Structural Engineering and Mechanics, *Vol. 62, No. 2 (2017) 171-178*
DOI: https://doi.org/10.12989/sem.2017.62.2.171

Vibration analysis of a beam on a nonlinear elastic foundation

M.M. Fatih Karahan* and Mehmet Pakdemirli

Department of Mechanical Engineering, Manisa Celal Bayar University, Muradiye 45140, Manisa, Turkey

(Received April 28, 2016, Revised December 21, 2016, Accepted January 2, 2017)

Abstract. Nonlinear vibrations of an Euler-Bernoulli beam resting on a nonlinear elastic foundation are discussed. In search of approximate analytical solutions, the classical multiple scales (MS) and the multiple scales Lindstedt Poincare (MSLP) methods are used. The case of primary resonance is investigated. Amplitude and phase modulation equations are obtained. Steady state solutions are considered. Frequency response curves obtained by both methods are contrasted with each other with respect to the effect of various physical parameters. For weakly nonlinear systems, MS and MSLP solutions are in good agreement. For strong hardening nonlinearities, MSLP solutions exhibit the usual jump phenomena whereas MS solutions are not reliable producing backward curves which are unphysical.

Keywords: beam on elastic foundation; direct perturbation method; Multiple Scales Lindstedt Poincare (MSLP) method; forced vibrations; strongly nonlinear systems

1. Introduction

Beams on elastic foundation have wide applications in many engineering problems. They are used extensively in structures such as aircrafts, boosters, missiles, vibrating machines, pipelines, buildings, bridges, railroad tracks etc. Several foundation models already exist in the literature. Nonlinear vibration of beams were studied by many researchers (Boyaci and Pakdemirli 1997, Oz *et al.* 1998, Pakdemirli 2001, Pakdemirli and Ozkaya 2003, Oz and Pakdemirli 2006, Ozhan and Pakdemirli 2009, Ozhan and Pakdemirli 2010, Ozhan and Pakdemirli 2012, Ghayesh and Paidoussis 2010, Ghayesh *et al.* 2011a, Ghayesh *et al.* 2011b, Ghayesh *et al.* 2012a, Ghayesh *et al.* 2012b, Ghayesh 2012a, Ghayesh 2012b, Boyaci 2006, Maccari 1999, Ozkaya and Tekin 2007, Ozkaya *et al.* 2008, Ding *et al.* 2012, Hosseini and Hosseini 2015).

Generally speaking, exact analytical solutions of nonlinear equations cannot be obtained for most of the cases. In the absence of exact solutions, the researchers resort to approximate analytical solution techniques which can be considered as the next best choice. Perturbation methods are one of the most common approximate techniques used in nonlinear vibrations. One of the major limitation of the perturbation methods is the need for a small parameter in the equations or artificial introduction of the small parameter to the equations. The classical perturbation methods are only efficient for solving weakly nonlinear systems because of this limitation. In recent years, many methods have been developed to determine valid solutions for strongly nonlinear systems such as the homotopy analysis method (Liao 2004), the modified Lindstedt Poincare method (Cheung 1991), the linearized perturbation method (He 2003), the Lindstedt Poincare method with frequency transformations (Hu 2004), improved and residue harmonic balance method (Wu *et al.* 2006, Leung and Guo 2011). Iterations techniques such as the modified Mickens procedure (Lim and Wu 2002), the perturbation iteration method (Pakdemirli 2015a), the variational iteration method (He 1999) were also used to solve approximately the nonlinear systems.

A relatively new method, namely the Multiple Scales Lindstedt Poincare (MSLP) method has been proven to be effective in solving nonlinear systems. For extremely high values of the perturbation parameter, the method produced compatible results with the numerical simulations for the ordinary differential systems (Pakdemirli *et al.* 2009, Pakdemirli and Karahan 2010, Pakdemirli *et al.* 2011, Pakdemirli 2015b, Pakdemirli and Sari 2015a, Pakdemirli and Sari 2015b, Karahan and Pakdemirli 2017). The direct application of the method to partial differential systems needs to be exploited in detail and one of the goals of this study is to show that the method produces better results than the classical versions.

To reach this aim, the mathematical model of a simply supported beam resting on a nonlinear elastic foundation is treated. The multiple scales (MS) and the multiple scales Lindstedt Poincare (MSLP) methods are applied directly to the governing partial differential equation to obtain approximate solutions. Primary resonance case is considered only. The amplitude and phase modulation equations and the steady state solutions are obtained. The effect of physical parameters on the nonlinear behavior are investigated via frequency response curves and the curves of MS and MSLP are contrasted with each other. It is shown that the MS and MSLP solutions are in qualitative and quantitative agreement for weakly nonlinear systems. For strong nonlinearities however, backward curves which do not have correspondence in physical systems occur in MS

*Corresponding author, Assistant Professor
E-mail: fatih.karahan@cbu.edu.tr

http://www.techno-press.org/?journal=sem&subpage=8 ISSN: 1225-4568 (Print), 1598-6217 (Online)

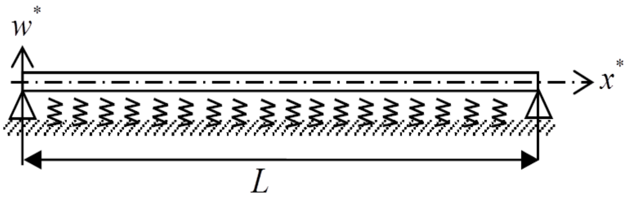

Fig. 1 A simply supported beam resting on a nonlinear elastic foundation

whereas the jump regions for MSLP do not exhibit such unphysical behavior.

2. Equation of motion

A simply supported Euler-Bernoulli beam resting on a nonlinear elastic foundation (Raju and Rao 1993 and Coskun and Engin 1999) is considered (Fig. 1).

For the beam w^* is the transverse displacement, A is the cross-section of the beam, I is the moment of inertia with respect to the neutral axis, l is the length of the beam, ρ is the density, E is Young's modulus, k_1 and k_2 are the linear and nonlinear coefficients of the elastic foundation, respectively.

