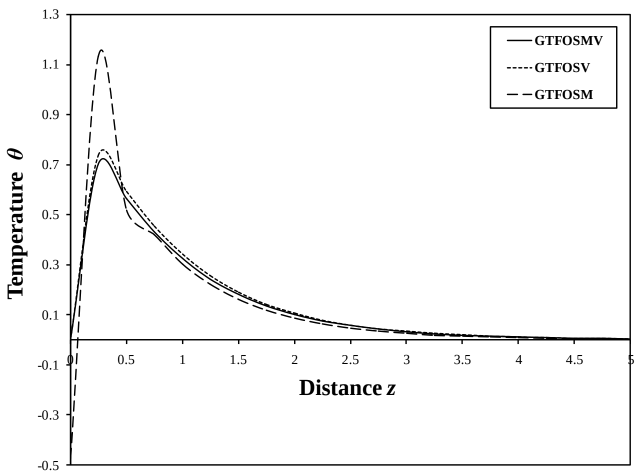

Fig. 10 Effect of magnetic field and viscosity on temperature θ at $\beta = 0.1$ and t=0.02

The effects of magnetic field and viscosity on the response of temperature field θ versus the distance z are illustrated in Fig. 10. As can be seen, the profiles of temperature field are alike for GTFOSMV and GTFOSV while it is quite different for GTFOSM. It shows apparently that the variations are limited in a finite region, which is in accordance with the second sound effect. Furthermore, the effect of viscosity is much pronounced on temperature field. Also, the absence of magnetic field causes to increase the field values but viscosity has both increasing and decreasing effects.

Figure 11 describes the behaviour of normal stress σ_{zz} versus distance z for the three different cases namely GTFOSMV, GTFOSV and GTFOSM. It is shown in figure that the curves of the stress field σ_{zz} vary from positive to negative and increase continuously to reach to the zero value beyond the heat wave front. Also, the stress distribution has non-zero values only in a bounded region of space. Outside this region, the values vanish identically which is in agreement with the experimental results. It is

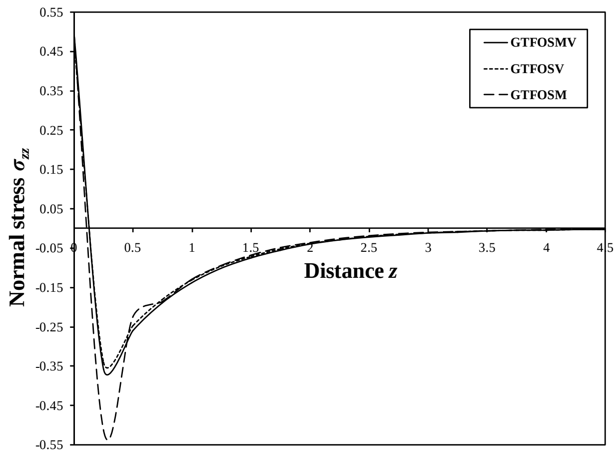

Fig. 11 Effect of magnetic field and viscosity on normal stress σ_{zz} at $\beta = 0.1$ and *t*=0.02

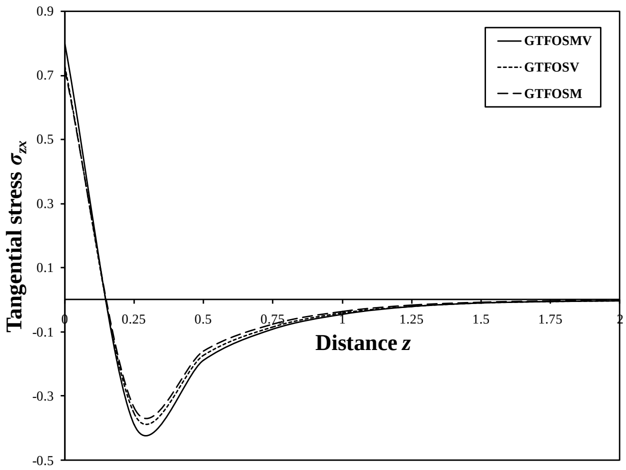

Fig. 12 Effect of magnetic field and viscosity on tangential stress σ_{zx} at $\beta = 0.1$ and *t*=0.02

further observed that the effect of viscosity is more pronounced on this field, however, in the absence of magnetic field, the amplitude of stress field decreases.

The effects of magnetic field and viscosity on the tangential stress σ_{zx} are displayed in Fig. 12. We have observed that the values of the stress field are maximum at the boundary surface for all the three cases GTFOSMV, GTFOSV and GTFOSM, which supports the mechanical boundary conditions. In addition, the magnitude of tangential stress σ_{zx} is pronouncedly reduced as the magnetic and viscous properties of the material are neglected (i.e., for GTFOSV and GTFOSM). However, all the curves are experiencing similar pattern.

