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Dynamic response of thin plates on time-varying elastic point supports

Mohammad A. Foyouzat* and Homayoon E. Estekanchi[a]

Department of Civil Engineering, Sharif University of Technology, Azadi Ave., Tehran, Iran

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Abstract. In this article, an analytical-numerical approach is presented in order to determine the dynamic response of thin plates resting on multiple elastic point supports with time-varying stiffness. The proposed method is essentially based on transforming a familiar governing partial differential equation into a new solvable system of linear ordinary differential equations. When dealing with time-invariant stiffness, the solution of this system of equations leads to a symmetric matrix, whose eigenvalues determine the natural frequencies of the point-supported plate. Moreover, this method proves to be applicable for any plate configuration with any type of boundary condition. The results, where possible, are verified upon comparison with available values in the literature, and excellent agreement is achieved.

Keywords: dynamic analysis; elastically point-supported plate; time-varying stiffness; eigenfunctions

1. Introduction

The analysis of plates resting on point supports has long been a problem of interest in engineering practice. Much of the interest likely stems from the extensive applications of point-supported plates to practical problems including the vibration of column-supported slabs, printed circuit boards, vibrator of cellular phones as well as other acoustic devices, bolted or spot-welded aircraft and ship bodies, to mention but a few. The problem is also of practical importance to the real-world structural designs, such as for eliminating internal resonance, and for modeling the vibration of electronic advertising boards and/or notice boards (Zhao *et al*. 2002). The literature contains a wealth of research on the free vibration analysis of plates supported by point supports, of which a brief survey is presented herein.

Cox and Boxer (1960) provided one of the early contributions to this topic and discussed the free vibration of a rectangular plate with point supports at the corners and free edge boundaries elsewhere, using a finite difference approach. Kerstens (1979) computed the first six natural frequencies of a point-supported rectangular plate for various aspect ratios by the modal constraint method. Fan and Cheung (1984) used a spline finite strip approach to treat rectangular plates having complex support conditions. Raju and Amba-Rao (1983) used the finite element method to determine an upper bound for the first few frequencies of a square plate resting on four symmetric point supports on the diagonals. Kim and Dickinson (1987) applied the Lagrange multiplier technique with a set of orthogonal polynomial functions as the basis functions in the Rayleigh-Ritz method to study the vibration of point-supported rectangular plates. Nowacki (1953) used the multiple domain approach to study the simply supported rectangular plates with one interior point support. Bapat and Suryanarayan (1989) used the flexibility function approach to study the free vibration of rectangular plates with multiple interior point supports. Liew and Lam (1994) generated a set of orthogonal functions by using the Gram-Schmidt algorithm to approximate the natural frequencies of point-supported rectangular plates. Wang *et al*. (1997) investigated the optimal location of internal rigid point supports in a symmetric laminated rectangular plate for maximum fundamental frequency, using the simplex method. By using the Galerkin method in conjunction with natural coordinates, Saadatpour *et al*. (2000) solved the free vibration problem of simply-supported plates of general shape with internal supports. Zhao *et al*. (2002), Xiang *et al*. (2002), and Wei *et al*. (2002) introduced the discrete singular convolution method to study the free vibration of rectangular plates with irregular internal supports. Narita and Hodgkinson (2005) used the layerwise optimization approach to maximize the fundamental frequencies of point-supported symmetrically laminated plates. They applied the Ritz method in order to solve the vibration problem. Hedayati *et al*. (2007) proposed a numerical method based on the Lagrange multiplier technique to solve the local buckling of plates with point supports. Altekin (2008) employed the Ritz approach to treat both the free vibration and the buckling of elliptical plates having point supports along the symmetric diagonals. Watkins and Barton (2010) evaluated the natural frequencies of elastically point-supported rectangular plates, using eigensensitivity analysis. Wu (2012) studied the free vibration of quadrilateral thick plates with internal columns, using a third-order shear deformation theory. Moreover, Watkins *et al*. (2010) conducted an experimental work in order to determine the natural frequencies for an elastically point-supported plate with attached masses. The static

*Corresponding author, Ph.D. Candidate
E-mail: foyozat_mohammadali@mehr.sharif.ir
[a]Professor
E-mail: stkanchi@sharif.edu

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analysis of a point-supported nonlinear super-elliptical plate under a uniform transverse pressure was also examined by Altekin (2014), using the Newton-Raphson method.

By looking at the studies conducted so far on the vibration of point-supported plates, it can be appreciated that two important topics of considerable practical and analytical importance are absent in the current literature. First, no study has as yet been carried out on the dynamic response of point-supported plates to an external exciting force, although there is an excessive literature on the free vibration analysis of plates with various shapes. Moreover, in this regard, research on the plates with elastic point supports is considerably limited in number as compared to the plates with rigid point supports. Second, in all of the previous studies, as an a priori assumption, the stiffness of the supports is considered as being time-invariant, while adopting a time-varying stiffness for the supports is a more realistic assumption due to the following discussion.

In recent years, the analysis of the civil and mechanical engineering structures whose properties, such as stiffness and mass, are time-variant has been receiving wide attention. It is because the dynamic characteristic properties of engineering structures often change over time during their service life (Udwadia and Jerath 1980, Loh and Tsaur 1988, Li 2000, 2002, Liu *et al*. 2008 a, b, Lin *et al*. 1990, Liu and Kujath 1994, Ghanem and Shinozuka 1995, Staszewski 1997, Shi and Law 2007, Basu *et al*. 2008, Bao 2012). For example, the mass distribution and stiffness of a vehicle-bridge interacted system constantly change over time when a vehicle or train passes through the bridge, and the stiffness of a cable often changes with the cable tension force (Barber *et al*. 2003, Wang *et al*. 2013). Moreover, in quite a few active and semi-active control devices, the time variation of the device stiffness plays an important role in mitigating structural responses (Soong 1990, Spencer and Nagarajaiah 2003). Therefore, the use of such control devices as point supports may be another potential application in which considering the time variation in the supports stiffness is inevitable in the model.

Composite steel-concrete members, commonly applied in building and bridge applications, also exhibit a time-dependent behavior. When a composite column is subjected to an axial load, its response will be time dependent (Bradford and Gilbert 1990). The time effects of composite steel-concrete columns, mainly used as supports for the deck slab of bridges, are put forward and investigated by Bradford and Gilbert (1990), Wang *et al*. (2011), Geng *et al*. (2012), and Ranzi *et al*. (2013). Their findings revealed that time effects can influence the ultimate response of the composite structure, including both the slab and the column supports.