The equation of motion for an Euler-Bernoulli beam with inclusion of the mid-plane stretching effect due to immovable boundaries is

$$\begin{aligned} &EI\frac{\partial^4 w^*}{\partial x^{*4}}+\rho A\frac{\partial^2 w^*}{\partial t^{*2}}+\mu^*\frac{\partial w^*}{\partial t^*}+k_1 w^*+k_2 w^{*3}\\ &=\frac{EA}{2L}\frac{\partial^2 w^*}{\partial x^{*2}}\int_0^L\left(\frac{\partial w^*}{\partial x^*}\right)^2 dx^*+F^*\cos\Omega^* t^* \end{aligned} \tag{1}$$

The model includes damping, external excitation and nonlinear elastic foundation terms. x^* and t^* are the spatial and time variable, respectively. The boundary conditions for the simply-simply supported beam are

$$\begin{aligned} &w^*\left(0,t^*\right)=0, \qquad \frac{\partial^2 w^*(0,t^*)}{\partial x^{*2}}=0,\\ &w^*\left(L,t^*\right)=0, \qquad \frac{\partial^2 w^*(L,t^*)}{\partial x^{*2}}=0 \end{aligned} \tag{2}$$

The equations are made dimensionless using the following definitions

$$w=\frac{w^*}{r}, \quad x=\frac{x^*}{L}, \quad t=\frac{t^*}{T} \tag{3}$$

where r is the radius of gyration of the beam cross-section. The final dimensionless system

$$\begin{aligned} &\frac{\partial^4 w}{\partial x^4}+\frac{\partial^2 w}{\partial t^2}+2\bar{\mu}\frac{\partial w}{\partial t}+\alpha_1 w+\alpha_2 w^3\\ &=\frac{1}{2}\frac{\partial^2 w}{\partial x^2}\int_0^1\left(\frac{\partial w}{\partial x}\right)^2 dx+\bar{F}\cos\Omega t \end{aligned} \tag{4}$$

and the non-dimensional parameters are

$$\begin{aligned} &2\bar{\mu}=\frac{\mu^* L^2}{rA\sqrt{E\rho}}, \quad \bar{F}=\frac{F^* L^4}{EIr},\\ &\Omega=\frac{\Omega^* L^2}{r}\sqrt{\frac{\rho}{E}}, \quad \alpha_1=\frac{k_1 L^4}{EI}, \quad \alpha_2=\frac{k_2 L^4}{EA} \end{aligned} \tag{5}$$

where T is a time scale chosen as

$$T=L^2\sqrt{\frac{\rho A}{EI}} \tag{6}$$

The associated non-dimensional boundary conditions read

$$\begin{aligned} &w\left(0,t\right)=0, \qquad \frac{\partial^2 w(0,t)}{\partial x^2}=0,\\ &w\left(1,t\right)=0, \qquad \frac{\partial^2 w(1,t)}{\partial x^2}=0 \end{aligned} \tag{7}$$

3. Analytical solutions

In this section, the multiple scales (MS) and the multiple scales Lindstedt Poincare (MSLP) methods are applied directly to the partial differential system to seek approximate solutions. In order to incorporate the stretching effects at order ε, a transformation of the dependent variable $u=\frac{w}{\sqrt{\varepsilon}}$ is introduced. For the case of primary resonances, the damping and the forcing terms are reordered so that they appear at the last order of approximation

$$\bar{\mu}=\varepsilon^2\mu, \quad \bar{F}=\varepsilon^2 f \tag{8}$$

where ε is the artificially introduced perturbation parameter. The partial differential system reduces to

$$\begin{aligned} &\frac{\partial^4 u}{\partial x^4}+\frac{\partial^2 u}{\partial t^2}+2\varepsilon^2\mu\frac{\partial u}{\partial t}+\alpha_1 u+\varepsilon\alpha_2 u^3\\ &=\frac{1}{2}\varepsilon\frac{\partial^2 u}{\partial x^2}\int_0^1\left(\frac{\partial u}{\partial x}\right)^2 dx+\varepsilon^2 f\cos\Omega t \end{aligned} \tag{9}$$

$$\begin{aligned} &u\left(0,t\right)=0, \qquad \frac{\partial^2 u(0,t)}{\partial x^2}=0,\\ &u\left(1,t\right)=0, \qquad \frac{\partial^2 u(1,t)}{\partial x^2}=0 \end{aligned} \tag{10}$$

3.1 The multiple scales method

Solutions are assumed to be of the form

$$\begin{aligned} &u(x,t;\varepsilon)=u_0(x,T_0,T_1,T_2)+\varepsilon u_1(x,T_0,T_1,T_2)\\ &+\varepsilon^2 u_2(x,T_0,T_1,T_2)+\ldots \end{aligned} \tag{11}$$

where $T_0=t$ is the usual fast time scale and $T_1=\varepsilon t$, $T_2=\varepsilon^2 t$ are the slow time scales. Time derivatives are defined as

$$\begin{aligned} &\frac{d}{dt}=D_0+\varepsilon D_1+\varepsilon^2 D_2+\ldots,\\ &\frac{d^2}{dt^2}=D_0^2+2\varepsilon D_0 D_1+\varepsilon^2(D_1^2+2D_0 D_2)+\ldots \end{aligned} \tag{12}$$

where $D_n = \dfrac{\partial}{\partial T_n}$. Substituting Eqs. (11) and (12) into Eq. (9) yield after separation

Order 1:

$$D_0^2 u_0 + u_0^{iv} + \alpha_1 u_0 = 0 \tag{13}$$

Order ε:

$$D_0^2 u_1 + u_1^{iv} + \alpha_1 u_1 = -2D_0 D_1 u_0 - \alpha_2 u_0^3 + \frac{1}{2} u_0'' \int_0^1 u_0'^2 dx \tag{14}$$

Order ε^2:

$$\begin{aligned} D_0^2 u_2 + u_2^{iv} + \alpha_1 u_2 &= -2D_0 D_1 u_1 - (D_1^2 + 2D_0 D_2) u_0 \\ &-2\mu D_0 u_0 - 3\alpha_2 u_0^2 u_1 + f \cos \Omega T_o \\ &+ u_0'' \int_0^1 u_0' u_1' dx + \frac{1}{2} u_1'' \int_0^1 u_0'^2 dx \end{aligned} \tag{15}$$

At order 1, the solution may be expressed as

$$u_0(x, T_0, T_1, T_2) = (A(T_1, T_2) e^{i\omega_{n0} T_0} + cc) Y(x) \tag{16}$$

where *cc* denotes the complex conjugates of the preceding terms. The mode shapes satisfy the following differential system

$$\begin{aligned} &Y^{iv} - \lambda^4 Y = 0 \\ &Y(0) = Y''(0) = Y(1) = Y''(1) = 0 \end{aligned} \tag{17}$$

where λ^4 is defined as

$$\lambda^4 = \omega_{n0}^2 - \alpha_1 \tag{18}$$

Solution of Eq. (17) is

$$\begin{aligned} &Y_n(x) = \sqrt{2} \sin n\pi x, \\ &\omega_{n0} = \sqrt{n^4 \pi^4 + \alpha_1}, \quad n = 1, 2, 3, .. \end{aligned} \tag{19}$$

where Y_n are the mode shapes and ω_{n0} are the natural frequencies. The eigenfunctions are normalized such that $\int_0^1 Y_n^2 dx = 1$.

