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Three-dimensional analysis of the natural vibration of the three-layered hollow sphere with middle layer made of FGM

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Abstract. This paper is a continuation of the investigations started in the paper by Akbarov, S.D., Guliyev, H.H and Yahnioglu, N. (2016) "Natural vibration of the three-layered solid sphere with middle layer made of FGM: three-dimensional approach", Structural Engineering and Mechanics, 57(2), 239-263, to the case where the three-layered sphere is a hollow one. Three-dimensional exact field equations of elastodynamics are employed for investigation and the discrete-analytical method is employed for solution of the corresponding eigenvalue problem. The FGM is modelled as inhomogeneous for which the modulus of elasticity, Poison's ratio and density vary continuously through the inward radial direction according to power law distribution. Numerical results on the natural frequencies are presented and discussed. These results are also compared with the corresponding ones obtained in the previous paper by the authors. In particular, it is established that for certain harmonics and for roots of certain order, the values of the natural frequency obtained for the hollow sphere can be greater (or less) than those obtained for the solid sphere.

Keywords: functionally graded material; three-layered hollow sphere; natural vibration; natural frequencies; torsional oscillation; spheroidal oscillation

1. Introduction

In the historical aspect, as noted by Love (1944), investigations on the natural vibration of the sphere were associated originally with interest in the oscillations of the earth and were started with the paper by Lamb (1882), in which the material of the sphere is assumed to be isotropic and homogeneous and the mathematical procedures are made in the Cartesian coordinates. Note that in this paper, the auto-model-similarity solution of the equations of elastodynamics for which all sought quantities are presented as a function of the distance of the point considered from the origin of the Cartesian coordinate system, is found. Consequently, Lamb's solution allows the values of the natural frequencies to be found but this solution does not allow the modes of the natural vibrations to be found from the standpoint of modern ideas. Despite the simplicity of Lamb's results, they can be estimated as fundamental in the dynamics of the spherical elastic body and these results have great significance not only in the theoretical, but also in the practical sense. Namely, in the paper by Lamb (1882), first it was established that the solid sphere has two types of uncoupled free vibrations, the first of which are torsional vibrations with rotatory motions of the sphere for which there is no radial displacement and no volumetric change. However, the second type of free vibration is characterized by the volumetric change of the sphere caused by the non-zero radial displacement and this type is called spheroidal vibration.

Later, the mathematical treatment by Lamb (1882) was developed by Chree (1889) by employing the spherical coordinates. Later on, the results by Lamb were developed and applied by many researchers to describe the vibration of the Earth generated by earthquakes. For instance, the papers by Sato and Usami (1962a, 1962b), Sato *et al*. (1962) were related to a detailed analysis of the natural frequencies and vibration modes of the homogeneous isotropic solid sphere, in which the corresponding earlier results were performed and tabulated. These results were also presented and discussed in the monograph by Eringen and Suhubi (1975).

The natural vibration of the solid sphere with initially uniform volumetric loading was a subject of the investigations by Guz (1985a, b) in which by utilizing the three-dimensional linearized theory of elastic waves in initially stressed bodies for incompressible and compressible bodies it was established that Lamb's results on the types of natural vibration of the sphere occur also for the initially stressed cases.

Natural vibration of the hollow sphere (or spherical shell) made of homogeneous and isotropic elastic material was investigated by Shah *et al*. (1969a, b) by utilizing the

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three-dimensional exact equations of the linear theory of elastodynamics. In these works, numerical results for a wide range of thickness-to-radius ratios were given in graphical form. A detailed analysis of these and other related results which are associated with the free oscillations of the earth, were given and discussed in the book by Lapwood and Usami (1981).

More complicated problems related to the vibration of the solid and hollow spheres and connected with the geometries and material properties of spheres were studied recently by Hasheminejad and Mirzaei (2011), Sharma *et al*. (2012) and others listed therein. At the same time, it should be noted that currently the study of the dynamics of the structural elements made of advanced materials such as FGM and piezoelectric materials are being developed intensively, as in the investigations carried out by Asemi *et al*. (2014), Ipek (2015), Yun *et al*. (2010), Asgari and Akhlaghi (2011), Ilhan and Koç (2015) and others listed therein.

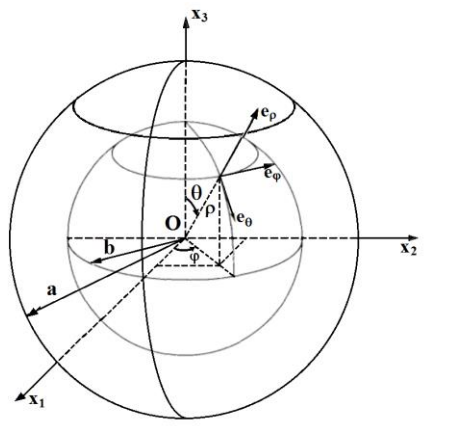
One of the main questions under investigation of the oscillations of the layered hollow and solid spheres is the accuracy of the theories which are applied under these investigations. In connection with this, in the paper by Grigorenko and Klina (1989), the accuracy of the approximate theories were examined and it is established that the accuracy of the approximate shell theories decreases with increasing of the ratio h/R, where h is the thickness and R is the radius of the middle surface of the sphere. In the paper by Jiang *et al*. (1996), the natural vibration of the layered hollow spheres is studied by using the three-dimensional exact equations of elastodynamics, and numerical results are presented in table form for the three-layered hollow sphere.

The paper by Chen and Ding (2001) deals with the investigation of the vibration of the layered hollow sphere, of which the materials of the layers are spherically isotropic (a special case of transversal isotropic materials) and homogeneous. Numerical results are presented and discussed for the three-layered case and the influence of the type of anisotropy of the layers' materials on the natural frequencies and vibration modes is established.

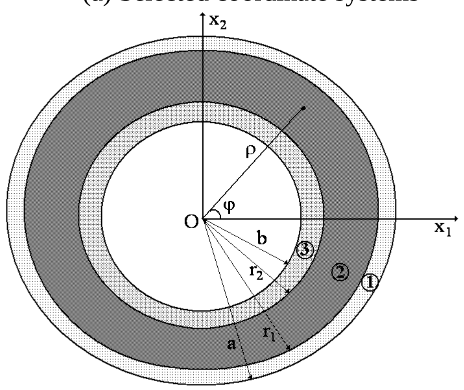
It follows from the modern level of studies on the bowels of the earth (see, for instance Anderson 2007) that the mechanical properties, such as the modulus of elasticity and density of the mantle material increase continuously from the crust to the core. At the same time, in modern layered hollow spheres, there are the cases in which the layers are made of Functionally Graded Materials (FGM) which give some advantages to these constructions in the application sense. Namely, these reasons require study of the dynamics of the layered hollow and solid spheres, the layers of which are made of FGM.

In the paper by Ye *et al*. (2014), some attempts were made for study of the three-dimensional vibration of a spherical shell which is obtained by cutting the complete hollow sphere by two parallel planes with arbitrary end conditions. The shell is modeled as a single-layered one with effective mechanical properties, the values of which change continuously in the thickness direction of the shell. These effective mechanical properties are determined through the mechanical properties of the ceramic and metal layers and their volumetric fraction in the shell, which also vary continuously through the thickness direction according to power law distribution. The three-dimensional elasticity relations are used in constructing the functional for employing the Rayleigh-Ritz method. The modified Fourier series with respect to all coordinates are applied for presenting the sought quantities and numerical results on the natural frequencies and the influence of the FGM properties on these frequencies are presented and discussed. The same approach was also employed in the papers by Jin *et al*. (2014a), Jin *et al*. (2014b) for investigation of the free vibration of arbitrarily thick rectangular plates made of FGM and for investigation of the vibration conical elastic shells made of orthotropic homogeneous material, respectively. Moreover, in the papers by Ye *et al*. (2016), Jin *et al*. (2015) the aforementioned method was developed and employed for study of the three-dimensional vibration of the functionally graded sandwich deep open spherical and cylindrical shells and functionally graded annular sector plates, respectively.

Recently, in the paper by Akbarov *et al*. (2016) the natural oscillation of the three-layered solid sphere with a middle layer made of FGM is examined. It is assumed that the materials of the core and outer layer of the sphere are homogeneous and isotropic elastic. Moreover, it is assumed that the modulus of elasticity, Poisson's ratio and density of the middle-layer material vary continuously through the



(a) Selected coordinate systems



(b) The cross section of the sphere at x_3=0

Fig. 1 Selected coordinate systems and cross section of the sphere inward radial direction according to power law distribution. The three dimensional exact equations and relations of linear elastodynamics are employed for the investigations. The discrete analytical method developed by Akbarov (2006, 2015) is applied for solution of the corresponding eigenvalue problem and numerical results on the natural frequencies related to the torsional and spheroidal oscillation modes are presented and discussed.

In the present work we attempt to further develop the foregoing investigations for the three-layered hollow cylinder, the middle layer of which is made of FGM. The investigations are made within the scope of all the assumptions accepted in the paper by Akbarov *et al*. (2016).

