Structural Engineering and Mechanics, *Vol. 61, No. 6 (2017) 721-736*
DOI: https://doi.org/10.12989/sem.2017.61.6.721

Vibration analysis of embedded size dependent FG nanobeams based on third-order shear deformation beam theory

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(Received July 13, 2015, Revised May 19, 2016, Accepted November 15, 2016)

Abstract. In this paper, free vibration characteristics of functionally graded (FG) nanobeams embedded on elastic medium are investigated based on third order shear deformation (Reddy) beam theory by presenting a Navier type solution for the first time. The material properties of FG nanobeam are assumed to vary gradually along the thickness and are estimated through the power-law and Mori-Tanaka models. A two parameters elastic foundation including the linear Winkler springs along with the Pasternak shear layer is in contact with beam. The small scale effect is taken into consideration based on nonlocal elasticity theory of Eringen. The nonlocal equations of motion are derived based on third order shear deformation beam theory through Hamilton's principle and they are solved applying analytical solution. According to the numerical results, it is revealed that the proposed modeling can provide accurate frequency results of the FG nanobeams as compared to some cases in the literature. The obtained results are presented for the vibration analysis of the FG nanobeams such as the influences of foundation parameters, gradient index, nonlocal parameter and slenderness ratio in detail.

Keywords: free vibration; third order shear deformation beam theory; functionally graded nanobeam; nonlocal elasticity theory

1. Introduction

Fast growing developments in materials engineering led to microscopically inhomogeneous spatial composite materials named Functionally graded materials (FGMs) which have extensive applications for various systems and devices, such as aerospace, aircraft, automobile and defense structures and most recently the electronic devices. According to the fact that FG materials have been placed in the category of composite materials, the volume fractions of two or more material constituents such as a pair of ceramic-metal are supposed to change continuously throughout the gradient directions. The FGM materials are made to take advantage of desirable features of its constituent phases, for example, in a thermal protection system, the ceramic constituents are capable to withstand extreme temperature environments due to their better thermal resistance characteristics, while the metal constituents provide stronger mechanical performance and diminishes the possibility of catastrophic fracture (Ebrahimi and Barati 2016a-c). Hence, possessing novel mechanical properties, FGMs have gained its applicability in several engineering fields, such as biomedical engineering, nuclear engineering and mechanical engineering.

Furthermore, noticeable development in the application of structural elements such as beams and plates with micro or nano length scale in micro/nano electro-mechanical systems (MEMS/ NEMS), due to their outstanding chemical, mechanical, and electrical properties, led to a provocation in modelling of micro/nano scale structures (Alizada and Sofiyev 2011, Ebrahimi *et al*. 2016, Ebrahimi and Hosseini 2016a, b, Ebrahimi and Nasirzadeh 2015, Ebrahimi and Barati 2016d, e). In these applications, it is observed that the size effect has a major role on dynamic behavior of material. After the invention of carbon nanotubes (CNTs) by Iijima (1991), nanoscale engineering materials have exposed to considerable attention in modern science and technology. These structures possess extraordinary mechanical, thermal, electrical and chemical performances that are superior to the conventional structural materials. Therefore nanostructures attract great interest by researchers based on molecular dynamics and continuum mechanics. The problem in using the classical theory is that the classical continuum mechanics theory does not take into account the size effects in micro/nano scale structures. The classical continuum mechanics over predicts the responses of micro/nano structures (Ebrahimi *et al*. 2016b, 2017). Another way to capture the size effects is to rely on molecular dynamic simulations which is considered as a powerful and accurate implement to study of structural components at nanoscale. But even the molecular dynamic simulation at nano scale is computationally exorbitant for modeling the nanostructures with large numbers of atoms. So a conventional form of continuum mechanics that can capture the small scale effect is required. Eringen's nonlocal elasticity theory is the most commonly used continuum mechanics theory that includes small scale effects with good accuracy to model micro/nano scale devices and systems. The nonlocal elasticity theory assumes that the

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ISSN: 1225-4568 (Print), 1598-6217 (Online)

stress state at a reference point is a function of the strain at all neighbor points of the body. Hence, this theory could take into consideration the effects of small scales. In order to successfully design of structures at nanoscale, it is very important to take all essential characteristics of their mechanical behaviors at this size regime. To achieve this goal, based on the nonlocal constitutive relation of Eringen, a number of studies have been conducted attempting to develop nonlocal beam models for predicting the mechanical responses of nanobeams. The potential of application of nonlocal Euler-Bernoulli beam theory to materials in micro and nano scale proposed by Peddieson *et al.* (2003) as the first researchers to suggest nonlocal elasticity theory to nanostructures. Then, the nonlocal elasticity theory gained considerable attention among the nanotechnology society and utilization of this theory extended in different mechanical analyses. Reddy (2007) formulated various available beam models, including the Euler-Bernoulli, Timoshenko, Reddy, and Levinson beam theories through nonlocal differential relations of Eringen. In other scientific work, Wang and Liew (2007) carried out the static analysis of micro and nano scale structures based on nonlocal continuum mechanics using Euler-Bernoulli and Timoshenko beam theory. Aydogdu (2009) presented a general nonlocal beam model for analysis bending, buckling, and vibration of nanobeams using different beam theories. Civalek *et al*. (2010) proposed formulation of the governing equations of nonlocal Euler-Bernoulli beams to investigate bending of cantilever microtubules via the differential quadrature method. In another study, Thai (2012) suggested a nonlocal higher order beam theory to study mechanical responses of nanobeams. Simsek (2014) proposed a non-classical beam model based on the Eringen's nonlocal elasticity theory for nonlinear vibration of nanobeams with various boundary conditions. Size-dependent nonlinear forced vibration analysis of magneto-electro-thermo-elastic Timoshenko nanobeams based upon the nonlocal elasticity theory is studied by Ansari *et al*. (2015). Moreover a semi-analytical vibration and buckling analysis of FG nanobeams is carried out by Ebrahimi *et al*. (2015a), Ebrahimi and Salari (2015a-e). Beneficial applications of functionally graded materials in micro/nano structures are greatly known in last years. To applying accurately this kinds of novel materials in MEMS/NEMS, their dynamic behaviors should be examined. Asghari *et al*. (2010, 2011) studied the vibration behavior of the functionally graded EBT and TBT microbeams using modified couple stress theory. The mechanical behaviors of FGM beams with axially or transversally power law distribution was examined by Alshorbagy *et al*. (2011). Ke and Wang (2011) exploited the small scale effects on dynamic stability of FGM microbeams based upon Timoshenko beam model. The free vibration of FG microbeams analyzed by Ansari *et al*. (2011) by using strain gradient TBT. They also concluded that the values of material graduation exponent have a remarkable influence in the vibration behavior of the functionally graded microbeams. Employing modified couple stress theory the nonlinear vibration of microbeams made of FGMs with von-Karman geometric nonlinearity presented by Ke *et al*. (2012). It was revealed that linear frequency as well as nonlinear frequency rise prominently since the thickness of microbeam and material length scale parameter are comparable. Recently, Eltaher *et al*. (2012, 2013a) presented a finite element analysis for free vibration of FG nanobeams using nonlocal EBT. They also exploited the static and stability responses of FG nanobeams based on nonlocal continuum theory (Eltaher *et al*. 2013b). More recently, using nonlocal TBT and EBT, Simsek and Yurtcu (2013) investigated bending and buckling of FG nanobeam by analytical method. Uymaz (2013) presented forced vibration analysis of functionally graded beams using nonlocal elasticity. Rahmani and Pedram (2014) analyzed the size effects on vibration of FG nanobeams based on nonlocal TBT. Nonlinear free vibration of FG nanobeams with fixed ends, i.e., simply supported-simply supported (SS) and simply supported-clamped (SC), using the nonlocal elasticity within the frame work of EBT with von kármán type nonlinearity is studied by Nazemnezhad and Hosseini-Hashemi (2014). Also, recently Hosseini-Hashemi *et al*. (2014) investigated free vibration of FG nanobeams with consideration surface effects and piezoelectric field using nonlocal elasticity theory. Most recently Ebrahimi and Salari (2015f) presented a semi-analytical method for vibrational and buckling analysis of FG nanobeams considering the physical neutral axis position. Niknam and Aghdam (2015) presented a closed form solution for both natural frequency and buckling load of nonlocal FG beams resting on nonlinear elastic foundation. Ebrahimi and Barati (2015) presented a nonlocal higher-order shear deformation beam theory for vibration analysis of size-Dependent FG nanobeams. Ebrahimi and Barati (2016f) presented dynamic modeling of a thermo-piezo-electrically actuated nanosize beam subjected to a magnetic field. Ebrahimi and Barati (2016g) presented vibration analysis of smart piezoelectrically actuated nanobeams subjected to magneto-electrical field in thermal environment. Thermal effects on vibration behavior of nonlocal temperature-dependent FGM nanobeams is investigated by Ebrahimi and Barati (2016h-j). Therefore, searching in literature reveals that, vibration analysis of FG nanobeams on elastic foundations are very limited. Various kinds of elastic foundation models for the sake of describing the interactions of the beam and foundation have proposed via scientists. Winkler or one-parameter elastic foundation is known as the simplest model which regards the foundation as a series of separated linear elastic springs without coupling effects between each other. The defect of Winkler's formulation is the behavioral inconsistency associated to the discontinuous deflections on the interacted surface area of the beam. Pasternak (1954) later introduced an incompressible vertical element as a shear layer which is physically realistic representation of the elastic medium and can take into account the transverse shear stresses due to interaction of shear deformation of the surrounding elastic medium. Thus, a more realistic and generalized representation of the elastic foundation is expected through a two-parameter foundation model.

It is apparent that most of the previous studies on mechanical analysis of FG nanobeams have been carried out based on Euler-Bernoulli and Timoshenko beam theories. It should be noted that the EBT fails to consider the influences of shear deformations. This theory is only applicable for slender beams and should not be applied for thick beams, and also it suppose that the transverse perpendicular to the neutral surface stays normal during and after bending, which indicates that the transversal shear strain is equal to zero Hence, the buckling loads and natural frequencies of thick beams are overestimated in which shear deformation effects are prominent. Timoshenko Beam Theory can enumerate the influences of shear deformations for thick beams with presumption of a constant shear strain state in the direction of beam thickness. So, as a disadvantage of this theory, a shear correction factor is required to properly demonstration of the deformation strain energy. To prevent using the shear correction factors, many higher-order shear deformation theories have been developed such as the third-order shear deformation theory proposed by Reddy (2007), the generalized beam theory proposed by Aydogdu (2009) and sinusoidal shear deformation theory of Touratier (1991). Reddy's third order beam theory (RBT) can be used with supposing the higher order longitudinal displacement variations of beam along the thickness. By verifying zero transverse shear stresses at the upper and lower surfaces of the beam, this theory captures both the microstructural and shear deformation effects. Therefore, The Reddy beam theory is more exact and provides better representation of the physics of the problem, which does not need any shear correction factors. This theory relaxes the limitation on the warping of the cross sections and allows cubic variations in the longitudinal direction of the beam, so it can produce adequate accuracy when applying for beam analysis. Therefore, a few studies have been performed to investigate the mechanical responses of FG micro/nano beams by using higher shear deformation beam theories. Rahmani and Jandaghian (2015) presented Buckling analysis of functionally graded nanobeams based on a nonlocal third-order shear deformation theory. A unified higher order beam theory which contains various beam theories as special cases for buckling of a FG microbeam embedded in elastic Pasternak medium is proposed by Simsek and Reddy (2013). Zhang *et al.* (2014) developed a size-dependent FG beam model resting on Winkler-Pasternak elastic foundation based on an improved third-order shear deformation theory and provided the analytical solutions for the bending, buckling and free vibration problems. By searching the literature, it is found that a work analyzing the buckling of embedded FG nanobeams using the third order shear deformation beam theory hasn't been yet published.

