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Reconstruction of structured models using incomplete measured data

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Abstract. The model updating problems, which are to find the optimal approximation to the discrete quadratic model obtained by the finite element method, are critically important to the vibration analysis. In this paper, the structured model updating problem is considered, where the coefficient matrices are required to be symmetric and positive semidefinite, represent the interconnectivity of elements in the physical configuration and minimize the dynamics equations, and furthermore, due to the physical feasibility, the physical parameters should be positive. To the best of our knowledge, the model updating problem involving all these constraints has not been proposed in the existed literature. In this paper, based on the semidefinite programming technique, we design a general-purpose numerical algorithm for solving the structured model updating problems with incomplete measured data and present some numerical results to demonstrate the effectiveness of our method.

Keywords: model updating problems; connectivity; positivity; positive semidefiniteness; semidefinite programming

1. Introduction

In many vibrating structures such as bridges, buildings, highways and so on, applying the finite element method to discretize the continuous structure, we can obtain the corresponding analytical finite element model

$$\left(\lambda^2 M + \lambda C + K\right) x = 0$$

where M,C,$K \in \Re^{n\times n}$ are the discrete mass matrix, damping matrix and stiffness matrix, respectively. However, the finite element model is only an approximate model of the continuous structure. Therefore, in general, natural frequencies (eigenvalues) and mode shapes (eigenvectors) of the analytical model and experimentally measured frequencies and mode shapes directly collected from the practical vibrating system do not match very well, and some actions should be taken. Model updating is a common method to improve the correlation between the analytical finite element models and measured data, and plays an important role in many applications, such as structured mechanics (Gladwell 2004), pole assignment problem (Kautsky *et al*. 1985).

In recent years, the model updating problem becomes more important and challenging, and many efficient numerical methods have been proposed, see the reviewed articles (Mottershead and Friswell 1993, Chu 1998) and the books (Gladwell 2004, Chu and Golub 2005, Friswell and Mottershead 1995, Gohberg *et al*. 1995) and the extensive references collected therein for general theory, algorithms and applications. However, a model updating problem without a structure is often trivial and meaningless (Chu and Golub 2005), and research results advanced thus far for the model updating problems can not address the structured problems very well. From a practical point of view, the matrices M,C,K are got from the practical physical system, which often has an inherent connectivity, it is therefore necessary to consider the reconstruction of the physical system under the connectivity constraint, which implies each entry of the coefficient matrices is the combination of some physical parameters, such as the mass, the damping coefficient, the stiffness coefficient in a mass-spring system, and the voltage, the resistance in an RLC electronic network and so on, and furthermore, it is reasonable to require these parameters are positive in the process of model updating. Therefore, the additional specified structure discussed in this paper involves the connectivity of the physical system and the positivity of the physical parameters. Since the measured data in applications are unavoidably polluted by unknown random and systematic errors, model updating techniques aim at fitting the given initial analytical model in such a way that the model behavior corresponds as closely as possible to the measured behavior. Generally, the structured model updating problem studied in this paper can be stated as follows

$$\begin{aligned}
\min \quad & \frac{\gamma_1}{2}\left\|M-\tilde{M}\right\|_F^2+\frac{\gamma_2}{2}\left\|C-\tilde{C}\right\|_F^2+\left\|K-\tilde{K}\right\|_F^2 \\
\text{s.t.} \quad & \left\|MX\Lambda^2+CX\Lambda+KX\right\|_F^2=0, \\
& \text{structure}\left(M,C,K\right)=\text{structure}\left(\tilde{M},\tilde{C},\tilde{K}\right), \\
& M=M^{\mathrm{T}}\succeq 0,\ C=C^{\mathrm{T}}\succeq 0,\ K=K^{\mathrm{T}}\succeq 0, \\
& \text{parameters}>0,
\end{aligned} \tag{1}$$

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where γ_1, γ_2 are weighting parameters, $\|\cdot\|$ is the Frobenius norm, $\tilde{M},\tilde{C},\tilde{K} \in \Re^{n\times n}$ are the given discrete mass matrix, damping matrix and stiffness matrix of the analytical model and $\Lambda=\mathrm{diag}(\lambda_1,\ldots,\lambda_l)\in \Re^{l\times l}$, $X\in \Re^{n\times l}$, consist of given natural frequencies and the corresponding mode shapes obtained by vibration tests, respectively. The constraint " $\|MX\Lambda^2+CX\Lambda+KX\|_F^2=0$ " means the reconstructed model should match the measured data (X, Λ). The constraint "structure (M,C,K)=structure $(\tilde{M},\tilde{C},\tilde{K})$ " means the combination patten of each entry of the matrices M,C,K is same as that of the matrices $\tilde{M},\tilde{C},\tilde{K}$. The constraint" $M=M^{\mathrm{T}}\succeq 0$, $C=C^{\mathrm{T}}\succeq 0$, $K=K^{\mathrm{T}}\succeq 0$ "means the matrices M,C,K are all symmetric and positive semidefinite. The constraint "parameters>0" means all physical parameters are positive. For convenience, we named these four constraints as "Equation constraint", "Connectivity constraint", "PSD constraint" and "Positivity constraint", respectively.

