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Non-stochastic interval factor method-based FEA for structural stress responses with uncertainty

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Abstract. The goal of this study is to evaluate behavior uncertainties of structures by using interval finite element analysis based on interval factor method as a specific non-stochastic tool. The interval finite element method, i.e., interval FEM, is a finite element method that uses interval parameters in situations where it is not possible to get reliable probabilistic characteristics of the structure. The present method solves the uncertainty problems of a 2D solid structure, in which structural characteristics are assumed to be represented as interval parameters. An interval analysis method using interval factors is applied to obtain the solution. Numerical applications verify the intuitive effectiveness of the present method to investigate structural uncertainties such as displacement and stress without the application of probability theory.

Keywords: interval parameters; interval factor method; interval finite element analysis; structural behavior; uncertainty; non-stochastic

1. Introduction

Physical parameters used to describe a structure are often uncertain, due to physical and geometrical uncertainties, or modeling inaccuracies. They are, for example, Young's modulus, Poisson's ratio, volumic mass or dimension of plates. Different methods may be used to solve these problems, in which these uncertain parameters are generally identified by random variables, such as Monte Carlo simulation (Samis and Davis 2014, Juan and Kimura 2014), a perturbation method (Wang and Qiu 2015), Neumann expansion series (Ramli and Jang 2014, Baricz *et al*. 2012), or a projection on homogeneous chaos (Galal 2014). But all these methods consider stochastic variables for which the density of probability (Ying *et al*. 2014, Hossenini and Shahabian 2014) is known. Unfortunately, the probabilistic approach cannot provide reliable results unless sufficient experimental data or statistical information is available to validate the assumptions about the probability densities of the random variables. In some cases, we can only obtain some range or lower and upper bounds of structural parameters.

Since the mid-1960s, the called interval analysis was introduced. Moore (1979), his co-workers, Alefeld and Herzberger (1983), and Rao and Berke (1997) have done the pioneering work. Mathematically, linear interval equations, nonlinear interval equations and interval eigenvalue problems in the method have been resolved partly. But because of the complexity of the algorithm, it is difficult to apply these results to practical engineering problems. In the pioneering study on non-probabilistic approaches, Koyluoglu and Ekishakoff (1998) exploited convex models to describe uncertainties. Therefore, the growing interest on non-probabilistic methods for non-deterministic analysis (Lee *et al*. 2016, Lee and Shin 2015) of structures with uncertain parameters is originated from criticism on the credibility of probabilistic analysis based on limited information. In this context, the application of the interval concepts on the non-deterministic analysis of structures provided very satisfactory results.

In addition, Gao (2007), Modares and Mullen (2008), Chen and Wang (2000), and other researchers (Rump 2012, Lee *et al*. 2008, Lee *et al*. 2010) have used interval set models in the study of the static response of structures with bounded uncertain parameters. In their studies, several important results have been obtained, using interval analysis and interval factor method. Their contribution was aimed to overcome the described drawbacks of probabilistic approaches in the solutions of a truss structure with interval factor under static known loads. The procedure is based on a factorization of the uncertain-but-bounded material parameters by using the concept of interval factor method. The procedure of interval factor method (IFM) is as follows. First, an interval variable such as structural parameters or loads is expressed as an interval factor multiplied by the mean value of this interval variable. Second, structural stiffness matrix is expressed as its deterministic value multiplied by the interval factors of the structural parameters. Finally, structural static responses are expressed as a function of these interval factors, and the computational expression for interval structural displacement and stress responses can be obtained by means of the interval operations. Therefore, the effect of the change of structural parameters and loads on the structural

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displacement and stress responses can be easily identified. This method allows for uncertainty in material parameters, geometric dimension and applied forces. In addition, the lower and upper bounds of structural displacement and stress responses can be obtained by using the computational expression given in this study. Once deterministic structural static responses are obtained by the traditional finite element analysis, computational work may be economical.