Apart from this, under certain circumstances, the structural stiffness varies over time due to accumulated damages under both service loads and environmental excitations as a result of inevitable material aging, fatigue, deterioration, etc., or sudden damage caused by accidents or natural disasters (Lin *et al*. 1990, Ghanem and Romeo 2000, Lin *et al*. 2005, Bao 2012). In such cases, a time-varying model may better capture the behavior of the real-world structure.

In this paper, a simple, yet general analytical-numerical approach is proposed in order to determine the dynamic response of plates resting on multiple elastic point supports. To the best of the authors' knowledge, it seems that the treatment and examples on the behavior of point-supported plates that are reported in the current literature have been limited to the free vibration response of plates with time-invariant supports. However, the method presented in the current study is readily applicable to the plates resting on time-varying elastic supports and subjected to an arbitrary exciting force. Moreover, the proposed solution can be exercised to treat step functions and discontinuous time variations in the support stiffness. This will be interesting since for example, the elimination of one or more supports at particular instants of time can then be modeled with no difficulty. At the same time, an illustrative example is presented in order to demonstrate how the work proceeds. In the provided example, the dynamic response of a plate on a time-varying spring, acted upon by a moving force, is studied for three different variations of stiffness with time. The support elimination is also examined in the example.

As far as the time-invariant problems are concerned, the proposed method can streamline the analysis of point-supported plates, as compared with the methods introduced in the preceding studies. In this particular case, by having the eigenfunctions of the plate in the absence of the point supports, one can easily determine the natural frequencies corresponding to the plate that is placed on the elastic point supports. In practice, the solution only requires finding the eigenvalues of a known symmetric matrix, which could be readily carried out on personal computers. Moreover, this method is applicable for any plate geometry with any type of boundary condition. Also, the methodology can potentially be generalized to evaluate the dynamic response of nanoplates resting on multiple elastic point supports, which has been received attention in recent years (e.g., see Akgöz and Civalek 2012, Farajpour *et al*. 2012, Akgöz and Civalek 2013).

Since the reported values in the literature are confined to time-invariant problems, the verification of the proposed method and the comparative study are performed only for the case of time-invariant stiffness. For the sake of coherence of the whole paper, the verification example is included in Appendix. In the mentioned example, the veracity of the results is corroborated through comparison with recognized solutions in the literature.

In the ensuing section, the case of a plate on a single elastic point support is treated first. The solution is then generalized for the case of multiple elastic point supports.

2. Problem statement and solution

Fig. 1 illustrates a thin plate with an arbitrary geometry having a constant thickness h and placed on a time-varying linear spring of stiffness $k_0(t)$ at point (x_0, y_0). As shown in Fig. 1, the plate is also acted upon by a transverse forcing action $P(x, y, t)$. Moreover, the plate could have any type of boundary condition. The governing partial differential equation of this plate could be expressed as

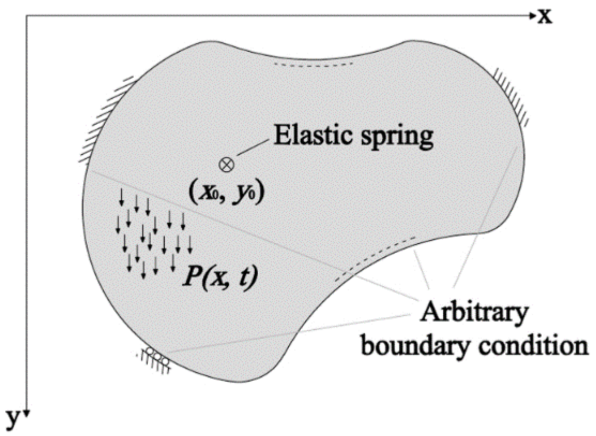


Fig. 1 Plate supported by a linear elastic spring

$$D\nabla^4 w+\rho h\frac{\partial^2 w}{\partial t^2} = -k_0(t)\delta(x-x_0)\delta(y-y_0)w+P(x,y,t) \tag{1}$$

where w is the deflection of the plate, ρ is the mass density of the material, which is assumed to be a constant value, D is the flexural rigidity of the plate, and δ indicates the Dirac delta function. Using the separation of variables technique, one may write the deflection of the plate in the following form

$$w(x,y,t)=\sum_{n=1}^{\infty}\Phi_n(x,y)F_n(t) \tag{2}$$

where Φ_n's are the known eigenfunctions of a plate with the same boundary conditions and with no spring attached to it. They have the form of

$$\nabla^4\Phi_n-\bar{\omega}_n^2\Phi_n=0 \tag{3}$$

and

$$\bar{\omega}_n^2=\frac{\omega_n^2\rho h}{D} \tag{4}$$

where ω_n's are the natural frequencies of the plate (without spring), and F_n's are amplitude functions, which remain to be calculated. The eigenfunctions along with the natural frequencies for different types plates with various boundary conditions may be found in (Leissa 1969, Soedel 1993).

In addition, one could rewrite the first term on the right-hand side of Eq. (1) in the form of a series as follows

$$-k_0(t)\delta(x-x_0)\delta(y-y_0)w = \sum_{n=1}^{\infty}\Phi_n(x,y)B_n(t) \tag{5}$$

where the time-dependent coefficients, $B_n(t)$, are to be determined. Multiplying both sides of Eq. (5) by $\Phi_{\mathrm{i}}(x,y)$ and making use of Eq. (2), one can reach

$$-k_0(t)\delta(x-x_0)\delta(y-y_0)\sum_{n=1}^{\infty}\Phi_{\mathrm{i}}(x,y)\Phi_n(x,y) \times F_n(t)=\sum_{n=1}^{\infty}\Phi_{\mathrm{i}}(x,y)\Phi_n(x,y)B_n(t) \tag{6}$$

Integrating Eq. (6) on the area of the plate, yields

$$-k_0(t)\int_A\left[\delta(x-x_0)\delta(y-y_0) \times\sum_{n=1}^{\infty}\Phi_{\mathrm{i}}(x,y)\Phi_n(x,y)F_n(t)\right]dA =\sum_{n=1}^{\infty}\left(\int_A\Phi_{\mathrm{i}}(x,y)\Phi_n(x,y)dA\right)B_n(t) \tag{7}$$

