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Free vibration behavior of viscoelastic annular plates using first order shear deformation theory

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Abstract. In this paper, an analytical procedure based on the perturbation technique is presented to study the free vibrations of annular viscoelastic plates by considering the first order shear deformation theory as the displacement field. The viscoelastic properties obey the standard linear solid model. The equations of motion are extracted for small deflection assumption using the Hamilton's principle. These equations which are a system of partial differential equations with variable coefficients are solved analytically with the perturbation technique. By using a new variable change, the governing equations are converted to equations with constant coefficients which have the analytical solution and they are appropriate especially to study the sensitivity analysis. Also the natural frequencies are calculated using the classical plate theory and finite elements method. A parametric study is performed and the effects of geometry, material and boundary conditions are investigated on the vibrational behavior of the plate. The results show that the first order shear deformation theory results is more closer than to the finite elements with respect to the classical plate theory for viscoelastic plate. The more results are summarized in conclusion section.

Keywords: perturbation technique; viscoelastic; annular plate; free vibration; first order shear deformation theory

1. Introduction

The viscoelastic materials are used in different industries like the national defense, civilian, vehicle technology, spacecraft attachments, composite structures and medical. For example, in medical science, the circular and annular viscoelastic plates can use as the contact lens based bioactive agent delivery system, which they are systems for delivery of ophthalmic drugs and other bioactive agents to the eye. By extending the application of these materials, the theory of viscoelasticity has become one of the important branches in the solid mechanics. The viscoelastic materials are fading memory materials capable of both storage and dissipation of energy. These materials exhibit a significant recovery response, i.e. the strain rate decreases with time in the absence of any externally applied stresses (creep and recovery). The rheological behavior of viscoelastic materials can be modeled by mechanical elements (spring and dashpot). Maxwell, Kelvin, Wiechert and Standard Linear Solid (*SLS*) are usual models for describing this behavior. Nagaya (1979) presented the vibrations of a viscoelastic annular plate having an eccentric circular inner boundary by Classical Plate Theory (*CPT*) using the numerical method. Baily and Chen (1987) studied the natural modes of linear viscoelastic circular plates by considering an extension of *CPT* which includes the effects of mechanical dissipation and rotary inertia without the effects of shear deformation. The governing equation was solved numerically. Wang and Tsai (1988) used the Finite Elements (*FE*) method for quasi-static and dynamic analysis of viscoelastic Mindlin plates by Maxwell and *SLS* models. Liu and Chen (1995) used a *FE* formulation based on the elasticity theory to study the vibrational behavior of isotropic and composite annular plates. So and Leissa (1998) applied the Ritz method by trigonometric functions in the three-dimensional theory of elasticity (3*D*) in order to obtain frequencies of thick circular and annular plates. Esmailzadeh and Jalali (1999) applied the *FE* method to study the nonlinear oscillations of viscoelastic simply supported rectangular plates by *CPT* assumption. Liew and Yang (2000) studied the vibrational characteristics of annular plates using the 3*D*. A polynomials-Ritz model was used in order to approximate the spatial displacements of the plates in cylindrical polar coordinates. Wang and Chen (2002) used the *FE* method to study the vibrations and damping of a composite annular plate with a viscoelastic mid-layer. Salehi and Aghaei (2005) investigated the dynamic large deflection analysis of non-axisymmetric circular viscoelastic plates by using the higher-order shear deformation theory and *SLS* viscoelastic material. The problem was solved by the finite difference technique. Dong (2008) investigated the free vibration of functionally graded (*FG*) annular plates with different boundary conditions using the Ritz method and the 3*D*. Hashemi et al. (2008) studied the free vibration of thick annular plates resting on elastic foundation with different boundary conditions and the 3*D*. The problem was assumed as linear, small strain in which the method of polynomials-Ritz was used in the solution. Hosseini *et al*. (2009) investigated the validity range of applicability of the *CPT* and the Mindlin plate theory, in comparison with the 3*D* for freely vibrating

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circular plates on the elastic foundation. They used the Ritz method for the solution. Nie and Zhong (2010) achieved dynamic analysis of multi-directional *FG* annular plates using the 3*D* and space-based differential quadrature (*DQ*) method for solving the equilibrium equations. Gupta (2010) presented the vibration analysis of a viscoelastic rectangular plate with varying thickness by considering the *CPT* assumption and Kelvin model. The governing equation was solved by the Ritz technique. Tahouneh and Yas (2012) presented the free vibration of thick *FG* annular sector plates resting on an elastic foundation based on the 3*D*, using *DQ* method. Khanna and Sharma (2012) presented a mathematical study for viscoelastic plates under elevated temperatures. They used the Ritz approach to calculate the fundamental frequency and deflection functions. Khanna and Sharma (2013) investigated the effect of thermal gradient on the vibration of square viscoelastic plates of varying thickness. They employed the Ritz technique to calculate the fundamental frequencies according to *CPT* assumption. Shariyat *et al*. (2013) studied the free vibration analysis of varying thickness viscoelastic circular *CPT* plates made of heterogeneous materials and resting on the elastic foundations. It was assumed that the viscoelastic material properties vary in the transverse and radial directions simultaneously. The complex modulus approach in conjunction with the correspondence principle was employed to obtain the solution by means of a power series solution. Tahouneh *et al*. (2013, 2014) investigated the free vibration of bidirectional *FG* annular plates on a Pasternark elastic foundation based on the 3*D* with different boundary conditions using the *DQ* and series solutions. Tahouneh and Yas (2014) presented the 3*D* for free vibration analysis of two-dimensional continuously graded carbon nanotube reinforced annular plates resting on an elastic foundation. The Eshelby-Mori-Tanaka approach and composed of two-dimensional *DQ* method were employed for the solution.

For the modal analysis of annular viscoelastic plates, the most authors have used the *CPT* for formulation and the numerical methods for solution. In this paper, the first order shear deformation theory (*FSDT*) is used as the displacement field to derive the governing equations of annular plates. The viscoelastic behavior obeys the *SLS* constitutive model in shear and elastic in bulk. These equations, which are four *coupled partial differential equations with variable coefficients*, are solved ***analytically*** with the ***perturbation*** technique to calculate the natural frequencies and damping coefficients. The results are compared with the *FE* method. By a parametric study, the effects of geometry, material properties and boundary conditions on the results are investigated.

2. Governing equations

Consider an isotropic homogeneous elastic annular plate with uniform thickness h, inner radius r_i and outer radius r_o. The plate geometry is defined in an orthogonal cylindrical coordinate system (r, θ, z). The origin of the coordinate system is taken at the center of the mid-plane as shown in Fig. 1. The in-plane displacement components of an

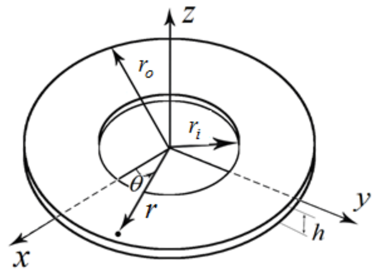

Fig. 1 Geometry and coordinate system of an annular plate

arbitrary point of the plate are U_r, U_θ, in the radial and circular directions and the out-of-plane component designated by U_z.