2. Formulation of the problem

Consider the three-layered hollow sphere and, with the center of the sphere, we associate the Cartesian coordinate system $Ox_1x_2x_3$ and spherical coordinate system $Or\theta\varphi$ (Fig. 1(a)). The cross section of sphere at x_3=0 and the parameters characterizing the structural geometry of this sphere are shown in Fig. 1(b). The outer and inner radius of the sphere (of the middle layer) will be denoted through a and b (r_1 and r_2), respectively. The values related to the outer, middle and inner layers will be indicated by the upper indices (1), (2) and (3), respectively.

Consider the field equations written in the spherical coordinate system shown in Fig. 1(a).

Equation of motion

$$\frac{\partial\sigma_{rr}^{(k)}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta r}^{(k)}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\sigma_{\varphi r}^{(k)}}{\partial\varphi}+ \frac{1}{r}\left(2\sigma_{rr}^{(k)}-\sigma_{\varphi\varphi}^{(k)}-\sigma_{\theta\theta}^{(k)}+\sigma_{\theta r}^{(k)}\cot\theta\right)=\rho\frac{\partial^2u_r^{(k)}}{\partial t^2}$$

$$\frac{\partial\sigma_{r\theta}^{(k)}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}^{(k)}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\sigma_{\varphi\theta}^{(k)}}{\partial\varphi}+ \frac{1}{r}\left(3\sigma_{r\theta}^{(k)}+\left(\sigma_{\theta\theta}^{(k)}-\sigma_{\varphi\varphi}^{(k)}\right)\cot\theta\right)=\rho\frac{\partial^2u_\theta^{(k)}}{\partial t^2}$$

$$\frac{\partial\sigma_{r\varphi}^{(k)}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\varphi}^{(k)}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\sigma_{\varphi\varphi}^{(k)}}{\partial\varphi}+ \frac{1}{r}\left(2\sigma_{r\varphi}^{(k)}+\sigma_{\varphi r}^{(k)}+\left(\sigma_{\theta\varphi}^{(k)}+\sigma_{\varphi\theta}^{(k)}\right)\cot\theta\right)=\rho\frac{\partial^2u_\varphi^{(k)}}{\partial t^2} \tag{1}$$

Elasticity relations

$$\sigma_{rr}^{(k)}=\lambda^{(k)}(\varepsilon_{rr}^{(k)}+\varepsilon_{\theta\theta}^{(k)}+\varepsilon_{\varphi\varphi}^{(k)})+2\mu^{(k)}\varepsilon_{rr}^{(k)}$$

$$\sigma_{\theta\theta}^{(k)}=\lambda^{(k)}(\varepsilon_{rr}^{(k)}+\varepsilon_{\theta\theta}^{(k)}+\varepsilon_{\varphi\varphi}^{(k)})+2\mu^{(k)}\varepsilon_{\theta\theta}^{(k)}$$

$$\sigma_{\varphi\varphi}^{(k)}=\lambda^{(k)}(\varepsilon_{rr}^{(k)}+\varepsilon_{\theta\theta}^{(k)}+\varepsilon_{\varphi\varphi}^{(k)})+2\mu^{(k)}\varepsilon_{\varphi\varphi}^{(k)}$$

$$\sigma_{r\theta}^{(k)}=2\mu^{(k)}\varepsilon_{r\theta}^{(k)},\quad \sigma_{\theta\varphi}^{(k)}=2\mu^{(k)}\varepsilon_{\theta\varphi}^{(k)},\sigma_{r\varphi}^{(k)}=2\mu^{(k)}\varepsilon_{r\varphi}^{(k)} \tag{2}$$

Strain-displacement relations

$$\varepsilon_{rr}^{(k)}=\frac{\partial u_r^{(k)}}{\partial r}\quad,\quad\varepsilon_{\theta\theta}^{(k)}=\frac{1}{r}\frac{\partial u_\theta^{(k)}}{\partial\theta}+\frac{1}{r}u_r^{(k)}$$

$$\varepsilon_{\varphi\varphi}^{(k)}=\frac{1}{r\sin\theta}\frac{\partial u_\varphi^{(k)}}{\partial\varphi}+\frac{1}{r}u_r^{(k)}+\frac{1}{r}u_\theta^{(k)}\cot\theta$$

$$\varepsilon_{r\theta}^{(k)}=\frac{1}{2}\left(\frac{\partial u_\theta^{(k)}}{\partial r}+\frac{1}{r}\frac{\partial u_r^{(k)}}{\partial\theta}-\frac{u_\theta^{(k)}}{r}\right)$$

$$\varepsilon_{\theta\varphi}^{(k)}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_\varphi^{(k)}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial u_\theta^{(k)}}{\partial\varphi}-\frac{u_\varphi^{(k)}}{r}\cot\theta\right)$$

$$\varepsilon_{r\varphi}^{(k)}=\frac{1}{2}\left(\frac{\partial u_\varphi^{(k)}}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial u_r^{(k)}}{\partial\theta}-\frac{u_\varphi^{(k)}}{r}\right) \tag{3}$$

In (1)-(3) conventional notation is used and it is assumed that k=1, 2 and 3.

As in the paper by Akbarov *et al*. (2016), we assume that the materials of the inner and outer layers are homogeneous isotropic, but the material of the middle layer is FG and isotropic, i.e., we assume that

$$\lambda^{(1)}=const_\lambda^{(1)}\quad,\quad\mu^{(1)}=const_\mu^{(1)},\rho^{(1)}=const_\rho^{(1)}$$

$$\lambda^{(3)}=const_\lambda^{(3)},\quad\mu^{(3)}=const_\mu^{(3)},\rho^{(3)}=const_\rho^{(3)}$$

$$\lambda^{(2)}=\lambda^{(2)}(r),\quad\mu^{(2)}=\mu^{(2)}(r)\quad\rho^{(2)}=\rho^{(2)}(r) \tag{4}$$

Specializations of the functions $\lambda^{(2)}=\lambda^{(2)}(r)$, $\mu^{(2)}=\mu^{(2)}(r)$ and $\rho^{(2)}(r)$ in (4) will be given below under consideration of the numerical results.

This completes the consideration of the field equations and relations which are used in the present investigations and also given in the paper by Akbarov *et al*. (2016). Now we consider formulation of the boundary and contact conditions. According to the nature of the problem under consideration, we assume that on the inner and the outer free surfaces of the sphere, i.e., at $r=b$ and $r=a$ (Fig. 1(b)), the following boundary conditions are satisfied

$$\sigma_{rr}^{(3)}\Big|_{r=b}=0,\quad\sigma_{r\theta}^{(3)}\Big|_{r=b}=0,\quad\sigma_{r\varphi}^{(3)}\Big|_{r=b}=0$$

$$\sigma_{rr}^{(1)}\Big|_{r=a}=0,\quad\sigma_{r\theta}^{(1)}\Big|_{r=a}=0,\quad\sigma_{r\varphi}^{(1)}\Big|_{r=a}=0 \tag{5}$$

Moreover, we assume that on the interface surfaces of the layers, i.e., at $r=r_2$ and $r=r_1$ (Fig. 1(b)), the following perfect contact conditions are satisfied

$$\sigma_{rr}^{(3)}\Big|_{r=r_2}=\sigma_{rr}^{(2)}\Big|_{r=r_2},\quad\sigma_{r\theta}^{(3)}\Big|_{r=r_2}=\sigma_{r\theta}^{(2)}\Big|_{r=r_2},$$

$$\sigma_{r\varphi}^{(3)}\Big|_{r=r_2}=\sigma_{r\varphi}^{(2)}\Big|_{r=r_2},\quad u_r^{(3)}\Big|_{r=r_2}=u_r^{(2)}\Big|_{r=r_2}$$

$$u_\theta^{(3)}\Big|_{r=r_2}=u_\theta^{(2)}\Big|_{r=r_2},\quad u_\varphi^{(3)}\Big|_{r=r_2}=u_\varphi^{(2)}\Big|_{r=r_2}$$

$$\sigma_{rr}^{(2)}\Big|_{r=r_1}=\sigma_{rr}^{(1)}\Big|_{r=r_1},\quad\sigma_{r\theta}^{(2)}\Big|_{r=r_1}=\sigma_{r\theta}^{(1)}\Big|_{r=r_1}$$

$$\sigma_{r\varphi}^{(2)}\Big|_{r=r_1}=\sigma_{r\varphi}^{(1)}\Big|_{r=r_1},\quad u_r^{(2)}\Big|_{r=r_1}=u_r^{(1)}\Big|_{r=r_1}$$

$$u_{\theta}^{(2)}\Big|_{r=r_1} = u_{\theta}^{(1)}\Big|_{r=r_1}, \quad u_{\varphi}^{(2)}\Big|_{r=r_1} = u_{\varphi}^{(1)}\Big|_{r=r_1} \tag{6}$$

This completes formulation of the problem on the natural vibration of the three-layered hollow sphere with middle layer made of FGM. The difference between the present formulation and the corresponding formulation given in the paper by Akbarov *et al*. (2016) consists of the first three conditions given in (5).