Little work has been done to the structured model updating problems due to the difficulty of handling all these constraints together. Several special versions of the optimization problem (1) have been studied. The existent methods, such as direct updating methods (Baruch 1978, Berman and Nagy 1983, Wei 1990), eigenstructure assignment techniques (Zimmerman and Widengren 1990, Datta 2002, Bai *et al*. 2010, Brahma and Datta 2009), optimization methods (Moreno *et al*. 2009, Bai *et al*. 2007, Chen 2014) mainly aim at model updating problems with the constraints "Equation constraint" and "PSD constraint" or only the constraint "Equation constraint". For the constraint "Connectivity constraint", the most simple connectivity requires the reconstructed coefficient matrices admit the same sparsity as the original coefficient matrices. In Kabe (1985), the Hadamard product is applied to solve the model updating problem with this sparsity constraint, however, this method can not ensure the positive semidefiniteness of the coefficient matrices and the positivity of the parameters. Furthermore, in Yuan (2012, 2013), the matrix linear variational inequality approach and the proximal-point method are respectively applied to solve the model updating problems with the sparsity constraint, however, these two methods can not keep the positivity of parameters. In Bai (2008), the sufficient and necessary conditions on the given eigenpairs so that the model updating problems with a special connectivity constraint and positivity constraint are solvable is presented. In Chu *et al*. (2007), the model updating problems with the constraints "Equation constraint", "Connectivity constraint" and "Positivity constraint" are considered. However, the methods in Bai (2008), Chu *et al*. (2007) do not consider the "PSD constraint" and only aim at special structures and can hardly be generalized to other systems. In Dong *et al*. (2009), a general and robust numerical method is presented to solve the structured quadratic inverse eigenvalue problems with the constraints "Connectivity constraint" and "Positivity constraint", however, the constraint "PSD constraint" can not preserved. In Lin *et al*. (2010), the semidefinite programming technique is proposed to solve the structured quadratic inverse eigenvalue problems, but this method relies on the interior point methods and probably can not handle large scale problems. The methods in Li (2002), Halevi and Bucher (2003), Sako and Kabe (2005) also consider the constraint "Connectivity constraint" but fail to guarantee the constraint "PSD constraint". The numerical methods for the parameter updating problems (Berman and Nagy 1983, Chen and Garbat 1980, Mottershead *et al*. 2011, Zhao *et al*. 2016) also can not be used to solve our problem due to the existence of the constraint "PSD constraint".

In this paper, we present a numerical method to solve the structured model updating problem with incomplete measured data, in which the solution to dynamic equations, the symmetry and positive semidefiniteness of the matrices, the inherent connectivity of the physical system and the positivity of the physical parameters are imposed as the constraints of the formulated optimization problem. Our method is based on semidefinite programming technique and can reduce the number of the variables greatly, and thus it can solve the structured model updating problems of large scale efficiently.

This paper is organized as follows. In Section 2, we apply QR decomposition technique to approximately solve a linear system, and then derive a constraint optimization problem with less variables, which is a necessary condition for the existence of a positive solution to a structured model updating problem with prescribed connectivity. The precise description of the algorithm also is presented in Section 2. The numerical tests are done in Section 3 to show the effectiveness of our algorithm.

2. Model reduction

Given a mass-spring system, applying the well-known Hook's law and the damping is negatively proportional to the velocity, the structure of the coefficient matrices M,C,K can be determined, (Dong *et al*. 2009, Johnson 2000, see). A typical way in the literature to describe the structure of a

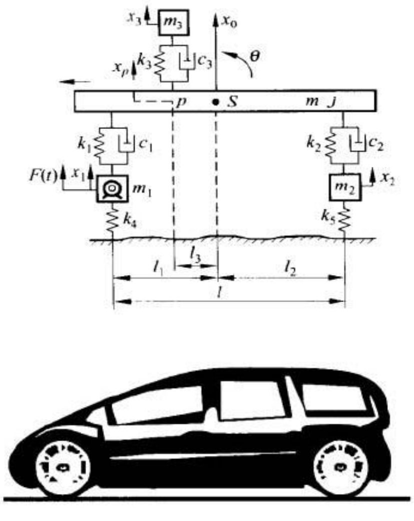

Fig. 1 An electric car model

general mass-spring system is that the mass matrix M, is diagonal, both the damping matrix C and the stiffness matrix K are symmetric. Moreover, the structures of the coefficient matrices M,C,K are related to the internal connectivity of the masses and springs. A different configuration of the connectivity leads to a different structure.