The first order solution is inserted into the right hand side of order ε equation leading to

$$\begin{aligned} D_0^2 u_1 + u_1^{iv} + \alpha_1 u_1 &= e^{i\omega_{n0} T_0} (-2i\omega_{n0} D_1 A Y - 3\alpha_2 A^2 \bar{A} Y^3 \\ &+ \frac{3}{2} A^2 \bar{A} Y'' \int_0^1 Y'^2 dx) - \alpha_2 (A^3 e^{3i\omega_{n0} T_o} + cc) Y^3 \\ &+ \frac{1}{2} (A^3 e^{3i\omega_{n0} T_o} + cc) Y'' \int_0^1 Y'^2 dx \end{aligned} \tag{20}$$

Since the homogenous part of Eq. (20) possesses a non-trivial solutions, the non-homogenous equations admits solutions only if a solvability condition is satisfied (Nayfeh 1981). At order ε solvability condition is

$$-2i\omega_{n0} D_1 A + \kappa_1 A^2 \bar{A} = 0 \tag{21}$$

where

$$\kappa_1 = -3\alpha_2 \int_0^1 Y^4 dx + \frac{3}{2} \int_0^1 Y Y'' dx \int_0^1 Y'^2 dx \tag{22}$$

A solution can be written at this order of the form

$$u_1(x, T_0, T_1, T_2) = (A^3(T_1, T_2) e^{3i\omega_{n0} T_0} + cc) \phi(x) \tag{23}$$

Substituting Eq. (23) into Eq. (14) yields

$$\begin{aligned} &\phi^{iv} - (9\omega_{n0}^2 - \alpha_1)\phi = -\alpha_2 Y^3 + \frac{1}{2} Y'' \int_0^1 Y'^2 dx \\ &\phi(0) = \phi''(0) = \phi(1) = \phi''(1) = 0 \end{aligned} \tag{24}$$

The mode shapes at this order are

$$\begin{aligned} \phi_n(x) = \frac{\sqrt{2}}{16} &\left(\left(\frac{n^4 \pi^4 + 3\alpha_2}{n^4 \pi^4 + \alpha_1} \right) \sin n\pi x \right. \\ &\left. + \left(\frac{\alpha_2}{9n^4 \pi^4 - \alpha_1} \right) \sin 3n\pi x \right) \end{aligned} \tag{25}$$

The solution at order ε^2 is

$$u_2(x, T_0, T_1, T_2) = \varphi(x, T_1, T_2) e^{i\omega_{n0} T_0} + U(x, T_0, T_1, T_2) \tag{26}$$

where φ and U are the functions for the secular and non-secular terms, respectively. For the primary resonances of the external excitation, the excitation frequency can be taken as

$$\Omega = \omega_{n0} + \varepsilon^2 \sigma \tag{27}$$

where σ is detuning parameter of order 1.

Inserting Eqs. (27), (26), (23) and (16) into Eq. (15) and considering only the terms producing secularities, one has

$$\begin{aligned} &\varphi^{iv} - (\omega_{n0}^2 - \alpha_1)\varphi = -2i\omega_{n0} Y (D_2 A + \mu A) + \frac{f}{2} e^{i\sigma T_2} \\ &+ A^3 \bar{A}^2 \left(\frac{\kappa_1^2}{4\omega_{n0}^2} Y - 3\alpha_2 Y^2 \phi + Y'' \int_0^1 Y' \phi' dx + \frac{1}{2} \phi'' \int_0^1 Y'^2 dx \right) \end{aligned} \tag{28}$$

with boundary conditions

$$\varphi(0) = \varphi''(0) = \varphi(1) = \varphi''(1) = 0 \tag{29}$$

The homogenous part of Eq. (28) has a non-trivial solution so that the non-homogenous part can be solved only if the following solvability condition is satisfied

$$\begin{aligned} &-2i\omega_{n0}(D_2 A + \mu A) + F e^{i\sigma T_2} \\ &+ A^3 \bar{A}^2 \left(\frac{\kappa_1^2}{4\omega_{n0}^2} + \kappa_2 + \kappa_3 + \kappa_4 \right) = 0 \end{aligned} \tag{30}$$

where

$$\begin{aligned} &F = \frac{f}{2} \int_0^1 Y dx, \quad \kappa_2 = -3\alpha_2 \int_0^1 Y^3 \phi dx, \\ &\kappa_3 = \int_0^1 Y Y'' dx \int_0^1 Y' \phi' dx, \quad \kappa_4 = \frac{1}{2} \int_0^1 Y \phi'' dx \int_0^1 Y'^2 dx \end{aligned} \tag{31}$$

For determining amplitude and phase modulations, for higher order solutions, the usual reconstitution method (Nayfeh 2005) will be employed. $D_2 A$ can be written as

$$2i\omega_{n0} D_2 A = -2i\omega_{n0} \mu A + F e^{i\sigma T_2} + \mathrm{K} A^3 \bar{A}^2 = 0 \tag{32}$$

where

$$\mathrm{K} = \frac{\kappa_1^2}{4\omega_{n0}^2} + \kappa_2 + \kappa_3 + \kappa_4 \tag{33}$$

The complex amplitude modulations are

$$2i\omega_{n0}\frac{dA}{dt} = 2i\omega_{n0}(\varepsilon D_1 A + \varepsilon^2 D_2 A) \tag{34}$$

or

$$2i\omega_{n0}\frac{dA}{dt} = \varepsilon\kappa_1 A^2 \bar{A} + \varepsilon^2(-2i\omega_{n0}\mu A + Fe^{i\sigma T_2} + \mathrm{K}A^3\bar{A}^2) \tag{35}$$