3. Method of solution

To solve the system of Eqs. (1)-(3) for the inner and outer layers of the sphere we use the following classical Lame (or Helmholtz) decomposition (see, for instance, Eringen and Suhubi 1975)

$$u_r^{(k)} = \frac{\partial\phi^{(k)}}{\partial r} + \frac{\partial^2 (r\chi^{(k)})}{\partial r^2} - r\nabla^2\chi^{(k)}$$

$$u_{\theta}^{(k)} = \frac{1}{r}\frac{\partial\phi^{(k)}}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial\psi^{(k)}}{\partial\varphi} + \frac{1}{r}\frac{\partial^2(r\chi^{(k)})}{\partial\theta\,\partial r}$$

$$u_{\varphi}^{(k)} = \frac{1}{r\sin\theta}\frac{\partial\phi^{(k)}}{\partial\varphi} - \frac{\partial\psi^{(k)}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial^2(r\chi^{(k)})}{\partial\varphi\,\partial r} \quad k=1,3 \tag{7}$$

The functions $\phi^{(k)}$, $\chi^{(k)}$ and $\psi^{(k)}$ are solutions to the following equations

$$\nabla^2\phi^{(k)} - \frac{1}{(c_1^{(k)})^2}\frac{\partial^2\phi^{(k)}}{\partial t^2} = 0,$$

$$\nabla^2\chi^{(k)} - \frac{1}{(c_2^{(k)})^2}\frac{\partial^2\chi^{(k)}}{\partial t^2} = 0$$

$$\nabla^2\psi^{(k)} - \frac{1}{(c_2^{(k)})^2}\frac{\partial^2\psi^{(k)}}{\partial t^2} = 0 \tag{8}$$

where

$$c_1^{(k)} = \sqrt{(\lambda^{(k)} + 2\mu^{(k)})/\rho^{(k)}}, \quad c_2^{(k)} = \sqrt{\mu^{(k)}/\rho^{(k)}}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2} \tag{9}$$

Solutions of the Eqs. (8) and (9) for the case under consideration are found as follows

$$\phi^{(k)}(r,\theta,\varphi,t) = \left[A^{(k)} j_n(\alpha^{(k)} r) + B^{(k)} y_n(\alpha^{(k)} r)\right]\times P_n^m(\cos\theta)\cos m\varphi e^{i\omega t}$$

$$\psi^{(k)}(r,\theta,\varphi,t) = \left[C^{(k)} j_n(\beta^{(k)} r) + D^{(k)} y_n(\beta^{(k)} r)\right]\times P_n^m(\cos\theta)\sin m\varphi e^{i\omega t}$$

$$\chi^{(k)}(r,\theta,\varphi,t) = \left[E^{(k)} j_n(\beta^{(k)} r) + F^{(k)} y_n(\beta^{(k)} r)\right]\times P_n^m(\cos\theta)\cos m\varphi e^{i\omega t}$$

$$\alpha^{(k)} = \omega/c_1^{(k)}, \quad \beta^{(k)} = \omega/c_2^{(k)} \tag{10}$$

In (10), $j_n(cr)$ and $y_n(cr)$ are spherical Bessel functions of the first and second kind and

$$j_n(cr) = \left(\frac{\pi}{2cr}\right)^{1/2} J_{n+1/2}(cr)$$

$$y_n(cr) = \left(\frac{\pi}{2cr}\right)^{1/2} Y_{n+1/2}(cr) \tag{11}$$

where $J_{n+1/2}(cr)$ and $Y_{n+1/2}(cr)$ are the Bessel functions of the first and the second kind with non-integer order, respectively. Moreover, $P_n^m(\cos\theta)$ in the expression (10) denotes the associated Legendre functions with m-th order and with n-th harmonic.

Thus, using the relations (11), (10) and (7) we obtain expressions for the displacements and, after substituting these expressions into the Eqs. (3) and (2), we determine the components of the stress tensor. For simplification of writing the obtained expressions, we introduce two sets of complete orthogonal functions in $[0, \pi]$ determined through the following expressions

$$X_{nm}(\theta) = P_n^m(\cos\theta)$$

$$Y_{nm}(\theta) = n\cot\theta P_n^m(\cos\theta) - \frac{n+m}{\sin\theta}P_{n-1}^m(\cos\theta) \tag{12}$$

Thus, using the notation (12) we can write the following expressions for the sought values

$$u_r^{(k)} = \frac{1}{r}\left\{A^{(k)}u_{11}^{(k)} + B^{(k)}u_{12}^{(k)} + E^{(k)}u_{31}^{(k)} + F^{(k)}u_{32}^{(k)}\right\}\times X_{nm}(\theta)\cos m\varphi e^{i\omega t}$$

$$u_{\theta}^{(k)} = \frac{1}{r}\left\{\left[A^{(k)}v_{11}^{(k)} + B^{(k)}v_{12}^{(k)} + E^{(k)}v_{31}^{(k)} + F^{(k)}v_{32}^{(k)}\right]\times Y_{nm}(\theta) + (C^{(k)}v_{21}^{(k)} + D^{(k)}v_{22}^{(k)})\frac{m}{\sin\theta}X_{nm}(\theta)\right\}\times \cos m\varphi e^{i\omega t}$$

$$u_{\varphi}^{(k)} = \frac{1}{r}\left\{\left[A^{(k)}v_{11}^{(k)} + B^{(k)}v_{12}^{(k)} + E^{(k)}v_{31}^{(k)} + F^{(k)}v_{32}^{(k)}\right]\times \frac{-m}{\sin\theta}X_{nm}(\theta) + (-C^{(k)}v_{21}^{(k)} - D^{(k)}v_{22}^{(k)})Y_{nm}(\theta)\right\}\times \sin m\varphi e^{i\omega t}$$

$$\sigma_{rr}^{(k)} = \frac{2\mu^{(k)}}{r^2}\left[A^{(k)}T_{111}^{(k)} + B^{(k)}T_{112}^{(k)} + E^{(k)}T_{131}^{(k)} + F^{(k)}T_{132}^{(k)}\right]X_{nm}(\theta)\cos m\varphi e^{i\omega t}$$

$$\sigma_{r\theta}^{(k)} = \frac{2\mu^{(k)}}{r^2}\left\{\left[A^{(k)}T_{411}^{(k)} + B^{(k)}T_{412}^{(k)} + E^{(k)}T_{431}^{(k)} + F^{(k)}T_{432}^{(k)}\right]Y_{nm}(\theta) + \left[-C^{(k)}T_{421}^{(k)} - D^{(k)}T_{422}^{(k)}\right]\frac{m}{\sin\theta}X_{nm}(\theta)\right\}\cos m\varphi e^{i\omega t}$$

$$\sigma_{r\varphi}^{(k)} = \frac{2\mu^{(k)}}{r^2}\left\{\left[A^{(k)}T_{411}^{(k)} + B^{(k)}T_{412}^{(k)} + E^{(k)}T_{431}^{(k)} + F^{(k)}T_{432}^{(k)}\right]\frac{m}{\sin\theta}X_{nm}(\theta) + \left[-C^{(k)}T_{421}^{(k)} - D^{(k)}T_{422}^{(k)}\right]Y_{nm}(\theta)\right\}\cos m\varphi e^{i\omega t} \quad (13)$$

where

$$u_{11}^{(k)} = nj_n(\alpha^{(k)}r) - \alpha^{(k)}rj_{n+1}(\alpha^{(k)}r)$$
$$u_{12}^{(k)} = ny_n(\alpha^{(k)}r) - \alpha^{(k)}ry_{n+1}(\alpha^{(k)}r)$$
$$u_{31}^{(k)} = n(n+1)j_n(\beta^{(k)}r), \quad u_{32}^{(k)} = n(n+1)y_n(\beta^{(k)}r)$$
$$v_{11}^{(k)} = j_n(\alpha^{(k)}r), \quad v_{12}^{(k)} = y_n(\alpha^{(k)}r),$$
$$v_{21}^{(k)} = j_n(\beta^{(k)}r), \quad v_{22}^{(k)} = y_n(\beta^{(k)}r)$$
$$v_{31}^{(k)} = (n+1)j_n(\beta^{(k)}r) - \beta^{(k)}rj_{n+1}(\beta^{(k)}r)$$
$$v_{32}^{(k)} = (n+1)y_n(\beta^{(k)}r) - \beta^{(k)}ry_{n+1}(\beta^{(k)}r)$$
$$T_{111}^{(k)} = (n^2 - n - \frac{1}{2}(\beta^{(k)})^2r^2)j_n(\alpha^{(k)}r) + 2\alpha^{(k)}rj_{n+1}(\alpha^{(k)}r)$$
$$T_{112}^{(k)} = (n^2 - n - \frac{1}{2}(\alpha^{(k)})^2r^2)y_n(\alpha^{(k)}r) + 2\alpha^{(k)}ry_{n+1}(\alpha^{(k)}r)$$
$$T_{131}^{(k)} = n(n+1)\left[(n-1)j_n(\beta^{(k)}r) - \beta^{(k)}rj_{n+1}(\beta^{(k)}r)\right]$$
$$T_{132}^{(k)} = n(n+1)\left[(n-1)y_n(\beta^{(k)}r) - \beta^{(k)}ry_{n+1}(\beta^{(k)}r)\right]$$
$$T_{411}^{(k)} = (n-1)j_n(\alpha^{(k)}r) - \alpha^{(k)}rj_{n+1}(\alpha^{(k)}r)$$
$$T_{412}^{(k)} = (n-1)y_n(\alpha^{(k)}r) - \alpha^{(k)}ry_{n+1}(\alpha^{(k)}r)$$
$$T_{421}^{(k)} = \frac{1}{2}r\left[(n-1)j_n(\beta^{(k)}r) - \beta^{(k)}rj_{n+1}(\beta^{(k)}r)\right]$$
$$T_{422}^{(k)} = \frac{1}{2}r\left[(n-1)y_n(\beta^{(k)}r) - \beta^{(k)}ry_{n+1}(\beta^{(k)}r)\right]$$
$$T_{431}^{(k)} = (n^2 - 1 - \frac{1}{2}(\beta^{(k)})^2r^2)j_n(\beta^{(k)}r) + \beta^{(k)}rj_{n+1}(\beta^{(k)}r)$$
$$T_{432}^{(k)} = (n^2 - 1 - \frac{1}{2}(\beta^{(k)})^2r^2)y_n(\beta^{(k)}r) + \beta^{(k)}ry_{n+1}(\beta^{(k)}r), \quad k = 1,3. \quad (14)$$