On taking into consideration the properties of Dirac delta function and the orthogonality of Φ_n's, Eq. (7) will be simplified as

$$\frac{-k_0(t)}{V_i}\Phi_i(x_0,y_0)\sum_{n=1}^{\infty}\Phi_n(x_0,y_0)F_n(t)=B_i(t) \tag{8}$$

where

$$V_i=\int_A\Phi_i(x,y)^2\,dA \tag{9}$$

Furthermore, the loading function can also be represented in a series in terms of the eigenfunctions, that is

$$P(x,y,t)=\sum_{n=1}^{\infty}\Phi_n(x,y)P_n(t) \tag{10}$$

Multiplying both sides of Eq. (10) by $\Phi_i(x,y)$, integrating it on the area of the plate, and employing the orthogonality property of Φ_n's, yield

$$\frac{1}{V_i}\int_A\Phi_i(x,y)P(x,y,t)dA=P_i(t) \tag{11}$$

where V_i has been defined in Eq. (9). Substituting Eqs. (5), (8), and (10) into Eq. (1) results in an equation for $F_n(t)$

$$D\left(\sum_n\nabla^4\Phi_n(x,y)F_n(t)\right)+\rho h\sum_n\Phi_n(x,y) \times\frac{d^2F_n(t)}{dt^2}=\sum_n\frac{-k_0(t)}{V_n}\Phi_n(x,y)\Phi_n(x_0,y_0) \times\sum_q\Phi_q(x_0,y_0)F_q(t)+\sum_n\Phi_n(x,y)P_n(t) \tag{12}$$

Applying Eq. (3), the preceding equation becomes

$$\sum_n \Phi_n(x,y)\left\{D\bar{\varpi}_n^2 F_n(t)+\rho h\frac{d^2F_n(t)}{dt^2}+\frac{k_0(t)}{V_n}\right.$$
$$\left.\times\Phi_n(x_0,y_0)\sum_q \Phi_q(x_0,y_0)F_q(t)-P_n(t)\right\}=0 \tag{13}$$

This equation must be satisfied for arbitrary (x, y) (i.e., each point of the plate), and this is possible only when

$$D\bar{\varpi}_n^2 F_n(t)+\rho h\frac{d^2F_n(t)}{dt^2}+\frac{k_0(t)}{V_n}$$
$$\Phi_n(x_0,y_0)\sum_{q=1}^{\infty}\Phi_q(x_0,y_0)F_q(t) \tag{14}$$
$$=P_n(t) \qquad (n=1,2,3,...)$$

This system is a linear system of coupled ordinary differential equations, and a standard numerical procedure could be employed to solve it. Retaining only the first N modes, dividing by ρh, and rearranging the outcome in the matrix form, results in

$$\{\ddot{F}(t)\}+\mathbf{K}(t)\{F(t)\}=\{P(t)\} \tag{15}$$

where dots are used in order to indicate differentiation with respect to time, and

$$\mathbf{K}(t)=\begin{bmatrix} \alpha_1\Phi_1^2+\beta_1 & \alpha_1\Phi_1\Phi_2 & \cdots & \cdots & \alpha_1\Phi_1\Phi_N \\ \alpha_2\Phi_2\Phi_1 & \alpha_2\Phi_2^2+\beta_2 & \cdots & \cdots & \alpha_2\Phi_2\Phi_N \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \alpha_N\Phi_N\Phi_1 & \alpha_N\Phi_N\Phi_2 & \cdots & \cdots & \alpha_n\Phi_N^2+\beta_N \end{bmatrix} \tag{16}$$

where

$$\alpha_i=\frac{k_0(t)}{\rho h V_i},\quad \beta_i=\frac{D\bar{\varpi}_i^2}{\rho h},\quad i=1,2,...,N \tag{17}$$

$$\Phi_i=\Phi_i(x_0,y_0) \tag{18}$$

$$\{F(t)\}^T=\{F_1(t),F_2(t),...,F_N(t)\}$$
$$\{\ddot{F}(t)\}^T=\left\{\frac{d^2F_1(t)}{dt^2},\frac{d^2F_2(t)}{dt^2},...,\frac{d^2F_N(t)}{dt^2}\right\} \tag{19}$$

$$\{P(t)\}^T=\frac{1}{\rho h}\{P_1(t),P_2(t),...,P_N(t)\} \tag{20}$$

It is very important to note that there is no need to calculate Eq. (9) in each mode, because it can always be normalized to a constant, say unity

$$V_i=\int_A \Phi_i(x,y)^2\,dA=1 \qquad i=1,2,3,\ldots \tag{21}$$

and

$$\alpha_1=\alpha_2=\alpha_3=\ldots=\alpha_N=\frac{k_0(t)}{\rho h} \tag{22}$$

Interestingly, for the special case of a spring with a time-invariant or constant stiffness, k_0, the general Eq. (15) reduces to a linear system of ordinary differential equations with constant coefficients. In this particular case, normalizing (i.e., Eq. (21)) makes possible a symmetric matrix **K**, indicating that the contribution of each mode can be studied individually. Further discussion on the case of time-invariant stiffness will be presented in the forthcoming section.

