Structural Engineering and Mechanics, *Vol. 61, No. 5 (2017) 685-691*
DOI: https://doi.org/10.12989/sem.2017.61.5.685

Free vibrations of AFG cantilever tapered beams carrying attached masses

Carlos A. Rossit[a], Diana V. Bambill* and Gonzalo J. Gilardi[b]

Department of Engineering, Institute of Applied Mechanics, (IMA), Universidad Nacional del Sur, Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Avenida Alem 1253, (8000) Bahía Blanca, Argentina

(Received July 21, 2016, Revised October 11, 2016, Accepted January 26, 2017)

Abstract. The free transverse vibrations of axially functionally graded (AFG) cantilever beams with concentrated masses attached at different points are studied in this paper. The material properties of the AFG beam, consisting of metal and ceramic, vary continuously in the axial direction according to an established law form. Approximated solutions for the title problem are obtained by means of the Ritz Method. The influence of the material variation on the natural frequencies of vibration of the functionally graded beam is investigated and compared with the influence of the variation of the cross section. The phenomenon of dynamic stiffening of beams can be observed in various situations. The accuracy of the procedure is verified through results available in the literature that can be represented by the model under study.

Keywords: vibration of beams; AFG beam; tapered beam; attached masses; Ritz method

1. Introduction

The situation of resistant structures supporting motors or engines attached to them is usual in technological applications. The operation of the machine alters the natural vibration and may introduce severe dynamic stresses on the beam. It is important, then, to know the natural frequencies of the coupled beam-mass system, in order to obtain a proper design of the structural element. In that case, the inertial effect magnifies the influence of the constituent materials. Beams are probably the most widespread component of those structures and a significant amount of papers has been written on the subject. Among them, the papers of Bapat and Bapat (1987) and Naguleswaran (2002) can be mentioned. Specifically, cantilever beams have countless applications including use of a cantilever - mass system as a dynamic absorber (La Malfa *et al*. 2000)

In recent years, it has become increasingly important to use advanced materials whose properties vary gradually in some of its dimensions (FGM). Such materials were first used by Japanese researchers posed as thermal barrier material in mid-eighties, Niino *et al*. (1987).

In present paper, beams of materials whose properties vary functionally along the axis (AFG) are studied.

Initially, the research established great progress in the field of elasticity theory and the study of plates and shells built with FGM. Advance in its application to beams (Functionally Graded Beams -FGB-) came later.

*Corresponding author, Professor
E-mail: dbambill@criba.edu.ar
[a]Professor
E-mail: carossit@criba.edu.ar
[b]Ph.D. Student
E-mail: gonzalo.gilardi@uns.edu.ar

A survey of the literature reveals that most of the early work on functionally graded beams has considered the gradation of the material properties in the thickness direction. Far fewer researchers have considered the variation of material properties in the axial direction (AFG). Probably because the problem becomes more complicated as variable coefficients appear in the governing differential equations.

Consequently, because of the mathematical difficulties, few analytical solutions have been obtained and for arbitrary specific gradients: It is worth mentioning the work of Elishakoff and his colleagues-Elishakoff (2000), Elishakoff and Candan (2001), Caliò and Elishakoff (2004) and (2005), Elishakoff (2005), Elishakoff and Guede (2005) and Wu, Wang and Elishakoff (2005)-, who by means of the semi-inverse method, solved various particular cases of vibrating AFG beams.

Huang and Li (2010) solved the problem by transforming the governing equation with variable coefficients in a Fredholm integral equation. Alshorbgy *et al*. (2011) investigated the dynamic characteristics of non-uniform beams with axially or transversely in height gradation of the material by means of the finite element method. Shahba and Rajasekaran (2012) studied longitudinal and transverse free vibration and buckling of AFG Euler-Bernoulli beams using the differential transform element method (DTEM) and differential quadrature element method of lowest-order (DQEL). Hein and Feklistova (2011) investigated the vibration of non-uniform and AFG beams with various boundary conditions and varying cross-sections using Haar wavelets. Çetin and Şimşek (2011), Chegenizadeh *et al*. (2014) studied statically and dynamically AFG beams embedded in an elastic medium. Şimşek and co-workers (2011, 2012) studied the dynamic behavior of AFG beams under the action of a moving load. Agköz and Civalek (2013) studied the free longitudinal vibrations of AFG bars on the basis of