The responses of normal displacement field *w* versus variable *z* for three different times (*t* = 0.01, 0.015, 0.02) are illustrated in Figure 13. It is clear that the displacement is maximum at the origin and decreases with the distance *z* for all the values of time *t*. Also, displacement field decreases with the increase of time t and the difference is more significant near the boundary surface.

The profile of strain *e* against distance *z* for three different values of times (*t* = 0.01, 0.015, 0.02) is presented

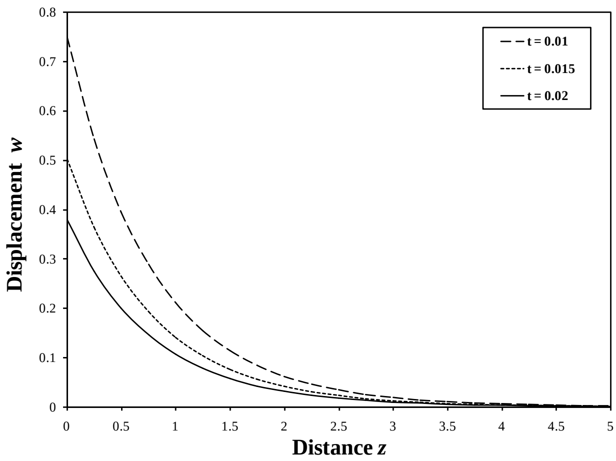

Fig. 13 Profile of displacement field *w* for *t*=0.01, 0.015, 0.02 at $\beta = 0.1$

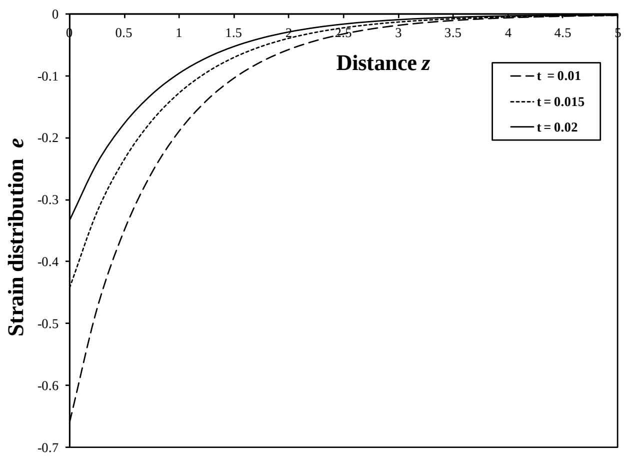

Fig. 14 Profile of strain distribution *e* for *t*=0.01, 0.015, 0.02 at $\beta = 0.1$

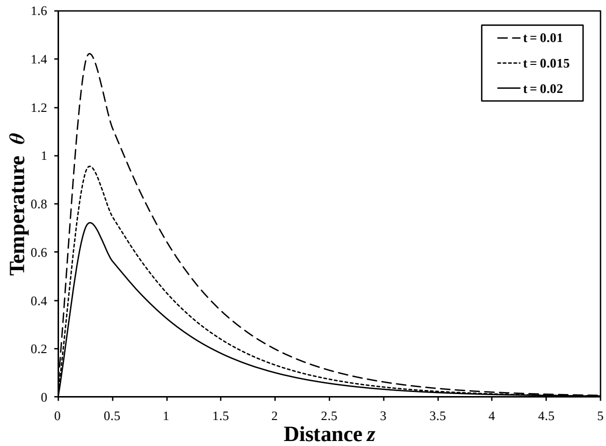

Fig. 15 Profile of temperature θ for *t*=0.01, 0.015, 0.02 at $\beta = 0.1$.

in Fig. 14. We have noted that the strain field follows similar pattern for all the values of time having difference in magnitude. This difference is maximum at the boundary of half-space and is lessening with the increase in distance.