Example 1. The coefficient matrices M,C,K corresponding to the electric car model depicted in Fig. 1 should be constructed as follow

$$M=\begin{pmatrix} m &&&& \\ & m\rho &&& \\ && m_1 && \\ &&& m_2 & \\ &&&& m_3 \end{pmatrix} \tag{2}$$

and

$$C=\begin{pmatrix} c_1+c_2+c_3 & -c_1l_1+c_2l_2-c_3l_3 & -c_1 & -c_2 & -c_3 \\ -c_1l_1+c_2l_2-c_3l_3 & c_1l_1^2+c_2l_2^2+c_3l_3^2 & c_1l_1 & -c_2l_2 & c_3l_3 \\ -c_1 & c_1l_1 & c_1 & 0 & 0 \\ -c_2 & -c_2l_2 & 0 & c_2 & 0 \\ -c_3 & c_3l_3 & 0 & 0 & c_3 \end{pmatrix} \tag{3}$$

$$K=\begin{pmatrix} k_1+k_2+k_3 & -k_1l_1+k_2l_2-k_3l_3 & -k_1 & -k_2 & -k_3 \\ -k_1l_1+k_2l_2-c_3l_3 & k_1l_1^2+k_2l_2^2+k_3l_3^2 & k_1l_1 & -k_2l_2 & k_3l_3 \\ -k_1 & k_1l_1 & k_1+k_4 & 0 & 0 \\ -k_2 & -k_2l_2 & 0 & k_2+k_5 & 0 \\ -k_3 & k_3l_3 & 0 & 0 & k_3 \end{pmatrix} \tag{4}$$

where l_1,l_2,l_3,ρ are all given values. Furthermore, the physical parameters m_1,m_2,m_3,m and $c_1,c_2,c_3,k_1...k_5$ are all required to be positive.

The matrix $MX\Lambda^2+CX\Lambda+KX$ in the first constraint of the problem (1) can be transformed to a vector. Now we consider to construct the vector through rewriting the (i,j)-th entry of the matrix $MX\Lambda^2+CX\Lambda+KX$. Let $m=(m_1,\dots,m_{n_1})$, $c=(c_1,\dots,c_{n_2})$, $k=(k_1,\dots,k_{n_3})$, and M_i,C_i,K_i be the i-th row of M,C,K, and let $(X\Lambda^2)^j$, $(X\Lambda)^j$, $(X)^j$ be j-th column of $X\Lambda,X\Lambda,X$, $M_i=(M_{st})_{n1\times n}$ $C_i=(C_{st})_{n2\times n}$,, $K_i=(K_{st})_{n3\times n}$, where M_{st}, C_{st}, K_{st} are respectively the coefficients of variables m_s,c_s,k_s $(1\le s\le n)$ in the t-th element of the vector M_i,C_i,K_i.

Example 2. For the matrices M,C,K in Example 1, let $m=(m_1,m_2,m_3,m)$, $c=(c_1,c_2,c_3)$, $k=(k_1,\dots,k_5)$, we have

$$M_3=(0,0,m_1,0,0),$$
$$C_3=(-c_1,c_1l_1,c_1,0,0),$$
$$K_3=(-k_1,k_1l_1,k_1+k_4,0,0),$$

and

$$M_3=\begin{pmatrix} 0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix},$$

$$C_3=\begin{pmatrix} -1&l_1&1&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{pmatrix},$$

$$K_3=\begin{pmatrix} -1&l_1&1&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&1&0&0\\0&0&0&0&0 \end{pmatrix}.$$

We must stress that a different configuration of the connectivity leads to different M_i,C_i,K_i, and these matrices can be constructed exploiting the rules which are well developed in the field of structural mechanics (Johnson 2000). On**ce we** obtain M_i,C_i,K_i, for the (i,j)-th entry of the matrix $MX\Lambda^2+CX\Lambda+KX$, we have

$$\begin{aligned} & M_i(X\Lambda^2)^i+C_i(X\Lambda)^i+K_i(X)_i \\ &= mM_i(X\Lambda^2)^i+cC_i(X\Lambda)^i+kK_i(X)^i \\ &= (m,c,k)\begin{pmatrix} M_i(X\Lambda^2)^i \\ C_i(X\Lambda)^i \\ K_i(X)^i \end{pmatrix} \\ &= ((M_i(X\Lambda^2)^i)^T,(C_i(X\Lambda)^i)^T,(K_i(X)^i)^T(m,c,k)^T. \end{aligned}$$

Thus the first constraint in the optimization problem (1) can be transformed to find the solution to

$$\|Ax\|_2^2=0 \tag{5}$$

where

$$A=\begin{pmatrix} (M_1(X\Lambda^2)^1)^T & (C_1(X\Lambda)^1)^T & (K_1(X)^1)^T \\ (M_1(X\Lambda^2)^l)^T & (C_1(X\Lambda)^l)^T & (K_1(X)^l)^T \\ (M_n(X\Lambda^2)^1)^T & (C_n(X\Lambda)^1)^T & (K_n(X)^1)^T \\ (M_n(X\Lambda^2)^l)^T & (C_n(X\Lambda)^l)^T & (K_n(X)^l)^T \end{pmatrix}, \tag{6}$$

$$x=\begin{pmatrix} x_1 \\ \vdots \\ x_{n_1+n_2+n_3} \end{pmatrix}=\begin{pmatrix} m^T \\ c^T \\ k^T \end{pmatrix}$$