In this study, the interval finite element analysis with interval parameters is investigated using interval factor method (IFM) proposed by Gao (2007) and Modares and Mullen (2014). Uncertainty problem of a plane stress structure with a given mesh is achieved to use the present method, in which structural physical parameters, geometry and applied forces are considered as interval variables. The structure responses are analyzed through two cases. Case 1 deals with consideration of interval changes in Young's modulus and static loads. Case 2 deals with another aspect of input parameter, i.e., geometry using 2 models: isotropically and anisotropically changes. The first one is based on an assumption that all dimensions of structure are scaled with the same ratio in geometrical change, while the second one considers scaling in one axis only. It has been shown that obtaining the bound on static responses of structures applied to the present method does not require a complicated procedure. The sensitivity of structure under changing of parameter can also be presented.

The outline of this study is as follows. In Section 2, the theory of interval arithmetic is described. Finite element method formulations using interval matrix derived from the interval arithmetic are shown in Section 3. By using the interval finite element method numerical applications for structural parameters such as Young's modulus, applied force, and geometry of plate structures are studied in Section 4 followed by the conclusions in Section 5.

2. Mathematic backgrounds of interval arithmetic

Assume that $I(R)$, $I(R^n)$ and $I(R^{n\times n})$ denote the sets of all closed real interval numbers, n dimension real interval vectors and $n\times n$ real interval matrices, respectively. R is the set of all real numbers. $X^I = [\underline{x}, \overline{x}]$ is a number of $I(R)$ and can be usually written in the following form

$$X^I = [X^C - \Delta X, X^C + \Delta X] \tag{1}$$

$$X^C = \frac{\bar{x} + \underline{x}}{2} \tag{2}$$

$$\Delta X = \frac{\bar{x} - \underline{x}}{2} \tag{3}$$

where X^C and ΔX denote the mean value (or midpoint value) of X^I and the uncertainty (or the maximum width) in X^I, respectively. In this study, an interval $\Delta X^I = [-\Delta X, +\Delta X]$ is called an uncertain interval. An arbitrary interval $X^I = [\underline{x}, \overline{x}]$ can also be written as the sum of its mean value and uncertain interval. Therefore Eq. (1) can be rewritten as

$$X^I = [X^C + \Delta X^I] \tag{4}$$

Eq. (4) can also be expressed as

$$\begin{aligned} X^I &= X^C\left[\left(1 - \frac{\Delta X}{X^C}\right), \left(1 + \frac{\Delta X}{X^C}\right)\right] \\ &= \left[\left(1 - \frac{\bar{x} - \underline{x}}{2X^C}\right), \left(1 + \frac{\bar{x} - \underline{x}}{2X^C}\right)\right] X^C \end{aligned} \tag{5}$$

Here, Gao (2007) introduces $X_F^I = [\underline{x_F}, \overline{x_F}]$ that is a number of $I(R)$ and let

$$X^I = X_F^I \cdot X^C \tag{6}$$

because X^C is the mean value of X^I and the uncertainty of X^I is denoted by X_F^I. Thus, X_F^I is called the interval factor of X^I in this study and it can be easily obtained that

$$\underline{x_F} = 1 - \frac{\bar{x} - \underline{x}}{2X^C} \tag{7}$$

$$\overline{x_F} = 1 + \frac{\bar{x} - \underline{x}}{2X^C} \tag{8}$$

$$\Delta X_F = \frac{\overline{x_F} - \underline{x_F}}{2} = \frac{\Delta X}{X^C} \tag{9}$$

Similar expressions exist for $n\times n$ interval matrix.