Note that in (13), the expressions for the stresses which enter the boundary and contact conditions have been written.

Using the expressions in (13) and (12) and boundary (5) and contact (6) conditions we obtain two uncoupled systems of algebraic equations, the first of which contains the unknown constants $A^{(k)}$, $B^{(k)}$, $E^{(k)}$ and $F^{(k)}$, but the second one contains the unknown constants $C^{(k)}$ and $D^{(k)}$. Analyzing the expressions obtained for the stresses $\sigma_{r\theta}^{(k)}$ and $\sigma_{r\varphi}^{(k)}$, and given in (13), it can be easily established that the aforementioned algebraic equations obtained with respect to these stresses coincide with each other. The same conclusion follows also from the expressions (13) for the displacements $u_{\theta}^{(k)}$ and $u_{\varphi}^{(k)}$, i.e. the algebraic equations obtained with respect to these displacements also coincide with each other. Hence, for obtaining the two uncoupled sets of systems of algebraic equations it is enough to use only the contact and boundary conditions written with respect to the stresses $\sigma_{rr}^{(k)}$ and $\sigma_{r\theta}^{(k)}$ (or $\sigma_{r\varphi}^{(k)}$), and displacements $u_r^{(k)}$ and $u_{\theta}^{(k)}$ (or $u_{\varphi}^{(k)}$).

It is evident that the foregoing (7)-(9) decomposition cannot be applied directly to the solution of the equations (1), (2) and (3) for the middle layer, which is made of FGM. In connection with this, as in the paper by Akbarov *et al*. (2016), we employ the discrete analytical method described in the references Akbarov (2006, 2015). According to this method, first, the middle layer with thickness $h_m=r_1-r_2$ is divided into M number of sublayers with thickness $h'_m=(r_1-r_2)/M$ and it is assumed that within the scope of each p-th ($1\le p\le M$) sublayer, the material is homogeneous and the Lame constants and density of this material are determined through the following expressions

$$\mu^{(2)p} = \mu^{(2)}(r)\Big|_{r=r_1+(p-1/2)h'_m}$$
$$\lambda^{(2)p} = \lambda^{(2)}(r)\Big|_{r=r_1+(p-1/2)h'_m}$$
$$\rho^{(2)p} = \rho^{(2)}(r)\Big|_{r=r_1+(p-1/2)h'_m} \quad (15)$$

Thereby, the solution to Eqs. (1), (2) and (3) for the middle layer, which are equations with variable coefficients, are reduced to the series of the same equations with constant coefficients determined according to the relations in (15) and the foregoing solution procedure employed for the face layers of the hollow sphere is applied to find the analytical expressions for each sublayer of the middle layer.

Supposing that between the sublayers, perfect contact conditions are satisfied, we can write the following

$$\sigma_{rr}^{(3)}\Big|_{r=r_2} = \sigma_{rr}^{(2)1}\Big|_{r=r_2}, \quad \sigma_{r\theta}^{(3)}\Big|_{r=r_2} = \sigma_{r\theta}^{(2)1}\Big|_{r=r_2}$$
$$\sigma_{r\varphi}^{(3)}\Big|_{r=r_2} = \sigma_{r\varphi}^{(2)1}\Big|_{r=r_2}, \quad u_r^{(3)}\Big|_{r=r_2} = u_r^{(2)1}\Big|_{r=r_2}$$
$$u_{\theta}^{(3)}\Big|_{r=r_2} = u_{\theta}^{(2)1}\Big|_{r=r_2}, \quad u_{\varphi}^{(3)}\Big|_{r=r_2} = u_{\varphi}^{(2)1}\Big|_{r=r_2},$$
$$\sigma_{rr}^{(2)1}\Big|_{r=r_2+h'_m} = \sigma_{rr}^{(2)2}\Big|_{r=r_2+h'_m}$$
$$\sigma_{r\theta}^{(2)1}\Big|_{r=r_2+h'_m} = \sigma_{r\theta}^{(2)1}\Big|_{r=r_2+h'_m}$$
$$\sigma_{r\varphi}^{(2)1}\Big|_{r=r_2+h'_m} = \sigma_{r\varphi}^{(2)2}\Big|_{r=r_2+h'}$$
$$u_r^{(2)1}\Big|_{r=r_2+h'_m} = u_r^{(2)2}\Big|_{r=r_2+h'_m}$$

$$u_{\theta}^{(2)1}\Big|_{r=r_2+h'_m} = u_{\theta}^{(2)2}\Big|_{r=r_2+h'_m}$$

$$u_{\varphi}^{(2)1}\Big|_{r=r_2+h'_m} = u_{\varphi}^{(2)2}\Big|_{r=r_2+h'_m}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sigma_{rr}^{(2)p-1}\Big|_{r=r_2+ph'_m} = \sigma_{rr}^{(2)p}\Big|_{r=r_2+ph'_m}$$

$$\sigma_{r\theta}^{(2)p-1}\Big|_{r=r_2+ph'_m} = \sigma_{r\theta}^{(2)p}\Big|_{r=r_2+ph'_m}$$

$$\sigma_{r\varphi}^{(2)p-1}\Big|_{r=r_2+ph'_m} = \sigma_{r\varphi}^{(2)p}\Big|_{r=r_2+ph'}$$

$$u_{r}^{(2)p-1}\Big|_{r=r_2+ph'_m} = u_{r}^{(2)p}\Big|_{r=r_2+ph'_m}$$

$$u_{\theta}^{(2)p-1}\Big|_{r=r_2+ph'_m} = u_{\theta}^{(2)p}\Big|_{r=r_2+ph'_m}$$

$$u_{\varphi}^{(2)p-1}\Big|_{r=r_2+ph'_m} = u_{\varphi}^{(2)p}\Big|_{r=r_2+ph'_m}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sigma_{rr}^{(2)M}\Big|_{r=r_1} = \sigma_{rr}^{(1)}\Big|_{r=r_1}, \quad \sigma_{r\theta}^{(2)M}\Big|_{r=r_1} = \sigma_{r\theta}^{(1)}\Big|_{r=r_1}$$

$$\sigma_{r\varphi}^{(2)M}\Big|_{r=r_1} = \sigma_{r\varphi}^{(1)}\Big|_{r=r_1}, \quad u_{r}^{(2)M}\Big|_{r=r_1} = u_{r}^{(1)}\Big|_{r=r_1}$$

$$u_{\theta}^{(2)M}\Big|_{r=r_1} = u_{\theta}^{(1)}\Big|_{r=r_1}, \quad u_{\varphi}^{(2)M}\Big|_{r=r_1} = u_{\varphi}^{(1)}\Big|_{r=r_1} \qquad (16)$$

where

$$u_r^{(2)p} = \frac{1}{r}\Big\{A^{(2)p}u_{11}^{(2)p} + B^{(2)p}u_{12}^{(2)p} + E^{(2)p}u_{31}^{(2)} + F^{(2)}u_{32}^{(2)p}\Big\}\times X_{nm}(\theta)\cos m\varphi e^{i\omega t}$$

$$u_{\theta}^{(2)p} = \frac{1}{r}\Big\{\Big[A^{(2)p}v_{11}^{(2)p} + B^{(2)p}v_{12}^{(2)p} + E^{(2)p}v_{31}^{(2)p} + F^{(2)p}v_{32}^{(2)p}\Big]Y_{nm}(\theta) + (C^{(2)p}v_{21}^{(2)p} + D^{(2)p}v_{22}^{(2)p})\frac{m}{\sin\theta}X_{nm}(\theta)\Big\}\cos m\varphi e^{i\omega t}$$