The objective function in the optimization problem (1) can be written as

$$f(x)\triangleq\frac{\gamma_1}{2}\sum_{i,j=1}^{n}(M_{ij}-\tilde{M}_{ij})^2+\frac{2}{2}\sum_{i,j=1}^{n}(C_{ij}-\tilde{C}_{ij})^2+\sum_{i,j=1}^{n}(K_{ij}-\tilde{K}_{ij})^2,$$

where M_{ij}, $\tilde{M}_{ij}$ (C_{ij}, $\tilde{C}_{ij}$, K_{ij}, $\tilde{K}_{ij}$) are respectively the (i,j)-th elements of the matrices M, $\tilde{M}$ (C, $\tilde{C}$, K, $\tilde{K}$). Since M_{ij}, C_{ij} and K_{ij} are the linear combinations of the parameters m,c,k the function $f(x)$ is a quadratic polynomial.

Let A_i, A_{n1+j}, $A_{n1+N2+k}$ be the coefficient matrices of the variables x_i, x_{n1+j}, $x_{n1+N2+k}$ in matrices M,C,K respectively, the matrices M,C,K can be rewritten as

$$M = x_1 A_1 + \ldots + x_{n_1} A_{n_1},$$
$$C = x_{n_1+1} A_{n_1+1} + \ldots + x_{n_1+n_2} A_{n_1+n_2},$$
$$K = x_{n_1+n_2+1} A_{n_1+n_2+1} + \ldots + x_{n_1+n_2+n_3} A_{n_1+n_2+n_3}.$$

Thus, the structured model updating problem is transformed to find the optimal solution of the following problem

$$\begin{aligned}
\min\ & f(x) \\
\text{s.t.}\ & \|Ax\|_2^2 = 0, \\
& x_1 A_1 + \ldots + x_{n_1} A_{n_1} = \left(x_1 A_1 + \ldots + x_{n_1} A_{n_1}\right)^{\mathrm{T}} \succeq 0, \\
& x_{n_1+1} A_{n_1+1} + \ldots + x_{n_1+n_2} A_{n_1+n_2} = \left(x_{n_1+1} A_{n_1+1} + \ldots + x_{n_1+n_2} A_{n_1+n_2}\right)^{\mathrm{T}} \succeq 0, \\
& x_{n_1+n_2+1} A_{n_1+n_2+1} + \ldots + x_{n_1+n_2+n_3} A_{n_1+n_2+n_3} \\
& = \left(x_{n_1+n_2+1} A_{n_1+n_2+1} + \ldots + x_{n_1+n_2+n_3} A_{n_1+n_2+n_3}\right)^{\mathrm{T}} \succeq 0, \\
& x_i > 0, i = 1, \ldots, n_1 + n_2 + n_3.
\end{aligned} \tag{7}$$

For the constraint "Equation constraint" in (7), since the measured data are inexact, the matrix A in (6) may be full rank and the solution to $\|Ax\|_2^2 = 0$ is zero. Furthermore, the solution to the optimization problem in (7) also is zero. However, in applications, it is more desirable to find an approximate positive solution to $\|Ax\|_2^2 = 0$. Performing the orthogonal-triangular decomposition of the matrix A, we get

$$A = QUP, \tag{8}$$

where Q is an orthogonal matrix, U is an upper triangular matrix, and the absolute values of its diagonal elements are decreasing, P is a permutation matrix. The function Ax is homogeneous, then the solution x to $\|Ax\|_2^2 = 0$ is satisfies $U\bar{x} = 0$, where $\bar{x} = Px$, and $\bar{x}_1, \ldots, \bar{x}_{n1+n2+n3}$ is a permutation of $x_1, \ldots, x_{n1+n2+n3}$. When the matrix U is nonsingular, (5) does not have a positive solution, therefore, we need to truncate the matrix U. Let U_e be the truncated matrix of rank r, the solution to $U_e\bar{x}=0$ is an approximate solution of $U\bar{x}=0$, and hence an approximate solution to $\|Ax\|_2^2 = 0$. Furthermore, we have

$$U_e \bar{x} = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1r} & \cdots & u_{1,n_1+n_2+n_3} \\ & u_{22} & \cdots & u_{2r} & \cdots & u_{2,n_1+n_2+n_3} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & u_{rr} & \cdots & u_{r,n_1+n_2+n_3} \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_{n_1+n_2+n_3} \end{pmatrix} = 0$$