3. Formulations of interval finite element analysis for plate structure

3.1 Interval matrix analysis

In the context of finite element analysis, the governing equation for displacements is

$$[\mathrm{K}]\,\{\mathrm{U}\} = \{\mathrm{f}\} = \{\mathrm{f}_1, \cdots, \mathrm{f}_i, \cdots, \mathrm{f}_n\}^T \tag{10}$$

where [K] is the stiffness matrix, {U} is the vector of displacements, and $\{f\} = \{f_1, \cdots, =f_i, \cdots, f_n\}^T$ is the vector of applied forces. Eq. (11) of global stiffness matrix can be rewritten as Eq. (12) as the combination of element stiffness matrix

$$[\mathrm{K}] = \sum_{e=1}^{n} [\mathrm{K}^{(e)}] \tag{11}$$

$$[\mathrm{K}^e]\mathrm{sym} = \frac{\mathrm{E}}{(1-\nu^2)}\begin{bmatrix} \frac{1}{2}-\frac{\nu}{6} & \frac{1}{8}(1+\nu) & \frac{1}{4}(-1-\frac{\nu}{3}) & \frac{1}{8}(-1+3\nu) & \frac{1}{4}(-1+\frac{\nu}{3}) & \frac{1}{8}(-1-\nu) & \frac{\nu}{6} & \frac{1}{8}(1-3\nu) \\ \vdots & \frac{1}{2}-\frac{\nu}{6} & \frac{1}{8}(1-3\nu) & \frac{\nu}{6} & \frac{1}{8}(-1-\nu) & \frac{1}{4}(-1+\frac{\nu}{3}) & \frac{1}{8}(-1+3\nu) & \frac{1}{4}(-1-\frac{\nu}{3}) \\ & & \frac{1}{2}-\frac{\nu}{6} & \frac{1}{8}(-1-\nu) & \frac{\nu}{6} & \frac{1}{8}(-1+3\nu) & \frac{1}{4}(-1+\frac{\nu}{3}) & \frac{1}{8}(1+\nu) \\ & & & \frac{1}{2}-\frac{\nu}{6} & \frac{1}{8}(1-3\nu) & \frac{1}{4}(-1-\frac{\nu}{3}) & \frac{1}{8}(1+\nu) & \frac{1}{4}(-1+\frac{\nu}{3}) \\ & & & & \frac{1}{2}-\frac{\nu}{6} & \frac{1}{8}(1+\nu) & \frac{1}{4}(-1-\frac{\nu}{3}) & \frac{1}{8}(-1+3\nu) \\ & & & & & \frac{1}{2}-\frac{\nu}{6} & \frac{1}{8}(1-3\nu) & \frac{\nu}{6} \\ & & & & & & \frac{1}{2}-\frac{\nu}{6} & \frac{1}{8}(-1-\nu) \\ & & & & & & & \frac{1}{2}-\frac{\nu}{6} \end{bmatrix} \tag{12}$$

where $[\mathrm{K}^e]$sym, E, and ν are e^{th} element's stiffness matrix, Young's modulus, and Poisson's ratio, respectively.

For interval analysis, it is necessary to construct deterministic parts of stiffness matrix. Stiffness matrix is derived from mean value of Young's modulus and plate's geometry (Belblidia *et al*. 2015). In this study, we consider two cases of interval analysis.

Table 1 Stress components of element 189 by interval change ratio of E and force

Element 189	$\sigma_{xx}{}^{lower}$	$\sigma_{xx}{}^{C}$	$\sigma_{xx}{}^{upper}$	$\sigma_{yy}{}^{lower}$	$\sigma_{yy}{}^{C}$	$\sigma_{yy}{}^{upper}$	$\sigma_{xy}{}^{lower}$	$\sigma_{xy}{}^{C}$	$\sigma_{xy}{}^{upper}$
$\Delta E_F = 0.06, \Delta f_F = 0.0$	-9631	-10138	-10654	-97315	-102440	-107560	325250	342370	359490
$\Delta E_F = 0.0, \Delta f_F = 0.06$	-9631	-10138	-10654	-97315	-102440	-107560	325250	342370	359490
$\Delta E_F = \Delta f_F = 0.06$	-9150	-10138	-11177	-92450	-102440	-112940	308990	342370	377460

Case 1: Effect of Young's modulus E and applied forces {f}

The structure is designed to be manufactured by the same material. E is assigned as constant. This implies a linear relationship between magnitude of stiffness matrix K and interval change ratio of Young's modulus ΔE_F. According to interval force, all applied forces varies simultaneously with the same interval change ratio Δf_F.