For the axisymmetric case, the displacement field is defined using the FSDT assumption, where the displacement components have linear variations with respect to z as the following

$$\begin{aligned} &U_r(r,z,t) = u_0(r,t) + z.u_1(r,t); \\ &U_z(r,z,t) = w_0(r,t) + z.w_1(r,t); \\ &U_\theta(r,z,t) = 0 \end{aligned} \tag{1}$$

Where u_0, w_0 denote the in-plane displacements in the mid-plane, z is distance from the mid-plane and u_0, u_1, w_0, w_1 are unknown functions which depend on the radial coordinate, r and the time parameter, t. The strain-displacement relations for small deformation are

$$\begin{aligned} &\varepsilon_r = \frac{\partial U_r}{\partial r} = \frac{\partial u_0}{\partial r} + z\,\frac{\partial u_1}{\partial r}; \\ &\varepsilon_\theta = \frac{U_r}{r} + \frac{1}{r}\frac{\partial U_\theta}{\partial \theta} = \frac{u_0 + zu_1}{r}; \\ &\gamma_{\theta z} = \frac{1}{r}\frac{\partial U_z}{\partial \theta} + \frac{\partial U_\theta}{\partial z} = 0; \\ &\gamma_{r\theta} = \frac{1}{r}\frac{\partial U_r}{\partial \theta} + \frac{\partial U_\theta}{\partial r} - \frac{U_\theta}{r} = 0; \\ &\gamma_{rz} = \frac{\partial U_z}{\partial r} + \frac{\partial U_r}{\partial z} = \frac{\partial w_0}{\partial r} + u_1 + z\,\frac{\partial w_1}{\partial r}; \\ &\varepsilon_z = \frac{\partial U_z}{\partial z} = w_1 \end{aligned} \tag{2}$$

The stress-strain relations, according to the Hooke’s law are as follows, (Sadd 2009)

$$\begin{aligned} &\sigma_r = (K - 2G/3)(\varepsilon_r + \varepsilon_\theta + \varepsilon_z) + 2G\,\varepsilon_r; \\ &\sigma_\theta = (K - 2G/3)(\varepsilon_r + \varepsilon_\theta + \varepsilon_z) + 2G\,\varepsilon_\theta; \\ &\sigma_z = (K - 2G/3)(\varepsilon_r + \varepsilon_\theta + \varepsilon_z) + 2G\,\varepsilon_z; \\ &\tau_{rz} = G\gamma_{rz};\ \tau_{\theta z} = 0;\ \tau_{r\theta} = 0 \end{aligned} \tag{3}$$

Where G and K are the shear and bulk modulus, respectively. The kinetic energy T and the strain energy U of an elastic plate are expressed as

$$T = \frac{1}{2}\int_0^{2\pi}\int_{r_i}^{r_o}\int_{-h/2}^{h/2} \rho\left(\dot{U}_r^2 + \dot{U}_\theta^2 + \dot{U}_z^2\right) r\, d\theta dz\, dr \tag{4a}$$

$$U=\frac{1}{2}\int_{0}^{2\pi}\int_{r_i}^{r_o}\int_{-h/2}^{h/2}\left(\sigma_r\varepsilon_r+\sigma_\theta\varepsilon_\theta+\sigma_z\varepsilon_z+\tau_{rz}\gamma_{rz}\right)r\,dz\,d\theta\,dr \tag{4b}$$

Where, ρ is the plate density (kg/m^3). By using the Hamilton's principle, the equations of motion and the boundary conditions are determined

$$\delta\int_{t_1}^{t_2}L\,dt=0;\; L=T-U \tag{4c}$$

From Eq. (4), four equations of motion in terms of stress resultants are determined as the following

$$\begin{aligned}
&\frac{\partial}{\partial r}(rN_r)-N_\theta-\rho r h\frac{\partial^2 u_0}{\partial t^2}=0;\\
&\frac{\partial}{\partial r}(rM_r)-M_\theta-rQ_r-\frac{\rho r h^3}{12}\frac{\partial^2 u_1}{\partial t^2}=0;\\
&\frac{\partial}{\partial r}(rQ_r)-\rho r h\frac{\partial^2 w_0}{\partial t^2}=0;\\
&\frac{1}{r}\frac{\partial}{\partial r}(rM_{rz})-N_z-\frac{\rho h^3}{12}\frac{\partial^2 w_1}{\partial t^2}=0
\end{aligned} \tag{5a}$$

The boundary conditions are

$$\left(r(N_r\delta u_0+M_r\delta u_1+Q_r\delta w_0+M_{rz}\delta w_1)\right)\Big|_{r_i}^{r_o}=0 \tag{5b}$$

And the stress resultants are defined as follows

$$\begin{aligned}
&(N_r,N_\theta,N_z,Q_r)=\int_{-h/2}^{h/2}(\sigma_r,\sigma_\theta,\sigma_z,K_s\tau_{rz})dz;\\
&(M_r,M_\theta,M_{rz})=\int_{-h/2}^{h/2}z\,(\sigma_r,\sigma_\theta,K_s\tau_{rz})dz;
\end{aligned} \tag{5c}$$

K_s is the shear correction factor which is assumed $\pi^2/12$, (Wang *et al*. 2000).

By Substituting Eq. (5c) into Eq. (5a), the equations of motion in terms of displacements are derived

$$\begin{aligned}
&(K+\frac{4}{3}G)\left(\frac{\partial}{\partial r}(r\frac{\partial u_0}{\partial r})-\frac{u_0}{r}\right)+(K-\frac{2}{3}G)r\frac{\partial w_1}{\partial r}\\
&-\rho r\frac{\partial^2 u_0}{\partial t^2}=0
\end{aligned} \tag{6a}$$

$$\begin{aligned}
&(K+\frac{4}{3}G)\frac{h^2}{12}\left(\frac{\partial}{\partial r}(r\frac{\partial u_1}{\partial r})-\frac{u_1}{r}\right)-K_sG\,r\,(u_1+\frac{\partial w_0}{\partial r})\\
&-\frac{\rho r h^2}{12}\frac{\partial^2 u_1}{\partial t^2}=0
\end{aligned} \tag{6b}$$

$$K_sG\frac{\partial}{\partial r}\left(r(\frac{\partial w_0}{\partial r}+u_1)\right)-\rho r\frac{\partial^2 w_0}{\partial t^2}=0 \tag{6c}$$

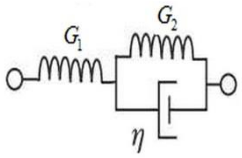

Fig. 2 Standard linear solid model (SLS), (Riande *et al*. 2000)

$$\begin{aligned}
&K_sG\frac{h^2}{12}\frac{\partial}{\partial r}(r\frac{\partial w_1}{\partial r})-(K-\frac{2}{3}G)\frac{\partial}{\partial r}(ru_0)-(K+\frac{4}{3}G)rw_1\\
&-\frac{\rho r h^2}{12}\frac{\partial^2 w_1}{\partial t^2}=0
\end{aligned} \tag{6d}$$

In viscoelastic analysis, it is usual to *separate* the deviatoric and dilatational parts of the stress components. For the deviatoric and dilatational parts we have $P_1\tau_{ij}=Q_1\gamma_{ij}$, and $P_2\sigma_{ii}=Q_2\varepsilon_{ii}$, respectively. P_1, Q_1, P_2, Q_2 are the viscoelastic operators, τ_{ij}, γ_{ij} denote the shear stress and strain, and σ_{ii}, ε_{ii} are the traces of stress and strain tensors. In the elastic case, the shear stress-strain relation is $\tau_{ij}=2G\varepsilon_{ij}$, so $G=Q_1/2P_1$ and the bulk modulus is $K=Q_2/3P_2$. We assume that the *viscoelastic material obeys the SLS model in **shear*** and *elastic in bulk* i.e., $K=K_0$ where K_0 is a constant (elastic bulk modulus). The viscoelastic operators are expressed as (Riande *et al*. 2000)

$$P_1=(\frac{1}{G_1}+\frac{1}{G_2})+\frac{\tau}{G_1}D;\;P_2=1;\;Q_1=2\left(1+\tau D\right);\;Q_2=3K_0 \tag{7}$$

Where $\tau=\eta/G_2$ is the relaxation time and $D=\partial/\partial t$ is the time derivative operator. By substituting G, K into Eq. (6) and applying the time derivative operator on the equations, the governing equations of motion for a viscoelastic plate are derived as the following

$$\begin{aligned}
&L_1[u_0,w_1,r,t,\partial/\partial t,\partial^2/\partial t^2,\partial^3/\partial t^3,\partial/\partial r,\\
&\partial^2/\partial r^2,\partial^3/\partial r^2\partial t,\partial^2/\partial r\partial t]=0
\end{aligned} \tag{8a}$$

$$\begin{aligned}
&L_2[u_1,w_0,w_1,r,t,\partial/\partial t,\partial^2/\partial t^2,\partial^3/\partial t^3,\partial/\partial r,\\
&\partial^2/\partial r^2,\partial^3/\partial r^2\partial t,\partial^2/\partial r\partial t]=0
\end{aligned} \tag{8b}$$

$$\begin{aligned}
&L_3[u_1,w_0,w_1,r,t,\partial/\partial t,\partial^2/\partial t^2,\partial^3/\partial t^3,\partial/\partial r,\\
&\partial^2/\partial r^2,\partial^3/\partial r^2\partial t,\partial^2/\partial r\partial t]=0
\end{aligned} \tag{8c}$$

$$\begin{aligned}
&L_4[u_0,w_1,r,t,\partial/\partial t,\partial^2/\partial t^2,\partial^3/\partial t^3,\partial/\partial r,\\
&\partial^2/\partial r^2,\partial^3/\partial r^2\partial t,\partial^2/\partial r\partial t]=0
\end{aligned} \tag{8d}$$

L_1, L_2, L_3, L_4 are differential operators. The explicit dimensionless form of these operators will be reported later.