Insertion of the polar form

$$A = \frac{1}{2}ae^{i\beta} \tag{36}$$

and separation into real and imaginary parts finally yield the amplitude and phase modulation equations

$$\dot{a} = \varepsilon^2\left(-\mu a + \frac{F}{\omega_{n0}}\sin\gamma\right) \tag{37}$$

$$\dot{\gamma} = \frac{\varepsilon\kappa_1 a^2}{8\omega_{n0}} + \varepsilon^2\left(\sigma + \frac{\mathrm{K}a^4}{32\omega_{n0}} + \frac{F}{a\omega_{n0}}\cos\gamma\right) \tag{38}$$

where dot denotes differentiation with respect to time t and phase γ is defined to be

$$\gamma = \sigma T_2 - \beta \tag{39}$$

For steady state solutions, $\dot{a} = \dot{\gamma} = 0$ in Eqs. (37) and (38). Frequency detuning parameter σ can be calculated by elimination of γ, which upon substitution into Eq. (27) yields the frequency response relation

$$\Omega = \omega_{n0} - \frac{\varepsilon\kappa_1}{8\omega_{n0}}a^2 + \varepsilon^2\left(\frac{-\mathrm{K}a^4}{32\omega_{n0}} \pm \sqrt{\frac{F^2}{a^2\omega_{n0}^2} - \mu^2}\right) \tag{40}$$

The approximate solution is

$$\begin{aligned} w(x,t) &= \sqrt{\varepsilon}a\cos(\Omega t - \gamma)Y(x) \\ &+ \varepsilon\sqrt{\varepsilon}\frac{a^3}{4}\cos\big(3(\Omega t - \gamma)\big)\phi(x) + O(\varepsilon^2) \end{aligned} \tag{41}$$

where real amplitudes a and phases γ are governed by Eqs. (37) and (38).

3.2 The multiple scales lindstedt poincare method

Details of the method can be found in Pakdemirli *et al.* (2009). First the time transformation

$$\tau = \omega t \tag{42}$$

is applied to Eq. (9)

$$\begin{aligned} &\omega^2\frac{\partial^2 u}{\partial\tau^2} + \frac{\partial^4 u}{\partial x^4} + 2\varepsilon^2\mu\omega\frac{\partial u}{\partial\tau} + \alpha_1 u + \varepsilon\alpha_2 u^3 \\ &= \frac{1}{2}\varepsilon\frac{\partial^2 u}{\partial x^2}\int_0^1\left(\frac{\partial u}{\partial x}\right)^2 dx + \varepsilon^2 f\cos\frac{\Omega}{\omega}T_0 \end{aligned} \tag{43}$$

Fast and slow time scales are

$$T_0 = \tau = \omega t, \quad T_1 = \varepsilon\tau = \varepsilon\omega t, \quad T_2 = \varepsilon^2\tau = \varepsilon^2\omega t \tag{44}$$

Using

$$\begin{aligned} \frac{d}{d\tau} &= D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots \\ \frac{d^2}{d\tau^2} &= D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2(D_1^2 + 2D_0 D_2) + \ldots \end{aligned} \tag{45}$$

and substituting the expansions

$$\begin{aligned} u(x,t;\varepsilon) &= u_0(x,T_0,T_1,T_2) + \varepsilon u_1(x,T_0,T_1,T_2) \\ &+ \varepsilon^2 u_2(x,T_0,T_1,T_2) + \ldots \end{aligned} \tag{46}$$

$$1 = \omega^2 - \varepsilon\omega_1 - \varepsilon^2\omega_2 \tag{47}$$

into Eq. (43) yields after separation

Order 1:

$$\omega^2(D_0^2 u_0 + u_0^{iv} + \alpha_1 u_0) = 0 \tag{48}$$

Order ε:

$$\begin{aligned} &\omega^2(D_0^2 u_1 + u_0^{iv} + \alpha_1 u_1) = -2\omega^2 D_0 D_1 u_0 + \omega_1 u_0^{iv} \\ &+ \alpha_1\omega_1 u_0 - \alpha_2 u_0^3 + \frac{1}{2}u_0''\int_0^1 u_0'^2 dx \end{aligned} \tag{49}$$

Order ε^2 :

$$\begin{aligned} &\omega^2(D_0^2 u_2 + u_2^{iv} + \alpha_1 u_2) = -2\omega^2 D_0 D_1 u_1 \\ &-\omega^2(D_1^2 + 2D_0 D_2)u_0 + \omega_1 u_1^{iv} + \omega_2 u_0^{iv} \\ &-2\mu\omega D_0 u_0 + \alpha_1\omega_1 u_1 + \alpha_1\omega_2 u_0 - 3\alpha_2 u_0^2 u_1 \\ &+ f\cos\frac{\Omega}{\omega}T_o + u_0''\int_0^1 u_0' u_1' dx + \frac{1}{2}u_1''\int_0^1 u_0'^2 dx \end{aligned} \tag{50}$$

Note that, the method requires expansion of the frequency in the above given form rather than the normal frequency expansion as usual in Lindstedt-Poincare method. At order 1, the solution may be expressed as

$$u_0(x,T_0,T_1,T_2) = (A(T_1,T_2)e^{i\omega_{n0}T_0} + cc)Y(x) \tag{51}$$

where *cc* denotes the complex conjugates of the preceding terms. The mode shapes satisfy the following differential system

$$\begin{aligned} &Y^{iv} - \lambda^4 Y = 0 \\ &Y(0) = Y''(0) = Y(1) = Y''(1) = 0 \end{aligned} \tag{52}$$

where λ^4 is defined to be

$$\lambda^4 = \omega_{n0}^2 - \alpha_1 \tag{53}$$

Solution for Eq. (52) is

$$Y_n(x) = \sqrt{2}\sin n\pi x, \quad \omega_{n0} = \sqrt{n^4\pi^4 + \alpha_1}, \quad n = 1,2,3,.. \tag{54}$$

where Y_n are the mode shapes and ω_{n0} are the natural frequencies. The eigenfunctions are normalized such that $\int_0^1 Y_n^2 dx = 1$. The first order solution is inserted into the right hand side of order ε equation leading to

$$\omega^2(D_0^2u_1+u_1^{iv}+\alpha_1u_1)=e^{i\omega_{n0}T_0}(-2i\omega_{n0}\omega^2D_1AY$$
$$+A\omega_1(Y^{iv}+\alpha_1Y)-3\alpha_2A^2\bar{A}Y^3+\frac{3}{2}A^2\bar{A}Y''\int_0^1Y'^2dx) \tag{55}$$
$$-\alpha_2(A^3e^{3i\omega_{n0}T_o}+cc)Y^3+\frac{1}{2}(A^3e^{3i\omega_{n0}T_o}+cc)Y''\int_0^1Y'^2dx$$