Therefore lower and upper bound of the force vector can be calculated as the multiplication between $\{f^C\}$ and scalar, i.e., interval change ratio Δf_F.

By using $\{U\}^* = [K]^{-1*}\{f\}$, $E^{(e)I} = E_F E^{(e)C}$, and $[K^e] = E_F[K^e]^*$, an interval formulation of Young's modulus and displacement is as follows

$$\{U\} \quad = \{U\}^*/(1 \pm E_F) \tag{13}$$

Similarly for $\{f\}^* = \{f\}^*/(1 \pm f_F)$, the interval formulation of force and displacement is written as

$$\{U\} \quad = \{U\}^*(1 \pm f_F) \tag{14}$$

Case 2: Effect of uncertainty geometry of 2D structure by changing Jacobian matrix

This part deals with the uncertainty in geometry of a given 2D structure, considering, in particular, isotropically and anisotropically changes in in-plane dimensions. In the first model, geometry of the determined structure is changed in all dimensions with the same scaling ratio. The second model deals with geometry scaling on one axis only, particularly, of horizontal direction.

Assume that the structure is constructed in a determined shape, then Jacobian matrix is constructed in this form: $J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & 0 \\ 0 & \frac{\partial y}{\partial \eta} \end{bmatrix}$ indicates that in global coordinates $\begin{Bmatrix} dx \\ dy \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} d\xi \\ \frac{\partial y}{\partial \eta} d\eta \end{bmatrix}$ where $\frac{\partial x}{\partial \xi}$ and $\frac{\partial y}{\partial \eta}$ are constants.

By multiplying $\frac{\partial x}{\partial \xi}$ and $\frac{\partial y}{\partial \eta}$ with interval change ratio, the uncertainty in simple geometry change J^{lower} and J^{upper} can be obtained as shown in Eqs. (15) and (16) in Section 4.2.

4. Numerical applications and discussion

4.1 Case 1: Effect of Young's modulus E and applied forces {f}

A discretized 2D solid structure with plane stress state with material properties of steel (Lee and Shin 2014, Lee and Shin 2015, Lee 2016) is modeled with 231 nodes and 200 elements. To present voids, density of elements at top right side of plate is equal to 0. Solid regions (black) have density values of 1.

The structural parameter is interval variable of Young's modulus E^C=2.1×10^8 KPa. The in-plane dimension can be seen in Fig. 1. Concentrated forces along the negative direction of Y-axis are applied on every node at the bottom side except at the fixed-end of the structure. The magnitude of the load is an interval variable, and its mean value is P=1×10^5 kN.

For standardization, the structure is initially modeled using bi-linear four node elements in natural coordinates. To obtain displacement vector, these elements later will be transformed into unit square elements (1×1 m^2) using the following Jacobian matrix $J = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$.

In order to investigate the effect of the change or uncertainty of interval Young's modulus and applied forces on the structural static stress, the values of interval change ratio ΔE_F=$\Delta E/E^C$ and Δf_F=$\Delta f/f^C$ of Young's modulus and applied forces are taken as different groups.