$$u_{\varphi}^{(2)p} = \frac{1}{r}\Big\{\Big[A^{(2)p}v_{11}^{(2)p} + B^{(2)p}v_{12}^{(2)p} + E^{(2)p}v_{31}^{(2)p} + F^{(2)p}v_{32}^{(2)p}\Big]\frac{-m}{\sin\theta}X_{nm}(\theta) + (-C^{(2)p}v_{21}^{(2)p} - D^{(2)p}v_{22}^{(2)p})Y_{nm}(\theta)\Big\}\sin m\varphi e^{i\omega t}$$

$$\sigma_{rr}^{(2)p} = \frac{2\mu^{(2)p}}{r^2}\Big[A^{(2)p}T_{111}^{(2)p} + B^{(2)p}T_{112}^{(2)p} + E^{(2)p}T_{131}^{(2)p} + F^{(2)p}T_{132}^{(2)p}\Big]X_{nm}(\theta)\cos m\varphi e^{i\omega t}$$

$$\sigma_{r\theta}^{(2)p} = \frac{2\mu^{(2)p}}{r^2}\Big\{\Big[A^{(2)p}T_{411}^{(2)p} + B^{(2)p}T_{412}^{(2)p} + E^{(2)p}T_{431}^{(2)p} + F^{(2)p}T_{432}^{(2)p}\Big]Y_{nm}(\theta) + \Big[-C^{(2)p}T_{421}^{(2)p} - D^{(2)p}T_{422}^{(2)p}\Big]\frac{m}{\sin\theta}X_{nm}(\theta)\Big\}\cos m\varphi e^{i\omega t}$$

$$\sigma_{r\varphi}^{(2)p} = \frac{2\mu^{(2)p}}{r^2}\Big\{\Big[A^{(2)p}T_{411}^{(2)p} + B^{(2)p}T_{412}^{(2)p} + E^{(2)p}T_{431}^{(2)p} + F^{(2)p}T_{432}^{(2)p}\Big]\frac{m}{\sin\theta}X_{nm}(\theta) + \Big[-C^{(2)p}T_{421}^{(2)p} - D^{(2)p}T_{422}^{(2)p}\Big]Y_{nm}(\theta)\Big\}\cos m\varphi e^{i\omega t} \qquad (17)$$

In (17) the following notation is used.

$$u_{11}^{(2)} = nj_n(\alpha^{(2)p}r) - \alpha^{(2)p}rj_{n+1}(\alpha^{(2)p}r)$$

$$u_{12}^{(2)p} = ny_n(\alpha^{(2)p}r) - \alpha^{(2)p}ry_{n+1}(\alpha^{(2)p}r)$$

$$u_{31}^{(2)p} = n(n+1)j_n(\beta^{(2)p}r)$$

$$u_{32}^{(2)p} = n(n+1)\,\mathrm{y}_n(\beta^{(2)p}r), \quad v_{11}^{(2)} = j_n(\alpha^{(2)p}r)$$

$$v_{12}^{(2)} = y_n(\alpha^{(2)p}r), \quad v_{21}^{(2)p} = j_n(\beta^{(2)p}r)$$

$$v_{22}^{(2)p} = y_n(\beta^{(2)p}r),$$

$$v_{31}^{(2)} = (n+1)j_n(\beta^{(2)p}r) - \beta^{(2)p}rj_{n+1}(\beta^{(2)p}r)$$

$$v_{32}^{(2)p} = (n+1)y_n(\beta^{(2)p}r) - \beta^{(2)p}ry_{n+1}(\beta^{(2)p}r)$$

$$T_{111}^{(2)p} = (n^2 - n - \frac{1}{2}(\beta^{(2)p})^2r^2)j_n(\alpha^{(2)p}r) + 2\alpha^{(2)p}rj_{n+1}(\alpha^{(2)p}r)$$

$$T_{112}^{(2)p} = (n^2 - n - \frac{1}{2}(\beta^{(2)p})^2r^2)y_n(\alpha^{(2)p}r) + 2\alpha^{(2)p}ry_{n+1}(\alpha^{(2)p}r)$$

$$T_{131}^{(2)p} = n(n+1)\Big[(n-1)j_n(\beta^{(2)p}r) - \beta^{(2)p}rj_{n+1}(\beta^{(2)p}r)\Big]$$

$$T_{132}^{(2)p} = n(n+1)\Big[(n-1)y_n(\beta^{(2)p}r) - \beta^{(2)p}ry_{n+1}(\beta^{(2)p}r)\Big]$$

$$T_{411}^{(2)p} = (n-1)j_n(\alpha^{(2)p}r) - \alpha^{(2)p}rj_{n+1}(\alpha^{(2)p}r)$$

$$T_{412}^{(2)p} = (n-1)y_n(\alpha^{(2)p}r) - \alpha^{(2)p}ry_{n+1}(\alpha^{(2)p}r)$$

$$T_{421}^{(2)p} = \frac{1}{2}r\Big[(n-1)j_n(\beta^{(2)p}r) - \beta^{(2)p}rj_{n+1}(\beta^{(2)p}r)\Big]$$

$$T_{422}^{(2)p} = \frac{1}{2}r\Big[(n-1)y_n(\beta^{(2)p}r) - \beta^{(2)p}ry_{n+1}(\beta^{(2)p}r)\Big]$$

$$T_{431}^{(2)p} = (n^2 - 1 - \frac{1}{2}(\beta^{(2)p})^2r^2)j_n(\beta^{(2)p}r) + \beta^{(2)p}rj_{n+1}(\beta^{(2)p}r)$$

$$T_{432}^{(2)p} = (n^2 - 1 - \frac{1}{2}(\beta^{(2)p})^2r^2)y_n(\beta^{(2)p}r) +$$

$$\beta^{(2)p} r y_{n+1}(\beta^{(2)p} r)$$
$$\alpha^{(k)p} = \omega / c_1^{(k)p}, \quad \beta^{(k)p} = \omega / c_2^{(k)p}$$
$$c_1^{(k)p} = \sqrt{(\lambda^{(k)p} + 2\mu^{(k)p}) / \rho^{(k)p}}$$
$$c_2^{(2)p} = \sqrt{\mu^{(2)p} / \rho^{(2)p}} \tag{18}$$

Taking the foregoing discussions and expressions into consideration, we obtain from (13) and (18) two uncoupled systems of algebraic equations from the boundary (5) and contact (16) conditions. The first (second) system contains the unknowns $A^{(1)}$, $B^{(1)}$, $E^{(1)}$, $F^{(1)}$, $A^{(2)1}$, $B^{(2)1}$, $E^{(2)1}$, $F^{(2)1}$,…, $A^{(2)M}$, $B^{(2)M}$, $E^{(2)M}$, $F^{(2)M}$, $A^{(3)}$, $B^{(3)}$, $E^{(3)}$ and $F^{(3)}$ ($C^{(1)}$, $D^{(1)}$, $C^{(2)1}$, $D^{(2)1}$,…, $C^{(2)M}$, $D^{(2)M}$, $C^{(3)}$ and $D^{(3)}$). Equating to zero the determinant of the coefficient matrix of the first and second group of equations separately, the following equations for determination of the frequency of the natural vibration are obtained.

$$\det\left(\gamma_{q_1 q_2}\right) = 0, \quad q_1; q_2 = 1, 2, ..., 4M + 8 \tag{19}$$

(for the spheroidal vibration) and

$$\det\left(\delta_{p_1 p_2}\right) = 0, \quad p_1; p_2 = 1, 2, ..., 2M + 4 \tag{20}$$

(for the torsional vibration).

The explicit expressions of the components γ_{q1q2} in (19) and of the components δ_{p1p2} in (20) can be easily determined from the expressions (13), (14), (17) and (18). The number M in the Eqs. (19) and (20) is determined in the numerical solution procedure of these equations from the convergence requirement of the numerical results.

This completes the consideration of the solution method.

4. Numerical results and discussions

As in the paper by Akbarov *et al*. (2016), to find the numerical solution to the Eqs. (19) and (20) we employ the well-known “bi-section” method and the functions $\lambda^{(2)}(r)$, $\mu^{(2)}(r)$, and $\rho^{(2)}(r)$ which characterize the functionally graded property of the middle-layer material of the sphere, are selected as follows:

$$E^{(2)}(r) = E_0^{(2)}(1 + \eta_1(a_1 r + b_1)^{n_1})^{m_1}$$
$$\nu^{(2)}(r) = \nu_0^{(2)}(1 + \eta_2(a_2 r + b_2)^{n_2})^{m_2}$$
$$\rho^{(2)}(r) = \rho_0^{(2)}(1 + \eta_3(a_3 r + b_3)^{n_3})^{m_3}$$
$$\lambda^{(2)}(r) = \frac{E^{(2)}(r)\nu^{(2)}(r)}{(1+\nu^{(2)}(r))(1-2\nu^{(2)}(r))}$$
$$\mu^{(2)}(r) = \frac{E^{(2)}(r)}{2(1+\nu^{(2)}(r))} \tag{21}$$

where a_k, b_k, n_k, m_k and η_k; (k=1,2,3) are real numbers.