At order ε, the solvability condition requires

$$-2i\omega_{n0}\omega^2D_1A+A\omega_1\Gamma_1-A^2\bar{A}\Gamma_2=0 \tag{56}$$

where

$$\Gamma_1=\int_0^1YY^{iv}dx+\alpha_1\int_o^1Y^2dx,$$
$$\Gamma_2=3\alpha_2\int_0^1Y^4dx-\frac{3}{2}\int_o^1YY''dx\int_o^1Y'^2dx \tag{57}$$

In the MSLP as outlined in Pakdemirli *et al.* (2009), first $D_1A=0$ is selected and if the frequency correction is real, this choice is admissible. If ω_1 turns out to be complex, then $D_1A\neq0$ which implies $\omega_1=0$ and secularities are eliminated by choosing D_1A. A complex ω_1 implies that there is amplitude variation and LP method fails to produce physical solutions (Nayfeh 1981). The method allows switching back and forth with MS and LP type of eliminating secularities thereby augmenting the advantages of both methods. For Eq. (56), selection of

$$D_1A=0 \;\Rightarrow\; A=A(T_2) \tag{58}$$

produces

$$\omega_1=A\bar{A}\frac{\Gamma_2}{\Gamma_1}=\frac{1}{4}a^2\frac{\Gamma_2}{\Gamma_1} \tag{59}$$

which is suitable because ω_1 is real.

A solution can be written at this order of the form

$$u_1(x,T_0,T_2)=(A^3(T_2)e^{3i\omega_{n0}T_0}+cc)\frac{\phi(x)}{\omega^2} \tag{60}$$

Substituting Eq. (60) into Eq. (49) yields

$$\phi^{iv}-(9\omega_{n0}^2-\alpha_1)\phi=-\alpha_2Y^3+\frac{1}{2}Y''\int_0^1Y'^2dx$$
$$\phi(0)=\phi''(0)=\phi(1)=\phi''(1)=0 \tag{61}$$

The mode shapes at this order are

$$\phi_n(x)=\frac{\sqrt{2}}{16}((\frac{n^4\pi^4+3\alpha_2}{n^4\pi^4+\alpha_1})\sin n\pi x$$
$$+(\frac{\alpha_2}{9n^4\pi^4-\alpha_1})\sin 3n\pi x) \tag{62}$$

The solution at order ε^2 is

$$u_2(x,T_0,T_2)=\varphi(x,T_2)e^{i\omega_{n0}T_0}+U(x,T_0,T_2) \tag{63}$$

where φ and U are the functions for the secular and non-secular terms, respectively. For primary resonances case, external excitation frequency can be taken as

$$\Omega=\omega(\omega_{n0}+\varepsilon^2\sigma) \tag{64}$$

Inserting Eqs. (64), (63), (60) and (51) into Eq. (50) and considering only the terms producing secularities, one has

$$\varphi^{iv}-(\omega_{0n}^2-\alpha_1)\varphi=-2i\omega_{n0}YD_2A+\frac{\omega_2A}{\omega^2}Y^{iv}$$
$$-\frac{2i\omega_{n0}\mu A}{\omega}Y+\frac{\alpha_1\omega_2A}{\omega^2}Y+\frac{f}{2\omega^2}e^{i\sigma T_2} \tag{65}$$
$$+\frac{A^3\bar{A}^2}{\omega^4}\left(-3\alpha_2Y^2\phi+Y''\int_0^1Y'\phi'dx+\frac{1}{2}\phi''\int_0^1Y'^2dx\right)$$

with boundary conditions

$$\varphi(0)=\varphi''(0)=\varphi(1)=\varphi''(1)=0 \tag{66}$$

The solvability condition is

$$-2i\omega_{n0}D_2A\int_0^1Y^2dx+\frac{\omega_2A}{\omega^2}\int_0^1YY^{iv}dx-\frac{2i\omega_{n0}\mu A}{\omega}\int_0^1Y^2dx$$
$$+\frac{\alpha_1\omega_2A}{\omega^2}\int_0^1Y^2dx+\frac{F}{\omega^2}e^{i\sigma T_2}+\frac{A^3\bar{A}^2}{\omega^4}(\Gamma_3+\Gamma_4+\Gamma_5)=0 \tag{67}$$

where

$$F=\frac{f}{2}\int_0^1Ydx,\quad \Gamma_3=-3\alpha_2\int_0^1Y^3\phi dx,$$
$$\Gamma_4=\int_0^1YY''dx\int_0^1Y'\phi'dx,\quad \Gamma_5=\frac{1}{2}\int_0^1Y\phi''dx\int_0^1Y'^2dx \tag{68}$$

D_2A cannot be selected as zero, since ω_2 would then be complex. Therefore the admissible choice is

$$\omega_2=0 \tag{69}$$

and the remaining equation is

$$-2i\omega_{n0}D_2A-\frac{2i\omega_{n0}\mu A}{\omega}+\frac{F}{\omega^2}e^{i\sigma T_2}+\frac{\Gamma}{\omega^4}A^3\bar{A}^2=0 \tag{70}$$

where

$$\Gamma=\Gamma_3+\Gamma_4+\Gamma_5 \tag{71}$$

Since the application processes are different, the solvability conditions are different for MS (i.e., Eq. (21)) and MSLP (i.e., Eq. (56)) methods. To write the solvability condition for MS, the reconstitution method taken into account. Note that in the first level of approximation in MSLP, the mechanism from LP is used, whereas at this level of approximation, the mechanism from MS is used. This choice of flexibility increases the success of the method.