In this work, the nonlocal beam model within the framework of third order shear deformation beam theory is developed for analysis of vibration of functionally graded nanobeam embedded on elastic foundations. Material properties of FG nanobeam are assumed to change continuously along the thickness according to two types of micromechanics models, namely, power-law model and Mori-Tanaka model. By using the Hamilton's principle the governing equations of motion are derived and Navier type solution method is used to solve the equations. The obtained results based on third order shear deformation beam theory

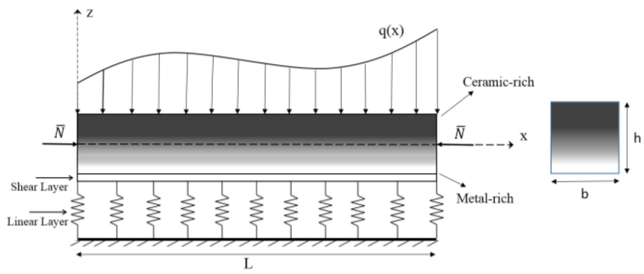

Fig. 1 Geometry and coordinates of FG nanobeam embedded on elastic foundation

are compared with those predicted by the previously published works to verify the accuracy of the present solution. Several numerical results are provided to indicate the influences of the gradient index, nonlocal parameter, Winkler and Pasternak parameters and slenderness ratio on the vibration behavior of FG nanobeams.

2. Governing equations

2.1 Power-law functionally graded material (P-FGM) beam

One of the most favorable models for FGMs is the power-law model, in which material properties of FGMs are supposed to change according to a power law about spatial coordinates. The coordinate system for FG nano beam is shown in Fig. 1. The FG nanobeam is assumed to be combination of ceramic and metal and effective material properties (P_f) of the FG beam such as Young's modulus E_f and mass density ρ are supposed to change continuously in the direction of z-axis (thickness direction) according to an power function of the volume fractions of the material constituents. So, the effective material properties, P_f can be stated as

$$P_f = P_c V_c + P_m V_m \tag{1}$$

Where subscripts m and c denote metal and ceramic, respectively and the volume fraction of the ceramic is associated to that of the metal in the following relation

$$V_c + V_m = 1 \tag{2a}$$

The volume fraction of the ceramic constituent of the beam is assumed to be given by

$$V_c = (\frac{z}{h} + \frac{1}{2})^P \tag{2b}$$

Here P is the power-law exponent which determines the material distribution through the thickness of the beam. Therefore, from Eqs. (1)-(2), the effective material properties of the FG nanobeam can be expressed as follows

$$E(z) = \left(E_c - E_m\right)\left(\frac{z}{h} + \frac{1}{2}\right)^p + E_m \tag{3a}$$

$$\rho(z) = \left(\rho_c - \rho_m\right)\left(\frac{z}{h} + \frac{1}{2}\right)^p + \rho_m \tag{3b}$$

$$\nu(z)=(\nu_c-\nu_m)\left(\frac{z}{h}+\frac{1}{2}\right)^p+\nu_m \tag{3c}$$

Additionally, in this study, Mori-Tanaka homogenization technique is also employed to model the effective material properties of the FG nanobeam. According to Mori-Tanaka homogenization technique the local effective material properties of the FG nanobeam such as effective local bulk modulus K_e and shear modulus μ_e can be calculated (Simsek and Reddy 2013)

$$\frac{K_e-K_m}{K_c-K_m}=\frac{V_c}{1+V_m(K_c-K_m)/(K_m+4\mu_m/3)} \tag{4a}$$

$$\frac{\mu_e-\mu_m}{\mu_c-\mu_m}=\frac{V_c}{1+V_m(\mu_c-\mu_m)/[(\mu_m+\mu_m(9K_m+8\mu_m)/(6(K_m+2\mu_m))]} \tag{4b}$$

Therefore from Eq. (4), the effective Young's modulus (E), Poisson's ratio (v) and mass density μ based on Mori-Tanaka scheme can be expressed by

$$E(z)=\frac{9K_e\mu_e}{3K_e+\mu_e} \tag{5a}$$

$$\nu(z)=\frac{3K_e-2\mu_e}{6K_e+2\mu_e} \tag{5b}$$

$$\rho(z)=\rho_c V_c+\rho_m V_m \tag{5c}$$

The shear modulus $G(z)$ of FG nanobeam with respect to both classical rule of mixture and Mori-Tanaka homogenization is defined as

$$G(z)=\frac{E(z)}{2(1+\nu(z))} \tag{6}$$

The material composition of FG nanobeam at the upper surface (z=+h/2) is supposed to be the pure ceramic and it changes continuously to the opposite side surface (z=-h/2) which is pure metal.

2.2 Kinematic relations

Based on the third order shear deformation (Reddy) beam theory, the displacement field at any point of the beam can be written as

$$u_x(x,z)=u(x)+z\varphi(x)-\alpha z^3(\varphi+\frac{\partial w}{\partial x}) \tag{7}$$

$$u_z(x,z)=w(x) \tag{8}$$

where $\alpha=\frac{4}{3h^2}$ and u and w are the longitudinal and the transverse displacements, φ is the rotation of the cross section at each point of the neutral axis. Nonzero strains of the Reddy beam model are expressed as follows

$$\varepsilon_{xx}=\varepsilon_{xx}^{(0)}+z\varepsilon_{xx}^{(1)}+z^3\varepsilon_{xx}^{(3)} \tag{9}$$

$$\gamma_{xz}=\gamma_{xz}^{(0)}+z^2\gamma_{xz}^{(2)} \tag{10}$$

Where

$$\varepsilon_{xx}^{(0)}=\frac{\partial u}{\partial x},\ \varepsilon_{xx}^{(1)}=\frac{\partial\varphi}{\partial x},\quad \varepsilon_{xx}^{(3)}=-\alpha(\frac{\partial\varphi}{\partial x}+\frac{\partial^2 w}{\partial x^2}) \tag{11}$$

$$\gamma_{xz}^{(0)}=\frac{\partial w}{\partial x}+\varphi,\ \gamma_{xz}^{(2)}=-\beta(\frac{\partial w}{\partial x}+\varphi) \tag{12}$$

And $\beta=\frac{4}{h^2}$. By using the Hamilton's principle, in which the motion of an elastic structure in the time interval $t1<t<t2$ is so that the integral with respect to time of the total potential energy is extremum

$$\int_0^t \delta(U-T+V)dt=0 \tag{13}$$

Here U is strain energy, T is kinetic energy and V is work done by external forces. The virtual strain energy can be calculated as

$$\delta U=\int_v \sigma_{ij}\delta\varepsilon_{ij}\,dV=\int_v(\sigma_{xx}\delta\varepsilon_{xx}+\sigma_{xz}\delta\gamma_{xz})dV \tag{14}$$

Substituting Eqs. (9) and (10) into Eq.(14) yields

$$\delta U=\int_0^L\begin{pmatrix}N\,\delta\varepsilon_{xx}^{(0)}+M\,\delta\varepsilon_{xx}^{(1)}+\\ P\,\delta\varepsilon_{xx}^{(3)}+Q\,\delta\gamma_{xz}^{(0)}+R\,\delta\gamma_{xz}^{(2)}\end{pmatrix}dx \tag{15}$$

In which the variables introduced in arriving at the last expression are defined as follows

$$N=\int_A\sigma_{xx}\,dA,\ M=\int_A\sigma_{xx}z\,dA,\ P=\int_A\sigma_{xx}z^3dA \quad Q=\int_A\sigma_{xz}\,dA,\ R=\int_A\sigma_{xz}z^2\,dA \tag{16}$$

The first variation of the work done by applied forces can be written in the form

$$\delta V=\int_0^L(\overline{N}\frac{\partial w}{\partial x}\frac{\partial\delta w}{\partial x}+q\,\delta w+f\,\delta u-k_w\,\delta w+k_p\frac{\partial^2\delta w}{\partial x^2}-N\,\delta\varepsilon_{xx}^{(0)}-M\frac{\partial\delta\varphi}{\partial x}+\alpha P\frac{\partial^2\delta w}{\partial x^2}-Q\delta\gamma_{xz}^{(0)})dx \tag{17}$$

where $\hat{M}=M-\alpha P,\ \hat{Q}=Q-\beta R$ and $\overline{N}$ is the applied axial compressive load and $q(x)$ and $f(x)$ are the transverse and axial distributed loads and k_w and k_p are linear and shear coefficient of elastic foundation. The first variation of the virtual kinetic energy can be written in the form

$$\delta K = \int_0^L I_0 \left(\frac{\partial u}{\partial t}\frac{\partial \delta u}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial \delta w}{\partial t} \right) + I_1\left(\frac{\partial u}{\partial t}\frac{\partial \delta\varphi}{\partial t} + \frac{\partial \varphi}{\partial t}\frac{\partial \delta u}{\partial t}\right) + I_2\frac{\partial \varphi}{\partial t}\frac{\partial \delta\varphi}{\partial t} + \alpha\Big[-I_3\frac{\partial u}{\partial t}\left(\frac{\partial^2 \delta w}{\partial x\,\partial t} + \frac{\partial \delta\varphi}{\partial t}\right) - I_3\frac{\partial \delta u}{\partial t}\left(\frac{\partial^2 w}{\partial x\,\partial t} + \frac{\partial \varphi}{\partial t}\right) - I_4\frac{\partial \varphi}{\partial t}\left(\frac{\partial \delta\varphi}{\partial t} + \frac{\partial^2 \delta w}{\partial x\,\partial t}\right) - I_4\frac{\partial \delta\varphi}{\partial t}\left(\frac{\partial \varphi}{\partial t} + \frac{\partial^2 w}{\partial x\,\partial t}\right) + \alpha I_6\left(\frac{\partial \varphi}{\partial t} + \frac{\partial^2 w}{\partial x\,\partial t}\right)\left(\frac{\partial \delta\varphi}{\partial t} + \frac{\partial^2 \delta w}{\partial x\,\partial t}\right)\Big] dA\,dx \quad (18)$$

In which I_0, I_1, I_2, I_3, I_4 and I_6 are mass inertia and defined as

$$(I_0, I_1, I_2, I_3, I_4, I_6) = \int_A (1, z, z^2, z^3, z^4, z^6)\rho dA \quad (19)$$

It is noticed from Eq. (19), for homogeneous nanobeams, $I_1=I_3=0$.