ISSN: 1225-4568 (Print), 1598-6217 (Online)

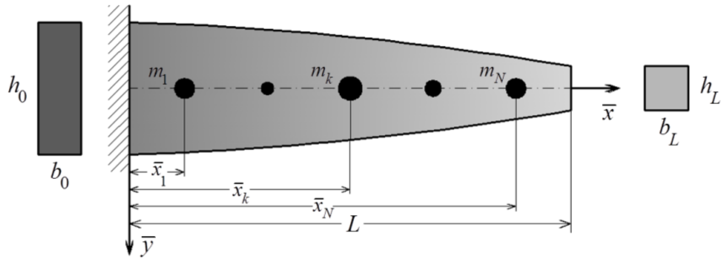

Fig. 1 AFG cantilever tapered beam with N masses attached at arbitrary points

strain gradient elasticity theory by using the Rayleigh-Ritz method. Recently, Gan *et al*. (2015) presented a finite element procedure to study AFG Timoshenko beams subjected to multiple moving points.

Apparently, according to a recent literature survey, there has not been any attempt to solve the problem of an **axially** functionally graded beam carrying an attached mass. In particular to the authors' knowledge, there are no natural frequency data in the literature for axially functionally graded, AFG, beams carrying an attached mass. Even, scarce information is available about homogeneous beams with variable cross section and attached masses, a very interesting paper of Wu and Hsieh (2000) can be quoted. In the case of cantilever beams, the papers by Auciello and Maurizi (1997), Chen and Lui (2006) and, recently Hozhabrossadati (2015) must be mentioned.

In the present paper, we describe the determination of the natural frequencies of vibration of a Bernoulli-Euler cantilever beam with varying rectangular cross section and made of axially functionally graded material, carrying attached masses at arbitrary positions, having into account their rotatory inertia (Fig. 1).

The well-known variational Ritz method (Ilanko and Monterrubio 2014) is employed to perform the analysis. The proposed approach exhibits an excellent accuracy for particular cases available in the literature.

2. Analytical approach

According to the classical Euler-Bernoulli beam theory, the energy functional J for a vibrating beam of length L carrying attached N masses m_k at positions $\bar{x}_k$ (see Fig. 1) is given by

$$J[V]=\frac{1}{2}\int_0^1 \frac{E(x)I(x)}{L^3}\left[\frac{d^2V(x)}{dx^2}\right]^2 dx$$
$$-\frac{1}{2}\omega^2\left\{\int_0^1 \rho(x)A(x)L[V(x)]^2 dx+\right. \tag{1}$$
$$\left.+\sum_{k=1}^{N} m_k\left[V(x_k)\right]^2+\sum_{i=1}^{N}\frac{m_k r_k^2}{L^2}\left[\frac{d\,V(x)}{dx}(x_k)\right]^2\right\}$$

where $x=\bar{x}/L$ is the dimensionless coordinate, $V(x)$ is the deflection, $A(x)$ is the varying cross section and $I(x)$ its second moment of area, the FGM density and Young's modulus are respectively $\rho(x)$ and $E(x)$; r_k is the radius of gyration of the mass m_k with respect to the neutral axis of the beam

As the material and geometric characteristics of the beam may be general, one can define

$$\begin{aligned} E(x)&=E_0\ f_E(x)\\ I(x)&=I_0\ f_I(x)\\ \rho(x)&=\rho_0\ f_\rho(x)\\ A(x)&=A_0\ f_A(x)\\ b(x)&=b_0\ f_b(x)\\ h(x)&=h_0\ f_h(x)\end{aligned} \tag{2}$$

Obviously: $f_A=f_b\times f_h$, $f_I=f_b\times f_h^3$

The subscript "0" refers to the cross section of the beam adopted as the reference section