The polar form $A=\frac{1}{2}ae^{i\beta}$ is substituted into the solvability condition and the real and imaginary parts are separated leading to

$$D_2a=-\frac{\mu}{\omega}a+\frac{F}{\omega_{n0}\omega^2}\sin\gamma \tag{72}$$

$$D_2\gamma=\sigma+\frac{\Gamma}{32\omega_{n0}\omega^4}a^4+\frac{F}{a\omega_{n0}\omega^2}\cos\gamma \tag{73}$$

where

$$\gamma = \sigma T_2 - \beta \tag{74}$$

For steady state solutions, $D_2 a=0$, $D_2\gamma=0$ and elimination of γ between Eqs. (72) and (73) yields

$$\sigma = \frac{-\Gamma}{32\omega_{n0}\omega^4}a^4 \pm \sqrt{\frac{F^2}{a^2\omega_{n0}^2\omega^4} - \left(\frac{\mu}{\omega}\right)^2} \tag{75}$$

From Eq. (64), frequency-response relation is

$$\Omega = \omega\left\{\omega_{n0} + \varepsilon^2\left(\frac{-\Gamma}{32\omega_{n0}\omega^4}a^4 \pm \sqrt{\frac{F^2}{a^2\omega_{n0}^2\omega^4} - \left(\frac{\mu}{\omega}\right)^2}\right)\right\} \tag{76}$$

where

$$\omega = \sqrt{1 + \varepsilon\frac{\Gamma_2}{4\Gamma_1}a^2} \tag{77}$$

Eq. (77) is obtained by substituting Eqs. (69) and (59) into (47). The approximate solution is

$$\begin{aligned} w(x,t;\varepsilon) &= \sqrt{\varepsilon}a\cos(\Omega t - \gamma)Y(x) \\ &+\varepsilon\sqrt{\varepsilon}\frac{a^3}{4\omega^2}\cos\left(3(\Omega t-\gamma)\right)\phi(x) + O(\varepsilon^2) \end{aligned} \tag{78}$$

The amplitude and phases are governed by the equations

$$\dot{a} = \varepsilon^2\left(-\mu a + \frac{F}{\omega_{n0}\omega}\sin\gamma\right) \tag{79}$$

$$\dot{\gamma} = \varepsilon^2\left(\omega\sigma + \frac{\Gamma}{32\omega_{n0}\omega^3}a^4 + \frac{F}{a\omega_{n0}\omega}\cos\gamma\right) \tag{80}$$

4. Comparisons of the results

For perturbation solutions to be valid, the correction term should be much smaller than the leading term. For both methods, the requirements are

$$\varepsilon\frac{a^2\phi(x)}{4Y(x)} << 1 \qquad \text{(MS)} \tag{81}$$

$$\varepsilon\frac{a^2\phi(x)}{4Y(x)\omega^2} << 1 \qquad \text{(MSLP)} \tag{82}$$

The only difference in the criteria is ω^2 being in the denominator of MSLP method. For strong nonlinearities, nonlinear elastic foundation coefficient (α_2) should be large. For MS method, taking the limit

$$\lim_{\alpha_2\to\infty}\varepsilon\frac{a^2\phi(x)}{4Y(x)} = \infty \tag{83}$$

yields infinity as expected. Hence, the MS method solution cannot be valid for large values of α_2. In contrast, for MSLP, the corresponding limit is

$$\lim_{\alpha_2\to\infty}\varepsilon\frac{a^2\phi(x)}{4\omega^2 Y(x)} \approx 0 << 1 \tag{84}$$

Which satisfies the perturbation requirement for large parameters.

In Figs. 2-7, frequency response curves obtained by the MS and the MSLP methods are contrasted with each other with respect to the effects of physical parameters such as the elastic foundation coefficients and the external excitation amplitude. In Figs. 2 and 3, the frequency response curves of both methods are obtained for various linear elastic foundation parameter (α_1) values when nonlinear elastic foundation parameter (α_2) is fixed. As α_1 increases, the frequency response curves of both methods shift to the right side with maximum amplitudes decreased.

The effect of cubic nonlinearity is amplified by increasing the nonlinear elastic foundation parameter (α_2) in Figs. 4 and 5. When the nonlinear elastic foundation

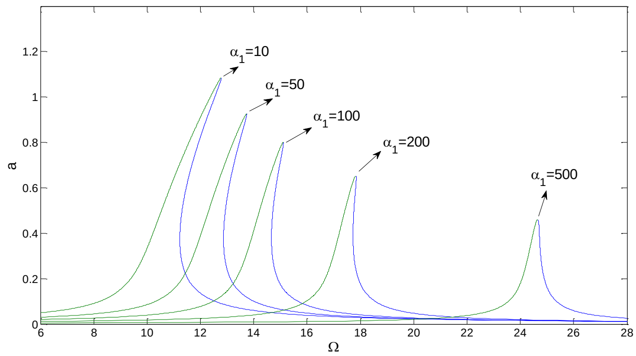

Fig. 2 Frequency response curves of MS method for various α_1 values. (n=1, ε=1, α_2=10, μ=0.2, f=5)

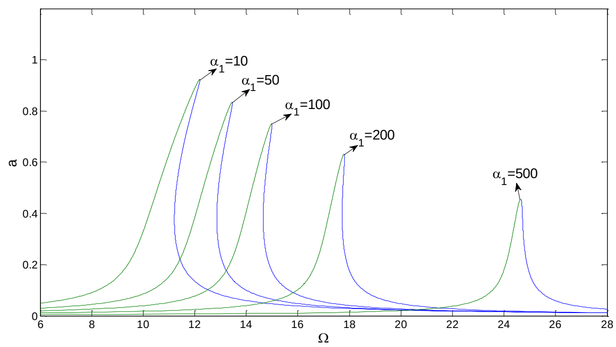

Fig. 3 Frequency response curves of MSLP method for various α_1 values. (n=1, ε=1, α_2=10, μ=0.2, f=5)

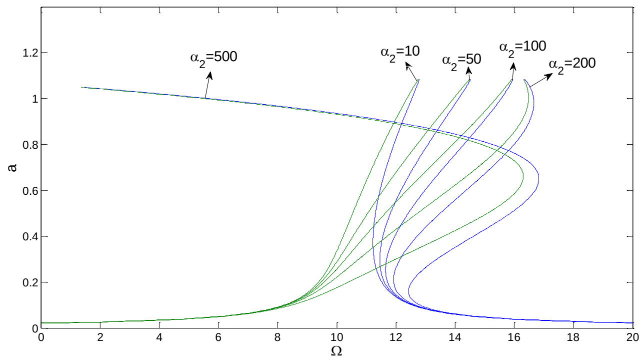

Fig. 4 Frequency response curves of MS method for various α_2 values. (n=1, ε=1, α_1=10, μ=0.2, f=5)