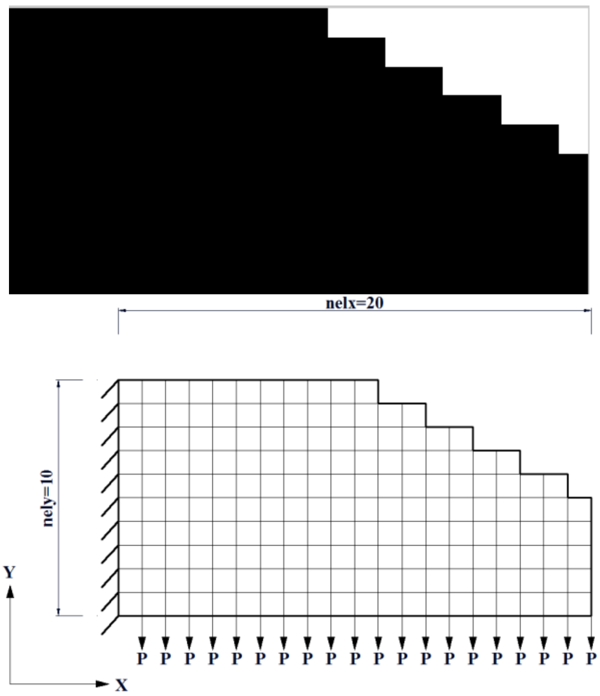

Fig. 1 2D structural solid, load and boundary conditions in global coordinates (X-Y) with a given mesh of finite elements

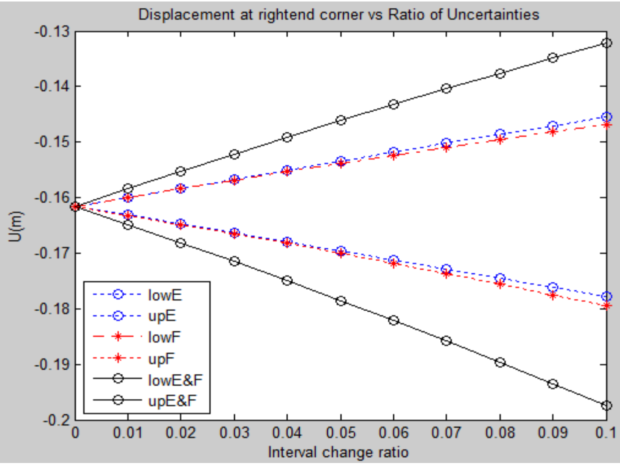

Fig. 2 Uncertainties of displacement with respect to interval change ratio

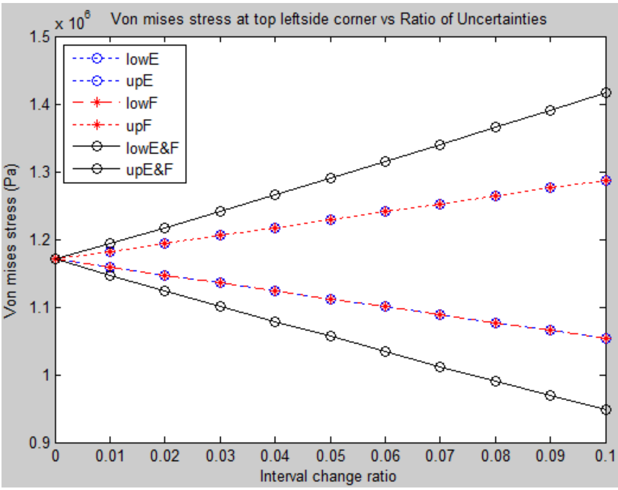

Fig. 3 Uncertainties of Von Mises stress with respect to interval change ratio

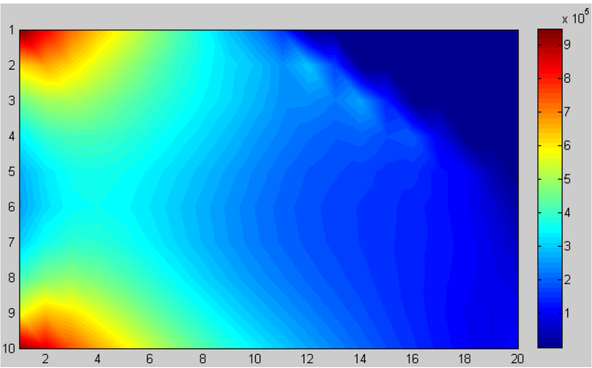

Fig. 4 Lower value of Von Mises stress distribution (ΔE_F=Δf_F=0.1)