Substituting Eq. (2) into Eq. (1), the functional can be expressed

$$J(V_a)=\frac{1}{2}\frac{E_0I_0}{L^3}\left\{\int_0^1 f_E\, f_I\left[\frac{d^2V_a}{dx^2}\right]^2 dx-\right.$$
$$-\Omega^2\left(\int_0^1 f_\rho\, f_A\, L[V(x)]^2\,dx+\right. \tag{3}$$
$$\left.\left.+\sum_{k=1}^{N}M_k\left[\left[V(x_k)\right]^2+c_k\left[\frac{dV(x)}{dx}(x_k)\right]^2\right]\right)\right\}$$

with: $\Omega=\omega L^2\sqrt{\dfrac{\rho_0A_0}{E_0I_0}}$, $M_k=\dfrac{m_k}{\rho_0A_0\,L}$, $c_k=\dfrac{r_k}{L}$

To apply the Ritz method, it is necessary to approximate the spatial component of the solution

$$V(x)\cong V_a(x)=\sum_{j=1}^{P}C_j\varphi_j \tag{4}$$

where φ_j are coordinate functions that satisfy the essential boundary conditions, C_j are arbitrary constants.

Following Ritz' procedure, the functional is minimized with respect to every arbitrary constant

$$\frac{\partial J\left[V_a(x)\right]}{\partial C_j}=0 \quad , \quad j=1,2.....P \tag{5}$$

Then a linear system of equations is formed

$$\mathbf{A}\left\{C_j\right\}=0 \tag{6}$$

which results in the following eigenvalue equation

$$\mathbf{A}=\mathbf{K} - \Omega^2 \mathbf{M} \tag{7}$$

where

$$k_{ij} = \int_0^1 f_E(x)\, f_I(x)\, \varphi_i'' \varphi_j''\, dx \tag{8}$$

$$m_{ij} = \int_0^1 f_\rho(x)\, f_A(x) \varphi_i \varphi_j\, dx + \sum_{k=1}^{N} M_k \left[\varphi_i(x_k)\varphi_j(x_k) + c^2 \varphi_i'(x_k)\varphi_j'(x_k)\right] \tag{9}$$

are the elements of matrices **K** and **M**, respectively

Then, the eigenvalue problem can be expressed as

$$\left|\mathbf{KM}^{-1} - \Omega^2\, \mathbf{I}\right| = \left|\mathbf{B}\text{-}\lambda\mathbf{I}\right| = 0 \tag{10}$$

where $\lambda=\Omega^2$ are the eigenvalues of matrix **B**

For the cantilever beam, the following coordinate functions are chosen

$$\left\{\varphi_j\right\}_{j=1}^{P} = \left\{x^{j+1}\right\}_{j=1}^{P} \tag{11}$$

which satisfy essential boundary conditions.

3. Numerical results

Since there were not found, in the technical literature, values of natural frequencies of vibration of AFG beams with attached masses in order to verify the accuracy of the proposed model, comparisons are made with particular cases available in the literature.

First, Table 1 compares values for a tapered Euler-Bernoulli beam made of axially functionally graded material studied by Shahba and Rajasekaran (2012).They obtained values for the first two natural frequency coefficients for a case that can be represented in the present model by adopting in Eqs. (2):

$$f_b = 1 - c_b\, x$$

$$f_h = 1 - c_h\, x$$

$$f_E = 1 + x$$

$$f_\rho = 1 + x + x^2$$

In all cases the calculations were done with P= 20 in Eq. (4), and the cross section at the clamped edge (x=0) is taken as the reference cross section.

As it can be seen, the agreement between the two sets of results is excellent.

Then, comparison is made with a homogeneous tapered cantilever beam carrying multiple point masses deeply studied by Chen and Liu (2006) by means of an analytical and numerical combined method proposed previously by Wu and Lin (1990).

The physical properties and dimensions of the beam studied are: Young´s modulus E=2.051×10^{11} N/m^2, mass density ρ=7850 kg/m^3, constant beam width b=0.1 m, beam length L=1.60 m, lineal variable beam depth: 0.08 m. at the free end and 0.40 m. at the clamped end.

Each attached mass has a magnitude of one-fifth of the actual total mass of the beam: 60.288 kg.

In the present model, according to the definition used for the relative magnitude of the mass, it must be adopted M_k=12/100 and c_k=0 for k= 1 to 5, and $f_i = \dfrac{\Omega_i}{2\pi l^2}\sqrt{\dfrac{E_0 I_0}{\rho_0 A_0}}$

Two particular situations in Fig. 1 are considered:

a) One point mass attached at the free end

b) Five equal point masses attached at coordinates: 0.2 m, 0.5 m, 0.8 m, 1.1 m and 1.4 m respectively

Table 2 shows the results:

Again, the accuracy is excellent. It is worth mentioning that the Ritz method gives upper bounds of the wanted values.