A convenient method to solve Eq. (15), using a computer program, is to transform the equation into a set of first-order linear equations, that is, to employ the state-space representation. The following equation expresses the state-space form of Eq. (15)

$$\{\dot{Q}(t)\}=\mathbf{A}(t)\{Q(t)\}+\mathbf{B}(t) \tag{23}$$

where

$$\{Q(t)\}=\begin{bmatrix}\{F(t)\}\\ \{\dot{F}(t)\}\end{bmatrix}_{2N\times1};\ \mathbf{A}(t)=\begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{K}(t) & \mathbf{0}\end{bmatrix}_{2N\times2N};$$
$$\mathbf{B}(t)=\begin{bmatrix}\mathbf{0}\\ \{P(t)\}\end{bmatrix}_{2N\times1} \tag{24}$$

and **I** is the identity matrix of order N. Built-in routines to solve Eq. (23) are available in virtually all mathematical software packages. Alternatively, Eq. (23) may be solved numerically by using the so-called transition matrix approach (Brogan 1991). In this regard, the state vector $\{Q(t)\}$, when the initial conditions are defined at $t=t_0$, can be expressed as

$$\{Q(t)\}=\boldsymbol{\chi}(t,t_0)\{Q(t_0)\}+\int_{t_0}^{t}\boldsymbol{\chi}(t,\tau)\mathbf{B}(\tau)\,\mathrm{d}\tau \tag{25}$$

where $\boldsymbol{\chi}(t,\tau)$ is the transition matrix. Assuming that $\{t_0, t_1, \ldots, t_k, \ldots\}$ is a set of discrete time points sufficiently close together, Eq. (25) can be used to write the solution at t_{k+1} by treating $\{Q(t_k)\}$ as the initial condition

$$\{Q(t_{k+1})\}=\boldsymbol{\chi}(t_{k+1},t_k)\{Q(t_k)\}$$
$$+\int_{t_k}^{t_{k+1}}\boldsymbol{\chi}(t_{k+1},\tau)\mathbf{B}(\tau)\,\mathrm{d}\tau \tag{26}$$

Eq. (26) is an approximating difference equation for $\{Q(t)\}$. An approximate solution to $\boldsymbol{\chi}(t,\tau)$ is

$$\boldsymbol{\chi}(t_{k+1},t_k)=\exp(\mathbf{A}(t_k)\Delta t_k) \tag{27}$$

where $\Delta t_k=t_{k+1}-t_k$ is an assumed time interval. Provided that

$\mathbf{A}(t_k)$ is invertible, a stepwise solution to Eq. (23) will found to be

$$\begin{aligned}\{Q(t_{k+1})\} &= \boldsymbol{\chi}(t_{k+1},t_k)\{Q(t_k)\} \\ &+ \left[\boldsymbol{\chi}(t_{k+1},t_k) - \mathbf{I}\right]\mathbf{A}^{-1}(t_k)\mathbf{B}(t_k)\end{aligned} \tag{28}$$

Further details about the transition matrix approach may be found in reference (Brogan 1991).

A similar approach can be applied for the case when the plate is supported by M time-varying linear springs of constants $k_0(t)$, $k_1(t)$, …, and $k_{M-1}(t)$ placed at points (x_0, y_0), (x_1, y_1), …, and (x_{M-1}, y_{M-1}), respectively. The governing equation of this plate may be expressed as

$$\begin{aligned} &D\nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} \\ &= -\left[\sum_{m=0}^{M-1} k_m(t)\,\delta(x - x_m)\,\delta(y - y_m)\right] w \\ &+ \mathcal{P}(x\ y, t)\end{aligned} \tag{29}$$

where all the parameters have the same definitions as in Eq. (1). Moreover, the deflection is assumed to have the form as shown in Eq. (2). In following the same procedure as discussed above for a single spring, one will arrive at a similar equation to Eq. (15) with matrix $\mathbf{K}(t)$ having the following general form

$$\mathbf{K}(t) = \frac{1}{\rho h}\begin{bmatrix} \sum_{m=0}^{M-1} k_m(t)\Phi_1^2(x_m, y_m) + \beta_1 \rho h & \cdots & \sum_{m=0}^{M-1} k_m(t)\Phi_1(x_m, y_m)\Phi_N(x_m, y_m) \\ \vdots & \ddots & \vdots \\ \sum_{m=0}^{M-1} k_m(t)\Phi_N(x_m, y_m)\Phi_1(x_m, y_m) & \cdots & \sum_{m=0}^{M-1} k_m(t)\Phi_N^2(x_m, y_m) + \beta_N \rho h \end{bmatrix} \tag{30}$$

with β_i 's being previously defined. Additionally, in deriving Eq. (30), use is made of Eq. (21), that is to say, Φ_i's must first be normalized in order to satisfy Eq. (21). This will guarantee the symmetry of matrix $\mathbf{K}$. It should be added that the state-space formulation introduced in Eqs. (23)-(28) is still valid for this general case.

The solution method presented herein involves direct use of the governing differential equation, so it does not assume the existence of a functional that is usually minimized as in other techniques. Therefore, the method actually covers a broader range of application as compared to the Rayleigh-Ritz technique, which has been widely used by numerous researchers (e.g., see Kim and Dickinson 1987, Liew and Lam 1994, Altekin 2008). One of the advantages of this method is that the proposed solution does not necessarily require the eigenfunctions as being described analytically. In the cases where the shape of the plate or the boundary conditions are such that the eigenfunctions cannot be analytically determined, the solution can be carried out by using numerical eigenfunctions in the equations. Numerical techniques to determine the eigenfunctions are well established in the literature.

Additionally, the method can be applied to multiple-support problems just as simple as for the problems with a single support without any significant increase in the computational cost. However, such techniques as the finite element method (e.g., Raju and Amba-Rao 1983) are computationally expensive, especially when the number of point supports is large, as they require finer meshing around the point supports in order to properly capture the response of the plate. Also, the proposed approach, as discussed previously, has no limit in terms of the boundary condition, the configuration of the plate, and the loading condition. Moreover, any time variation in the support stiffness, even a discontinuous variation, may be readily treated by using the proposed method.A possible weakness of the proposed solution, however, may be referred to the selection of the appropriate number of modes, N, which cannot be determined beforehand. To proceed with the solution, a small number of modes is selected in the first step, and then the convergence is checked by increasing the number of considered modes in the subsequent steps. The iteration terminates once the convergence takes place. Despite this, since the convergence of the solution is fast, this may not be considered as a serious weakness.

3. Special case of time-invariant stiffness

In view of Eq. (15), when the spring stiffness is not time-varying, the frequency equation corresponding to the free vibration of the plate, i.e., when $\{P(t)\} = \{0\}$, would be the characteristic equation of matrix $\mathbf{K}$, that is

$$\det\left(\mathbf{K} - \Omega^2 \mathbf{I}\right) = 0 \tag{31}$$

where $\mathbf{K}$ is obtained from Eq. (30) with variable t dropped, $\mathbf{I}$ is the identity matrix of order N, and Ω^2 indicates the square of the natural frequency of vibration of the plate, which is basically the eigenvalue of matrix $\mathbf{K}$. For the special case of a single point support, $\mathbf{K}$ may be substituted from Eq. (16). In regard to the symmetry of matrix $\mathbf{K}$, Eq. (31) has N real roots for Ω^2. These N roots determine the N natural frequencies of vibration of the system.