The main aim of the present numerical investigations is to determine how the functionally graded properties of the middle-layer material (as in the paper by Akbarov *et al*. 2016) and the dimension of the inner radius of the three-layered hollow sphere act on its spheroidal and torsional vibration. Before consideration of the main numerical results, we test the algorithm and PC programs used for calculating the numerical results, which are composed by the authors and realized in MATLAB. For this purpose we consider the numerical results obtained for the case where the middle-layer material of the hollow sphere is homogeneous like the outer and inner layers. Consider the case, which was also considered in the paper by Jiang *et al*. (1996), according to which, we assume that $\eta_1=\eta_2=\eta_3=0$ in (21) and $E^{(2)}/E^{(1)}=\rho^{(2)}/\rho^{(1)}=3$, $E^{(3)}/E^{(1)}=\rho^{(3)}/\rho^{(1)}=5$, $\nu^{(1)}=\nu^{(2)}=\nu^{(3)}=0.3$, $r_1/a=0.8$, $r_2/a=0.6$ and $b/a=0.4$.

Introduce the dimensionless frequency

$$\Omega = \omega a \sqrt{\rho^{(1)} / \mu^{(1)}} \tag{22}$$

and consider the results given in Tables 1 and 2 which show the first five values of Ω obtained for the first six harmonic numbers (n=0,1,…,5) for the torsional and spheroidal vibration modes, respectively, of the sphere under the foregoing values of the problem parameters. Note that in these tables, the corresponding results obtained in the paper by Jiang *et al*. (1996) are also presented (lower numbers). The tables show that the results obtained by employing the present algorithm coincide almost completely with the corresponding ones obtained in the paper by Jiang *et al*. (1996). This confirms the validity of the algorithm and programs used in the present investigation.

Now we attempt to discuss the effect of an increase (or a decrease) in the values of the modulus of elasticity under fixed values of the materials’ densities, as well as the effect of an increase (or a decrease) in the values of the materials’ densities under fixed values of the modulus of elasticity in the inward radial direction on the values of the natural frequencies. Consider the case where $h^{(1)}/a$=0.1, $h^{(2)}/a$=0.5, $h^{(3)}/a$=0.2 and $\nu^{(1)}=\nu^{(2)}=\nu^{(3)}$=0.3 where $\nu^{(k)}$ is Poisson’s ratio of the k-th material. We analyze the results given in Tables 3 and 4 which illustrate the influence of the change of the modulus of elasticity and the change of the densities in the aforementioned cases, respectively, on the values of Ω

Table 1 Natural frequencies $\Omega = \omega a \sqrt{\rho^{(1)} / \mu^{(1)}}$ of the torsional vibration mode obtained in the present paper (upper number) and in the paper by Jiang *et al*. (1996) (lower number)

n	$\Omega\left(=\omega a\sqrt{\rho^{(1)}/\mu^{(1)}}\right)$				
	1	2	3	4	5
0	<u>5.4835</u> 5.4832	<u>9.7525</u> 9.7525	<u>16.077</u> 16.078	<u>21.485</u> 21.486	<u>25.557</u> 25.558
1	<u>5.9255</u> 5.9254	<u>10.005</u> 10.006	<u>16.285</u> 16.286	<u>21.588</u> 21.589	<u>25.645</u> 25.645
2	<u>2.7775</u> 2.7776	<u>6.7325</u> 6.7323	<u>10.508</u> 10.508	<u>16.695</u> 16.695	<u>21.793</u> 21.793
3	<u>4.2635</u> 4.2637	<u>7.7845</u> 7.7840	<u>11.252</u> 11.253	<u>17.293</u> 17.294	<u>22.097</u> 22.097
4	<u>5.5515</u> 5.5511	<u>8.9565</u> 8.9568	<u>12.226</u> 12.227	<u>18.063</u> 18.063	<u>22.499</u> 22.499
5	<u>6.7375</u> 6.7379	<u>10.159</u> 10.160	<u>13.403</u> 13.403	<u>18.981</u> 18.981	<u>22.997</u> 22.997

Table 2 Natural frequencies $\Omega = \omega a\sqrt{\rho^{(1)}/\mu^{(1)}}$ of the spheroidal vibration mode obtained in the present paper (upper number) and in the paper by Jiang *et al*. (1996) (lower number)

n	$\Omega\left(=\omega a\sqrt{\rho^{(1)}/\mu^{(1)}}\right)$ 1	2	3	4	5
0	4.6895	5.4835	9.7525	11.599	16.077
	4.6892	5.4832	9.7525	11.599	16.078
1	4.0895	7.5365	9.8365	12.237	16.160
	4.0896	7.5362	9.8366	12.238	16.160
2	2.4735	5.1495	8.7905	10.842	13.147
	2.4736	5.1491	8.7907	10.842	13.147
3	4.0875	6.6345	9.4525	12.672	14.065
	4.0871	6.6342	9.4524	12.672	14.066
4	5.7115	8.0685	10.285	14.319	15.036
	5.7119	8.0681	10.286	14.320	15.037
5	7.2215	9.4335	11.413	15.246	16.515
	7.2218	9.4333	11.414	15.246	16.515

(22) for the torsional and spheroidal vibration modes. These results are obtained for the first six harmonics and the first three roots are presented for each harmonic.

Thus, it follows from these results that an increase (a decrease) in the values of the modulus of elasticity in the inward radial direction causes an increase (a decrease) in the values of the natural frequency Ω. Also, these results show that an increase (a decrease) in the values of the densities in the inward radial direction causes a decrease (an increase) in the values of Ω. These conclusions agree in the qualitative sense with the corresponding ones obtained in the paper by Akbarov *et al*. (2016). Moreover, these conclusions give some orientation for estimation and explanation of the numerical results which are obtained for the case where the material of the middle layer of the sphere is FG. It should be noted that the results obtained in the cases where the modulus of elasticity and densities are changed simultaneously can also be explained and estimated according to the foregoing conclusions.

Now we consider the results which illustrate the influence of the change of the inner radius of the hollow cylinder on the values of the natural frequencies. For this purpose we consider the numerical results obtained for various values of $h^{(3)}/a$ in the cases indicated in Table 3. These results are shown by graphs given in Figs. 2, 3, 4 and 5 which are constructed in the cases where $\{E^{(2)}/E^{(1)}=0.5, E^{(3)}/E^{(1)}=0.3\}$, $\{E^{(2)}/E^{(1)}=0.7, E^{(3)}/E^{(1)}=0.5\}$, $\{E^{(2)}/E^{(1)}=3, E^{(3)}/E^{(1)}=5\}$ and $\{E^{(2)}/E^{(1)}=7, E^{(3)}/E^{(1)}=9\}$ respectively for $h^{(3)}/a$=0.1, 0.3 and 0.38. Note that in these figures the graphs grouped by letters *a*, *b* and *c* (by letters *d*, *e* and *f*) relate to the torsional (spheroidal) vibration of the hollow sphere and show the natural vibration frequencies for the first, second and third roots respectively.

As we assume that $h^{(1)}/a$=0.1 and $h^{(2)}/a$=0.5, therefore, according to well-known mechanical considerations, the values of Ω must approach the corresponding values of Ω obtained for the corresponding solid sphere (i.e., in the case

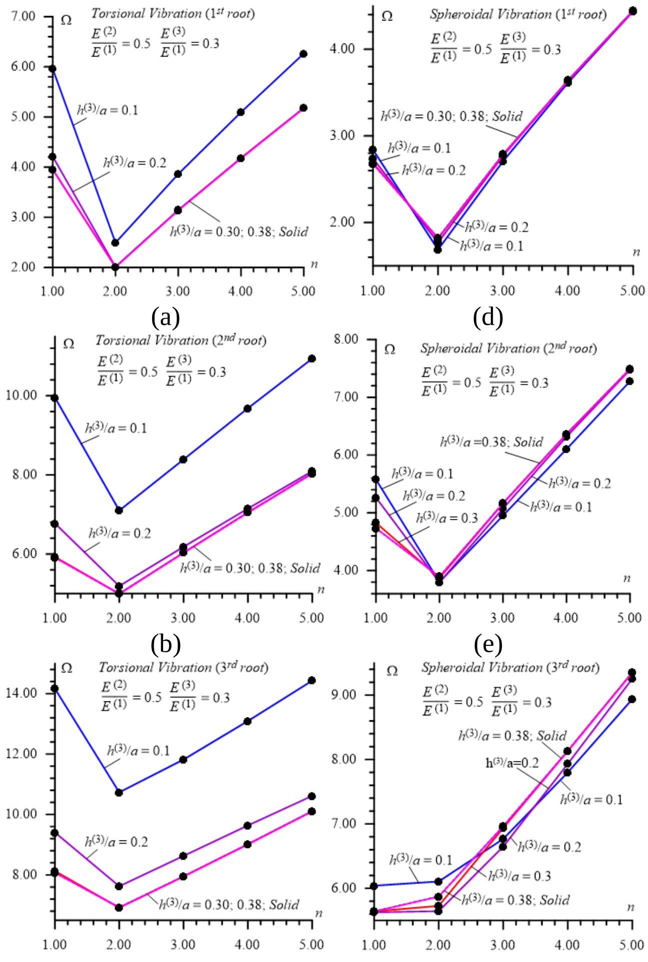


Fig. 2 Dependence of the natural vibration frequency on the harmonics of vibration under $E^{(2)}/E^{(1)}$=0.5 and $E^{(3)}/E^{(1)}$=0.3