By Substituting Eqs. (15),(17) and (18) into Eq.(13) and setting the coefficients of δu, δw and $\delta\varphi$ to zero, the following Euler-Lagrange equation can be obtained

$$\frac{\partial N}{\partial x} + f - I_0\frac{\partial^2 u}{\partial t^2} - \hat{I}_1\frac{\partial^2 \varphi}{\partial t^2} + \alpha I_3\frac{\partial^3 w}{\partial x\partial t^2} = 0 \quad (20a)$$

$$\frac{\partial \hat{M}}{\partial x} - \hat{Q} - \hat{I}_1\frac{\partial^2 u}{\partial t^2} - \hat{I}_2\frac{\partial^2 \varphi}{\partial t^2} + \alpha\hat{I}_4\left(\frac{\partial^2 \varphi}{\partial t^3} + \frac{\partial^3 w}{\partial x\partial t^2}\right) = 0 \quad (20b)$$

$$\frac{\partial \hat{Q}}{\partial x} + q - \overline{N}\frac{\partial^2 w}{\partial x^2} + \alpha\frac{\partial^2 P}{\partial x^2} - I_0\frac{\partial^2 w}{\partial t^2} - \alpha I_3\frac{\partial^3 u}{\partial x\,\partial t^2} - \alpha I_4\frac{\partial^3 \varphi}{\partial x\,\partial t^2} + \alpha^2 I_6\left(\frac{\partial^3 \varphi}{\partial x\,\partial t^2} + \frac{\partial^4 w}{\partial x^2\partial t^2}\right) - k_w w + k_p\frac{\partial^2 w}{\partial x^2} = 0 \quad (20c)$$

where $\hat{I}_1 = I_1 - \alpha I_3, \hat{I}_2 = I_2 - \alpha I_4, \hat{I}_4 = I_4 - \alpha I_6$.

2.3 The nonlocal elasticity model for FG nanobeam

According to Eringen nonlocal elasticity model (Eringen and Edelen 1972), the stress state at a point inside a body is regarded to be function of strains of all points in the neighbor regions. For homogeneous elastic solids the nonlocal stress-tensor components σ_{ij} at each point x in the solid can be defined as

$$\sigma_{ij}(x) = \int_\Omega \alpha(|x' - x|, \tau)\, t_{ij}(x')\, d\Omega(x') \quad (21)$$

where $t_{ij}(x')$ are the components available in local stress tensor at point x which are associated to the strain tensor components ε_{kl} as

$$t_{ij} = C_{ijkl}\varepsilon_{kl} \quad (22)$$

The concept of Eq. (19) is that the nonlocal stress at any point is weighting average of local stress of all points in the near region that point, the size that is related to the nonlocal kernel $\alpha(|x'-x|, \tau)$. Also $|x'-x|$ is Euclidean distance and τ is a constant as follows

$$\tau = \frac{e_0 a}{l} \quad (23)$$

which indicates the relation of a characteristic internal length, (for instance lattice parameter, C-C bond length and granular distance) and a characteristic external length, l (for instance crack length and wavelength) using a constant, e_0, dependent on each material. The value of e_0 is experimentally estimated by comparing the scattering curves of plane waves and atomistic dynamics. According to (Eringen and Edelen, 1972) for a class of physically admissible kernel $\alpha(|x'-x|, \tau)$ it is possible to represent the integral constitutive relations given by Eq. (22) in an equivalent differential form as

$$(1-(e_0 a)\nabla^2)\sigma_{kl} = t_{kl} \quad (24)$$

where ∇^2 is the Laplacian operator. Thus, the scale length e_0a conside the influences of small scles on the response of nano-structures. The magnitude of the small scale parameter relies on several parameters including mode shapes, boundary conditions, chirality and the essence of motion. The parameter $e_0 = (\pi^2 - 4)^{1/2}/2\pi \cong 0.39$ was given by Eringen (1983). Also, Zhang *et al*. (2005) found the value of 0.82 nm for nonlocal parameter when they compared the vibrational results of simply supported single-walled carbon nanotubes with molecular dynamics simulations. The nonlocal parameter, μ, is experimentally obtained for various materials; for instance, a conservative estimate of μ<4 (nm)^2 for a single-walled carbon nanotube is proposed (Wang 2005). It is worth mentioning that this magnitude is dependent of size and chirality, because the properties of carbon nanotubes are extensively confirmed to be dependent of chirality. There is no serious study conducted to determining the value of small scale to simulate mechanical behavior of FG micro/nanobeams (Eltaher *et al*. 2012). Hence all researchers who worked on size-dependent mechanical behavior of functionally graded nanobeams on the basis the nonlocal elasticity method investigated the influence of small scale parameter on mechanical behaviour of FG nanobeams by changing the value of the small scale parameter. So, for a material in the one-dimension case, the constitutive relations of nonlocal theory can be expressed as

$$\sigma_{xx} - (e_0 a)^2\frac{\partial^2 \sigma_{xx}}{\partial x^2} = E\,\varepsilon_{xx} \quad (25)$$

$$\sigma_{xz} - (e_0 a)^2\frac{\partial^2 \sigma_{xz}}{\partial x^2} = G\,\gamma_{xz} \quad (26)$$

where σ and ε are the nonlocal stress and strain, respectively. For a nonlocal FG beam, Eqs. (25) and (26) can be written as

$$\sigma_{xx} - \mu \frac{\partial^2 \sigma_{xx}}{\partial x^2} = E(z)\varepsilon_{xx} \tag{27}$$

$$\sigma_{xz} - \mu \frac{\partial^2 \sigma_{xz}}{\partial x^2} = G(z)\gamma_{xz} \tag{28}$$

where $\mu=(e_0a)^2$. Integrating Eqs. (27) and (28) over the beam's cross-section area, we obtain the force-strain and the moment-strain of the nonlocal Reddy FG beam theory can be obtained as follows

$$\begin{aligned} & N - \mu \frac{\partial^2 N}{\partial x^2} = A_{xx} \frac{\partial u}{\partial x} + \\ & (B_{xx} - \alpha E_{xx}) \frac{\partial \varphi}{\partial x} - \alpha E_{xx} \frac{\partial^2 w}{\partial x^2} \end{aligned} \tag{29}$$

$$\begin{aligned} & M - \mu \frac{\partial^2 M}{\partial x^2} = \\ & B_{xx} \frac{\partial u}{\partial x} + (D_{xx} - \alpha F_{xx}) \frac{\partial \varphi}{\partial x} - \alpha F_{xx} \frac{\partial^2 w}{\partial x^2} \end{aligned} \tag{30}$$

$$\begin{aligned} & P - \mu \frac{\partial^2 P}{\partial x^2} = \\ & E_{xx} \frac{\partial u}{\partial x} + (F_{xx} - \alpha H_{xx}) \frac{\partial \varphi}{\partial x} - \alpha H_{xx} \frac{\partial^2 w}{\partial x^2} \end{aligned} \tag{31}$$

$$Q - \mu \frac{\partial^2 Q}{\partial x^2} = (A_{xz} - \beta D_{xz})(\frac{\partial w}{\partial x} + \varphi) \tag{32}$$

$$R - \mu \frac{\partial^2 R}{\partial x^2} = (D_{xz} - \beta F_{xz})(\frac{\partial w}{\partial x} + \varphi) \tag{33}$$

In which the cross-sectional rigidities are defined as follows

$$\begin{aligned} & (A_{xx}, B_{xx}, D_{xx}, E_{xx}, F_{xx}, H_{xx}) = \\ & \int_A E(z)(1, z, z^2, z^3, z^4, z^6) dA \end{aligned} \tag{34}$$

$$(A_{xz}, D_{xz}, F_{xz}) = \int_A G(z)(1, z^2, z^4) dA \tag{35}$$

The explicit relation of the nonlocal normal force can be derived by substituting for the second derivative of N from Eq. (20a) into Eq. (29) as follows

$$\begin{aligned} & N = A_{xx} \frac{\partial u}{\partial x} + K_{xx} \frac{\partial \varphi}{\partial x} - \alpha E_{xx} \frac{\partial^2 w}{\partial x^2} \\ & + \mu(-\frac{\partial f}{\partial x} + I_0 \frac{\partial^3 u}{\partial x \partial t^2} + \hat{I}_1 \frac{\partial^3 \varphi}{\partial x \partial t^2} - \alpha I_3 \frac{\partial^4 w}{\partial x^2 \partial t^2}) \end{aligned} \tag{36}$$

Eliminating $\hat{Q}$ from Eqs. (20.b) and (20.c), we obtain the following equation

$$\begin{aligned} & \frac{\partial^2 \hat{M}}{\partial x^2} = -\alpha \frac{\partial^2 P}{\partial x^2} - q + \overline{N} \frac{\partial^2 w}{\partial x^2} + \\ & I_0 \frac{\partial^2 w}{\partial t^2} + I_1 \frac{\partial^3 u}{\partial x \partial t^2} + I_2 \frac{\partial^3 \varphi}{\partial x \partial t^2} \\ & -\alpha I_4 (\frac{\partial^3 \varphi}{\partial x \partial t^2} + \frac{\partial^4 w}{\partial x^2 \partial t^2}) + k_w w - k_p \frac{\partial^2 w}{\partial x^2} \end{aligned} \tag{37}$$

Also the explicit relation of the nonlocal bending moment can be derived by substituting for the second derivative of M from Eq. (20b) into Eq. (30) and using Eqs. (30) and (31) as follows

$$\begin{aligned} & M = K_{xx} \frac{\partial u}{\partial x} + I_{xx} \frac{\partial \varphi}{\partial x} - \alpha J_{xx} (\frac{\partial \varphi}{\partial x} + \frac{\partial^2 w}{\partial x^2}) \\ & + \mu(-\alpha \frac{\partial^2 P}{\partial x^2} - q + \frac{\partial}{\partial x}(\overline{N} \frac{\partial w}{\partial x}) + k_w w \\ & -k_p \frac{\partial^2 w}{\partial x^2} + I_0 \frac{\partial^2 w}{\partial t^2} + I_1 \frac{\partial^3 u}{\partial x \partial t^2} + \\ & I_2 \frac{\partial^3 \varphi}{\partial x \partial t^2} - \alpha I_4 (\frac{\partial^3 \varphi}{\partial x \partial t^2} + \frac{\partial^4 w}{\partial x^2 \partial t^2})) \end{aligned} \tag{38}$$

where

$$\begin{aligned} & K_{xx} = B_{xx} - \alpha E_{xx}, \\ & I_{xx} = D_{xx} - \alpha F_{xx}, \; J_{xx} = F_{xx} - \alpha H_{xx} \end{aligned} \tag{39}$$