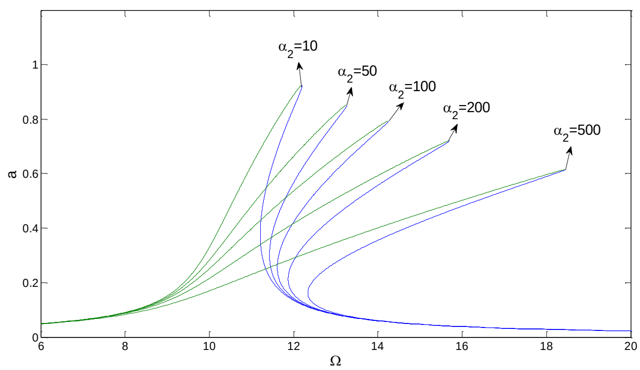

Fig. 5 Frequency response curves of MSLP method for various α_2 values. (n=1, ε=1, α_1=10, μ=0.2, f=5)

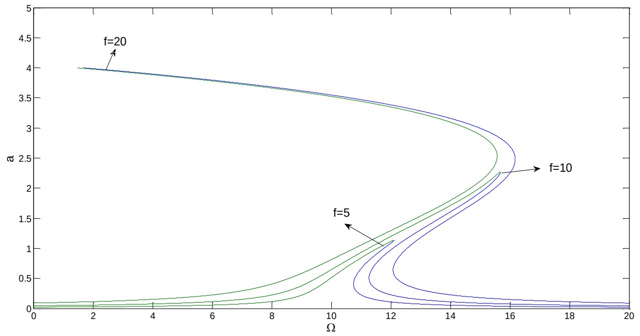

Fig. 6 Frequency response curves of MS method for various f values. (n=1, ε=1, α_1=10, α_2=1, μ=0.2)

coefficient (α_2) is increased from 10 to 100, for the MS and the MSLP, the multivalued regions considerably increase as expected. If the nonlinear elastic foundation coefficient is further increased (α_2=200−500), the MS solutions produce a backward curve which is unphysical. However, this anomalous behavior is not observed for the MSLP, as expected.

Figs. 6 and 7 depict the effect of external excitation amplitude (f) on the frequency response curves for the MS and the MSLP methods. In Figure 6, the frequency response curves of the MS method are plotted for various external excitation amplitude values. Increasing the external excitation amplitude results in the backward curves for the MS method. As can be seen from Fig. 7, reliable results can still be obtained from the MSLP method for high excitation amplitudes.

5. Conclusions

Nonlinear vibrations of a simply supported beam resting on a nonlinear elastic foundation are treated using the MS and the MSLP methods. The case of primary resonances of the external excitation is investigated. The amplitude and the phase modulation equations are obtained from which the steady state solutions are retrieved. The effects of the physical parameters on the nonlinear behavior are investigated using the frequency response curves of both methods. The following conclusions can be listed as follows.

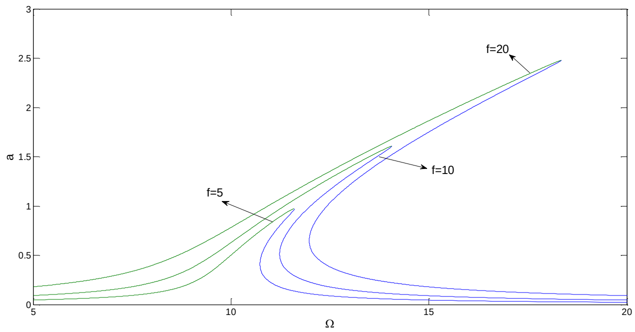

Fig. 7 Frequency response curves of MSLP method for various f values. (n=1, ε=1, α_1=1, α_2=1, μ=0.2)

1) The MSLP method is successfully applied to partial differential equation.
2) MS and MSLP are in good agreement when the parameter values are within the range of weak nonlinearity.
3) Comparisons of the frequency response curves of the MS and the MSLP reveal that the MSLP solutions are reliable for high excitation amplitudes and for high nonlinear elastic coefficients whereas the MS produces unphysical backward curves for such high parameter values.
4) MSLP definitely improves the classical and well established MS solutions to have a wider range of validity.

A further study would be to apply MSLP method to quadratic and cubic partial differential equations. The nonlinearities arising in partial differential equations are classified using a suitable operator notation and general solution algorithms were developed for the models previously (Pakdemirli 1994, Pakdemirli and Boyaci 1995, Pakdemirli 2001).

References

Boyaci, H. (2006), “Vibrations of stretched damped beams under non-ideal boundary conditions”, *Sadhana-Acad. P. Eng. S.*, **31**(1), 1-8.

Boyaci, H. and Pakdemirli, M. (1997), “A comparison of different versions of the method of multiple scales for partial differential equations”, *J. Sound Vib.*, **204**(4), 595-607.

Cheung, Y.K., Chen, S.H. and Lau, S.L. (1991), “A modified Lindstedt Poincare method for certain strongly nonlinear oscillators”, *Int. J. Nonlin. Mech.*, **26** (3-4), 367-378.

Coskun, I. and Engin H. (1999), “Nonlinear vibrations of a beam on an elastic foundation”, *J. Sound Vib.*, **223**, 335-354.

Ding, H., Chen, L.Q. and Yang, S.P. (2012), “Convergence of Galerkin truncation for dynamic response of finite beams on nonlinear foundations under a moving load”, *J. Sound Vib.*, **331**, 2426-2442.

Ghayesh M.H. (2012b), “Stability and bifurcations of an axially moving beam with an intermediate spring support”, *Nonlin. Dyn.*, **69**, 193-210

Ghayesh, M.H, Amabili, M. and Paidoussis, M.P. (2012b), “Nonlinear vibrations and stability of an axially moving beam with an intermediate spring support: two-dimensional analysis”, *Nonlin. Dyn.*, **70 (1)**, 335-354

Ghayesh, M.H. (2012a), “Nonlinear dynamic response of a simply-supported Kelvin-Voigt viscoelastic beam, additionally supported by a nonlinear spring", *Nonlin. Anal. Real World Appl*., **13**, 1319-1333.

Ghayesh, M.H. and Paidoussis, M.P. (2010), "Three-dimensional dynamics of a cantilevered pipe conveying fluid, additionally supported by an intermediate spring array", *Int. J. Nonlin. Mech*., **45**, 507-524.

Ghayesh, M.H., Kazemirad, S. and Darabi, M.A. (2011b), "A general solution procedure for vibrations of systems with cubic nonlinearities and nonlinear/time-dependent internal boundary conditions", *J. Sound Vib.*, **330**, 5382-5400.