and then

$$\begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_r \end{pmatrix} \underline{\underline{\Delta}} - ST \begin{pmatrix} \bar{x}_{r+1} \\ \vdots \\ \bar{x}_{n1+n2+n3} \end{pmatrix}, \tag{9}$$

where

$$S = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1r} \\ & u_{22} & \cdots & u_{2r} \\ & & \ddots & \vdots \\ & & & u_{rr} \end{pmatrix}^{-1} \quad T = \begin{pmatrix} u_{1,r+1} & u_{1,r+2} & \cdots & u_{1,n_1+n_2+n_3} \\ u_{2,r+1} & u_{2,r+2} & \cdots & u_{2,n_1+n_2+n_3} \\ & \cdots & \cdots & \\ u_{r,r+1} & u_{r,r+2} & \cdots & u_{r,n_1+n_2+n_3} \end{pmatrix}.$$

The solution $\bar{x}$ is obtained from the truncated linear system $U_e\bar{x}=0$ and just an approximate solution to the linear system $Ax=0$. Theorem 4.1 in Dong *et al*. (2009) states how the truncation of the matrix U affect the accuracy of the solution.

For the constraint "PSD constraint" in (7), applying the variable substitution (9), we have

$$x = P\bar{x} = P \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_{n1+n2+n3} \end{pmatrix} \underline{\underline{\Delta}} \begin{pmatrix} g_1(\bar{x}_{r+1}, \ldots, \bar{x}_{n1+n2+n3}) \\ \vdots \\ g_{n1+n2+n3}(\bar{x}_{r+1}, 333, \bar{x}_{n1+n2+n3}) \end{pmatrix}, \tag{10}$$

where $g_i(\bar{x}_{r+1}, \ldots, \bar{x}_{n1+n2+n3})$ $(1 \le i \le n_1+n_2+n_3)$ are all linear functions in the variables $\bar{x}_{r+1}, \ldots, \bar{x}_{n1+n2+n3}$, and all three constraints have the following forms

$$\begin{aligned}
& g_1 A_1 + \ldots + g_{n1} A_{n1} = (g_1 A_1 + \ldots + g_{n1} A_{n1})^T \succeq 0, \\
& g_{n1+1} A_{n1+1} + \ldots + g_{n1+n2} A_{n1+n2} = \\
& (g_{n1+1} A_{n1+1} + \ldots + g_{n1+n2} A_{n1+n2})^T \succeq 0, \\
& g_{n1+n2+1} A_{n1+n2+1} + \ldots + g_{n1+n2+n3} A_{n1+n2+n3} \\
& = (g_{n1+n2+.1} A_{n1+n2+1} + \ldots + g_{n1+n2+n3} A_{n1+n2+n3})^T \succeq 0.
\end{aligned} \tag{11}$$

For the constraint "Positivity constraint" in (7), applying the variable substitution (9), we have

$$ST \begin{pmatrix} \bar{x}_{r+1} \\ \vdots \\ \bar{x}_{n1+n2+n3} \end{pmatrix} < 0, \quad \bar{x}_{r+i} > 0, i = 1, \ldots, n_1 + n_2 + n_3 - r. \tag{12}$$

For the objective function in (7), since $\bar{x}$ only is a permutation of x, we have

$$\begin{aligned} f(x) = g(\bar{x}) \underline{\underline{\Delta}} f(P\bar{x}) = f(g_1(\bar{x}_{r+1}, \ldots, \bar{x}_{n1+n2+n3}), \ldots, \\ g_{n1+n2+n3}(\bar{x}_{r+1}, \ldots, \bar{x}_{n1+n2+n3})). \end{aligned} \tag{13}$$

Thus, combining (11), (12) and (13), the structured model updating problem can be transformed to the following problem with $n_1+n_2+n_3-r$ variables

$$\begin{aligned}
\min\ & g(\bar{x}) \\
s.t.\ & g_1 A_1 + \ldots g_{n1} A_{n1} = (g_1 A_1 + \ldots + g_{n1} A_{n1})^T \succeq 0, \\
& g_{n1+1} A_{n1+1} + \ldots + g_{n1+n2} A_{n1+n2} = (g_{n1+1} A_{n1+1} + \ldots \\
& + g_{n1+n2} A_{n1+n2})^T \succeq 0, \\
& g_{n1+n2+1} A_{n1+n2+1} + \ldots + g_{n1+n2+n3} A_{n1+n2+n3})^T \succeq 0. \\
& ST \begin{pmatrix} \bar{x}_{r+1} \\ \vdots \\ \bar{x}_{n1+n2+n3} \end{pmatrix} < 0, \quad \bar{x}_{r+1} > 0, i = 1, \ldots, n_1 + n_2 + n_3 - r.
\end{aligned} \tag{14}$$

where $g_1, \ldots, g_n+n_2+n_3$ are defined as (10). And the physical parameters

$$\begin{aligned} (m, c, k)^T = P(g_1(\bar{x}^*_{r+1}, \ldots, \bar{x}^*_{n1+n2+n3}), \ldots, \\ g_{n1+n2+n3}(\bar{x}^*_{r+1}, \ldots, \bar{x}^*_{n1+n2+n3}))^T, \end{aligned}$$

where $(\bar{x}^*_{r+1}, \ldots, \bar{x}^*_{n1+n2+n3})$ is the optimal solution of the optimization problem (14).