3. Analytical solution

The perturbation technique is used for analytical solution. Before using this method, it is necessary to convert the equations to dimensionless form. We define the following dimensionless quantities

$$r^*=\frac{r}{a},\ h^*=\frac{h}{h_0},\ t^*=\frac{t}{t_0},\ u_0^*(r^*,t^*)=\frac{u_0(\mathrm{r,t})}{h_0},$$
$$u_1^*(r^*,t^*)=u_1(r,t),\ w_0^*(r^*,t^*)=\frac{w_0(r,t)}{h_0}, \tag{9a}$$
$$w_1^*(r^*,t^*)=w_1(r,t)$$

By substituting Eq. (9a) into governing equations Eq. (8), the following dimensionless parameters are appeared

$$\varepsilon=\frac{h_0}{a},\ e=(\frac{h_0/t_0}{c})^2,\ \beta=\frac{\tau}{t_0},\ G_0^*=K_0(\frac{1}{G_1}+\frac{1}{G_2}),$$
$$G_1^*=\frac{K_0}{G_1},\ c=\sqrt{K_0/\rho} \tag{9b}$$

Where r^* and t^* are dimensionless location and time, respectively. u_0^*, u_1^*, w_0^*, w_1^* are dimensionless displacement components. a, h_0 and t_0 are characteristic radius, thickness and time, respectively, which are defined as $a=r_o$, $h=h_0$ and $t_0=a/c$. c is a quantity with speed dimension. ε is a small parameter which is considered as the *perturbation parameter*. By using Eq. (9) and defining $\boldsymbol{X=(r^*-1)/\varepsilon}$, the dimensionless form of the governing equations (in terms of displacement) are derived. The method of multiple scale in perturbation technique is used for the solution. The new scale $T_0=t^*$, $T_1=\varepsilon t^*$ is defined. We have (Nayfeh 1993)

$$\frac{\partial}{\partial t^*}=\frac{\partial}{\partial T_0}+\varepsilon\frac{\partial}{\partial T_1},\ \ \frac{\partial^2}{\partial t^{*2}}=\frac{\partial^2}{\partial T_0^2}+2\varepsilon\frac{\partial^2}{\partial T_0\partial T_1},$$
$$\frac{\partial^3}{\partial t^{*3}}=\frac{\partial^3}{\partial T_0^3}+3\varepsilon\frac{\partial^3}{\partial T_0^{\,2}\partial T_1} \tag{10}$$

By substituting Eqs. (9), (10) into Eq. (8), the dimensionless form of the governing equations (in terms of displacement components) are derived as the following

$$eq_1:\ 6L_{11}[u_0^*,w_1^*]+6\varepsilon(2L_{11}[u_0^*,w_1^*].X\ +A_{12}[u_0^*,w_1^*])+$$
$$\varepsilon^2 f_{11}(u_0^*,w_1^*,\varepsilon)=0$$
$$L_{11}[u_0^*,w_1^*]=g_{32}[u_0^*,a_0,a_1]+g_{33}[w_1^*,b_0,b_1]-eg_{51}[u_0^*]$$
$$A_{12}[u_0^*,w_1^*]=\frac{\partial}{\partial X}g_{35}[u_0^*,\beta a_1,a_0]-2e\frac{\partial^2}{\partial T_0\partial T_1}g_{35}[u_0^*, \tag{11a}$$
$$\frac{3}{2}\beta G_1^*,G_0^*]+\beta(a_1\frac{\partial^3 u_0^*}{\partial X^{\,2}\partial T_1}+b_1\frac{\partial^2 w_1^*}{\partial X\,\partial T_1})$$

$$eq_2:\ 3L_{12}[u_1^*,w_0^*]+3\varepsilon(2L_{12}[u_1^*,w_0^*].X\ +A_{22})+$$
$$\varepsilon^2 f_{12}(u_1^*,w_0^*,\varepsilon)=0$$
$$L_{12}[u_1^*,w_0^*]=h^{*2}(g_{32}[u_1^*,a_0,a_1]-e\,g_{51}[u_1^*])-$$
$$12K_s(\frac{\partial}{\partial X}g_{35}[w_0^*,\beta,1]+g_{35}[u_1^*,\beta,1])$$
$$A_{22}=-3eh^{*2}\frac{\partial^2}{\partial T_0\partial T_1}g_{35}[u_1^*,\beta G_1^*,\frac{2}{3}G_0^*]+ \tag{11b}$$
$$\beta h^{*2}a_1\frac{\partial}{\partial X}(\frac{\partial^2 u_1^*}{\partial X\,\partial T_1}+\frac{\partial u_1^*}{\partial T_0})+h^{*2}a_0\frac{\partial u_1^*}{\partial X}-$$
$$12K_s\beta\frac{\partial}{\partial T_1}(\frac{\partial w_0^*}{\partial X}+u_1^*)$$

$$eq_3: L_{13}[u_1^*,w_0^*]+\varepsilon(L_{13}[u_1^*,w_0^*].X\ +A_{32})+$$
$$\varepsilon^2 f_{13}(u_1^*,w_0^*,\varepsilon)=0$$
$$L_{13}[u_1^*,w_0^*]=K_s(g_{32}[w_0^*,1,1]+\frac{\partial g_{35}[u_1^*,\beta,1]}{\partial X})-eg_{51}[w_0^*]$$
$$A_{32}=K_s(g_{35}[u_1^*,\beta,1]+\frac{\partial}{\partial X}g_{35}[w_0^*,\beta,1]+ \tag{11c}$$
$$\beta\frac{\partial^2}{\partial X\,\partial T_1}(u_1^*+\frac{\partial w_0^*}{\partial X}))-2e\frac{\partial^2}{\partial T_0\partial T_1}g_{35}[w_0^*,\frac{3}{2}\beta G_1^*,G_0^*]$$

$$eq_4: L_{14}[u_0^*,w_1^*]+\varepsilon(L_{14}[u_0^*,w_1^*].X\ +A_{42})+$$
$$\varepsilon^2 f_{14}(u_0^*,w_1^*,\varepsilon)=0;$$
$$L_{14}[u_0^*,w_1^*]=h^{*2}(K_s g_{32}[w_1^*,1,1]-eg_{51}[w_1^*])-$$
$$12(\frac{\partial}{\partial X}g_{35}[u_0^*,\beta b_1,b_0]-2g_{35}[w_1^*,\beta a_1,a_0])$$
$$A_{42}=K_s h^{*2}g_{33}[w_1^*,1,1]- \tag{11d}$$
$$2eh^{*2}\frac{\partial^2}{\partial T_0\partial T_1}g_{35}[w_1^*,\frac{3}{2}\beta G_1^*,G_0^*]-12g_{35}[u_0^*,\beta b_1,b_0]+$$
$$K_s\beta h^{*2}\frac{\partial^3 w_1^*}{\partial X^{\,2}\partial T_1}-12\beta\frac{\partial}{\partial T_1}(a_1 w_1^*+b_1\frac{\partial u_0^*}{\partial X})$$