Due to the quantity and variability of the parameters involved in the description of the behavior of these kinds of structures, just a few representative cases will be considered to demonstrate the convenience of the procedure.

In all cases, there are taken into account the quantity and positions of masses of Table 2 in order to facilitate comparisons to evaluate the influence of variation in the height of the cross section and properties of the material.

Table 1 Frequency coefficients $\Omega_i = \omega_i\, l^2 \sqrt{\dfrac{\rho_0 A_0}{E_0 I_0}}$ for a tapered cantilever beam of AFG material

C_h	C_b	0		0.2		0.4		0.6		0.8	
		Ω_1	Ω_2	Ω_1	Ω_2	Ω_1	Ω_2	Ω_1	Ω_2	Ω_1	Ω_2
0	Present	2.42556	18.6041	2.60542	19.0041	2.85075	19.5303	3.21368	20.2958	3.83105	21.6759
	Shahba *et al*.	2.4256	18.6041	2.6054	19.0041	2.8507	19.5303	3.2137	20.2958	3.8310	21.6759
0.2	Present	2.50506	17.3802	2.68633	17.7501	2.93357	18.2379	3.29935	18.9501	3.92194	20.2432
	Shahba *et al*.	2.5051	17.3801	2.6863	17.7501	2.9336	18.2379	3.2993	18.9501	3.9219	20.2432
0.4	Present	2.61547	16.0705	2.79874	16.4092	3.04857	16.8571	3.41810	17.5139	4.04714	18.7164
	Shahba *et al*.	2.6155	16.0705	2.7987	16.4092	3.0486	16.8571	3.4181	17.5139	4.0471	18.7164
0.6	Present	2.78355	14.6508	2.96994	14.9567	3.22368	15.3627	3.59847	15.9616	4.23553	17.0694
	Shahba *et al*.	2.7836	14.6508	2.9699	14.9567	3.2236	15.3627	3.5985	15.9616	4.2355	17.0694
0.8	Present	3.08711	13.1142	3.27943	13.3849	3.54015	13.7466	3.92322	14.2848	4.56946	15.2954
	Shahba *et al*.	3.0871	13.1142	3.2794	13.3849	3.5401	13.7466	3.9232	14.2848	4.5695	15.2955

Table 2 Frequency values for a tapered homogeneous cantilever beam with attached masses

Number of point masses		1		5	
		Chen and Liu	Present	Chen and Liu	Present
Natural frequencies f_i (Hz)	f_1	569.6279	569.3747	613.2201	613.1940
	f_2	2508.895	2503.714	2525.538	2524.883
	f_3	6743.232	6710.268	6366.500	6356.546
	f_4	13408.53	13289.00	12184.03	12116.38
	f_5	22570.17	22240.74	16089.95	15928.03

Table 3 Frequency coefficients for a uniform homogeneous beam with attached masses

Number of point masses	c	Natural frequency coefficients Ω_i				
		Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
0	0	3.51602	22.0345	61.6972	120.902	199.860
1	0	2.88547	19,0573	54.9751	109.981	184.480
	0.1	2.87595	18.2857	48.9752	88.7585	142.480
	0.2	2.84756	16.1534	38.0611	74.3913	132.695
5	0	2.82078	17.7766	50.0195	98.5845	165.116
	0.1	2.79557	16.7531	43.7182	77.7529	112.439
	0.2	2.72349	14.4833	33.3422	52.1344	66.2409

Table 4 Frequency coefficients for a linear tapered homogeneous beam

Number of point masses	c	Natural frequency coefficients Ω_i				
		Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
0	0	4.29249	15.7427	36.8846	68.1164	109.594
1	0	2.46957	10.8594	29.1047	57.6388	96.4655
	0.1	2.41889	7.94847	15.5177	34.6076	65.1943
	0.2	2.24807	4.89120	13.8757	34.0758	64.9337
5	0	2.65963	10.9513	27.5704	52.5597	69.0851
	0.1	2.59868	9.30890	19.4349	34.7070	58.2398
	0.2	2,43031	6.70613	13.2538	24.0551	42.4725