By substituting for the second derivative of Q from Eq. (20c) into Eq. (32), and using Eqs. (32) and (33) the following expression for the nonlocal shear force is derived

$$\begin{aligned} & \hat{Q} = \overline{A}_{xz} (\frac{\partial w}{\partial x} + \varphi) + \\ & \mu(\overline{N} \frac{\partial^3 w}{\partial x^3} - \alpha \frac{\partial^3 P}{\partial x^3} - \frac{\partial q}{\partial x} + k_w \frac{\partial w}{\partial x} - k_p \frac{\partial^3 w}{\partial x^3}) \\ & + \mu(I_0 \frac{\partial^3 w}{\partial x \partial t^2} + \alpha I_3 \frac{\partial^4 u}{\partial x^2 \partial t^2} + \alpha I_4 \frac{\partial^4 \varphi}{\partial x^2 \partial t^2} - \\ & \alpha^2 I_6 (\frac{\partial^5 w}{\partial x^3 \partial t^2} + \frac{\partial^4 \varphi}{\partial x^2 \partial t^2})) \end{aligned} \tag{40}$$

where

$$\begin{aligned} & \overline{A}_{xz} = A^*_{xz} - \beta I^*_{xz}, \\ & A^*_{xz} = A_{xz} - \beta D_{xz}, \; I^*_{xz} = D_{xz} - \beta F_{xz} \end{aligned} \tag{41}$$

Now we use M and Q from Eqs. (38) and (40) and the identity

$$\begin{aligned} & \alpha \frac{\partial^2}{\partial x^2}(P - \mu \frac{\partial^2 P}{\partial x^2}) = \alpha(E_{xx} \frac{\partial^3 u}{\partial x^3} + F_{xx} \frac{\partial^3 \varphi}{\partial x^3} \\ & -\alpha H_{xx} (\frac{\partial^3 \varphi}{\partial x^3} + \frac{\partial^4 \varphi}{\partial x^4})) \end{aligned} \tag{42}$$

The nonlocal governing equations of third order shear deformation FG nanobeam in terms of the displacement can be derived by substituting for N, M and Q from Eqs. (36), (38) and (40), respectively , into Eq. (20) as follows

$$\begin{aligned} & A_{xx} \frac{\partial^2 u}{\partial x^2} + K_{xx} \frac{\partial^2 \varphi}{\partial x^2} - \alpha E_{xx} \frac{\partial^3 w}{\partial x^3} + \\ & \mu(-\frac{\partial^2 f}{\partial x^2} + I_0 \frac{\partial^4 u}{\partial x^2 \partial t^2} + I_1 \frac{\partial^4 \varphi}{\partial x^2 \partial t^2} \\ & -\alpha I_3 \frac{\partial^4 \varphi}{\partial x^2 \partial t^2} - \alpha I_3 \frac{\partial^5 w}{\partial x^3 \partial t^2}) + f \\ & - I_0 \frac{\partial^2 u}{\partial t^2} - \hat{I}_1 \frac{\partial^2 \varphi}{\partial t^2} + \alpha I_3 \frac{\partial^3 w}{\partial x \partial t^2} = 0 \end{aligned} \tag{43}$$

$$K_{xx}\frac{\partial^2 u}{\partial x^2}+I_{xx}\frac{\partial^2\varphi}{\partial x^2}-\alpha J_{xx}(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2 w}{\partial x^3})$$
$$-\bar{A}_{xz}(\varphi+\frac{\partial w}{\partial x})-\hat{I}_2\frac{\partial^2\varphi}{\partial t^2}$$
$$+\alpha\hat{I}_4(\frac{\partial^2\varphi}{\partial t^2}+\frac{\partial^3 w}{\partial x\partial t^2})-\hat{I}_1\frac{\partial^2 u}{\partial t^2}+\mu(\hat{I}_1\frac{\partial^4 u}{\partial x^2\partial t^2}+\hat{I}_2\frac{\partial^4\varphi}{\partial x^2\partial t^2} \tag{44}$$
$$-\alpha\hat{I}_4(\frac{\partial^4\varphi}{\partial x^2\partial t^2}+\frac{\partial^5 w}{\partial x^3\partial t^2}))=0$$

$$\bar{A}_{xz}\left(\frac{\partial\varphi}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+$$
$$\mu(\bar{N}\frac{\partial^4 w}{\partial x^4}-\frac{\partial^2 q}{\partial x^2}+k_w\frac{\partial^2 w}{\partial x^2}-k_p\frac{\partial^4 w}{\partial x^4})$$
$$+q-\bar{N}\frac{\partial^2 w}{\partial x^2}-k_w w+k_p\frac{\partial^2 w}{\partial x^2}+$$
$$\alpha(E_{xx}\frac{\partial^3 u}{\partial x^3}+J_{xx}\frac{\partial^3\varphi}{\partial x^3}-\alpha H_{xx}\frac{\partial^4 w}{\partial x^4})$$
$$-I_0\frac{\partial^2 w}{\partial t^2}-\alpha I_3\frac{\partial^3 u}{\partial x\partial t^2}-\alpha I_4\frac{\partial^3\varphi}{\partial t^2\partial x} \tag{45}$$
$$+\alpha^2 I_6(\frac{\partial^3\varphi}{\partial t^2\partial x}+\frac{\partial^4 w}{\partial t^2\partial x^2})+\mu(I_0\frac{\partial^4 w}{\partial x^2\partial t^2}$$
$$+\alpha I_3\frac{\partial^5 u}{\partial x^3\partial t^2}+\alpha I_4\frac{\partial^5\varphi}{\partial t^2\partial x^3}$$
$$-\alpha^2 I_6(\frac{\partial^5\varphi}{\partial t^2\partial x^3}+\frac{\partial^6 w}{\partial t^2\partial x^4}))=0$$

3. Solution procedures

Here, on the basis the Navier method, an analytical solution of the governing equations for free vibration of a simply supported FG nanobeam is presented. To satisfy governing equations of motion and the simply supported boundary condition, the displacement variables are adopted to be of the form

$$u(x,t)=\sum_{n=1}^{\infty}U_n\cos(\frac{n\pi}{L}x)e^{i\omega_n t} \tag{46}$$

$$w(x,t)=\sum_{n=1}^{\infty}W_n\sin(\frac{n\pi}{L}x)e^{i\omega_n t} \tag{47}$$

$$\varphi(x,t)=\sum_{n=1}^{\infty}\varphi_n\cos(\frac{n\pi}{L}x)e^{i\omega_n t} \tag{48}$$

where (U_n, W_n, φ_n) are the unknown Fourier coefficients to be determined for each n value. Boundary conditions for simply supported beam are as Eq. (49)

$$u(0)=0\ ,\ \frac{\partial u}{\partial x}(L)=0$$
$$w(0)=w(L)=0\ ,\ \frac{\partial\varphi}{\partial x}(0)=\frac{\partial\varphi}{\partial x}(L)=0 \tag{49}$$

Substituting Eqs. (46)-(48) into Eqs. (43)-(45) respectively, leads to Eqs. (47)-(49)

$$(-A_{xx}(\frac{n\pi}{l})^2+I_0(1+\mu(\frac{n\pi}{l})^2)\omega^2{}_n)U_n+$$
$$(-K_{xx}(\frac{n\pi}{l})^2+\hat{I}_1((1+\mu(\frac{n\pi}{l})^2\omega_n^2)\phi_n$$
$$+(\alpha E_{xx}(\frac{n\pi}{l})^3-\alpha I_3(\frac{n\pi}{l})\omega^2{}_n- \tag{50}$$
$$\alpha I_3\mu(\frac{n\pi}{l})^3\omega^2{}_n)W_n=0$$

$$(-K_{xx}(\frac{n\pi}{l})^2+\hat{I}_1\omega^2{}_n(\frac{n\pi}{l})^2)U_n+(-I_{xx}(\frac{n\pi}{l})^2$$
$$+\alpha J_{xx}(\frac{n\pi}{l})^2-\bar{A}_{xz}+\hat{I}_2\omega^2{}_n-\alpha\hat{I}_4\omega^2{}_n+\mu((\hat{I}_2-\alpha\hat{I}_4)$$
$$\omega^2{}_n(\frac{n\pi}{l})^2)\phi_n+(\alpha J_{xx}(\frac{n\pi}{l})^3-\bar{A}_{xz}(\frac{n\pi}{l}) \tag{51}$$
$$-\alpha\hat{I}_4\omega^2{}_n(\frac{n\pi}{l})+\mu\alpha\hat{I}_4\omega^2{}_n(\frac{n\pi}{l})^3)W_n$$

$$(\alpha E_{xx}(\frac{n\pi}{l})^3-\alpha I_3(\frac{n\pi}{l})\omega^2{}_n-\mu\alpha I_3(\frac{n\pi}{l})^3\omega^2{}_n)U_n$$
$$+(-\bar{A}_{xz}(\frac{n\pi}{l})+J_{xx}(\frac{n\pi}{l})^3$$
$$-\alpha\hat{I}_4(\frac{n\pi}{l})\omega^2{}_n+\mu(-\alpha\hat{I}_4(\frac{n\pi}{l})^3\omega^2{}_n))\phi_n+$$
$$(-\bar{A}_{xz}(\frac{n\pi}{l})^2-k_w(1+\mu(\frac{n\pi}{l})^2)$$
$$-k_p(\frac{n\pi}{l})^2(1+\mu(\frac{n\pi}{l})^2)+ \tag{52}$$
$$\bar{N}(\frac{n\pi}{l})^2(1+\mu(\frac{n\pi}{l})^2)-\alpha^2(\frac{n\pi}{l})^4+I_0(\frac{n\pi}{l})^2$$
$$+\alpha^2 I_6(\frac{n\pi}{l})^2\omega^2{}_n+\mu(I_0(\frac{n\pi}{l})^2\omega^2{}_n$$
$$+\alpha^2 I_6(\frac{n\pi}{l})^4\omega^2{}_n))W_n=0$$

By setting the determinant of the coefficient matrix of the above equations, the analytical solutions can be obtained from the following equations

$$([K]-\omega^2[M])\begin{bmatrix}U_n\\W_n\\\phi_n\end{bmatrix}=0 \tag{53}$$

where [K] is stiffness matrix and [M] is the mass matrix. By setting this polynomial to zero, we can find natural frequencies ω_n.