Where

$$g_{33}[y,a,b]=a\frac{\partial y}{\partial X}+\beta b\frac{\partial^2 y}{\partial X\,\partial T_0};\ g_{32}[y,a,b]=$$
$$\frac{\partial}{\partial X}g_{33}[y,a,b];\ g_{35}[y,a,b]=a\frac{\partial y}{\partial T_0}+b.y$$
$$g_{51}[y]=\beta G_1^*\frac{\partial^3 y}{\partial T_0^{\,3}}+G_0^*\frac{\partial^2 y}{\partial T_0^{\,2}}; \tag{11e}$$
$$a_0=G_0^*+\frac{4}{3};\ a_1=G_1^*+\frac{4}{3};\ b_0=G_0^*-\frac{2}{3};\ b_1=G_1^*-\frac{2}{3}$$
$$u_0^*=u_0^*(X,T_0,T_1);\ u_1^*=u_1^*(X,T_0,T_1);$$
$$w_0^*=w_0^*(X,T_0,T_1);\ w_1^*=w_1^*(X,T_0,T_1)$$

f_{11}, f_{12}, f_{13}, f_{14} are functions of dependent variables which do not appear in our selected expansions (Eq. (12)). A straightforward expansion is considered for the solution as the following

$$\begin{aligned} u_0^*(X,T_0,T_1;\varepsilon) &= u_{00}^*(X,T_0,T_1) + \varepsilon u_{01}^*(X,T_0,T_1); \\ u_1^*(X,T_0,T_1;\varepsilon) &= u_{10}^*(X,T_0,T_1) + \varepsilon u_{11}^*(X,T_0,T_1); \\ w_0^*(X,T_0,T_1;\varepsilon) &= w_{00}^*(X,T_0,T_1) + \varepsilon w_{01}^*(X,T_0,T_1); \\ w_1^*(X,T_0,T_1;\varepsilon) &= w_{10}^*(X,T_0,T_1) + \varepsilon w_{11}^*(X,T_0,T_1) \end{aligned} \tag{12}$$

We substitute Eq. (12) into Eq. (11) and equate the terms with the same power of ε to zero. The governing equations with order-zero are extracted as follows

$$eq_1 : L_{11}[u_{00}^*, w_{10}^*] = 0; \quad eq_4 : L_{14}[u_{00}^*, w_{10}^*] = 0 \tag{13a}$$

$$eq_2 : L_{12}[u_{10}^*, w_{00}^*] = 0; \quad eq_3 : L_{13}[u_{10}^*, w_{00}^*] = 0 \tag{13b}$$

Note that Eqs. (6), (8) were *four coupled equations with variable coefficients*. By using the parameter X, one can convert the governing equations with *variable coefficients* to a system of equations with *constant coefficients* in each order of ε. Eq. (13) are two homogenous system of coupled partial differential equations with *constant* coefficients. Eq. (13a) result the radial frequencies and Eq. (13b) compute the transverse motion frequencies.

The governing equations with order-one are as the following

$$\begin{aligned} & eq_1 : L_{11}[u_{01}^*, w_{11}^*] = FF_1; \quad eq_4 : L_{14}[u_{01}^*, w_{11}^*] = FF_4; \\ & FF_1 = -2L_{11}[u_{00}^*, w_{10}^*].X - A_{12}[u_{00}^*, w_{10}^*]; \\ & FF_4 = -L_{14}[u_{00}^*, w_{10}^*].X - A_{42}[u_{00}^*, w_{10}^*] \end{aligned} \tag{14a}$$

$$\begin{aligned} & eq_2 : L_{12}[u_{11}^*, w_{01}^*] = FF_2; \; eq_3 : L_{13}[u_{11}^*, w_{01}^*] = FF_3; \\ & FF_2 = -2L_{12}[u_{10}^*, w_{00}^*].X - A_{22}[u_{10}^*, w_{00}^*]; \\ & FF_3 = -L_{13}[u_{10}^*, w_{00}^*].X + A_{32}[u_{10}^*, w_{00}^*] \end{aligned} \tag{14b}$$

Eq. (14) are two non-homogeneous system of coupled partial differential equations with *constant* coefficients. The homogenous parts of Eqs. (13), (14) are the same. In general, the boundary conditions obey Eq. (5b). The boundary conditions for special cases in order-zero and one, have been defined as the following

Clamped (C)

$$U_r = 0, \; U_z = 0 \rightarrow \begin{cases} O(\varepsilon^0) : u_{00}^* = 0, \; u_{10}^* = 0, \; w_{00}^* = 0, \; w_{10}^* = 0 \\ O(\varepsilon^1) : u_{01}^* = 0, \; u_{11}^* = 0, \; w_{01}^* = 0, \; w_{11}^* = 0 \end{cases} \tag{14c}$$

Simply support (S), (type 1)

$$U_z = 0, rN_r = 0, rM_r = 0 \rightarrow \begin{cases} O(\varepsilon^0) : \dfrac{\partial u_{00}^*}{\partial X} = 0, \dfrac{\partial u_{10}^*}{\partial X} = 0, w_{00}^* = 0, \; w_{10}^* = 0 \\ O(\varepsilon^1) : \dfrac{\partial u_{01}^*}{\partial X} = \alpha_{11} u_{00}^*, \dfrac{\partial u_{11}^*}{\partial X} = \alpha_2 u_{10}^*, w_{01}^* = 0, \\ w_{11}^* = 0 \end{cases} \tag{14d}$$

Simply support (S),(type 2):

$$U_z\big|_{z=0} = 0, \; U_r\big|_{z=0} = 0, rN_r = 0, rM_{rz} = 0 \rightarrow \begin{cases} O(\varepsilon^0) : u_{00}^* = 0, \; \dfrac{\partial u_{10}^*}{\partial X} = 0, w_{00}^* = 0, \; \dfrac{\partial w_{10}^*}{\partial X} = 0 \\ O(\varepsilon^1) : u_{01}^* = 0, \; \dfrac{\partial u_{11}^*}{\partial X} = \alpha_3 u_{10}^*, w_{01}^* = 0, \; \dfrac{\partial w_{11}^*}{\partial X} = 0 \end{cases} \tag{14e}$$

Free (F):

$$rN_r = 0, rM_r = 0, rQ_r = 0, rM_{rz} = 0 \rightarrow \begin{cases} O(\varepsilon^0) : \dfrac{\partial u_{10}^*}{\partial X} = 0, \dfrac{\partial w_{00}^*}{\partial X} + u_{10}^* = 0, \; \dfrac{\partial w_{10}^*}{\partial X} = 0, \\ \gamma_{11} \dfrac{\partial u_{00}^*}{\partial X} + \gamma_{22} w_{10}^* = 0 \\ O(\varepsilon^1) : \dfrac{\partial w_{11}^*}{\partial X} = 0, \; \dfrac{\partial w_{01}^*}{\partial X} + u_{11}^* = 0, \dfrac{\partial u_{11}^*}{\partial X} = \alpha_5 u_{10}^*, \\ \dfrac{\partial u_{01}^*}{\partial X} + \alpha_4 w_{11}^* = \alpha_6 w_{01}^* + \alpha_7 u_{00}^* \end{cases} \tag{14f}$$

$$\gamma_{11} = 6a_0 + 6i\beta\omega a_1, \; \gamma_{22} = 6b_0 + 6i\beta\omega b_1; \; i = \sqrt{-1}$$

Where α_{11}, $\alpha_2, \ldots$, α_7 are functions of X, ω, K_s and material properties of the plate. The reported results are based on type 1 formulation for a simply supported edge. For a circular (solid) plate, the boundary conditions at the center are

$$U_r = 0, rQ_r = 0, rM_{rz} = 0 \rightarrow \begin{cases} O(\varepsilon^0) : u_{00}^* = 0, \; u_{10}^* = 0, \; \dfrac{\partial w_{00}^*}{\partial X} = 0, \; \dfrac{\partial w_{10}^*}{\partial X} = 0 \\ O(\varepsilon^1) : u_{01}^* = 0, \; u_{11}^* = 0, \; \dfrac{\partial w_{01}^*}{\partial X} = 0, \; \dfrac{\partial w_{11}^*}{\partial X} = 0 \end{cases} \tag{14g}$$