In our algorithm, to find a positive solution, we often have to truncate the matrix U in (8), however, in many cases, we do not know how to truncate the matrix U. If we truncate the matrix too much, then we might miss a more

Table 1 True eigenpairs

	1	2	3	4
λ_j	-8.3531655419	-6.4981553681	-0.3043306852	-0.1026321280
x_j	0.1197150942	0.1538898261	0.18693339163	-1
	-0.1197150942	0.1538898261	-0.1869333916	-1
	-0.0387088666	0	0.1367498450	0
	0	-0.0774192048	0	0.0038356771
	-0.0122873455	0	-1	0

	5,6	7,8	9,10
λ_j	$-0.0220591068\pm1.0618348626i$	$-0.2870361602\pm0.8563423876i$	$-0.2185228687\pm0.2333626416i$
x_j	$-0.1331430470\mp0.4145412462i$	$-0.3208892811\pm0.0219877872i$	$-0.0515597052\mp0.1632009434i$
	$-0.1331430470\mp0.4145412462i$	$0.3208892811\mp0.0219877872i$	$0.0515597052\pm0.1632009434i$
	0	$-0.7132866341\mp0.2867133659i$	$-0.0028037930\mp0.1605881350i$
	$-0.0946888694\mp0.8630392663i$	0	0
	0	$0.0287378149\pm0.3187586742i$	$0.8904814639\pm0.1095185361i$

Table 2 Approximation error

s	$\|m-m_e\|_F$	$\|c-c_e\|_F$	$\|k-k_e\|_F$	Residual	PSD	Positive
1	2.36×10^{-10}	2.45×10^{-10}	3.73×10^{-10}	2.56×10^{-16}	Yes	Yes
2	4.14×10^{-10}	3.34×10^{-10}	2.51×10^{-9}	3.42×10^{-15}	Yes	Yes
3	6.98×10^{-10}	4.86×10^{-10}	3.53×10^{-9}	1.78×10^{-10}	Yes	Yes
4	1.09×10^{-9}	2.05×10^{-9}	9.76×10^{-10}	2.34×10^{-10}	Yes	Yes
6	3.10×10^{-9}	3.57×10^{-9}	6.12×10^{-9}	4.53×10^{-10}	Yes	Yes
8	5.35×10^{-9}	5.67×10^{-9}	7.65×10^{-9}	6.25×10^{-10}	Yes	Yes
10	7.21×10^{-9}	8.01×10^{-9}	9.34×10^{-9}	8.13×10^{-10}	Yes	Yes

accurate solution, and if the truncation is too little, we can not get an approximate solution. Since the absolute values of the diagonal elements of U are decreasing, it is reasonable to dynamically truncate the matrix U from the bottom to the top, that is, we solve (14) from r=rand(A) to r=1, where rand(A) is the rank of the matrix A, and

- if for some r, the optimization problem (14) is solvable, then we claim that we get an approximate solution of the structured model updating problem.
- if for all r, the optimization problem (14) has no solution, then we state the structured model updating problem has no solution.

Our proposed method in this paper is carried out in three steps: formulate the optimization problem in (1); construct the matrix A and approximately solve the problem $\|Ax\|_2^2=0$ through the QR decomposition technique; reduce the optimization problem in (1) and solve the reduced optimization problem by the semidefinite programming technique. The following is the algorithm for reconstructing a structured model according to the measured data and the connectivity of the physical system.

Algorithm 3.1: Model Reconstruction:

Input the connectivity of the physical system and measured data (X, Λ)

- Step 1. Construct the structure of the coefficient matrices M,C,K based on the connectivity.
- Step 2. Calculate $MX\Lambda^2+CX\Lambda+KX$ and vectorize it to the vector Ax.
- Step 3. Perform QR decomposition $A=QUP$.
- from $r=\text{rank}(A)$ to $r=1$
  - Obtain the truncated matrix U_e and solve the linear system U_ePx=0;
  - Reduce the optimization problem in (1);
  - Solve the reduced optimization problem (14) by the free software Yalmip;
  - If for some r, the optimization problem (14) has a solution, then stop.
- Step5. Output the solution or state the reconstruction is impossible.

There is a limitation in the stated procedures in that the structure of the model is known and, thus, in order to allow for the correction of a larger model, it is necessary to obtain an estimate of the model structure.