Moreover, the mode shapes could be readily determined from the eigenvectors of matrix $\mathbf{K}$, as follows. Suppose $\{q^{(i)}\}$ to be the eigenvector of matrix $\mathbf{K}$ corresponding to eigenvalue $\Omega_i{}^2$, so that $\{F\} = \{q^{(i)}\}\sin\Omega_i t$ is a solution for Eq. (15). Therefore, in referring to Eq. (2), the i-th mode shape of the elastically supported plate, $\Psi_i(x, y)$, would be

$$\Psi_i(x, y) = \sum_{n=1}^{N} \Phi_n(x, y)\, q_n^{(i)} \tag{32}$$

where $q_n^{(i)}$'s are the components of vector $\{q^{(i)}\}$. Eq. (32) could be rewritten in the form of an inner product of two vectors

$$\Psi_i(x,y)=\{\Phi(x,y)\}\cdot\{q^{(i)}\} \tag{33}$$

where

$$\{\Phi(x,y)\}^T=\{\Phi_1(x,y),\Phi_2(x,y),\ldots,\Phi_N(x,y)\} \tag{34}$$

As noted earlier, when the spring stiffness is not time-varying, the contribution of each mode can be accounted for independently. Therefore, the solution of Eq. (15) may be given by a linear combination of $\{q^{(i)}\}$'s, that is

$$\{F(t)\}=\sum_{i=1}^{N}T_i(t)\{q^{(i)}\} \tag{35}$$

where, by virtue of the orthogonality property of $\{q^{(i)}\}$'s, $T_i(t)$'s are determined from the following equation

$$\ddot{T}_i(t)+\Omega_i^2\,T_i(t)=\frac{f_i(t)}{\left\|\{q^{(i)}\}\right\|^2} \tag{36}$$

where

$$f_i(t)=\{q^{(i)}\}\cdot\{P(t)\},\ \|\{q^{(i)}\}\|^2=\{q^{(i)}\}\cdot\{q^{(i)}\} \tag{37}$$

If $\{q^{(i)}\}$'s are normalized so that $\|\{q^{(i)}\}\|^2=1$, the right-hand side of Eq. (36) will simply be reduced to $f_i(t)$. On substituting Eq. (35) into Eq. (2), and applying Eq. (32), the deflection function for the case of time-invariant spring stiffness would become

$$w(x,y,t)=\sum_{n=1}^{N}\Psi_n(x,y)T_n(t) \tag{38}$$

where $T_n(t)$'s are determined from Eq. (36).

It is worth mentioning that the proposed method can be applied to plates of various configurations. Despite this, since most of the problems in practice are concerned with rectangular plates, the examples and results presented in the current paper are limited to the rectangular configuration.

4. Illustrative example

Consider a simply supported square plate resting on a point spring at its center, as shown in Fig. 2. The spring may be considered to act as a control device. Also, the plate is subjected to a moving concentrated force of magnitude $P_0=10\text{ kN}$, travelling at a uniform velocity of $v=10\text{ m/s}$ along line $y=0.25a$ from $x=0$ to $x=a$. Other relevant data are as follows: $a=50\text{ m}$, $\rho=2.8e+3\text{ kg/m}^3$, $h=1.1\text{ m}$, $D=2.625e+9\text{ N.m}$, and $(x_0/a, y_0/a)=(0.5,0.5)$.

The loading function of the problem can be expressed as

$$P(x,y,t)=P_0\delta(x-vt)\delta(y-0.25a) \tag{39}$$

where δ indicates the Dirac delta function. The plate is initially at rest, i.e., $w(x,y,0)=0$ and $\partial w(x,y,0)/\partial t=0$. Moreover, the stiffness of the centrally located spring is time-varying. The object is to determine the deflection of the plate, right under the moving force, as a function of the moving force position. In order to demonstrate the versatility of the method, the problem is to be solved for three different variations of device stiffness with time, namely:

a) Time-invariant: $k(t)=1.0\times10^9$ N/m

b) Linear variation: $k(t)=1.0\times10^9+4.0\times10^8t$ N/m

c) Harmonic variation: $k(t)=1.0\times10^9+4.0\times10^8\sin(0.8\pi t)$ N/m

In addition, the response of the system in the case when the spring with a linear variation is suddenly eliminated at $t=1.5$ s is to be investigated. In this case, the stiffness can be defined as follows

$$k(t)=\left(1.0\times10^9+4.0\times10^8t\right)\times[1-H(t-1.5)]\text{N/m} \tag{40}$$

where $\mathrm{H}(\cdot)$ stands for the Heaviside function.

The well-known shape functions as well as the natural frequencies of the plate under study, when no spring is attached to it, are (Leissa 1969, Soedel 1993)

$$\Phi_i(x,y)=\Phi_{mn}(x,y)=\frac{2}{a}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a} \tag{41}$$

$$\omega_i=\omega_{mn}=\frac{\pi^2}{a^2}\sqrt{\frac{D}{\rho h}}\left(m^2+n^2\right),\ i,m,n=1,2,3,\ldots \tag{42}$$

To begin with, one should determine the non-homogeneous term in Eq. (15). Using Eq. (11), one will get

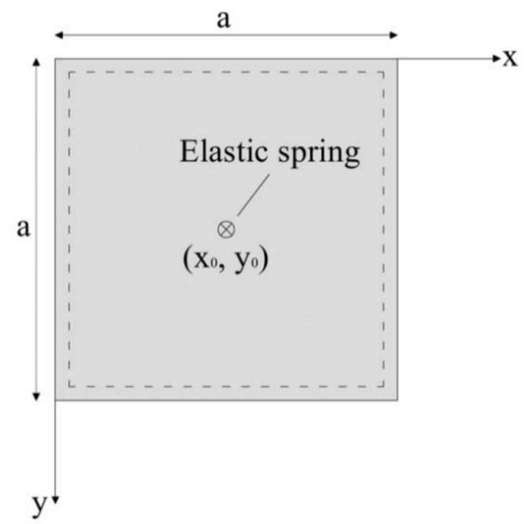
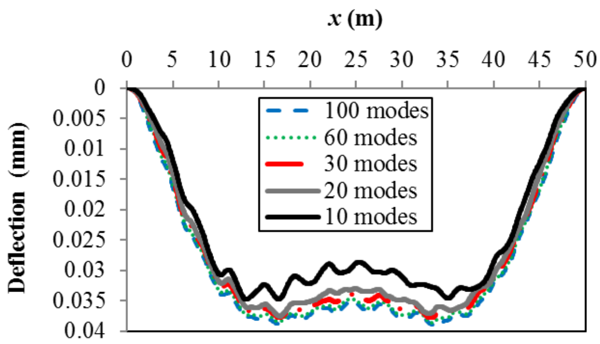