4. Frequency analysis

4.1 Order-zero

The solution of Eq. (13b) can express as

$$\begin{aligned} & \begin{Bmatrix} u_{10}^*(X,T_0,T_1) \\ w_{00}^*(X,T_0,T_1) \end{Bmatrix} = \{V_n(T_1)\} \exp(m_n X + i\omega T_0); \\ & \{V_n(T_1)\} = C_n(T_1) \begin{Bmatrix} B_{1n} \\ B_{2n} \end{Bmatrix} \end{aligned} \tag{15a}$$

Where m_n is the eigenvalues and $\{V_n\}$ is the eigenvector. ω is the complex dimensionless frequency. By substituting Eq. (15a) into Eq. (13b), a system of algebraic equation is obtained as the following

$$\begin{aligned} & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} B_{1n} \\ B_{2n} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}; a_{11} = -K_s m_n (1 + i\beta\omega); \\ & a_{12} = \frac{h^{*2} m_n^{\;2}}{12}(a_0 + i\beta a_1 \omega) - K_s(1 + i\beta\omega) \\ & + \frac{e h^{*2}\omega^2}{12}(G_0^* + i\beta G_1^*\omega); \; a_{21} = m_n^{\;2}(c_0 + i\beta c_1\omega) + \\ & e\omega^2(G_0^* + i\beta G_1^*\omega); a_{22} = K_s m_n (1 + i\beta\omega) \end{aligned} \tag{15b}$$

For a non-trivial solution, the determinant of the matrix coefficient must be vanished i.e., $a_{11}.a_{22}-a_{12}.a_{21}=0$ which is known as the dispersion equation and it is a relation between m_n and ω. It is an order-four algebraic equation and its roots (m_1, m_2, m_3, m_4) are functions of ω. Then the eigenvectors are determined from Eq. (15b). $\{V_n\}$ is the eigenvector corresponds to the eigenvalue m_n. The total solution is as the following

$$\begin{Bmatrix} u_{10}^*(X,T_0,T_1) \\ w_{00}^*(X,T_0,T_1) \end{Bmatrix} = \sum_{n=1}^{4} C_n(T_1) \begin{Bmatrix} B_{1n} \\ B_{2n} \end{Bmatrix} exp(m_n X + i\omega T_0) \quad (15c)$$

By applying the boundary conditions, a set of homogenous algebraic equations as $[ax]\{C\}=\{0\}$ is formed where vector $\{C\}$ contains the constants $C_1(T_1)$, $C_2(T_1)$, $C_3(T_1)$, $C_4(T_1)$ and the matrix $[ax]$ contains the coefficients of equations. For a non-zero solution, the determinant of the coefficients matrix (i.e., $[ax]$) is equated to zero which results a *complicated* algebraic equation in terms of ω. The roots of this equation are the dimensionless complex frequencies which are obtained from the relative minimum of the absolute value of matrix determinant. Also $C_2(T_1)$, $C_3(T_1)$, $C_4(T_1)$ are determined in terms of $C_1(T_1)$.

4.2 Order-one

By substituting Eqs. (15c) into (14b), the non-homogenous part of Eq. (14b) i.e., FF_2, FF_3 are obtained. The general form of these parts are as the following

$$\begin{aligned} FF_2 &= \left(K_{21} C_1(T1) + K_{22} \frac{dC_1}{dT_1} \right) exp(i\,\omega T_0); \\ FF_3 &= \left(K_{31} C_1(T1) + K_{32} \frac{dC_1}{dT_1} \right) exp(i\,\omega T_0) \end{aligned} \quad (16)$$

K_{21}, K_{22}, K_{23}, K_{24} are known functions of X, ω. From Eqs. (14b), (15c), (16), it is clear that Eq. (14b) have secular terms and before finding its particular solution as a uniform expansion, it is necessary to remove its secularity. For this purpose, we define the adjoint functions for these equations. Consider two coupled homogenous differential equations as the following and two adjoint functions $\varphi_1(x)$, $\varphi_2(x)$. We multiply the first equation in $\varphi_1(x)$ and the second one in $\varphi_2(x)$ and integrated over the total domain

$$\begin{aligned} &Eq_{20} : L_{20}[y1(x), y2(x)] = 0;\ Eq_{21} : L_{21}[y1(x), y2(x)] = 0 \\ &\int_a^b (Eq_{20}.\phi_1(x) + Eq_{21}.\phi_2(x))dx = 0 \end{aligned} \quad (17a)$$

Where, L_{20}, L_{21} are differential operators. The part by part integration results the adjoint equations (related to homogenous part) and appropriated boundary conditions for $\varphi_1(x)$, $\varphi_2(x)$

$$\begin{aligned} Eq_{30} &: L_{30}[\phi_1(x), \phi_2(x)] = 0; \\ Eq_{31} &: L_{31}[\phi_1(x), \phi_2(x)] = 0 \end{aligned} \quad (17b)$$

L_{30}, L_{31} are differential operators which are not the same with L_{20}, L_{21} necessarily. By solving Eqs. (17b), the adjoint functions are obtained. Now, the same procedure is applied on Eq. (14b) (order-one) i.e.

$$\begin{aligned} &\int_a^b (L_{12}[u_{11}^*, w_{01}^*]_2.\phi_1(x) + L_{13}[u_{11}^*, w_{01}^*].\phi_2(x))dx = \\ &\int_a^b (FF_2.\phi_1(x) + FF_3.\phi_2(x))dx \end{aligned} \quad (18a)$$

Which it is resulted

$$\frac{dC_1}{dT_1} - \alpha_1 C_1 = 0 \rightarrow C_1(T1) = C_0\, exp(\alpha_1 T_1) \quad (18b)$$

Where α_1 is a constant complex value. So the dependency of C_1 to T_1 is determined and the non-homogeneous parts of Eq. (14b) are removed and the solution Eq. (15c) is defined completely. So Eq. (15c) can write as

$$\begin{aligned} &\begin{Bmatrix} u_{10}^* \\ w_{00}^* \end{Bmatrix} = C_0 \{Y_m(X)\}_{2*1} exp(\alpha_1 T_1 + i\,\omega T_0) = \\ &C_0 \{Y_m(X)\} exp(i\,\omega_0 T_0) exp(\alpha T_0); \\ &\omega_0 = \omega + \mathrm{Im}(\varepsilon.\alpha_1); \alpha = \mathrm{Re}(\varepsilon.\alpha_1) \end{aligned} \quad (19)$$

Im, *Re* stand of real and imaginary parts of a quantity. A similar procedure can apply on Eqs. (14a).

5. CPT plate

For *CPT*, the equation of motion in the polar coordinate system is as follows

$$\begin{aligned} &D_0 \nabla^4 w_0 + \rho h \frac{\partial^2 w_0}{\partial t^2} = 0;\ \nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r});\ D_0 = \frac{Eh^3}{12(1-\nu^2)}; \\ &E = \frac{9KG}{3K+G};\ \nu = \frac{3K-2G}{6K+2G} \end{aligned} \quad (20)$$