Table 5 Frequency coefficients for a quadratic tapered homogeneous beam

Number of point masses	c	Natural frequency coefficients Ω_i				
		Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
0	0	4.76281	19.9475	46.8110	86.0549	137.967
1	0	3.13723	13.2651	35.5197	71.0597	119.501
	0.1	3.09420	9.73446	18.1845	41.2880	79.1301
	0.2	2.93478	5.81291	16.3535	40.7512	78.8805
5	0	3.19588	14.5899	36.7244	69.8795	94.7423
	0.1	3.14343	12.7900	27.0165	47.6306	80.3455
	0.2	2.99608	9.54393	18.5847	33.3465	53.9520

The weight of the beam is considered in relation to the uniform beam of steel, which is taken as reference material

$$W_b = \frac{g\int_0^L \rho\, b\, h\, dx}{\rho_{st} g b_0 h_0 L} \tag{12}$$

The rotatory inertia of the attached masses will also be considered by means of the coefficient c.

As a reference, the case of a homogeneous cantilever beam of uniform cross sections (W_b=1) is evaluated in Table 3.

3.1 Variations in the cross section:

In order to evaluate the influence of the variation of the cross section, two situations are considered. In all cases, h_o the height of the cross section at the clamped edge (x=0) is the same value as the height of the uniform beam, while h_L=0.2h_0 (at x=1) and h_o=0.25L. The width of the section b_o remains constant.

3.1.1 Linear variation

First, the case of a homogeneous cantilever beam whose height varies linearly: $f_h(x)=1+\left(h_L/h_0-1\right)x$, ($W_b$=0.6), is presented in Table 4:

As it is shown in Table 4, all values decrease, except the fundamental frequency of the bare beam, which increases 22%. However, it should be noted that the weight of the beam has diminished by 40%.

3.1.2 Quadratic variation

Consider now the case of a homogeneous cantilever beam when the height of the cross section diminishes according to a quadratic law: $f_h(x)=1+\left(h_L/h_0-1\right)x^2$, ($W_b$=0.733)

Results are shown in Table 5. In this case, in all situations the fundamental frequency increases; meaningfully (35%) for the bare beam, and between 3 and 13 percent when the masses are attached. Higher frequencies are less than the values of the uniform beam but clearly higher than those of the beam with linear variation.

3.2 Variations in the material

In order to evaluate the influence of the material composition, an AFG beam of uniform cross section is analyzed.

The inhomogeneous material, with gradient compositional variation of the constituents, varies in the longitudinal direction of the beam. Properties of AFG materials, like mass density ρ, Young's modulus E, shear modulus G, continuously vary in the axial direction.

For example, a generic material property $P(x)$ is assumed to vary along the beam axis x with a power law relation

$$P(x) = P_a\left[1+\frac{(P_b-P_a)}{P_a}\left(x\right)^n\right] \tag{13}$$

where P_a and P_b are properties of material "a" and material "b", respectively. They are the constituents of the

Table 6 Frequency coefficients for a uniform beam with AFG material (St-Al) varying linearly

Number of point masses	c	Natural frequency coefficients Ω_i				
		Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
0	0	4.84745	30.1784	84.4097	165.343	273.269
1	0	3.63945	24.9071	73.1570	147.788	249.226
	0.1	3.62457	23.8377	64.8324	118.066	190.461
	0.2	3. 58043	20.9484	50.2585	99.1100	177.935
5	0	3.49942	22.7537	64.6523	127.796	202.344
	0.1	3.45989	21.2017	55.0943	96.6527	139.852
	0.2	3.34833	17.9389	40.7078	63.3601	79.9092

Table 7 Frequency coefficients for a uniform beam with AFG material (Al-St) varying linearly

Number of point masses	c	Natural frequency coefficients Ω_i				
		Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
0	0	4.78561	30.0225	84.2107	165.133	273.073
1	0	4.23038	27.0913	77.1388	153.109	255.575
	0.1	4.22038	26.1676	69.6304	125.482	199.362
	0.2	4.19044	23.5080	54.4067	104.060	184.390
5	0	4.18394	25.9031	72.4193	142.327	233.000
	0.1	4.15772	24.7621	65.2887	118.287	177.961
	0.2	4.08165	22.0508	52.0354	83.6246	111.390