The computational results of the lower bound, upper bound, and mean value of vertical displacement (462th components of U matrix) of the bottom right side corner node are shown in Fig. 2. The lower bound, upper bound, and mean value of Von Mises stress of the top left side corner element are shown in Fig. 3. The mean values correspond to interval factor equal to 0. The mean value of

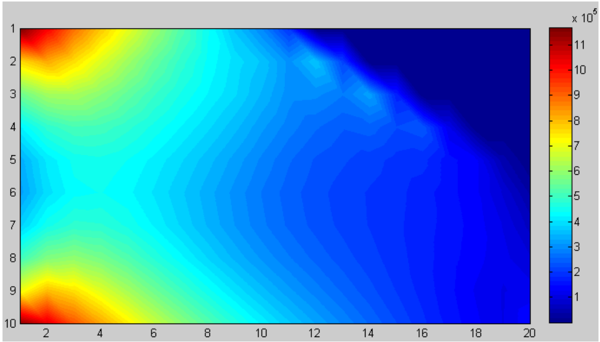

Fig. 5 Von Mises stress distribution corresponds to E^C and f^C (ΔE_F=Δf_F=0.0)

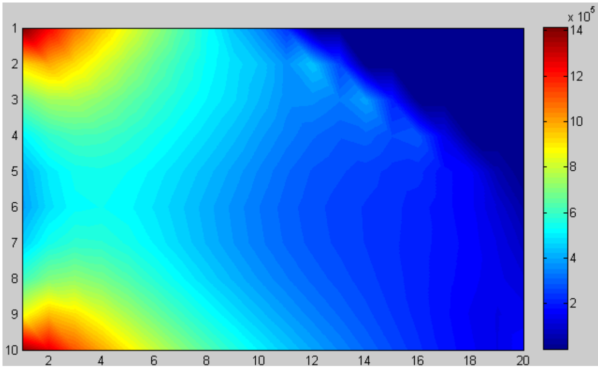

Fig. 6 Upper bound value of Von Mises stress distribution (ΔE_F=Δf_F=0.1)

Von Mises stress distributed over the plate and the value corresponding to ΔE_F=Δf_F=0.1 are shown in Figs. 4, 5, and 6. A summary of stress *x-x* and *y-y* of element 189 (nelx=19, nely=9) with ΔE_F=Δf_F=0.05 are shown in Table 1.

As can be seen in Figs. 2 and 3, uncertainties also increase, as interval change ratio increases. Additionally, it can be found that the tendency of nonlinear curves of the uncertainties may occur depending on interval change ratio.

4.2 Case 2: Effect of uncertainty geometry of 2D structure by changing Jacobian matrix

This example uses the same structure as in example 1 with the same boundary conditions of E=2.1×10^8 KPa and P=1×10^5 kN. A similar mesh is used in modeling the whole system.

In order to investigate the effect of the change or uncertainty of geometry on the structural static stress, the interval change ratio ΔJ_F is used. ΔJ_F is multiplied to both J_{11} and J_{22} components of Jacobian matrix in case of isotropic geometry change $J = (1 \pm \Delta J_F) * \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$.

Under the condition of isotropic geometry change, Fig. 7 shows the equivalent stress in according to mean value of parameter considered. Then, the computational results of lower and upper bound Von Mises stress distributed over the plate and the value corresponding to ΔJ_F=0.9 are shown in Figs. 8 and 9.