By substituting $G=Q_1/2P_1$ and $K=Q_2/3P_2$ in Eq. (20), the flexural rigidity D_0 is obtained for a viscoelastic plate. For a plate which is viscoelastic in shear and elastic in bulk and using Eq. (7) for the *SLS* model, the governing equation for a viscoelastic *CPT* plate is determined. By inserting Eq. (9), the governing equation is converted to a dimensionless form as the following

$$\begin{aligned} &eq: L_{15}[w_0^*] + \varepsilon(3L_{15}[w_0^*].X + A_{52}[w_0^*]) + \varepsilon^2 f_{15}(w_0^*, \varepsilon) = 0 \\ &L_{15}[w_0^*] = \frac{\partial^4}{\partial X^4}(d_6 \frac{\partial^2 w_0^*}{\partial T_0^2} + d_5 \frac{\partial w_0^*}{\partial T_0} + d_4 w_0^*) + d_{40} \frac{\partial^4 w_0^*}{\partial T_0^4} + \\ &d_3 \frac{\partial^3 w_0^*}{\partial T_0^3} + d_2 \frac{\partial^2 w_0^*}{\partial T_0^2}; \\ &A_{52}[w_0^*] = \frac{\partial}{\partial T_1}(4d_{40} \frac{\partial^3 w_0^*}{\partial T_0^3} + 3d_3 \frac{\partial^2 w_0^*}{\partial T_0^2} + 2d_2 \frac{\partial w_0^*}{\partial T_0} + \\ &d_5 \frac{\partial^4 w_0^*}{\partial X^4} + 2d_6 \frac{\partial^5 w_0^*}{\partial X^4 \partial T_0}) + 2\frac{\partial^3}{\partial X^3}(d_6 \frac{\partial^2 w_0^*}{\partial T_0^2} + d_5 \frac{\partial w_0^*}{\partial T_0} + \\ &d_4 w_0^*) \end{aligned} \quad (21a)$$

$$d_6 = 3\beta^2 h^{*3} G_1^* a_3;\ d_5 = 3\beta h^{*3} G_1^* (a_2 + a_3);\ d_4 = 3h^{*3} G_1^* a_2;$$
$$d_{40} = 9e\beta^2 h^* G_1^{*2} a_1; d_3 = 9e\beta h^* G_1^* (G_1^* a_0 + G_0^* a_1); \tag{21b}$$
$$a_2 = G_0^* + \frac{1}{3}; a_3 = G_1^* + \frac{1}{3}\ ;\ d_2 = 9eh^* G_0^* G_1^* a_0$$

By applying the perturbation technique as explained before, the natural frequencies are determined.

6. Numerical analysis

Ansys 12 *FE* package has been used for numerical (*FE*) analysis. The selected element is *Solid-Shell 3D* element which has the viscoelastic feature capability and it includes eight-node with three degrees of freedom at each node (Ansys User Manual 2009). The boundary conditions are considered as clamped and simply supported at the edges of annular plate. The viscoelastic property has been defined using Prony's series. The geometrical and material properties of the plate have been listed in Table 1. The viscoelastic parameters values are selected from Marynowski (2005).

It should be noticed that the density of the plate which is the mass density per unit area must be divided to thickness for presented solution. Also the bulk modulus (K_0) and elastic modulus (E) have been defined as the following:

$$K_0 = \frac{2}{3} G_s \frac{1+\nu}{1-2\nu}; G_s = \begin{Bmatrix} G_1 & for\ \eta \neq 0 \\ \dfrac{G_1 G_2}{G_1 + G_2} & for\ \eta = 0 \end{Bmatrix}; E = 2(1+\nu) G_s \tag{22}$$

We used Eq. (22) to convert the frequencies of viscoelastic cases to dimensionless form.

7. Parametric study

To check the validity of the presented solution, we consider some case studies. The dimensionless frequency of the system is a complex parameter and it can be writen as ω_0+iα (Eq. (19)). Its real part (ω_0) stands for the free vibration frequency and we call it the natural frequency in this text. The imaginary part (α) defines the rate of decay in amplitude of vibration and we name it the damping ratio. At first we investigate the natural frequency sensitivity to input data. For this purpose, a parametric study is performed. The analytical results were calculated using the mathematical environment Maple 15.

By setting $\tau \to 0$ the solution of a viscoelastic plate can approach to an elastic plate. In this case (elastic case), the equivalent shear modulus of the plate is calculated using Eq. (22) for η=0. In Table 2, the transverse frequencies of the elastic plates with clamped-clamped (*C-C*) edges has been presented for r_i/r_o=0.3 and different values of r_o/h. The results have been reported with the *FSDT*, *CPT* (Hagedorn 2007) which has used the Bessel functions, *FE* (Ansys), *CPT* (Eq. (21)) and the Liu and Chen (1995) which has used the *FE* method based on the theory of elasticity. By increasing r_o/h, the dimensionless frequencies decrease. The difference percentage of the results have been calculated too. The difference percentage has been defined as *diff*(A_1)=abs(((A_1-A_{FE}) / A_{FE})*100) where A_{FE} stands for the *FE* results, A_1 is the analytical value and *abs* stands of absolute value of the quantity. According to Table 2, for large values of r_o/h, the difference percentage with the *CPT* is smaller than the *FSDT* i.e., the *FSDT* does not have any advantage for thin plates. For small values of r_o/h, the *FSDT* is closer than the *CPT* to the FE especially in the second modes. The *FSDT* results are in a good agreement with the Liu and Chen (1995) for thicker plates and the *CPT* results are close to the Liu and Chen for thinner plates. So, the presented solution method predicts the correct results as we expected for elastic plate. All the other tables and graphs are for the viscoelastic case with the *FE* (Ansys), *CPT* (Eq. (21)) and *FSDT* (Eqs. (13), (14)). In Tables 3, the frequencies of viscoelastic plate have been reported. The results of the *CPT* have a large discrepancy with the *FE* for small values of r_o/h especially in the second mode. For a constant value of r_o/h, by increasing r_i/r_o, the dimensionless frequencies increase (Fig. 3). Also for a constant value of r_i/r_o, the obtained frequency with the *FE* is smaller than the *FSDT* and *CPT*. The calculation showed that this result is not correct for all boundary conditions (The results of the other boundary conditions are reported later).

The presented formulation can use for the plate with different boundary conditions. Tables 4 present the first and second frequencies of an ELASTIC annular plate for different boundary conditions. The first letter stands of the inner boundary and the second is for the outer boundary e.g., *S-F* is related to the simply supported at the inner edge and free at the outer one. Tables 4 compare the exact

Table 1 geometrical and material properties

Outer radius (m)	r_o=0.15
Poisson's ratio	ν=0.25
Viscoelastic modulus (Pa)	G_1=5.354e9, G_2=4.504e9
Viscosity coefficient (Pa.s)	η=2.673e10
Density (Kg/m^2)	0.22

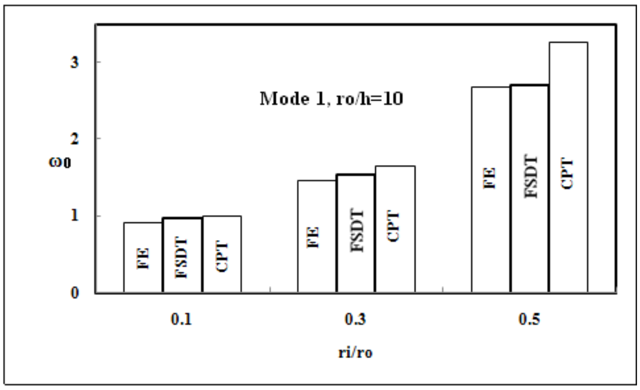

Fig. 3 Effect of *ri/ro* on frequencies of viscoelastic annular plates (C-C)

Table 2 Dimensionless natural frequencies (ω_0) for *C-C* ELASTIC annular plate r_i/r_o=0.3

	Mode 1				Mode 2			
r_o/h→	5	10	20	50	5	10	20	50
FE	2.2746	1.4626	0.8010	0.3296	4.9133	3.5800	2.1259	0.9045
FSDT	2.3414	1.5365	0.8502	0.3514	4.9419	3.6990	2.2372	0.9606
diff(FSDT)	2.8	5.0	6.1	6.6	0.6	3.3	5.2	6.2
CPT (Eq. (21))	3.3347	1.6672	0.8337	0.3335	9.1919	4.5959	2.2980	0.9192
diff(CPT-Eq. (21))	31.7	14.0	4.1	1.2	87.1	28.4	8.1	1.6
CPT (Bessel)	3.3116	1.6558	0.8279	0.3312	9.1555	4.5779	2.2889	0.9156
diff(CPT- Bessel)	45.5	13.2	3.4	0.5	86.3	27.9	7.7	1.2
Liu		1.5119	0.8139	0.3315		3.7070	2.1641	0.9103
diff(Liu)		3.4	1.6	0.6		3.5	3.3	0.6

Table 3a Dimensionless natural frequencies (ω_0) for *C-C* viscoelastic annular plate r_i/r_o=0.1