Table 8 Frequency coefficients for a uniform beam with AFG material (Al-St) varying quadratically

Number of point masses	c	Natural frequency coefficients Ω_i				
		Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
0	0	5.14333	33.6805	94.2912	184.503	304.710
1	0	4.51349	30.3102	86.1815	170.757	284.768
	0.1	4.50246	29.1460	76.6237	136.698	218.872
	0.2	4.46941	25.8205	58.8790	114.593	204.518
5	0	4.45544	28.5057	79.5376	156.038	253.136
	0.1	4.42640	27.1601	71.1662	128.152	192.345
	0.2	4.34227	24.0168	56.0502	89.4112	118.633

Table 9 Frequency coefficients for a tapered AFG beam (Al-St), height and material varying quadratically

Number of point masses	c	Natural frequency coefficients Ω_i				
		Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
0	0	7.06861	29.0973	67.8828	124.822	200.283
1	0	5.16753	20.4007	52.9015	104.570	175.033
	0.1	5.09747	14.4902	27.5847	61.9386	117.211
	0.2	4.81317	8.58708	25.1197	61.1326	116.819
5	0	5.31489	23.2589	57.0182	98.7379	147.534
	0.1	5.24014	20.5669	43.7810	80.2447	129.946
	0.2	5.02701	15.5660	31.1930	58.8238	99.6792

inhomogeneous material of the beam; n is the material non-homogeneity parameter and $P(\bar{x})$ is a typical material property such as ρ, E or G. Note that for x=0 the entire section is of material "*a*", and for x=1 the whole section is of material "*b*". The percentage content of material "*a*" along the beam increases as n increases. When n=1 the composition changes linearly through the length L, while n=1/2 or n=2 corresponds to a quadratic distribution,

In the calculations, the AFG material made of steel and aluminum oxide Al_2O_3 (alumina) proposed by Su *et al*. (2013) is used. Their Young modulus and density are:

$$E_{St} = 210GPa\,;\ \ \rho_{St} = 7800kg/m^3\,;\ \ E_{Al} = 390GPa\,;$$

$$\rho_{Al} = 3960kg/m^3\,;\ \ \nu_{St} = \nu_{Al} = 0.30$$

The relationships between material properties are: $E_{Al}/E_{St} = 1.857$ for Young's modulus and $\rho_{Al}/\rho_{St} = 0.508$ for the density. Note that the alumina, more rigid, is lighter than steel.

In all evaluated cases, in order to facilitate comparisons will be considered E_0=E_{St} and ρ_0=ρ_{St}

3.2.1 Linear variation

A linear distribution of the materials is considered, n=1 in Eq. (13), and two possibilities are explored:

a) Material *a* in Eq. (13) is steel and material *b* is alumina. (W_b=0.754)

Then, at x=0, the section is of steel and at x=1the section is entirely of alumina

b) Material *a* in Eq.(13) is alumina and material *b* is steel. (W_b=0.754)

Then, at x=0, the section is of alumina and at x=1 the section is entirely of steel

All frequency values (Tables 6 and 7) are higher than those corresponding to the uniform homogeneous beam. It should also be noted that this is achieved with a reduction in weight of almost 25%

Comparison of values in Tables 6 and 7 shows that for the case of the beam without attached masses, the frequency coefficients of the composition a) (St-Al) are slightly higher than b) (Al-St). But when acting concentrated masses attached, frequencies for b) composition (Al-St) are clearly higher in all situations. The reason is that due to the greater rigidity of the beam near the clamped end, decreases the amplitude of the displacements of the points of application of the attached masses and consequently decreases the kinetic energy that masses add to energy balance.

3.2.2 Quadratic variation

It is considered then a quadratic variation of the material, n=2 in Eq. (13), and due to the results of the above case is taken into account only the situation where the material *a* is alumina and *b* is steel (W_b=0.672). Values are shown in Table 8

As can be seen, the frequency coefficients are higher in all cases than those obtained for a linear variation in the material composition and with an even smaller weight

3.3 Variation in section and material

Finally, the effect caused by varying both the section and the material composition is analyzed. To this end, the two situations that had provided the highest frequency values are combined: quadratic variation in the height of the cross section (*3.1.2*) and quadratic variation in the material composition Al-St: *3.2.2* (W_b=0.458)

Again, variation in the cross section increases the fundamental frequency in all cases and decrease higher frequencies. The reduction in weight is considerable: 54% compared to uniform steel beam and 32% with respect to the AFG beam -case *3.2.2*-.