under consideration to the values of Ω given in Table 3 of the paper by Akbarov *et al*. (2016)) as $h^{(3)}/a\to0.4$. Thus, comparison of the results given in Figs. 2-5 with each other and with the corresponding ones given in Table 3 of the paper by Akbarov *et al*. (2016) proves the foregoing prediction and the results obtained in the case where $h^{(3)}/a$=0.38 almost coincide completely with the corresponding ones given in Table 3 of the paper by Akbarov *et al*. (2016). Comparison of the results also shows that this approach may be close “from below” as well as close “from above”. In other words, for certain harmonics and for certain order of roots, the values of Ω obtained for the hollow sphere can be greater (or less) than the corresponding ones obtained for the solid sphere. Moreover, these results suggest that it is impossible to make any conclusion on the values of the natural vibration of the hollow sphere using the corresponding results obtained for the corresponding solid sphere. Namely, this conclusion illustrates the significance and necessity of the investigation of the natural vibration of the hollow sphere to which the present work is devoted. The results given in Figs. 2, 3, 4 and 5 show clearly that almost in all cases under consideration the natural vibration frequency Ω has its minimum under n=2.

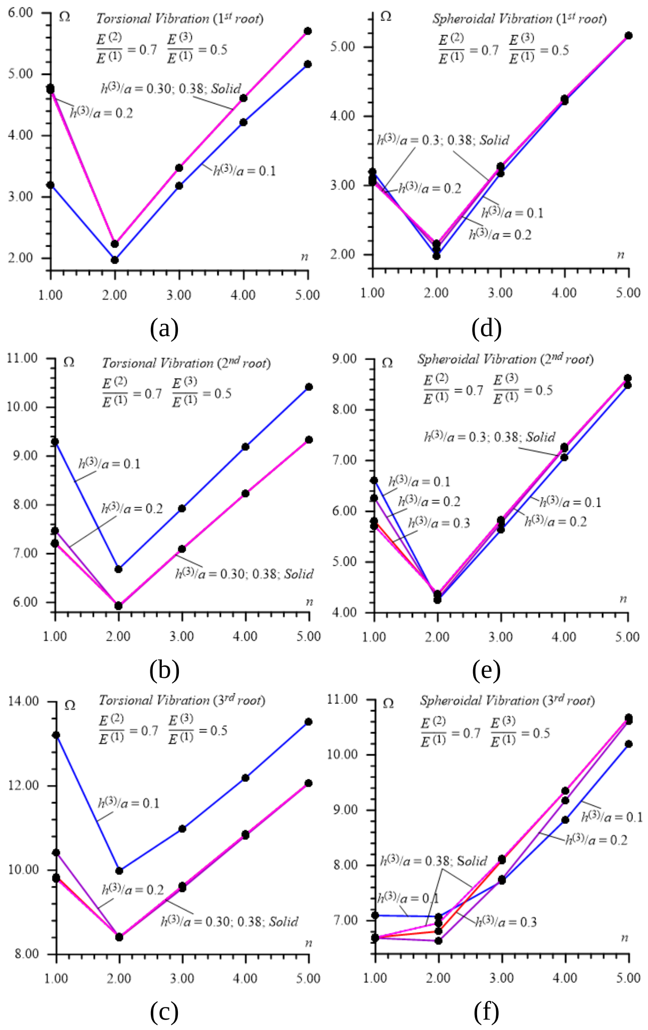


Fig. 3 Dependence of the natural vibration frequency on the harmonics of vibration under $E^{(2)}/E^{(1)}$=0.7 and $E^{(3)}/E^{(1)}$=0.5

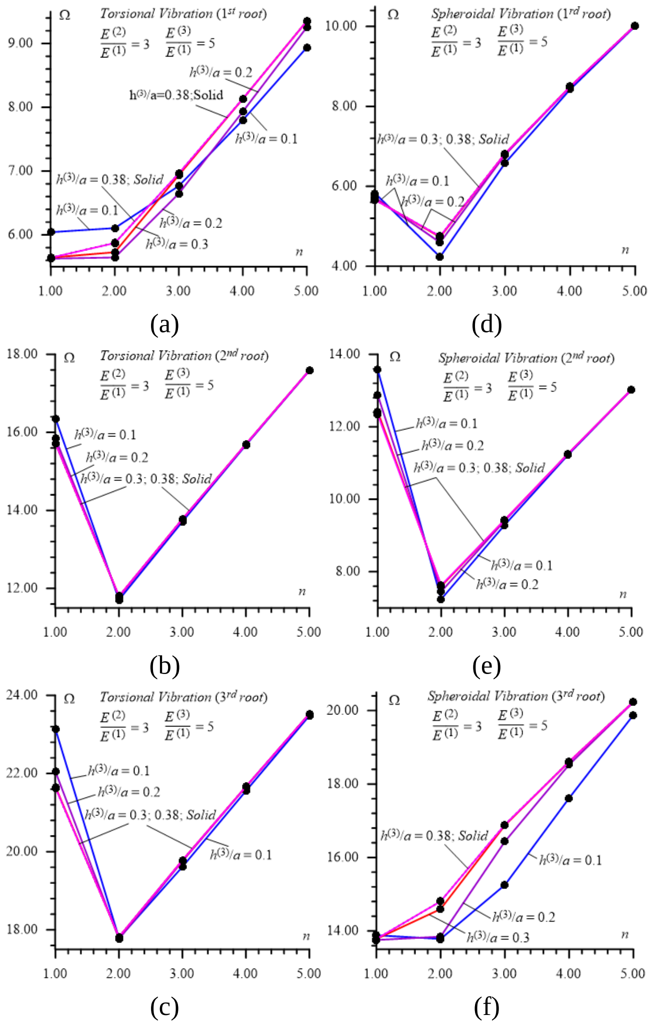


Fig. 4 Dependence of the natural vibration frequency on the harmonics of vibration under $E^{(2)}/E^{(1)}$=3 and $E^{(3)}/E^{(1)}$=5

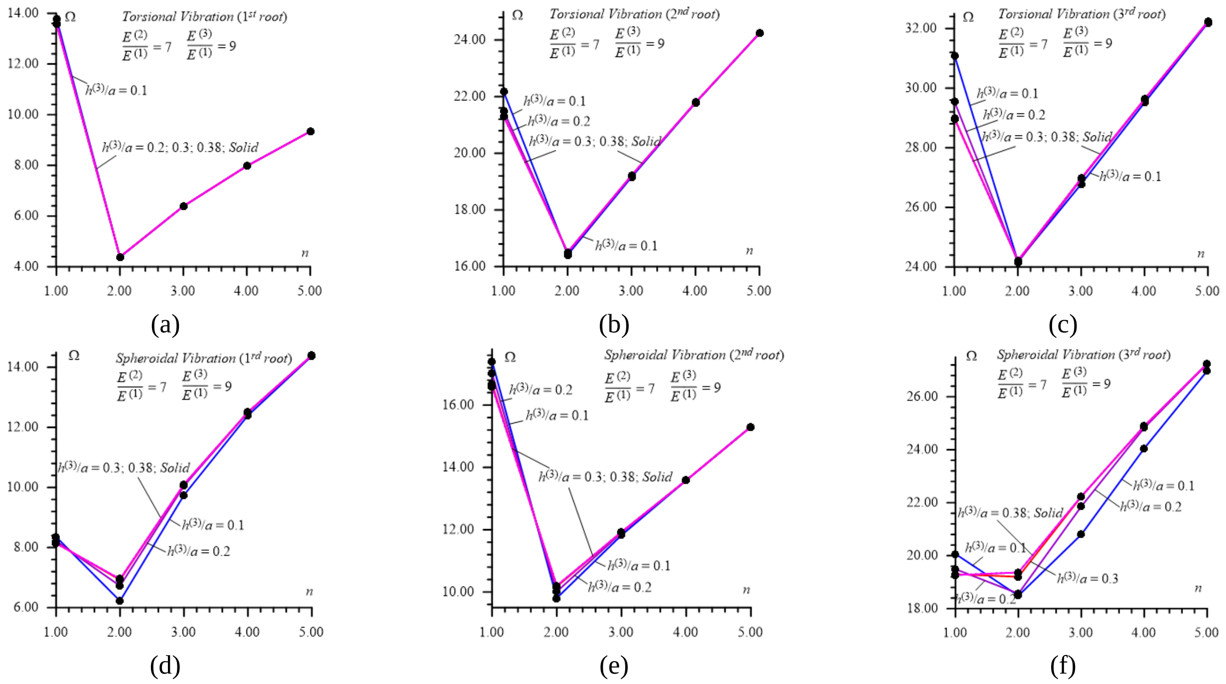


Fig. 5 Dependence of the natural vibration frequency on the harmonics of vibration under $E^{(2)}/E^{(1)}$=7 and $E^{(3)}/E^{(1)}$=9