3. Illustrative numerical examples

In this section, we give some examples to state our method has the high specificity in determining a structured model updating problem is not solvable and high sensitivity in predicting a problem is solvable. All the computations are carried out on a Dell PC with an Intel Core(TM) CPU of 2.40GHz, 8GB of memory. The package is written in MATLAB. The free MATLAB-based toolbox Yalmip can serve as a simple but powerful tool to handle our optimization problem.

Table 3 Approximation error

s	$\|m-m_e\|_F$	$\|c-c_e\|_F$	$\|k-k_e\|_F$	Residual	PSD	Positive
1	5.21×10^{-5}	6.23×10^{-5}	5.32×10^{-6}	8.34×10^{-16}	Yes	Yes
2	1.36×10^{-4}	2.43×10^{-4}	7.89×10^{-5}	5.76×10^{-16}	Yes	Yes
3	1.02×10^{-4}	0.14×10^{-3}	5.01×10^{-4}	3.28×10^{-5}	Yes	Yes
4	1.78×10^{-4}	3.12×10^{-4}	2.12×10^{-4}	4.58×10^{-5}	Yes	Yes
6	6.21×10^{-4}	5.21×10^{-4}	1.67×10^{-5}	3.23×10^{-5}	Yes	Yes
8	2.12×10^{-4}	6.23×10^{-5}	1.97×10^{-4}	2.78×10^{-5}	Yes	Yes
10	2.34×10^{-4}	7.54×10^{-5}	2.45×10^{-5}	3.46×10^{-5}	Yes	Yes

Example 3. For the physical system in Example 1, we set $l_1=l_2=l_3=\rho=1$ and randomly generate the vectors m,c,k

$$m=[0.58526409,\ 0.54972360,\ 0.91719366,\ 0.28583908],$$
$$c=[0.75720022,\ 0.75372909,\ 0.38044584],$$
$$k=[0.56782164,\ 0.07585428,\ 0.05395011,\ 0.53079755,\ 0.77916723].$$

then the matrices M,C,K in (2) and (3) are generated.
The resulting pencil has 6 complex conjugate eigenpairs and 4 real eigenpairs, see Table 1.

3.1 Test 1

We first reconstruct the masses, damping coefficients and stiffness coefficients from the exact eigenpairs $\{(\lambda_i,x_j)\}_{j=1}^{s}$ for s=1,2,3,4,6,8,10 through solving the optimization problem in (1). Since $\{(\lambda_i,x_j)\}_{j=1}^{s}$ are exact eigenpairs, there must exist a positive solution to the optimization problem in (1). Tables 2 presents the residuals of the reconstructed parameters m_e,c_e,k_e and $\max\limits_{1\le j\le s}\|(\lambda_j^2 M_e+\lambda_j C_e+K_e)x_i\|_P$.

The residual of parameters are all smaller than 10^{-8}, which states the reconstructed parameters agree with the true parameters at the ninth decimal. The residual $\max\limits_{1\le j\le s}\|(\lambda_j^2 M_e+\lambda_j C_e+K_e)x_i\|_P$, denoted by "Residual", is smaller than 10^{-9}, which states reconstructed coefficient matrices M,C,K have a good coincidence with the given eigennpairs. The "Yes" in columns "PSD" and "Positive" in Table 2 show our reconstructed coefficient matrices are positive semidefinite and the reconstructed parameters are all positive. Furthermore, these results hold independent of the number of the given eigenpairs. Therefore, the results in Table 2 state our method is reliable and independent of the number of the given eigenpairs.

3.2 Test 2

In this subsection, we consider the problem with the noisy eigenpair. We obtain the perturbed eigenpairs through truncating the exact eigenpairs at the fifth decimal. Taking the perturbed data as the measured frequencies and mode shapes, we apply our method to reconstruct the parameters and obtain a solution, see Table 3.

For a given data nearby the true eigenpairs in Table 1, it is still not clear that there exists a quadratic model with positive semidefinite coefficient matrices and prescribed connectivity structure can match the data, our method presents a numerical justification to the problem. When s=1,2, the matrix A in (6) is of size 5×12 and 10×12, therefore, the matrixin (8) is singular and no truncation happens in the process U of finding the solution to the problem $\|Ax\|_2^2=0$, and thus the residual $\max\limits_{1\le j\le s}\|(\lambda_j^2 M_e+\lambda_j C_e+K_e)x_i\|_P$. is approximately zero. However, when $s\ge3$, the matrix A is of size $5s\times12$, and its number of rows is larger than that of columns, and since the measured data are noisy, the matrix U in (8) is nonsingular and the truncation has to be done to get a nonzero solution, and thus the residual $\max\limits_{1\le j\le s}\|(\lambda_j^2 M_e+\lambda_j C_e+K_e)x_i\|_P$. is not zero, the maximal value of the residual $\|(M_e\tilde{\lambda}_j^2+C_e\tilde{\lambda}_j+K_e)\tilde{x}_i\|$ is of the order 10^{-4}, which is almost same as the order of the truncation.