4. Conclusions

The results obtained indicate that variations in the composition of the AFG material have a uniform impact on the modification of the natural frequencies of the beam. Instead, by varying its cross section, the fundamental frequency increases and higher decrease.

This leads to the conclusion that the use of such materials is a reliable means when it is necessary to increase the values of natural frequencies.

The classical, variational method of Ritz has been successfully used to obtain an approximate, yet accurate, solution to a difficult elastodynamics problem from which the authors has not found data in the literature.

Acknowledgments

The present study has been sponsored by the Universidad Nacional del Sur, UNS, the Consejo Nacional de Investigaciones Científicas y Técnicas, CONICET and the Comisión de Investigaciones Científicas, Buenos Aires Province (CIC). It was developed in the Department of Enginering and Institute of Applied Mechanics, IMA.

References

Akgöz, B. and Civalek, Ö. (2013), "Longitudinal vibration analysis of strain gradient bars made of functionally graded materials (FGM)", *Compos. Part B*, **55**, 263-268.

Alshorbgy, A.E., Eltaher, M.A. and Mahmoud, F.F. (2011), "Free vibration characteristics of a functionally graded beam by finite element method", *Appl. Math. Model.*, **35**, 412-425.

Auciello, N.M. and Maurizi, M.J. (1997), "On the natural vibrations of tapered beams with attached inertia elements", *J. Sound Vib.*, **199**(3), 522-530.

Bapat, C.N. and Bapat, C. (1987), "Natural Frequencies of a beam with non-classical boundary conditions and concentrated masses", *J. Sound Vib.*, **112**(1), 177-182.

Caliò, I. and Elishakoff, I (2005), "Closed-form solutions for axially graded beam-columns", *J. Sound Vib.*, **280**, 1083-1094.

Caliò, I. and Elishakoff, I. (2004), "Closed-form trigonometric solutions for inhomogeneous beam-columns on elastic foundation", *Int. J. Struct. Stab. Dyn.*, **4**, 139-146.

Çetin, D. and Şimşek, M. (2011), "Free vibration of an axially functionally graded pile with pinned ends embedded in Winkler-Pasternak elastic medium", *Struct. Eng. Mech.*, **40**, 583-594.

Chabraborty, A., Gopalakrishnan, S. and Reddy, J.N. (2003), "A new beam finite element for the analysis of functionally graded materials", *Int. J. Mech. Sci.*, **45**, 519-539.

Chegenizadeh, A., Ghadimi, B., Nikraz, H. and Şimşek, M. (2014), "A novel two-dimensional approach to modelling functionally graded beams resting on a soil medium", *Struct. Eng. Mech.*, **51**(5), 727-741.

Chen, D.W. and Liu, T.L. (2006), "Free and forced vibrations of a tapered cantilever beam carrying multiple point masses", *Struct. Eng. Mech.*, **23**(2), 209-216.

Elishakoff, I. (2000), "A selective review of direct, semi-inverse and inverse eigenvalue problems for structures described by differential equations with variable coefficients", *Arch. Comput. Meth. Eng.*, **7**(4), 387-461.

Elishakoff, I. (2005), *Eigenvalues of Inhomogeneous Structures: Unusual Closed-Form Solutions of Semi-Inverse Problems*, CRC Press, Boca Raton, USA.

Elishakoff, I. and Candan, S. (2001), "Apparently first closed-form solution for vibrating inhomogeneous beams", *Int. J. Solid. Struct.*, **38**(19), 3411-3441.

Elishakoff, I. and Guede, Z. (2004), "Analytical polynomial solutions for vibrating axially graded beams", *Mech. Adv. Mater. Struct.*, **11**, 517-533.

Gan, B.S., Trinh, T.H., Le, T.H. and Nguyen, D.K. (2015), "Dynamic response of non-uniform Timoshenko beams made of axially FGM subjected to multiple moving point loads", *Struct. Eng. Mech.*, **53**(5), 981-995.