Fig. 2 A simply supported square plate supported by a linear elastic spring

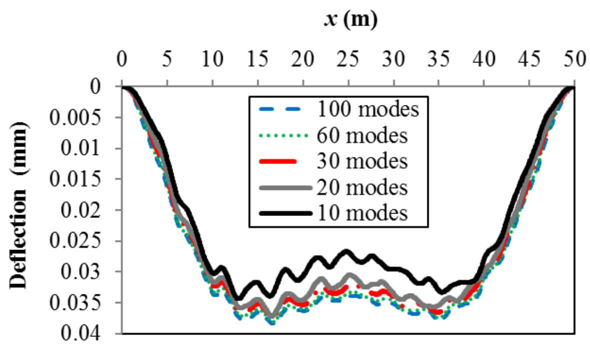
$$P_i(t) = P_{mn}(t) = \frac{2}{a}\int_0^a \int_0^a \left(\sin\frac{m\pi x}{a} \sin\frac{n\pi y}{a} \right)$$
$$\times P_0 \delta(x - vt)\delta(y - 0.25a)dxdy \qquad (43)$$
$$= \frac{2P_0}{a} \sin\frac{m\pi vt}{a} \sin\frac{n\pi}{4}$$

Making use of Eq. (43) and considering an appropriate number of modes, Eq. (15) can be solved to give $F_n(t)$'s. Any standard numerical method may be applied in order to solve the resulting system of differential equations. In this example, the fourth order Runge-Kutta method is employed to this end. Having determined $F_n(t)$'s, the deflection response could be readily attained, using Eq. (2), wherein $\Phi_n(x,y)$'s should be substituted from Eq. (41). By doing so, the deflection of the plate, right under the moving concentrated force, is obtained for the three aforementioned variations of stiffness, of which the results are plotted in Fig. 3. In this figure, the improvement in the results is also demonstrated, as the number of the considered modes increases.

It can be observed that, for all the three studied cases, it is sufficient to consider only the first 60 modes to obtain reasonable results, and the effect of higher modes becomes less pronounced. In order to make a better comparison between the solutions corresponding to the three foregoing

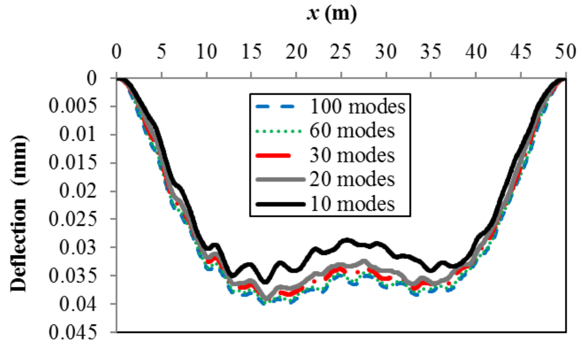


(a) time-invariant stiffness



(b) linearly varying stiffness

Fig. 3 Participation of different number of modes in the deflection of the plate under the moving force for three cases



(c)harmonically varying stiffness

Fig. 3 Continued

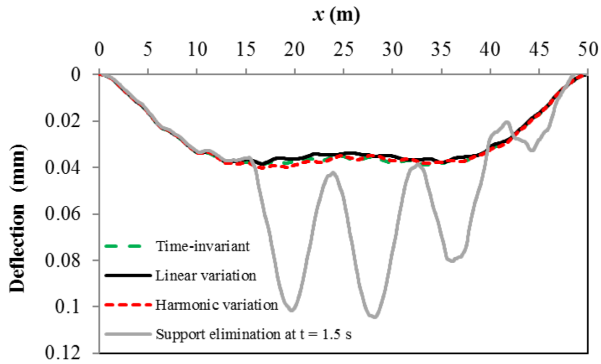


Fig. 4 Final results for the deflection of the plate under the moving force for the time-invariant, linearly varying, and harmonically varying stiffness

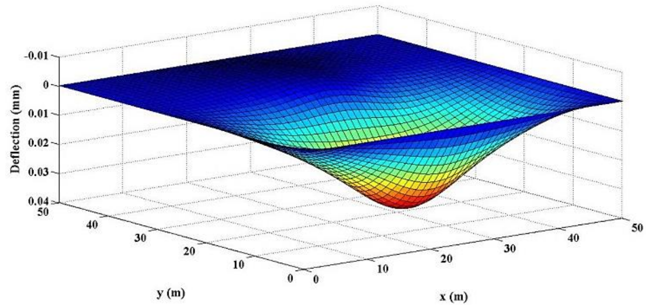


Fig. 5 Deflected geometry of the plate at the instant when the moving force is at one-half of the span. (Plotted for the case of a linearly varying stiffness)

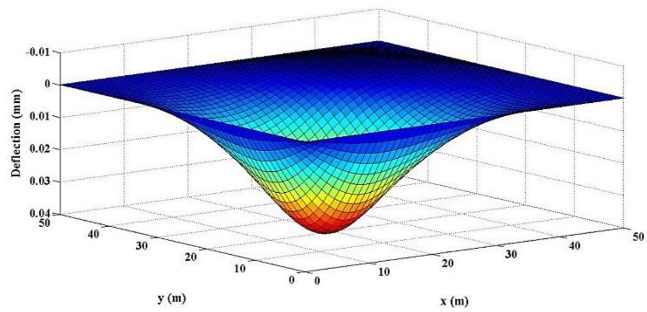


Fig. 6 Deflected geometry of the plate at the instant when the moving force is at one-fourth of the span. (Plotted for the case of a linearly varying stiffness)

stiffness variations, the final results of the three cases are shown in a single plot in Fig. 4. In this figure, the deflection under the moving force when the linearly varying spring is suddenly removed at t=1.5 s is also shown. In this case, the maximum deflection is about three times as large as that obtained in the case when the spring is not removed.