4. Numerical results and discussions

Through this section, the effects of FG distribution, nonlocality effect and mode number on the natural frequencies of the FG nanobeam will be figured out. The

Table 1 Material properties of FGM constituents

Properties	Steel	Alumina (Al_2O_3)
E	210 (GPa)	390 (GPa)
ρ	7800 (kg/m^3)	3960 (kg/m^3)
ν	0.3	0.3

FG nanobeam is a combination of Steel and Alumina (Al_2O_3) where their properties are given in Table 1. The following dimensions for the beam geometry is considered: L (length)=10000 nm, b (width)=1000 nm (Eltaher *et al*. 2012, Rahmani and Pedram 2014). Also, for better presentation of the results the following dimensionless quantities are adopted

$$\hat{\omega} = \omega L^2 \sqrt{\frac{\rho_c A}{EI_c}},\ K_w = k_w \frac{L^4}{E_c I},\ K_p = k_p \frac{L^2}{E_c I} \qquad (54)$$

where $I=bh^3/12$ is the moment inertia of the beam's cross section. To evaluate the correctness of the non-dimensional frequencies predicted by the present method, the natural frequencies of simply supported FG nanobeam with various nonlocal parameters and gradient indexes are compared with the results presented by Eltaher *et al*. (2012) For Euler-Bernoulli FG nanobeams and Rahmani and Pedram (2014) which has been obtained by analytical method for FG Timoshenko nanobeam. The reliability of the presented method and procedure for FG nanobeam may be concluded from Table 2; where the results are in an excellent agreement as values of non-dimensional fundamental frequency are consistent with presented analytical solution.

The variation of the first three non-dimensional frequencies of FG nanobeam based on power-law and Mori-Tanaka models for various gradient indexes (p=0.0.2,1,5), nonlocal parameters, foundation parameters (K_p=0,5,10, K_w=0,25,50,100) and slenderness ratios is tabulated in Tables 3-5, respectively. The obtained results for Mori-Tanaka model and power-law model are referred to as MT-FGM and PL-FGM, respectively. It can be seen from the results of the tables that the non-dimensional frequencies predicted by power-law model are greater than that of Mori-Tanaka homogenization scheme, due to the fact that FG nanobeam becomes more flexible according to Mori-Tanaka homogenization scheme than with respect to power-law model at a constant gradient index. Also, when p=0 the results of Mori-Tanaka and power-law models are exactly same because the nanobeam is full ceramic. Therefore, the difference in results of these two models becomes prominent when the gradient index value is higher than p=0. Also it should be noted from Tables 3-5 that, when the gradient index increases the non-dimensional frequencies reduces at a constant nonlocal parameter. Moreover, by fixing gradient index and increasing nonlocal parameter the non-dimensional frequencies decreases. Furthermore, it must be cited that when the Winkler or Pasternak parameters increase the non-dimensional frequencies increases which indicates the stiffening influence of foundation parameters on the FG nanobeam.

Table 6 presents the variation of dimensionless

Table 2 Comparison of the nondimensional fundamental frequency for an S-S FG nanobeam with various gradient indexes when L/h=20

	P=0			P=0.5		
μ	EBT (Eltaher *et al.* 2012)	TBT (Rahmani and Pedram 2014)	Present RBT	EBT (Eltaher *et al.* 2012)	TBT (Rahmani and Pedram 2014)	Present RBT
0	9.8797	9.8296	9.82957	7.8061	7.7149	7.71546
1	9.4238	9.3777	9.377686	7.4458	7.3602	7.36078
2	9.0257	8.9829	8.982894	7.1312	7.0504	7.0509
3	8.6741	8.6341	8.634103	6.8533	6.7766	6.77714
4	8.3607	8.323	8.323021	6.6057	6.5325	6.53296
5	8.0789	8.0433	8.043309	6.383	6.3129	6.31342
	P=1			P=5		
0	7.0904	6.9676	6.967613	6.0025	5.9172	5.916152
1	6.7631	6.6473	6.6473	5.7256	5.6452	5.644175
2	6.4774	6.3674	6.367454	5.4837	5.4075	5.406561
3	6.2251	6.1202	6.120217	5.2702	5.1975	5.196632
4	6.0001	5.8997	5.899708	5.0797	5.0103	5.0094
5	5.7979	5.7014	5.701436	4.9086	4.8419	4.841049

frequency of thick embedded FG nanobeam for both the third-order and Euler-Bernoulli beam model at K_w=25. K_p=5. This table shows that the difference in frequency results of classical and higher order beam theory becomes more prominent as the thickness increases. EBT present larer frequencies than third order beam model by neglecting the shear deformation effects, especially at L/h=5.

The effect of existence of elastic foundation on the first non-dimensional frequencies of FG nanobeam versus gradient index at L/h=20 is presented in Fig. 2, respectively so that the variation of the non-dimensional frequency with and without elastic foundation according to both power-law and Mori-Tanaka methods are compared with each other. It is observable from the figure that the non-dimensional frequency of FG nanobeam embedded in elastic medium are greater than that of FG nanobeam without elastic foundation. Since when the foundation parameters increase the nanobeam becomes more rigid. Also, the Mori-Tanaka scheme determines lower values for the non-dimensional frequencies with comparing to the power-law model. The reason is that, Mori-Tanaka model produces lower values for Young's modulus than the power-law model, and that gives rise to a more flexible structure. Furthermore, usually the differences of the frequencies between PL and MT models are insignificant, specifically for lower gradient indexes. Also, it is found that, the first non-dimensional frequency reduces with rigorous rate where the gradient index changes from 0 to 2 than that where gradient index changes from 2 to 10. Also it is observable that increasing nonlocal parameter shows a decreasing effect on the first non-dimensional frequency. Therefore, as a consequence, the existence of nonlocality and Winkler foundation softens and stiffens the structure, respectively. Fig. 3, shows the variation of the first non-dimensional frequency of S-S FG nanobeam versus slenderness ratio (L/h) when K_p=5, K_w=25

Table 3 Variation of the first non-dimensional fundamental frequency for an embedded S-S FG nanobeam with various gradient indexes and nonlocal parameters when L/h=20

(K_w, K_p)	μ	Gradient index							
		0		0.2		1		5	
		PL-FGM	MT-FGM	PL-FGM	MT-FGM	PL-FGM	MT-FGM	PL-FGM	MT-FGM
(0,0)	0	9.82957	9.82957	8.66012	8.55348	6.96748	6.89192	5.91610	5.86622
	1	9.37769	9.37769	8.26200	8.16026	6.64717	6.57509	5.64413	5.59654
	2	8.98289	8.98289	7.91418	7.81672	6.36733	6.29828	5.40651	5.36093
	3	8.6341	8.63410	7.60688	7.51321	6.1201	6.05373	5.19659	5.15277
	4	8.32302	8.32302	7.33281	7.24252	5.89959	5.83562	5.00936	4.96712
(25,0)	0	11.0259	11.0259	9.82210	9.72819	8.08363	8.01859	6.98564	6.94347
	1	10.625	10.6250	9.47294	9.38433	7.80924	7.74797	6.75686	6.71718
	2	10.2782	10.2782	9.17115	9.08718	7.57247	7.51449	6.55968	6.52218
	3	9.97482	9.97482	8.90733	8.82746	7.36579	7.31074	6.38776	6.35219
	4	9.7068	9.70680	8.67444	8.59824	7.18363	7.13118	6.23640	6.20254
(25,5)	0	13.0697	13.0697	11.7841	11.7059	9.92500	9.87209	8.71994	8.68621
	1	12.7334	12.7334	11.4947	11.4218	9.70282	9.65357	8.53776	8.50641
	2	12.4455	12.4455	11.2473	11.1789	9.51329	9.46721	8.38257	8.35328
	3	12.1961	12.1961	11.0332	10.9688	9.34962	9.30630	8.24873	8.22124
	4	11.9779	11.9779	10.8461	10.7852	9.20679	9.16592	8.13208	8.10618
(25,10)	0	14.8346	14.8346	13.4632	13.3948	11.4746	11.4288	10.1625	10.1336
	1	14.5391	14.5391	13.2106	13.1472	11.2830	11.2406	10.0066	9.97986
	2	14.2877	14.2877	12.9959	12.9367	11.1204	11.0810	9.87449	9.84967
	3	14.0710	14.0710	12.8111	12.7556	10.9807	10.9438	9.76113	9.73794
	4	13.8823	13.8823	12.6503	12.5981	10.8594	10.8247	9.66276	9.64099
(50,0)	0	12.1045	12.1045	10.8605	10.7756	9.06335	9.00539	7.91190	7.87470
	1	11.7405	11.7405	10.5457	10.4662	8.81950	8.76529	7.71065	7.67592
	2	11.4276	11.4276	10.2755	10.2006	8.61055	8.55961	7.53846	7.50586
	3	11.1556	11.1556	10.0407	9.96994	8.42937	8.38130	7.38935	7.35863
	4	10.9166	10.9166	9.83473	9.76757	8.27066	8.22514	7.25890	7.22985
(50,5)	0	13.9917	13.9917	12.6626	12.5899	10.7380	10.6891	9.47820	9.44720
	1	13.6780	13.6780	12.3937	12.3261	10.5330	10.4876	9.31087	9.28215
	2	13.4104	13.4104	12.1646	12.1014	10.3587	10.3164	9.16878	9.14202
	3	13.1793	13.1793	11.9670	11.9076	10.2086	10.1689	9.04658	9.02153
	4	12.9776	12.9776	11.7946	11.7387	10.0779	10.0406	8.94034	8.91680
(50,10)	0	15.6530	15.6530	14.2384	14.1738	12.1846	12.1416	10.8201	10.7930
	1	15.3732	15.3732	13.9999	13.9400	12.0044	11.9646	10.6738	10.6488
	2	15.1356	15.1356	13.7974	13.7417	11.8517	11.8147	10.5501	10.5269
	3	14.9313	14.9313	13.6235	13.5714	11.7207	11.6862	10.4441	10.4224
	4	14.7536	14.7536	13.4724	13.4234	11.6071	11.5747	10.3522	10.3319

for different gradient indexes based on Mori-Tanaka model. It is observed from the figure that, the first non-dimensional frequency increases with increase in slenderness ratio. But this observation is more accurate when slenderness ratio is less than L/h=20. So, it must be concluded that the influence of slenderness ratio on the non-dimensional frequency for the values larger than L/h=20 is not sensible. The variation of the first non-dimensional frequency of S-S MT-FGM nanobeam with Winkler parameter for various nonlocal parameters and gradient indexes is presented in Fig. 4. It is seen that with an increase in the Winkler parameter the first non-dimensional frequency increases for all gradient indexes. Also, it is seen that increasing the gradient index results in reduction in the first non-dimensional frequency at any Winkler and nonlocal parameters. In addition, it is worth noting that, the nonlocal parameter possess a softening effect and when increases the first non-dimensional frequency reduces for all gradient indexes. The variation of the first non-dimensional frequency of S-S FG nanobeam versus Pasternak parameter K_p and various gradient indexes and nonlocal parameters at K_w=25 is plotted in Fig. 5. It can be seen that with increasing the Pasternak parameter the non-dimensional frequency increases for all gradient indexes and nonlocal parameters. Also, it is observed that increasing the gradient index yields in increment of the first non-dimensional