Hein, H. and Feklistova, L. (2011), "Free vibrations of non-uniform and axially functionally graded beams using Haar wavelets", *Eng. Struct.*, **33**(12), 3696-3701.

Hozhabrossadat, S.M. (2015), "Exact solution for free vibration of elastically restrained cantilever non-uniform beams joined by a mass-spring system at the free end", *IES J. Part A: Civil Struct. Eng.*, **8**(4), 232-239.

Huang, Y. and Li, X.F. (2010), "A new approach for free vibration of axially functionally graded beams with non-uniform cross-section", *J. Sound Vib.*, **329**, 2291-2303.

Ilanko, S. and Monterrubio, L.E. (2014), *The Rayleigh-Ritz Method for Structural Analysis*, John Wiley & Sons Inc., London, U.K.

Karami, G., Malekzadeh, P. and Shahpari, S.A. (2003), "A DQEM for vibration of shear deformable non-uniform beams with general boundary conditions", *Eng. Struct.*, **25**, 1169-1178.

Karnovsky, I.A. and Lebed, O.I. (2004), *Non-classical Vibrations of Arches and Beams*, McGraw-Hill Inc., New York, USA.

La Malfa, S., Laura, P.A.A., Rossit, C.A. and Alvarez, O. (2000), "Use of a dynamic absorber in the case of a vibrating printed circuit board of complicated boundary shape", *J. Sound Vib.*, **230**(3), 721-724.

Li, X.F. (2008), "A unified approach for analyzing static and dynamic behaviors of functionally graded Timoshenko and Euler-Bernoulli beams", *J. Sound Vib.*, **318**, 1210-1229.

Lu, C.F. and Chen, W.Q. (2005), "Free vibration of orthotropic functionally graded beams with various end conditions", *Struct. Eng. Mech.*, **20**, 465-476.

Mahamood, R.M., Akinlabi, E.T., Shukla, M. and Pityana, S. (2012), "Functionally graded material: an overview", *Proceedings of the World Congress on Engineering 2012*, London, U.K., July.

Naguleswaran, S. (2002), "Transverse vibrations of an Euler-Bernoulli uniform beam carrying several particles", *Int. J. Mech. Sci.*, **44**, 2463-2478.

Niino, M., Hirai, T. and Watanabe, R. (1987), "The functionally gradient materials", *J. JPN Soc. Compos. Mater.*, **13**, 257-264.

Shahba, A. and Rajasekaran, S. (2012), "Free vibration and stability of tapered Euler-Bernoulli beams made of axially functionally graded materials", *Appl. Math. Model.*, **36**, 3094-3111.

Shahba, A., Attarnejad, R., Marvi, M.T. and Hajilar, S. (2011), "Free vibration and stability analysis of axially functionally graded tapered Timoshenko beams with classical and non-classical boundary conditions", *Compos. Part B*, **42**, 801-808.

Şimşek, M., Kocatürk, T. and Akbas, S.D. (2011), "Dynamics of an axially functionally graded beam carrying a moving harmonic load", *16th International Conference on Composite Structures*, Porto, Portugal, June.

Şimşek, M., Kocatürk, T. and Akbas, S.D. (2012), "Dynamic behavior of an axially functionally graded beam under action of a moving harmonic load", *Compos. Struct.*, **94**, 2358-2364.

Su, H., Banerjee, J.R. and Cheung, C.W. (2013), "Dynamic stiffness formulation and free vibration analysis of functionally graded beams", *Compos. Struct.*, **106**, 854-862.

Wang, S.S. (1983), "Fracture mechanics for delamination problems in composite materials", *J. Compos. Mater.*, **17**(3), 210-223.

Wu, J.S. and Hsieh, M. (2000), "Free vibration analysis of non-uniform beam with multiple point masses", *Struct. Eng. Mech.*, **9**(5), 449-467.

Wu, J.S. and Lin, T.L. (1990), "Free vibration analysis of a uniform cantilever beam with point masses by an analytical-and-numerical-combined method", *J. Sound Vib.*, **136**¸ 201-203.

Wu, L., Wang, Q. and Elishakoff, I. (2005), "Semi-inverse method for axially functionally graded beams with an anti-symmetric vibration mode", *J. Sound Vib.*, **284**, 1190-1202.

CC