Furthermore, the deflected geometry of the plate, for the case of a linearly varying stiffness, is displayed in Figs. 5-6, at two different instants, that is, when the moving force is at one-half and at one-fourth of the span, respectively.

5. Conclusions

The present study offers a simple analytical-numerical method in order to investigate the dynamic response of plates with elastic point supports. Moreover, this approach is general, to the extent that the stiffness of the elastic supports can vary as an arbitrary function of time.

Discontinuous functions can also be incorporated in the solution. This is particularly interesting when dealing with problems with one or more supports eliminated at specific instants of time. The use of active or semi-active control devices with time-varying stiffness as point supports may be another potential application in which considering the time variation in the supports stiffness is inevitable in the model. The solution is based on orthogonal functions, and the results indicate that the governing differential equation can be transformed into a system of linear ordinary differential equations. Making use of this method, the dynamic response of the point-supported plate could be determined for an arbitrary loading function, using the mode shapes of the same plate in the absence of point supports. Furthermore, the approach can be applied for any plate geometry. An illustrative example has also been presented to demonstrate the procedure.

In the case of time-invariant stiffness, the general system reduces to a linear system of ordinary differential equations with constant coefficients, and the contribution of each mode could be studied individually. Therefore, when dealing with the free vibration problem, in this particular case, the solution requires finding the eigenvalues of a known symmetric matrix, which is simple enough to carry out on personal computers. In addition, the convergence is very fast with any desirable accuracy.

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Appendix: Verification example

Consider the simply supported square plate in Fig. 2, resting on a time-invariant spring. The object of this section is to study the variation of the natural frequencies of the plate with the stiffness and location of the spring. The relevant data are: $a = 1\,m$, $\rho = 7.85e+3\ kg/m^3$, $h = 4\,cm$, $D = 1.23e+7\ kgf.cm$, and $(x_0/a, y_0/a) = (0.5, 0.5)$.

The shape functions as well as the natural frequencies of this plate, in the absence of the spring, are obtained from Eqs. (41)-(42), respectively. Note that the shape functions in Eq. (41) are multiplied by an additional normalizing coefficient, i.e., $2/a$, so that Eq. (21) will be satisfied. This guarantees the symmetry of matrix **K**.

In considering only the first 200 modes, Eq. (31) can be solved for $N = 200$. Having solved this equation for various values of k_0, the frequencies can be obtained as a function of the spring constant. The effect of the spring constant on the vibration frequency is shown in Fig. 7. In this figure, the frequency and the spring constant are expressed in terms of non-dimensional parameters $\lambda = \sqrt{\rho h \Omega^2 a^4 / D}$ and $\kappa = k_0 a^2 / D$, respectively. Fig. 7 reveals that as the spring constant increases, the frequencies of the plate also increase. In addition, when the spring constant is larger than about 10^3, no significant change will occur in the frequency parameter. Interestingly, it can be observed that, when the spring constant approaches infinity, the frequency parameters of both the first and the second modes converge to the common value of 49.3480. This observation is in line with the previously reported results in references (Johns and Nataraja 1972, Liew and Lam 1994).

For large values of spring constant ($\kappa > 10^3$), the spring behaves like a rigid point support. Table 1 represents the frequency parameters corresponding to the first six modes of a centrally supported square plate. Moreover, the results are compared with the values available from the literature. It is discernible from Table 1 that there is an excellent agreement between the present results and those from other studies.

Furthermore, Liew and Lam (1994), by applying a different approach, as referred in section 1, studied the variation of the frequency parameter with spring stiffness and produced a similar diagram to the one shown in Fig 7. Comparing the results that are shown in Fig. 7 with their diagram, a very close agreement is observed between them. In making use of Eq. (32), the first four mode shapes of the square plate under study with a centrally located rigid point support are obtained, whose shapes are plotted in Fig. 8. Upon comparison of Fig. 8(a) and Fig. 8(b), it is observed that, introducing an appropriate transformation of the coordinate axes, the shapes of the first and second modes can be transformed to each other. This observation confirms the aforementioned convergence of the first and second frequency parameters, as the spring constant approaches infinity, to a single value, i.e., 49.3480.

The approximation of the natural frequencies could be further improved by retaining a higher number of modes. Table 2 depicts the improvement in the successive approximations of the frequencies, when the number of the considered modes, N, increases. As can be observed in this table, except for the third mode, the approximation of the frequencies does not improve for $N \geq 7$, so that the calculated values for $N = 7$ are exact. For the third mode, however, acceptable values are obtained for N's beyond 50.

As the second part of the example, the variation of the frequency parameter with the location of the interior spring is to be investigated for the aforementioned square plate. Fig. 9(a) displays this variation for the first six modes in the case when $\kappa = 100$, and the interior spring is moving along line $y/a = y_0/a = 0.6$. Moreover, Fig. 9(b) illustrates the corresponding results for the case of a rigid point support. As expected, in general, the value of the

Table 1 Values of the frequency parameter, λ, for the first six modes of a square plate with a centrally located rigid point support

	Mode number					
Reference	1	2	3	4	5	6
Present study	49.3480	49.3480	52.9329	78.9568	98.6960	128.3049
LL*	49.3480	49.3480	53.2697	78.9568	98.6983	128.3049
KD*	49.3480	49.3480	53.1700	78.9568	98.6962	128.3049
FC*	49.35	49.35	52.78	78.96	98.71	128.32
N*	49.3	49.3				
JN*	53.4	53.4				

LL=Liew and Lam (1994); KD=Kim and Dickinson (1987); FC=Fan and Cheung (1984); N=Nowacki (1953); JN=Johns and Nataraja (1972)

Table 2 Rate of convergence of the frequency parameter for a square plate with a centrally located rigid point support