In case of anisotropic change, i.e., the structure change in horizontal dimension ΔJ_F is only multiplied to J_{11}. Lower and upper bound formulations of J are as follows, respectively

$$J^{lower} = (1 - \Delta J_F) * \begin{bmatrix} 1/2 & 0 \\ 0 & 1/[(1 - \Delta J_F) * 2] \end{bmatrix} \quad (15)$$

$$J^{upper} = (1 + \Delta J_F) * \begin{bmatrix} 1/2 & 0 \\ 0 & 1/[(1 + \Delta J_F) * 2] \end{bmatrix} \quad (16)$$

In anisotropic case, the value of $J_{11}>1$ indicates that elements in global coordinates are not squares. Rectangles with dimension $(1\pm\Delta J_F)$ m×1 m, indicate that the structure is horizontally scaled up or down. The computational results of lower and upper bound Von Mises stress distributed over the plate and the value corresponding to ΔJ_F=0.5 and 0.9 are shown in Figs. 10, 11, 12, and 13, respectively. As can be seen in Figs. 7 to 13, uncertainties of anisotropic case are larger than those of isotropic case.

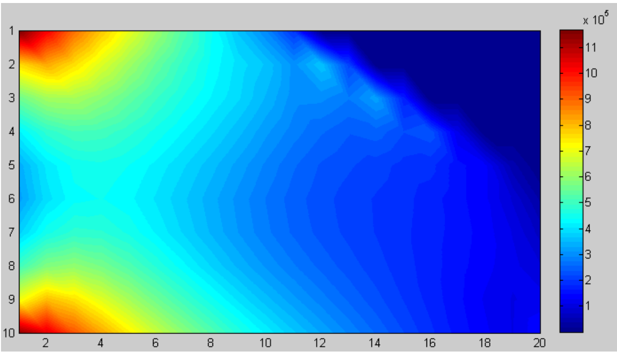

Fig. 7 Von Mises stress distribution corresponds to ΔJ_F=0.0 (mean value)

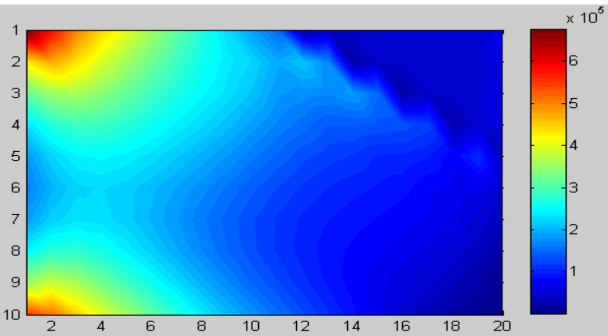

Fig. 8 Lower bound of Von Mises stress distribution corresponds to ΔJ_F=0.9 (isotropic case)

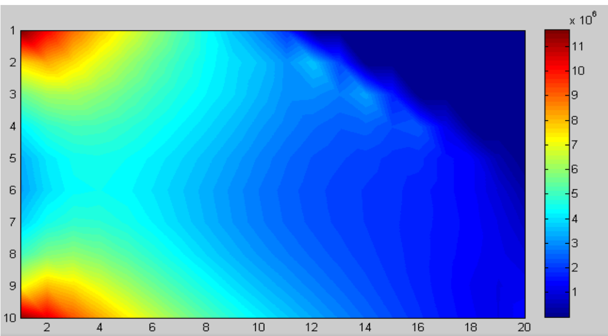

Fig. 9 Upper bound of Von Mises stress distribution corresponds to ΔJ_F=0.9 (isotropic case)

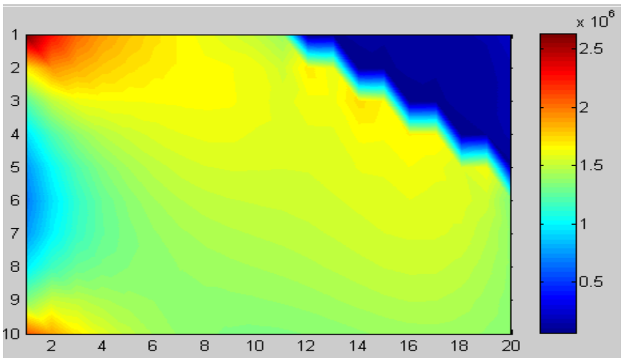

Fig. 10 Lower bound of Von Mises stress distribution corresponds to ΔJ_F=0.5 (anisotropic case with horizontally shortened structure)