	Mode 1				Mode 2			
r_o/h→	5	10	20	50	5	10	20	50
FE	1.5616	0.9156	0.4891	0.1987	3.5159	2.3364	1.3243	0.5479
FSDT	1.6039	0.9778	05221	0.2131	3.5491	2.4500	1.3973	0.5844
diff (FSDT)	2.7	6.8	6.8	7.3	0.9	4.9	5.5	6.7
CPT(Eq. (21))	2.0172	1.0085	0.5043	0.2017	5.5601	2.7804	1.3902	0.5561
diff(CPT)	29.2	10.2	3.1	1.5	58.1	19.0	5.0	1.5

Table 3b Dimensionless natural frequencies (ω_0) for *C-C* viscoelastic annular plate r_i/r_o=0.3

	Mode 1				Mode 2			
r_o/h→	5	10	20	50	5	10	20	50
FE	2.2738	1.4615	0.8006	0.3295	4.9122	3.5774	2.1247	0.9043
FSDT	2.3414	1.5365	0.8501	0.3514	4.9419	3.6988	2.2373	0.9606
diff (FSDT)	3.0	5.1	6.2	6.7	0.6	3.4	5.3	6.2
CPT (Eq. (21))	3.3347	1.6673	0.8337	0.3335	9.1919	4.5961	2.2983	0.9192
diff (CPT)	46.7	14.0	4.1	1.2	87.1	28.5	8.2	1.7

Table 3c Dimensionless natural frequencies (ω_0) for *C-C* viscoelastic annular plate r_i/r_o=0.5

	Mode 1				Mode 2			
r_o/h→	5	10	20	50	5	10	20	50
FE	3.6966	2.6821	1.5530	0.6493	7.5178	6.2508	4.0486	1.7890
FSDT	3.7073	2.7127	1.6092	0.6845	7.4263	6.1279	4.0902	1.8569
diff(FSDT)	0.3	1.1	3.6	5.4	1.2	2.0	1.0	3.8
CPT(Eq21)	6.5356	3.2681	1.6340	0.6535	18.0166	9.0075	4.5038	1.8017
diff(CPT)	76.8	21.9	5.2	0.7	139.7	44.1	11.2	0.7

solution using the Bessel functions for the *CPT* case (Hagedorn *et al*. 2007) and the presented method with the perturbation technique (Eq. (21)). There is a good agreement between the results or it shows that the presented formulations work properly for these boundary conditions. The calculations showed that in the presented formulation, the equations order-zero (Eq. (13)) are sufficient to find the frequencies for the plates with the same boundary conditions (i.e., *S-S*, *C-C*, *F-F*) but for non-similar boundary conditions (i.e., *S-F*, *F-S*, *C-S*, *S-C*,…), the order-zero presents the same frequencies for the non-similar cases e.g., *S-F* and *F-S* (which did not report here). So it is necessary to consider the order-one (Eq. (14)) too or the order-zero, does not have the sufficient convergence. The natural frequencies of the viscoelastic annular plates with different boundary conditions have been reported in Tables 5. These results have been compared with the *FE* and *CPT* results. In the most cases, the *FSDT* is closer than to the *FE* with respect to the *CPT*. The reported frequencies relate to the bending mode. In some cases, there are the rigid body frequencies which did not report here. It is seen that the natural frequencies for the *C-C* case is larger than the *S-S* as we expected. It is due to increasing the stiffness of the plate. Also, the difference percentage for *S-S* is smaller than the *C-C* boundary conditions.

Fig. 4 shows the effect of relaxation time on the natural

Table 4a Dimensionless natural frequencies (ω_0) for different boundary conditions of an annular ELASTIC plate r_i/r_o=0.3, r_o/h=10, E=0.6115e10, ν=0.25 (Mode 1)

	Mode 1								
Support→	C-C	S-S	C-S	S-C	C-F	F-C	S-F	F-S	F-F
CPT (Bessel)	1.6558	0.7719	1.0913	1.2390	0.2408	0.4215	1.1539	1.3609	1.8451
CPT(Eq. (21))	1.6672	0.7355	1.0938	1.2041	0.2058	0.3182	1.0938	1.2041	1.6672

Table 4b Dimensionless natural frequencies (ω_0) for different boundary conditions of an annular ELASTIC plate r_i/r_o=0.3, r_o/h=10, E=0.6115e10, ν=0.25 (Mode 2)

	Mode 2								
Support→	C-C	S-S	C-S	S-C	C-F	F-C	S-F	F-S	F-F
CPT (Bessel)	4.5779	2.9883	3.6641	3.8124	1.5501	1.9015	3.7373	3.9354	4.7735
CPT(Eq. (21))	4.5959	2.9421	3.6710	3.7754	1.5337	1.7502	3.6712	3.7753	4.5959

Table 5a Dimensionless natural frequencies (ω_0) for different boundary conditions of an annular viscoelastic plate r_i/r_o=0.3, r_o/h=10 (Mode 1)

	Mode 1								
Support→	C-C	S-S	C-S	S-C	C-F	F-C	S-F	F-S	F-F
FE	1.4618	0.7359	1.0036	1.1386	0.2366	0.4118	1.0814	1.2820	1.7155
FSDT	1.5365	0.6978	1.0640	1.1186	0.2151	0.3308	0.9811	1.2100	1.6454
diff(FSDT)	5.1	5.2	6	2	8.9	19.6	9.3	5.6	4.1
CPT(Eq. (21))	1.6672	0.7355	1.0938	1.2041	0.2058	0.3182	1.0938	1.2041	1.6672
diff(CPT)	14	0.05	9	5.7	13	22	1.1	7.8	2.8

Table 5b Dimensionless natural frequencies (ω_0) for different boundary conditions of an annular viscoelastic plate r_i/r_o=0.3, r_o/h=10 (Mode 2)

	Mode 2								
Support→	C-C	S-S	C-S	S-C	C-F	F-C	S-F	F-S	F-F
FE	3.5778	2.5840	3.0190	3.1285	1.3936	1.7171	3.1747	3.3604	4.1790
FSDT	3.6988	2.4640	3.1830	3.2276	1.4507	1.6596	3.0317	3.4242	4.0551
diff(FSDT)	3.4	4.6	5.4	3	4.1	3.3	4.5	1.9	3
CPT(Eq. (21))	4.5961	2.9422	3.6710	3.7751	1.5337	1.7502	3.6710	3.7751	4.5959
diff(CPT)	28.4	13.8	21.6	20.7	10	1.9	15.6	12.3	10

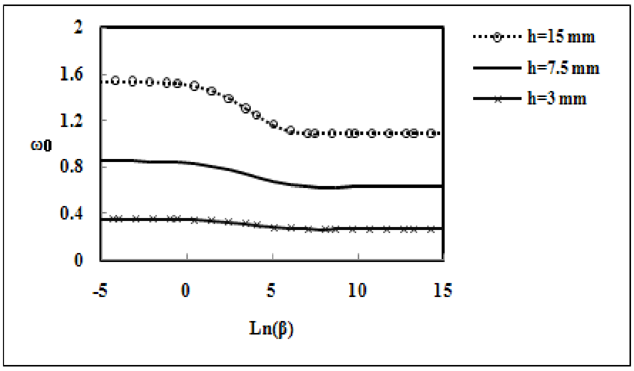

Fig. 4 Variation of natural frequency (ω_0) with relaxation time (*C-C*, Mode 1, r_i/r_o=0.3)

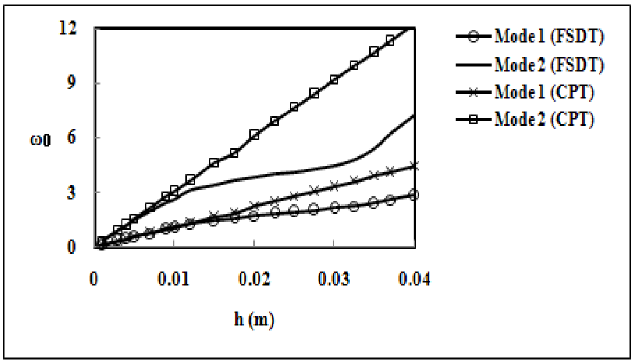

Fig. 5 Variation of natural frequency (ω_0) with thickness for different modes (*C-C*, r_i/r_o=0.3)

frequency of *C-C* annular plates with different thicknesses. The sensitivity of the natural frequency to relaxation time restricted to the small values of η. Fig. 5 shows the effect of thickness on the natural frequency of *C-C* annular plates. By increasing the thickness, the natural frequency increases and the growth of its slope is more on the second mode in the studied range. In addition, for small values of thickness, the difference between *CPT* and *FSD*T results are small especially in Mode 1.