Table 4 Variation of the second non-dimensional fundamental frequency for an embedded S-S FG nanobeam with various gradient indexes and nonlocal parameters when L/h=20

(K_w, K_p)	μ	Gradient index							
		0		0.2		1		5	
		PL-FGM	MT-FGM	PL-FGM	MT-FGM	PL-FGM	MT-FGM	PL-FGM	MT-FGM
(0,0)	0	38.8521	38.8521	34.2318	33.8103	27.5337	27.2299	23.3508	23.1549
	1	32.8974	32.8974	28.9852	28.6283	23.3137	23.0564	19.7719	19.6060
	2	29.0429	29.0429	25.5891	25.2741	20.5821	20.3550	17.4553	17.3089
	3	26.2878	26.2878	23.1616	22.8764	18.6296	18.4240	15.7994	15.6669
	4	24.1923	24.1923	21.3153	21.0529	17.1446	16.9554	14.5400	14.4180
(25,0)	0	39.1701	39.1701	34.5422	34.1245	27.8352	27.5347	23.6428	23.4494
	1	33.2723	33.2723	29.3511	28.9987	23.6690	23.4156	20.1159	19.9529
	2	29.4669	29.4669	26.0029	25.6929	20.9837	20.7610	17.8440	17.7008
	3	26.7554	26.7554	23.6179	23.3383	19.0723	18.8716	16.2278	16.0988
	4	24.6997	24.6997	21.8103	21.5539	17.6247	17.4407	15.0044	14.8863
(25,5)	0	41.5951	41.5951	36.9012	36.5104	30.1098	29.8322	25.8324	25.6557
	1	36.0957	36.0957	32.0940	31.7719	26.3063	26.0786	22.6491	22.5046
	2	32.6214	32.6214	29.0635	28.7863	23.9190	23.7238	20.6577	20.5344
	3	30.1945	30.1945	26.9507	26.7059	22.2611	22.0893	19.2789	19.1706
	4	28.3889	28.3889	25.3816	25.1615	21.0341	20.8801	18.2610	18.1643
(25,10)	0	43.8863	43.8863	39.1182	38.7498	32.2242	31.9650	27.8504	27.6867
	1	38.7138	38.7138	34.6203	34.3218	28.7024	28.4938	24.9262	24.7952
	2	35.4966	35.4966	31.8312	31.5782	26.5315	26.3557	23.1317	23.0217
	3	33.2801	33.2801	29.9145	29.6940	25.0471	24.8945	21.9091	21.8141
	4	31.6510	31.6510	28.5090	28.3131	23.9632	23.8281	21.0190	20.9352
(50,0)	0	39.4854	39.4854	34.8498	34.4358	28.1334	27.8362	23.9312	23.7401
	1	33.6430	33.6430	29.7125	29.3644	24.0190	23.7694	20.4541	20.2938
	2	29.8848	29.8848	26.4101	26.1049	21.3778	21.1592	18.2244	18.0843
	3	27.2150	27.2150	24.0655	23.7912	19.5051	19.3088	16.6452	16.5195
	4	25.1968	25.1968	22.2943	22.0435	18.0921	17.9129	15.4549	15.3402
(50,5)	0	41.8922	41.8922	37.1893	36.8016	30.3857	30.1107	26.0966	25.9217
	1	36.4377	36.4377	32.4248	32.1061	26.6217	26.3967	22.9500	22.8075
	2	32.9994	32.9994	29.4284	29.1547	24.2655	24.0731	20.9872	20.8658
	3	30.6025	30.6025	27.3439	27.1026	22.6329	22.4640	19.6315	19.5252
	4	28.8225	28.8225	25.7987	25.5821	21.4272	21.2761	18.6329	18.5382
(50,10)	0	44.1680	44.1680	39.3901	39.0242	32.4822	32.2251	28.0957	27.9334
	1	39.0328	39.0328	34.9272	34.6314	28.9917	28.7852	25.1999	25.0703
	2	35.8443	35.8443	32.1647	31.9144	26.8443	26.6705	23.4264	23.3178
	3	33.6507	33.6507	30.2692	30.0513	25.3782	25.2276	22.2200	22.1263
	4	32.0405	32.0405	28.8810	28.6875	24.3090	24.1759	21.3429	21.2604

frequency at any Pasternak parameter. With Comparison of this figure with Fig. 4, it is specified that the influence of the Winkler parameter (K_w) on the non-dimensional frequency is less prominent than that of the Pasternak parameter (K_p). Figs. 6 and 7 depict the influences of mode number as well as foundation parameters, K_w and K_p on the variation of the non-dimensional frequency of S-S FG nanobeam with changing of nonlocal parameter at L/h=20, p=0.2. It is seen from the figures that, for the higher modes the influence of nonlocal parameter is more sensible. Also it is observable that increasing foundation parameters has affected all modes of FG nanobeam and increases the values of non-dimensional frequencies.

5. Conclusions

Free vibration characteristics of the third order shear deformable FG nanobeams embedded on elastic medium are investigated on the basis of nonlocal elasticity theory in

Table 5 Variation of the third non-dimensional fundamental frequency for an embedded S-S FG nanobeam with various gradient indexes and nonlocal parameters when L/h=20

(K_w, K_p)	μ	Gradient index							
		0		0.2		1		5	
		PL-FGM	MT-FGM	PL-FGM	MT-FGM	PL-FGM	MT-FGM	PL-FGM	MT-FGM
(0,0)	0	85.7679	85.7679	75.5751	74.6448	60.7624	60.0743	51.4360	51.0080
	1	62.4156	62.4156	54.9981	54.3211	44.2184	43.7177	37.4314	37.1199
	2	51.4723	51.4723	45.3553	44.7970	36.4656	36.0527	30.8686	30.6117
	3	44.8023	44.8023	39.4779	38.9920	31.7402	31.3808	26.8685	26.6449
	4	40.1951	40.1951	35.4183	34.9823	28.4763	28.1538	24.1055	23.9049
(25,0)	0	85.9111	85.9111	75.7149	74.7863	60.8983	60.2117	51.5681	51.1412
	1	62.6123	62.6123	55.1900	54.5154	44.4050	43.9064	37.6127	37.3028
	2	51.7106	51.7106	45.5878	45.0324	36.6916	36.2812	31.0882	30.8332
	3	45.0759	45.0759	39.7449	39.2622	31.9996	31.6431	27.1205	26.8990
	4	40.4998	40.4998	35.7156	35.2833	28.7651	28.4459	24.3861	24.1878
(25,5)	0	88.4171	88.4171	78.1568	77.2574	63.2639	62.6033	53.8606	53.4524
	1	66.0088	66.0088	58.4951	57.8587	47.5975	47.1327	40.6990	40.4132
	2	55.7749	55.7749	49.5377	49.0267	40.4968	40.1254	34.7587	34.5313
	3	49.6859	49.6859	44.2198	43.7861	36.3000	35.9862	31.2604	31.0690
	4	45.5751	45.5751	40.6367	40.2568	33.4836	33.2097	28.9201	28.7538
(25,10)	0	90.8539	90.8539	80.5247	79.6518	65.5442	64.9068	56.0595	55.6676
	1	69.2388	69.2388	61.6231	61.0191	50.5889	50.1518	43.5672	43.3007
	2	59.5625	59.5625	53.1951	52.7192	43.9739	43.6320	38.0770	37.8700
	3	53.9031	53.9031	48.2817	47.8844	40.1423	39.8586	34.9128	34.7420
	4	50.1392	50.1392	45.0230	44.6800	37.6145	37.3709	32.8338	32.6879
(50,0)	0	86.0541	86.0541	75.8545	74.9276	61.0339	60.3489	51.6999	51.2741
	1	62.8084	62.8084	55.3813	54.7090	44.5908	44.0942	37.7931	37.4847
	2	51.9479	51.9479	45.8192	45.2666	36.9162	36.5084	31.3062	31.0530
	3	45.3478	45.3478	40.0101	39.5306	32.2569	31.9033	27.3701	27.1508
	4	40.8023	40.8023	36.0105	35.5817	29.0511	28.7351	24.6634	24.4675
(50,5)	0	88.5560	88.5560	78.2920	77.3941	63.3944	62.7352	53.9868	53.5795
	1	66.1948	66.1948	58.6756	58.0412	47.7709	47.3077	40.8658	40.5812
	2	55.9950	55.9950	49.7508	49.2419	40.7005	40.3309	34.9539	34.7278
	3	49.9327	49.9327	44.4583	44.0269	36.5271	36.2151	31.4773	31.2872
	4	45.8441	45.8441	40.8961	40.5186	33.7295	33.4577	29.1544	28.9894
(50,10)	0	90.9891	90.9891	80.6559	79.7844	65.6702	65.0340	56.1807	55.7897
	1	69.4161	69.4161	61.7945	61.1921	50.7521	50.3163	43.7231	43.4576
	2	59.7686	59.7686	53.3936	52.9194	44.1615	43.8211	38.2553	38.0492
	3	54.1307	54.1307	48.5002	48.1047	40.3477	40.0655	35.1071	34.9373
	4	50.3838	50.3838	45.2573	44.9160	37.8337	37.5915	33.0403	32.8954

conjunction with Navier analytical method. Eringen's theory of nonlocal elasticity together with third order shear deformation (Reddy) beam theory are used to model the FG nanobeam. Mechanical properties of the FG nanobeams considered to change over the thickness based to the power law and Mori-Tanaka models. The governing nonlocal differential equations are derived by implementing Hamilton's principle. Correctness of the results is examined using available date in the literature. Finally, through some numerical examples, the influence of different parameters including nonlocal parameter, foundation parameters, material graduation, mode number and slenderness ratio on fundamental frequencies of FG nanobeams are investigated. It is found that, with an increase of Winkler or Pasternak parameter, the FG nanobeam becomes more rigid and the non-dimensional frequencies increase. Also, it is revealed that existence of nonlocality has a notable decreasing influence on the non-dimensional frequencies of FG nanobeams, which indicates the prominence of the nonlocal effect. So, it must be mentioned that properly selection of the values of the nonlocal parameter is also vital to ensure the accuracy of the nonlocal beam models. It should also be cited that the power-law and Mori-Tanaka indexes have a remarkable influence on the vibrational responses of FG nanobeams. In addition, often the differences of the frequencies between PL and MT models is very small,

Table 6 Variation of the non-dimensional fundamental frequency for an embedded thick FG nanobeam with various gradient indices and nonlocal parameters when K_w=25, K_p=5

	μ	L/h=5				L/h=10			
		p=0	p=0.2	p=1	p=5	p=0	p=0.2	p=1	p=5
EBT	0	12.8952	11.5387	9.72586	8.57126	13.0520	11.687	9.85632	8.67455
present		12.5986	11.2967	9.53389	8.39316	12.9658	11.6157	9.79716	8.62067
EBT	1	12.5624	11.2577	9.50969	8.39308	12.7152	11.4024	9.63732	8.49425
present		12.2869	11.0335	9.33239	8.22889	12.6348	11.336	9.58235	8.44427
EBT	2	12.2776	11.0176	9.32531	8.24131	12.4269	11.1593	9.45054	8.34069
present		12.0203	10.8087	9.1607	8.08915	12.3515	11.0971	9.39919	8.29408
EBT	3	12.0309	10.8099	9.16611	8.11043	12.1772	10.9489	9.28926	8.20827
present		11.7896	10.6144	9.01258	7.96875	12.1062	10.8904	9.24108	8.16460

specifically for lower gradient indexes. Hence, considering PL and MT models, it is concluded that with the increment of gradient index the natural frequencies reduce.