	Number of retained modes (N)					
Mode No.	7	20	50	100	150	200
Mode 1	49.3480	49.3480	49.3480	49.3480	49.3480	49.3480
Mode 2	49.3480	49.3480	49.3480	49.3480	49.3480	49.3480
Mode 3	59.2040	55.7037	53.8130	53.2488	53.0191	52.9329
Mode 4	78.9568	78.9568	78.9568	78.9568	78.9568	78.9568
Mode 5	98.6960	98.6960	98.6960	98.6960	98.6960	98.6960
Mode 6	128.3049	128.3049	128.3049	128.3049	128.3049	128.3049

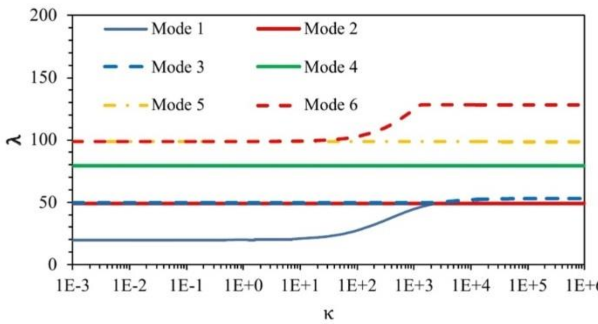
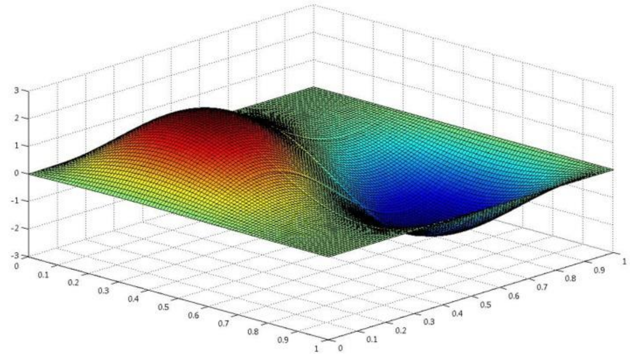
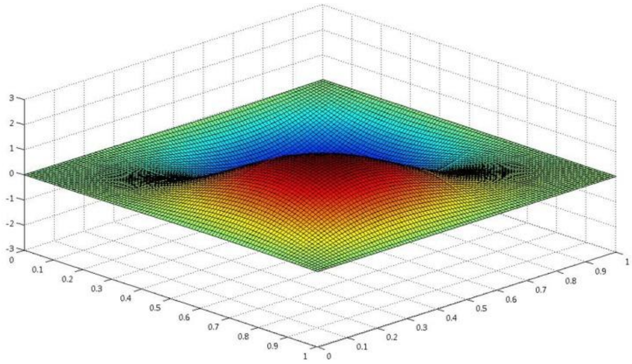


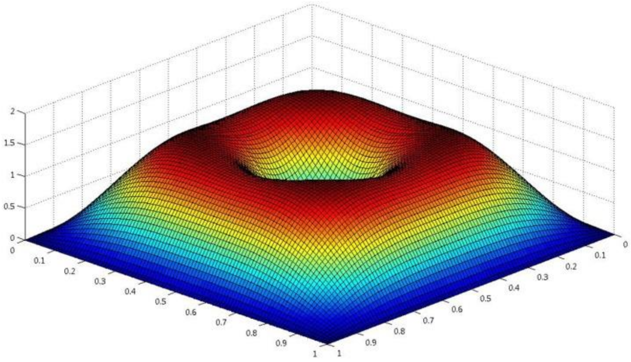
Fig. 7 Variation of the first six frequency parameters with the spring constant for a simply supported square plate with a centrally located elastic point support



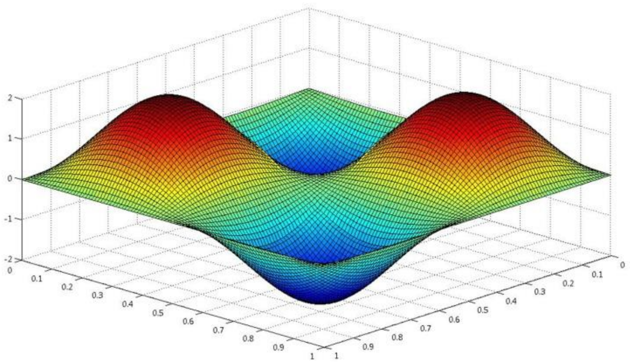
(a) 1st mode



(b) 2nd mode



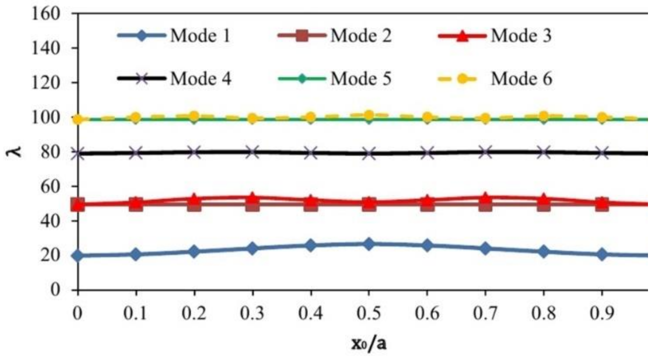
(c) 3rd mode



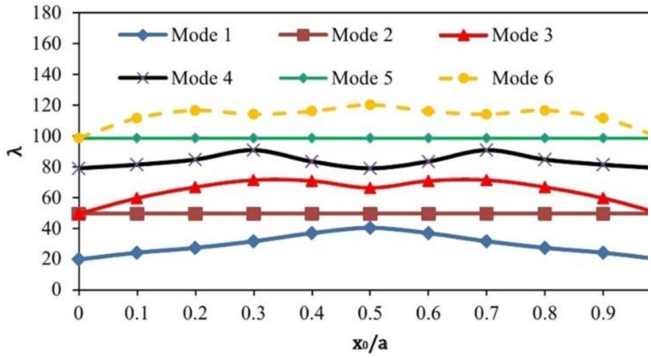
(d) 4th mode

Fig. 8 Mode shape plots for a simply supported square plate with a centrally located rigid point support

frequency parameter decreases as the spring moves towards the edges. The results so obtained for the case of a rigid point support have been compared with the results available in (Bapat and Suryanarayan 1989) and are found to be in a very good agreement.



(a) $\kappa = 100$



(b) κ approaches infinity (rigid point support)

Fig. 9 Variation of the first six frequency parameters of a square plate with the location of interior spring along line $y_0/a = 0.6$