In Table 6, the frequencies and damping of the system for different values of η ($2.6972 \leq \eta \leq 2.6972e10$) have been

Table 6 Dimensionless Transverse complex frequencies (ω_0+ i.α) for a *C-C* annular plate and different values of viscoelastic parameter η(r_i/r_o=0.3, r_o/h=10)

	n→	10	8	6	5	4	3	2	0
Mode1	α	-1.2211e-7	-1.2211e-5	-1.2212e-3	-3.2113e-2	-0.14872	-1.9510e-2	-7.3890e-4	-7.390e-6
	ω_0	1.5365	1.5365	1.5365	1.5299	1.3460	1.2465	1.2462	1.2492
Mode2	α	-1.4040e-7	-1.4040e-5	-1.4040e-3	-3.8056e-2	-0.34591	-0.13806	-1.4267e-2	-4.926e-4
	ω_0	3.6988	3.6988	3.6987	3.6974	3.5411	2.9009	2.8746	2.8754

Table 7. Dimensionless Radial complex frequencies (ω_0+ i.α) for an *C-C* annular plate and different values of viscoelastic parameter η(r_i/r_o=0.3, r_o/h=10)

	n→	10	8	6	5	4	3	2	0
Mode1	α	-1.5001e-7	-1.5001e-5	-1.5001e-3	-5.2030e-2	-0.40621	-0.36914	-4.1004e-2	-1.3335e-4
	ω_0	5.7026	5.7026	5.7025	5.6981	5.6007	4.3358	4.2570	4.2564
Mode2	α	-1.5186e-7	-1.5186e-5	-1.5186e-3	-5.2182e-2	-0.44533	-1.4054	-1.6355e-1	-5.3574e-4
	ω_0	11.3844	11.3844	11.3844	11.3839	11.3427	9.0805	8.4623	8.4580

listed. The results are related to Eq. (14b) which we called it the transverse motion. In the range of 0≤*n*≤4, by increasing the viscoelastic parameter, the damping of the system increases or the viscoelastic parameter can reduce the rate of amplitude vibration but the more increasing of “*n*”, does not affect on the damping of the system. Note that for the large values of η, the system approach to an elastic system which has the shear modulus G_1 (Fig. 2) or the large values of η, can convert the dashpot element to a rigid link. The value of ω_0 is nearly constant for large values of *n* but in 0≤*n*≤4, it can affect on the natural frequency. For these values, it is necessary to consider the terms with order-one (i.e., Eq. (14b)) to calculate the frequencies. Also, it is seen that the frequencies of elastic case (Table 2, which is just related to order-zero) are not nearly the same as Table 6 for *n*=0. In Fig. 6, the vibrational amplitude of the system for especial value of damping has been shown.

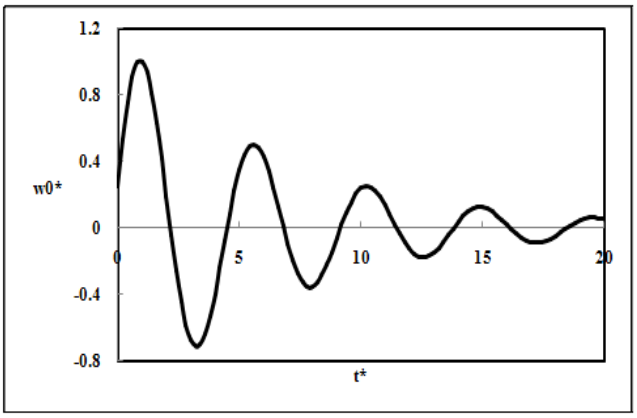

Fig. 6 Transverse response of middle radius (η=2.6729e4 Pa)

As mentioned before, the *FSDT* can represent the radial frequencies, too (Eq. (14a)). The radial natural frequencies of a *C-C* plate have been reported in Table 7 for different values of “*n*”. The radial frequencies values are large in comparison with respect to the transverse frequencies. These results cannot calculate with the *CPT* formulation. Also the similar conclusion of Table 7 can express for this Table. Fig. 7 shows the mode shapes for the first and second modes in the case of simply supported and clamped plates.

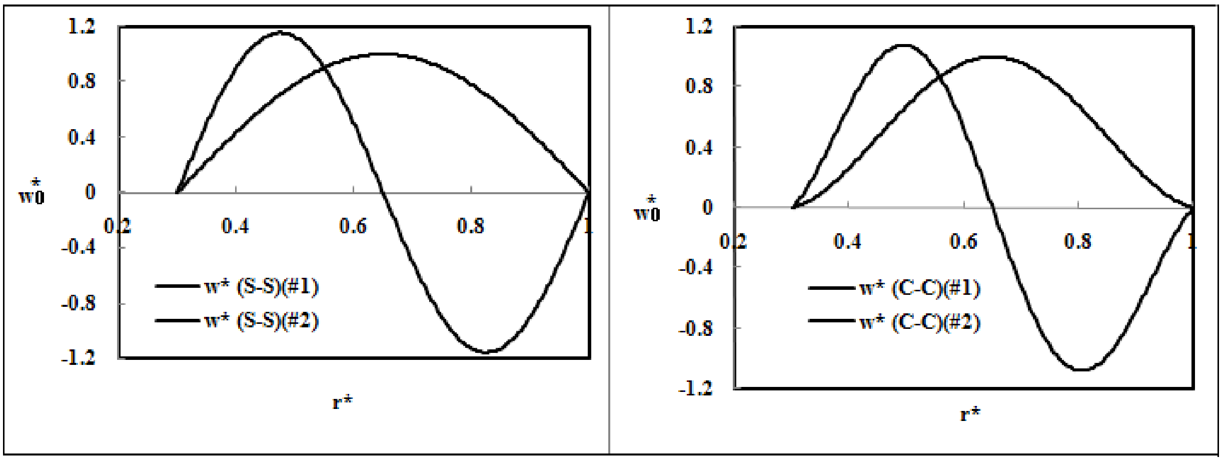

Fig. 7 Mode shapes 1 and 2 for *S-S* and *C-C* annular plate

8. Conclusions

A mathematical approach has been applied to investigate the free vibration of viscoelastic annular plates by considering the first order shear deformation theory as the displacement field. The governing equations were solved *analytically* by the *perturbation technique*. A sensitivity analysis was performed to investigate the influences of boundary conditions, radius ratio, relaxation time, viscoelastic parameters and thickness on the transverse natural frequency of the plate. The following conclusions can obtain for the studied cases:

- Using a new variable change, the governing equations which were a system of differential equations with variable coefficients, were converted to equations with constant coefficients. The new form of equations has a closed-form solution in each order of ε.
- The calculations can be performed using a simple code in a mathematical environment such as Maple and it is not necessary to construct a *FE* model. In the other word, the calculations can be performed faster than the *FE*.
- For smaller values of r_o/h, the *FSDT* is more closer than *CPT* to the *FE* results, especially in the second mode.
- For larger values of r_o/h, the *FSDT* does not have any advantage with respect to the *CPT*.
- For a constant value of r_o/h, by increasing r_i/r_o, the dimensionless frequencies increase.
- For the plates with *C-C* edges, the natural frequencies are larger than the *S-S* results for a viscoelastic plate. Also the difference percentage for *S-S* is smaller than the *C-C* boundary conditions.
- The *FSDT* can represent the radial frequencies, too.
- The sensitivity of the natural frequency to the relaxation time is restricted to the small values of η.
- By increasing the thickness, the natural frequency increases, especially in the second mode.
- The presented method can use for different boundary conditions.

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