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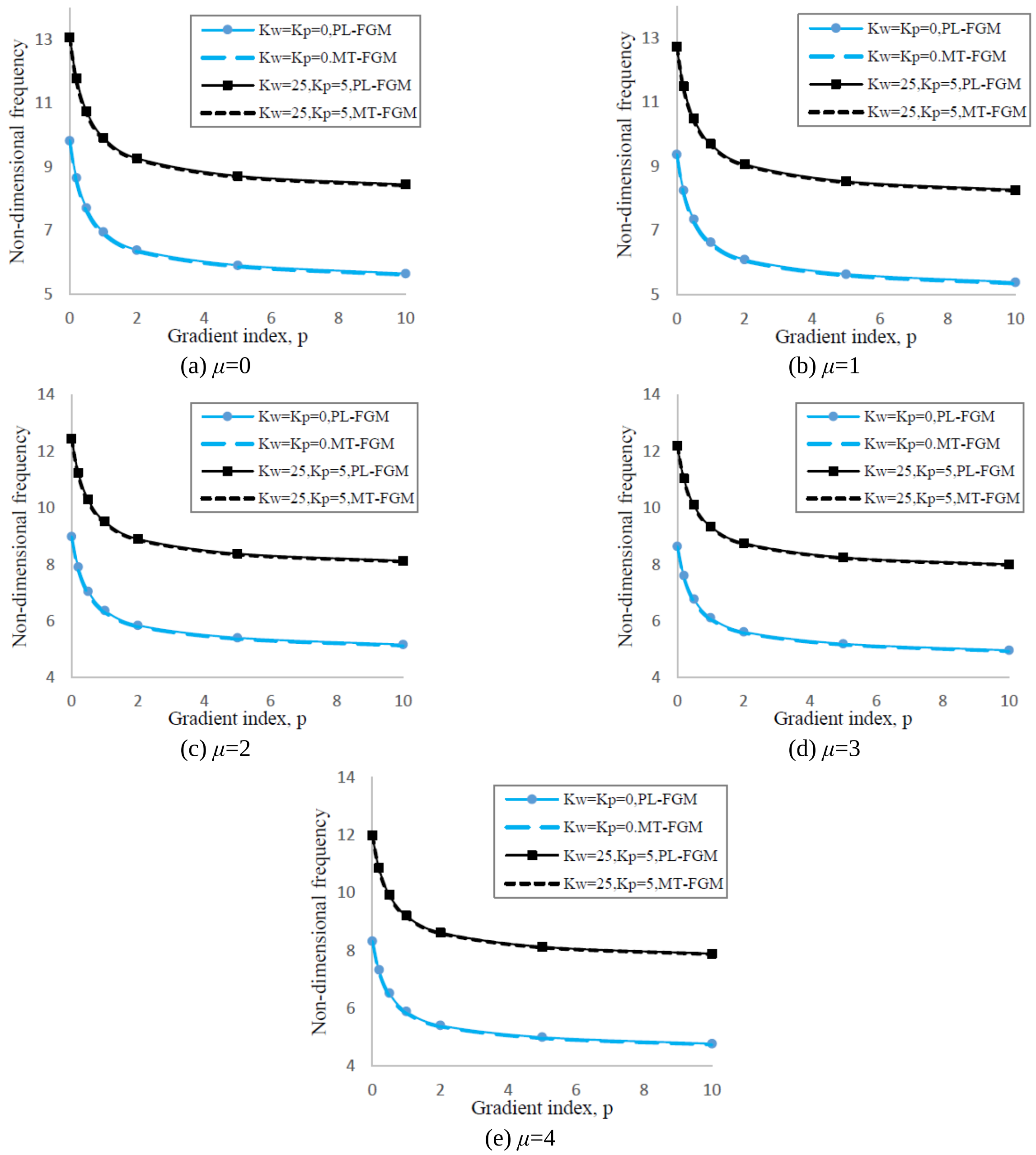


Fig. 2 The effect of presence of elastic foundation and the estimation method of material properties on the first non-dimensional frequency with gradient index for different nonlocal parameters when L/h=20

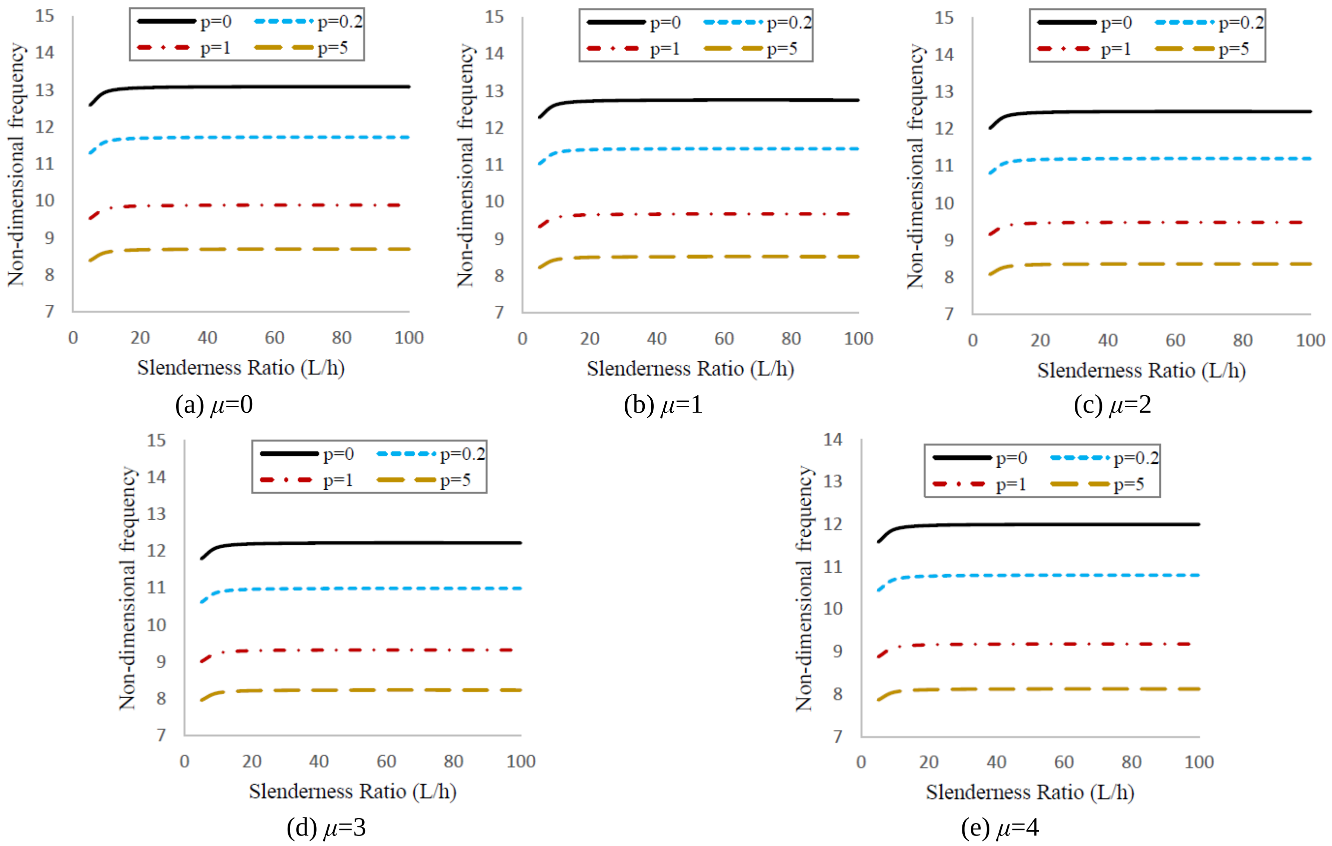


Fig. 3 The comparison of the first non-dimensional frequency of S-S FG nanobeam in the case of Mori-Tanaka model versus slenderness ratio for different gradient indexes and nonlocal parameters at K_p=5, K_w=25