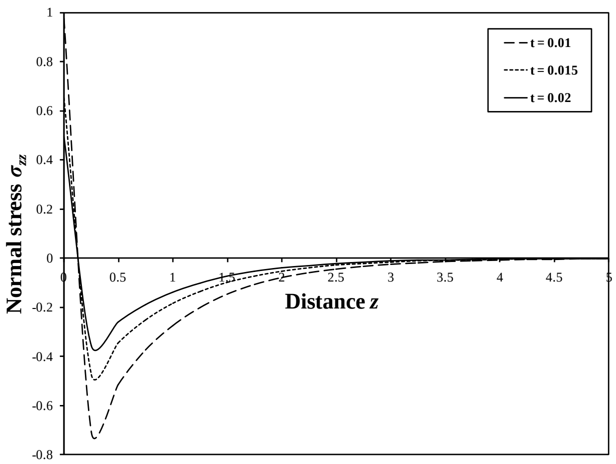

Fig. 16 Profile of normal stress σ_{zz} for t=0.01, 0.015, 0.02 at $\beta = 0.1$

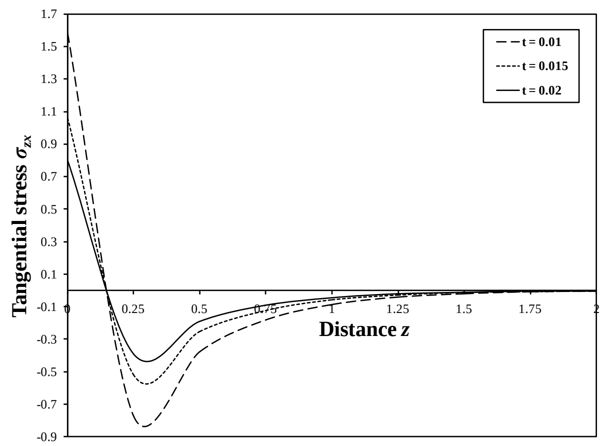

Fig. 17 Profile of tangential stress σ_{zx} for t=0.01, 0.015, 0.02 at $\beta = 0.1$

Moreover, figure also suggests that time t has decreasing effect on the strain field.

The distribution of temperature θ against distance z for three different times (t = 0.01, 0.015, 0.02) is depicted in Fig. 15. For all the values of time, the profiles of θ firstly increase rapidly in the range $0 \le z < 0.5$ to attain maximum values, then decrease smoothly in the range $0.5 \le z \le 5$ and afterwards vanish to zero value. It can be further inferred that the position at which the temperature field vanishes corresponds to the heat wave front, which is in support of previous findings that the heat wave traverses at finite speed. Finally, it is demonstrated that the profiles of temperature distribution decrease due to the increment in time t.

The dynamical variations of normal stress σ_{zz} and tangential stress σ_{zx} versus distance z for three different times (t = 0.01, 0.015, 0.02) are described in Figs. 16 and 17 respectively. It is clear from these figures that the curves corresponding to different times are experiencing qualitatively similar behaviour. As expected, the stress fields have maximum strength at the boundary surface which is consistent to the physical boundary conditions. Initially stress fields are in tensile mode, thereafter these fields become compressive with the passage of time. It is also pertinent to mention here that the increase in time t causes to decrease the magnitude of these stresses.

8. Conclusions

The dynamical interactions of a magneto-thermo-viscoelastic half-space have been investigated in the context of the fractional order strain theory of generalized thermo-elasticity without energy dissipation. In the present model, the Duhamel-Neumann's stress-strain relation is modified by introducing fractional order differential operator of the strain. Utilizing the fractional order strain model, we are able to characterize the material properties more flexibly contrary to classical formulation. However, from the analysis of the results obtained in this study, we can conclude the following remarks:

- The thermoelastic responses in the solid half-space are restricted to a limited region and outside this region, the responses vanish identically. This confirms that second sound wave phenomenon is manifested in all the figures.
- The fractional order strain parameter β (for fixed $\tau = 0.02$) has significant effect on the response of the physical quantities which clearly shows its importance in describing the behaviour of these physical quantities. However, the fractional parameter β and mechanical relaxation time τ have both increasing and decreasing effects on the physical fields.
- In fractional order strain theory, the characteristics of a material depend on the order of fractional order strain operator, hence, the fractional parameter β is a new indicator to provide knowledge about the time history of the deformation of the materials.
- The effect of the viscosity is found to be very much prominent on the physical fields. Moreover, as viscoelastic properties of the material are neglected from the medium then the variations of strain e, temperature θ and normal stress σ_{zz} behave distinctly from others.
- We can deduce from the figures that the nature of variations of all the physical quantities are very much similar in the presence and absence of magnetic field. It is observed that as the magnetic field is neglected from the medium then the magnitude of the thermoelastic fields w, e, σ_{zz} and σ_{zx} is pronouncedly reduced while it increases the magnitude of temperature θ.
- Stress fields have maximum absolute values near the surface of the body, which is consistent to the physical boundary conditions.
- All the physical fields show qualitatively similar pattern for different values of time t with a significant difference in magnitude. Although, the numerical values of all the fields decrease with increase in the values of time t.

The use of fractional order strain model in the study of the wave phenomenon in thermoelastic solid is rather limited. To address this issue, here, we have performed a two-dimensional study to find the dynamical behaviour of an infinite magneto-thermo-viscoelastic medium constituted by fractional order material. The fractional order strain problem studied herein may be applicable in the fields of biomechanics, biomedical problems and skin tissues where knowledge of such changes would enable early diagnostic monitoring for the onset of disease and better assessment of the effectiveness of new drugs or therapies.

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