Ghayesh, M.H., Kazemirad, S. and Reid, T. (2012a), "Nonlinear vibrations and stability of parametrically exited systems with cubic nonlinearities and internal boundary conditions: A general solution procedure", *Appl. Math. Model*., **36**, 3299-3311.

Ghayesh, M.H., Paidoussis, M.P. and Modarres-Sadeghi, Y. (2011a), "Three-dimensional dynamics of a fluid-conveying cantilevered pipe fitted with an additional spring-support and an end-mass", *J. Sound Vib*., **330**, 2869-2899.

He, J.H. (1999), "Variational iteration method: a kind of nonlinear analytical technique: some examples", *Int. J. Nonlin. Mech*., **34**(4), 699-708.

He, J.H. (2003), "Linearized perturbation technique and its applications to strongly nonlinear oscillators", *Comput. Math. Appl.*, **45**, 1-8.

Hosseini, L. and Hosseini, S.A.A. (2015), "A general analytical approximation for nonlinear vibrations analysis of continuous systems using renormalization group method", *Appl. Math. Model*., **39**, 182-193.

Hu, H. (2004), "A classical perturbation technique which is valid for large parameters", *J. Sound Vib.*, **269**, 409-412.

Karahan, M.M.F. and Pakdemirli, M. (2017), "Free and forced vibrations of the strongly nonlinear cubic-quintic Duffing oscillators", *Z. Naturforsch.*, **72**(1), 59-69.

Leung, A.Y.T. and Guo, Z. (2011), "Residue harmonic balance approach to limit cycles of non-linear jerk equations", *Int. J. Nonlin. Mech.*, **46**, 898-906.

Liao S.J. (2004), "On the homotopy analysis method for nonlinear problems", *Appl. Math. Comput.*, **147**, 499-513.

Lim, C.W. and Wu, B.S. (2002), "A modified Mickens procedure for certain non-linear oscillators", *J. Sound Vib.*, **257**, 202-206.

Maccari, A. (1999), "The asymptotic perturbation method for nonlinear continuous systems", *Nonlin. Dyn*., **19**, 1-18.

Nayfeh, A.H. (1981), *Introduction to Perturbation Techniques*, John Wiley&Sons, New York, USA.

Nayfeh, A.H. (2005), "Resolving controversies in the application of the method of multiple scales and the generalized method of averaging", *Nonlin. Dyn*., **40**, 61-102.

Oz, H.R. and Pakdemirli, M. (2006), "Two-to-one internal resonances in a shallow curved beam resting on an elastic foundation", *Acta Mech.*, **185**, 245-260.

Oz, H.R., Pakdemirli, M., Ozkaya, E. and Yılmaz, M. (1998), "Non-linear vibrations of a slightly curved beam resting on a non-linear elastic foundation", *J. Sound Vib*., **212**(2), 295-309.

Ozhan, B.B. and Pakdemirli, M. (2009), "A general solution procedure for the forced vibrations of a continuous system with cubic nonlinearities: Primary resonance case", *J. Sound Vib*., **325**, 894-906

Ozhan, B.B. and Pakdemirli, M. (2010), "A general solution procedure for the forced vibrations of a continuous system with cubic nonlinearities: three-to-one internal resonances with external excitation", *J. Sound Vib*., **329**(13), 2603-2615.

Ozhan, B.B. and Pakdemirli, M. (2012), "Principal parametric resonances of a general continuous system with cubic nonlinearities", *Appl. Math. Comput.*, **219**, 2412-2423.

Ozkaya, E. And Tekin, A. (2007), "Non linear vibrations of stepped beam system under different boundary conditions", *Struct. Eng. Mech*., **27**(3), 333-345

Ozkaya, E., Bagdatli, S.M. and Oz, H.R. (2008), "Nonlinear transverse vibrations and 3:1 internal resonances of a beam with multiple supports", *J. Vib. Acoust.*, **130**, 021013-1-021013-11.

Pakdemirli M. (2015b), "Precession of a planet with the multiple scales Lindstedt-Poincare Technique", *Z. Naturforsch*., **70**, 829-834.

Pakdemirli, M. (1994) "A comparison of two perturbation methods for vibrations of systems with quadratic and cubic nonlinearities", *Mech. Res. Commun*., **21**, 203-208.

Pakdemirli, M. (2001), "Vibrations of continuous systems with a general operator notation suitable for perturbative calculations", *J. Sound Vib*., **246**, 841- 851.

Pakdemirli, M. (2015a), "Perturbation-iteration method for strongly nonlinear vibrations", *J Vib Control.*, DOI: 10.1177/1077546315586647.

Pakdemirli, M. and Boyaci, H. (1995), "Comparison of direct-perturbation methods with discretization-perturbation methods for nonlinear vibrations", *J. Sound Vib.*, **186**, 837-845.

Pakdemirli, M. and Karahan, M.M.F. (2010), "A new perturbation solution for systems with strong quadratic and cubic nonlinearities", *Math. Meth. Appl. Sci.*, **33**, 704-712.

Pakdemirli, M. and Ozkaya, E. (2003), "Three-to-one internal resonances in a general cubic non-linear continuous system", *J. Sound Vib*., **268**, 543-553.

Pakdemirli, M. and Sari, G. (2015a), "Solution of quadratic nonlinear problems with multiple scales Lindstedt-Poincare Method" *Math. Comput. Appl*., **20**, 137-150.

Pakdemirli, M. and Sari, G. (2015b), "Perturbation solutions of the quintic Duffing equation with strong nonlinearities", *Commun. Numer. Anal*., **2015**, 82-89.

Pakdemirli, M., Karahan, M.M.F. and Boyacı, H. (2009), "A new perturbation algorithm with better convergence properties: Multiple Scales Lindstedt Poincare method", *Math. Comput. Appl*., **14**(1), 31-44.

Pakdemirli, M., Karahan, M.M.F. and Boyacı, H. (2011), "Forced vibrations of strongly nonlinear systems with multiple scales Lindstedt Poincare Method", *Math. Comput. Appl.*, **16**(4), 879-889.

Raju, K.K. and Rao, G.V. (1993), "Effect of a non-linear elastic foundation on the mode shapes in stability and vibration problems of uniform columns/beams", *J. Sound Vib.*, **16**, 369-371.

Wu, B.S., Lim, C.W. and Sun, W.P. (2006), "Improved harmonic balance approach to periodic solutions of nonlinear jerk equations", *Phys. Lett. A.*, **354**, 95-100.

CC