3.3 Test 3

The above two tests show that if a solution or a nearby solution ever exists, our method will find it. However, in practical applications, we do not know whether the measured data are feasible in advance. In this test, we will prove our software can serve as a numerical tool to indicate a structured model updating problem is not solvable.

We randomly generate one real number and two pairs of complex conjugate numbers

$$\Lambda=(0.1839,\ 0.6557\pm0.7060i,\ 0.0357\pm0.0318i)$$

as the given frequencies and one real vector and two pairs of complex conjugate vectors

$$X=\begin{pmatrix} 0.9340 & 0.3371\pm0.7482i & 0.1656\pm0.1524i \\ 0.1299 & 0.1622\pm0.4505i & 0.6020\pm0.8258i \\ 0.5688 & 0.7943\pm0.0838i & 0.2630\pm0.5383i \\ 0.4694 & 0.3112\pm0.2290i & 0.6541\pm0.9961i \\ 0.0119 & 0.5285\pm0.9133i & 0.6892\pm0.0782i \end{pmatrix}.$$

as the given mode shapes, whose real part and imaginary part are all uniformly distributed random numbers on the interval [0,1]. Applying our method, we find all parameter values are of order 10^{-16}, it implies all reconstructed parameters are zero and there does not exist such a structured coefficient matrices M,C,K.

In fact, the coefficient matrices M,C,K have a special structure, and hence the set consisting of the eigenvalues and the eigenvectors of the quadratic pencil $\lambda^2M+\lambda C+K$is a zero-measure subset of $C\times C^n$. Therefore, it is reasonable

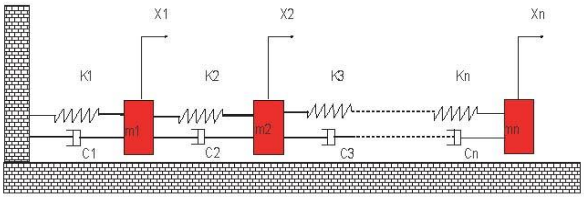

Fig. 2 A damped mass-spring system

Table 4 Results of problems with different size

n	# data	$\|\|m-m_e\|\|_F$	$\|\|c-c_e\|\|_F$	$\|\|k-k_e\|\|_F$	Time	Residual
10	6	7.26×10^{-12}	1.25×10^{-11}	1.79×10^{-11}	1.12(s)	1.39×10^{-12}
50	30	3.39×10^{-12}	7.38×10^{-12}	7.98×10^{-12}	3.01(s)	1.73×10^{-12}
100	60	$.27\times10^{-12}$	4.17×10^{-12}	2.85×10^{-12}	12.87(s)	2.86×10^{-12}
150	90	9.85×10^{-12}	2.26×10^{-11}	2.34×10^{-11}	60.83(s)	6.82×10^{-10}
200	120	3.56×10^{-12}	9.31×10^{-10}	9.77×10^{-10}	208.45(s)	2.13×10^{-112}

that we can not find (m,c,k) for randomly generated (Λ,X).

Example 4. The coefficient matrices M,C,K corresponding to the mass-spring system depicted in Fig. 2 should be constructed as follows:

$$M=\begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{pmatrix}, C=\begin{pmatrix} c_1+c_2 & -c_2 & & & \\ -c_2 & c_2+c_3 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & c_{n-1}+c_n & -c_n \\ & & & -c_n & c_n \end{pmatrix},$$

$$K=\begin{pmatrix} k_1+k_2 & -k_2 & & & \\ -k_2 & k_2+k_3 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & k_{n-1}+k_n & -k_n \\ & & & -k_n & k_n \end{pmatrix}$$

In this test, we consider to apply our algorithm to solve the structured model updating problems of different sizes n. We take the first 30% of eigenpairs as the measured data and reconstruct the parameters by our algorithm.

Table 4 shows our algorithm can handle the structured model updating problems of large size due to the variable reduction in the process of solving the structured model updating problems.

4. Conclusions

In many applications, the structured model updating problems are becoming more important and challenging due to the introducing of the special constraints: the inner connectivity of the system and the positivity of the physical parameters.

The main contribution of this paper is presenting an efficient and robust numerical method to solve the structured model updating problem, in which the solution to dynamic equations, the symmetry and positive semidefiniteness of the coefficient matrices, the interconnectivity of the physical system and the positivity of the physical parameters are all imposed as the constraints. And our method has no limitation on the number of measured data and can handle any physical system if the structure of the corresponding coefficient matrices can be presented. Through the returned residual estimate given by our method, we can confidently determine whether a structured model updating problem is solvable, and if yes, we present the reconstructed parameters.

We must stress that, in this paper, we mainly focus on reconstructing the physical parameters based on the known model structure. However, obtaining an estimate of the model structure is the limitation in our proposed algorithm and for very large problem it could be time consuming. We hope we could do some further work on the estimation of the model structure in the near future.

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