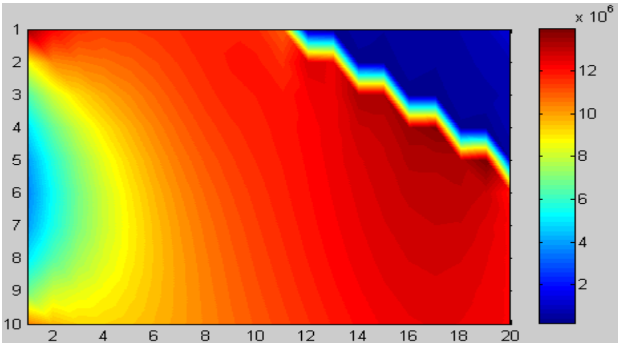

Fig. 11 Upper bound of Von Mises stress distribution corresponds to ΔJ_F=0.5 (anisotropic case with horizontally shortened structure)

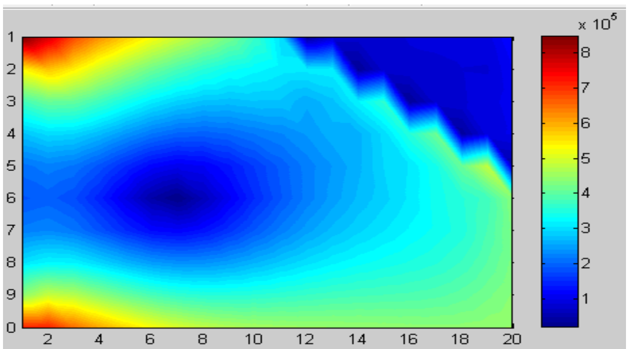

Fig. 12 Lower bound of Von Mises stress distribution corresponds to ΔJ_F=0.9 (anisotropic case with horizontally lengthened structure)

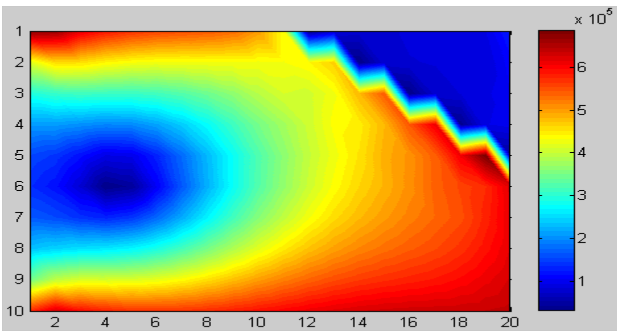

Fig. 13 Upper bound of Von Mises stress distribution corresponds to ΔJ_F=0.9 (anisotropic case with horizontally lengthened structure)

According to the results of the numerical examples, it may be observed that:

(1) The uncertainty of Young's modulus and applied forces produce the same effect on displacement and yield stress of a 2D structure.

(2) A linear change in values of Young's modulus and applied forces also results in a linear change in displacement and yield stress of structure. And along with an increase in interval change ratios, the uncertainty of structural stresses also increases.

(3) The uncertainty of geometry with isotropic change affects the magnitude of structural stress only while anisotropic change affects both the magnitude of structural response and the distribution of Von Mises stress over the in-plane surface. Note that when the maximum change of stress with bounded parameters can be calculated, the probability of each stress may be determined to be small usually.

5. Conclusions

In this study, the effect of uncertainty in the material parameters, structural dimensions and applied forces on the uncertainty of the structural static stresses of a plane stress structure is presented to use a technique called the interval factor method. The lower bound, upper bound, mean value and interval change ratio of displacement and yield stress response of a 2D structural solid with interval parameters can be obtained expediently. This method will also be applied to the interval static response analysis of further types of interval structures.

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