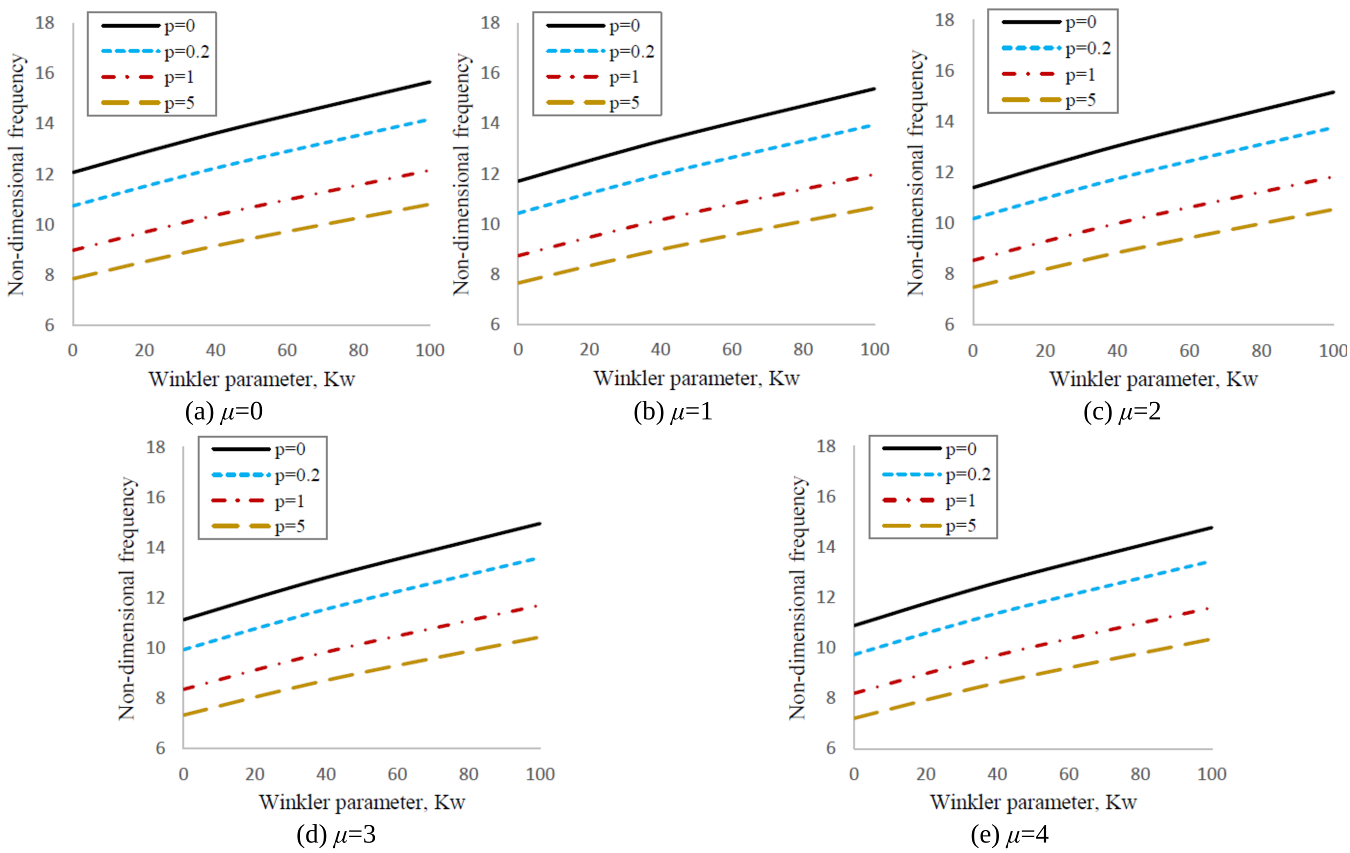


Fig. 4 The variation of the first non-dimensional frequency of S-S FG nanobeam with Winkler parameter and gradient index for different nonlocal parameters at L/h=20 and K_p=5

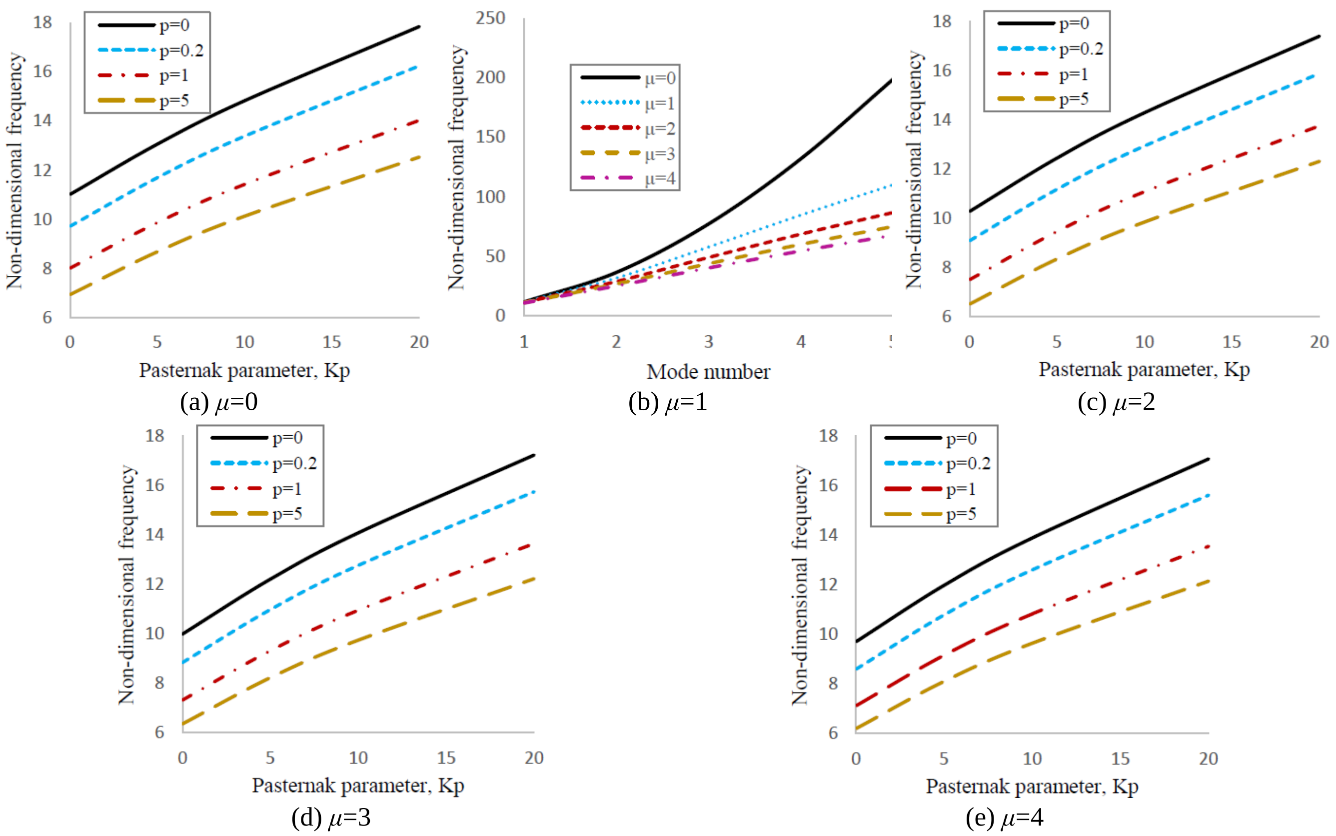


Fig. 5 The variation of the first non-dimensional frequency of S-S FG nanobeam with Pasternak parameter and gradient index for different nonlocal parameters at L/h=20 and K_w=25

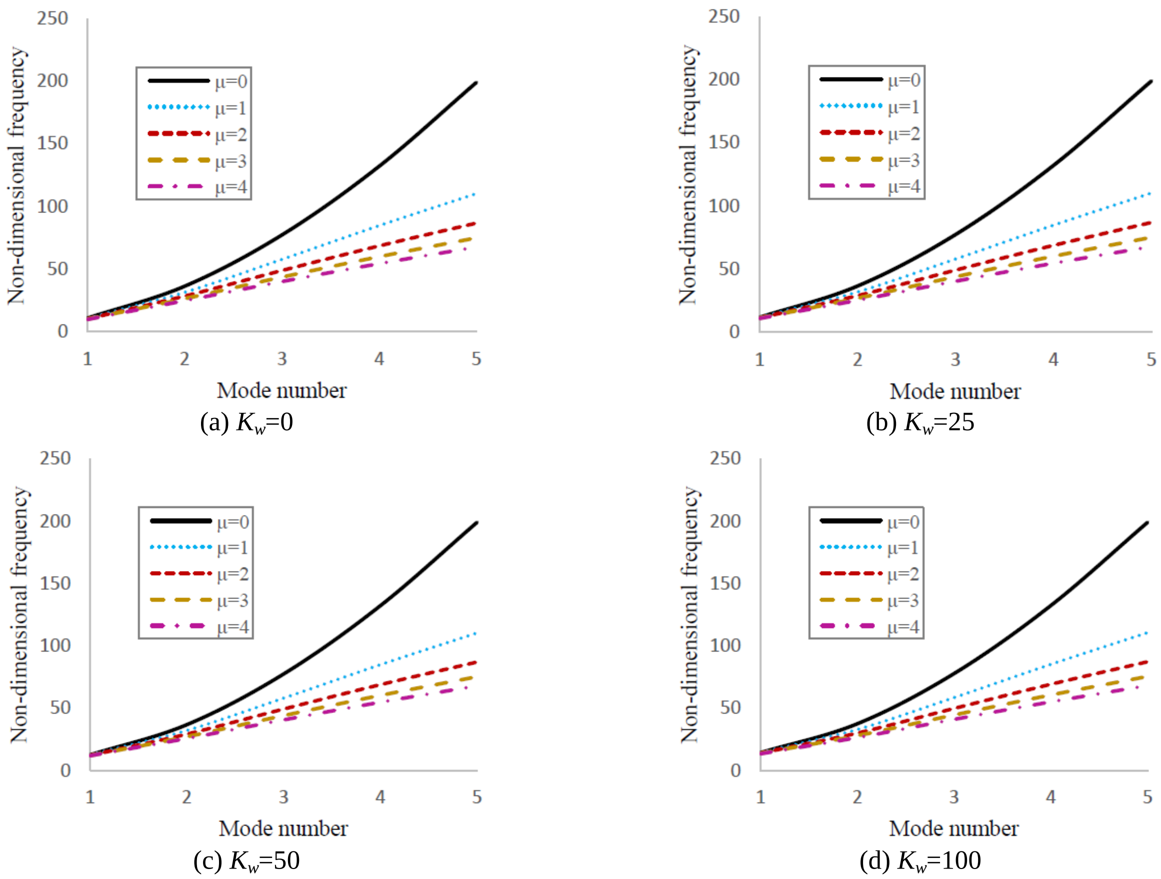
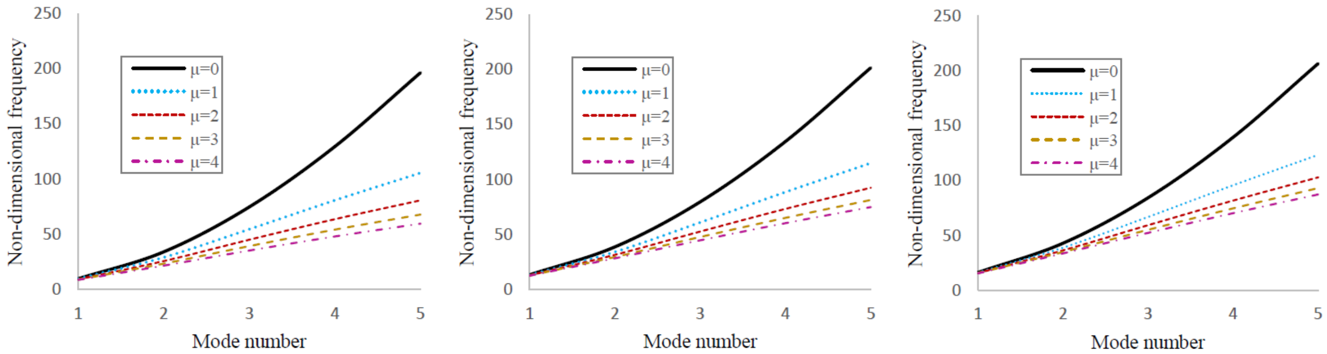


Fig. 6 The variation of the non-dimensional frequency of S-S FG nanobeam with mode number and nonlocal parameter for different Winkler parameters at L/h=20, p=0.2 and K_p=5



(a) K_p=0 (b) K_p=10 (c) K_p=20

Fig. 7 The variation of the non-dimensional frequency of S-S FG nanobeam with mode number and nonlocal parameter for different Pasternak parameters at L/h=20, p=0.2 and K_w=25

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