Table 3 The influence of the change of the modulus of elasticity in the inward radial direction on the values of the natural frequencies $\Omega = \omega a\sqrt{\rho^{(1)}/\mu^{(1)}}$ obtained for the three-layered piecewise homogeneous hollow sphere in the case where $h^{(3)}/a$=0.2

$\frac{E^{(2)}}{E^{(1)}}$	$\frac{E^{(3)}}{E^{(1)}}$	n	Torsional vibration			Spheroidal vibration		
			1	2	3	1	2	3
0.5	0.3	1	4.2115	6.7745	9.4075	2.7415	5.2655	5.6275
		2	2.0125	5.1905	7.6255	1.7685	3.7945	5.6455
		3	3.1435	6.1835	8.6305	2.7715	5.0745	6.6475
		4	4.1805	7.1485	9.6375	3.6405	6.3165	7.9325
		5	5.1795	8.0945	10.616	4.4465	7.4755	9.2555
0.7	0.5	1	4.7865	7.4745	10.414	3.1075	6.2685	6.6855
		2	2.2335	5.9145	8.4025	2.0905	4.2755	6.6355
		3	3.4795	7.0925	9.5745	3.2605	5.7705	7.7575
		4	4.6145	8.2345	10.821	4.2535	7.2405	9.1815
		5	5.7035	9.3375	12.061	5.1705	8.6195	10.621
0.9	0.7	1	5.4205	8.5215	11.832	3.4245	7.1025	7.5875
		2	2.4165	6.6795	9.5895	2.3675	4.6665	7.4815
		3	3.7465	7.9755	10.910	3.6795	6.2865	8.7275
		4	4.9505	9.2255	12.284	4.7805	7.8885	10.267
		5	6.0995	10.434	13.629	5.7915	9.4175	11.767
1	1	1	5.7995	9.3105	12.901	3.5825	7.6065	8.0775
		2	2.5005	7.1175	10.479	2.5575	4.8895	8.0385
		3	3.8645	8.4385	11.843	3.9175	6.5655	9.4575
		4	5.0945	9.7115	13.197	5.0435	8.2045	11.079
		5	6.2655	10.950	14.507	6.0835	9.7745	12.542
3	5	1	9.6835	15.846	22.046	5.7205	12.877	13.754
		2	3.4755	11.780	17.770	4.5955	7.4515	13.845
		3	5.2075	13.772	19.757	6.7885	9.4055	16.435
		4	6.6775	15.697	21.660	8.4895	11.255	18.527
		5	8.0085	17.592	23.529	10.023	13.028	20.228
5	7	1	11.930	19.116	26.395	7.1265	15.524	17.010
		2	4.0065	14.483	21.465	5.7695	8.9315	16.651
		3	5.9055	16.921	23.915	8.6005	10.921	19.568
		4	7.4555	19.257	26.264	10.733	12.720	22.142
		5	8.8155	21.519	28.553	12.559	14.416	24.315
7	9	1	13.621	21.490	29.541	8.2175	17.017	19.492
		2	4.3875	16.489	24.158	6.7225	10.012	18.566
		3	6.3885	19.216	26.945	10.053	11.905	21.861
		4	7.9755	21.796	29.622	12.4995	13.5855	24.8245
		5	9.3395	24.2415	32.2325	14.3955	15.3055	27.2365

Table 4 The influence of the change of the material densities in the inward radial direction on the values of the natural frequencies $\Omega = \omega a\sqrt{\rho^{(1)}/\mu^{(1)}}$ obtained for the three-layered piecewise homogeneous hollow sphere in the case where $h^{(3)}/a$=0.2

$\frac{\rho^{(2)}}{\rho^{(1)}}$	$\frac{\rho^{(3)}}{\rho^{(1)}}$	n	Torsional vibration			Spheroidal vibration		
			1	2	3	1	2	3
0.5	0.3	1	7.9665	13.909	19.623	4.4975	10.504	11.091
		2	2.7705	9.5745	15.298	3.2475	5.8225	11.495
		3	4.2295	11.291	16.837	4.8875	7.5595	13.234
		4	5.5165	13.026	18.456	6.1805	9.2935	14.976
		5	6.7225	14.752	20.137	7.3295	11.006	16.612
0.7	0.5	1	6.9165	11.659	16.120	4.0805	9.2305	9.4765
		2	2.6765	8.3175	12.930	2.9195	5.4165	9.8355
		3	4.1105	9.7835	14.313	4.4285	7.1455	11.393
		4	5.3875	11.249	15.728	5.6495	8.8595	12.994
		5	6.5915	12.701	17.172	6.7575	10.537	14.462
0.9	0.7	1	6.2405	10.311	14.116	3.7665	8.3235	8.4955
		2	2.5585	7.5155	11.474	2.6715	5.0775	8.7885
		3	3.9495	8.8345	12.740	4.0725	6.7655	10.219
		4	5.1985	10.140	14.014	5.2265	8.4265	11.738
		5	6.3845	11.430	15.299	6.2875	10.036	13.129
1	1	1	5.7995	9.3105	12.901	3.5825	7.6065	8.0775
		2	2.5005	7.1175	10.479	2.5575	4.8895	8.0385
		3	3.8645	8.4385	11.843	3.9175	6.5655	9.4575
		4	5.0945	9.7115	13.197	5.0435	8.2045	11.079
		5	6.2655	10.950	14.507	6.0835	9.7745	12.542
3	5	1	3.1085	5.2405	7.5445	2.2155	4.2995	4.7955
		2	1.6755	4.0335	5.7645	1.5775	3.1085	4.5695
		3	2.6085	5.0265	6.4755	2.4735	4.2135	5.3775
		4	3.4585	5.9775	7.3255	3.2485	5.3135	6.4485
		5	4.2725	6.8615	8.2735	3.9845	6.3145	7.6055
5	7	1	2.5965	4.2655	6.1255	1.8065	3.5415	3.8315
		2	1.3135	3.3055	4.7185	1.2525	2.5255	3.7725
		3	2.0425	4.0475	5.3305	1.9605	3.4005	4.4345
		4	2.7065	4.7525	6.0395	2.5785	4.2455	5.2995
		5	3.3405	5.4155	6.7905	3.1705	5.0005	6.2025
7	9	1	2.2665	3.6915	5.2765	1.5635	3.0825	3.2815
		2	1.1095	2.8565	4.0955	1.0705	2.1805	3.2835
		3	1.7255	3.4685	4.6335	1.6735	2.9245	3.8555
		4	2.2845	4.0505	5.2415	2.2025	3.6325	4.5975
		5	2.8185	4.6025	5.8675	2.7105	4.2635	5.3545

Thus, we consider the numerical results related to the case where the material of the middle layer of the sphere is FG. Assume that $h^{(1)}/a$=0.1, $h^{(2)}/a$=0.5, $h^{(3)}/a$=0.2 and $\nu^{(1)}=\nu^{(2)}=\nu^{(3)}$=0.3. First, we illustrate some fragments of the results which show convergence with respect to the number M of the sublayers into which the middle layer is divided. These fragments are given in Table 5 for the torsional and spherical vibration modes in the case where $a_1=-14/a$, b_1=14.6, η_1=1, η_2=0, η_3=0, $E^{(3)}/E^{(1)}$=12 and $n_1=m_1$=1. Consequently, we assume that the density and Poisson's ratio of the material of the middle layer are constants, but the modulus of elasticity changes linearly through the thickness, so that the ratio $E^{(2)}(r)/E_0^{(2)}$ increases from $(E^{(2)}(r)/E_0^{(2)})\big|_{r=0.9a}=3$ until $(E^{(2)}(r)/E_0^{(2)})\big|_{r=0.4a}=10$ in the inward radial direction. Analyses of the results given in Table 8 show the high effectiveness of the approach used in the convergence sense with respect to the sublayers' number M. Taking these and many other results, which are not given here, into consideration allows us to conclude that it is enough to take M=21 to obtain numerical results with very high accuracy. Taking this conclusion into account, we assume that M=21 under obtaining all numerical results which will be discussed below

Now we consider the numerical results related namely to the case where the middle layer material is FG and assume that $\nu^{(1)}=\nu^{(2)}=\nu^{(3)}$=0.3, i.e., assume that η_2=0. Moreover, as above, assume that $h^{(1)}/a$=0.1, $h^{(2)}/a$=0.5, $h^{(3)}/a$=0.2, $E^{(3)}/E^{(1)}$=9 and suppose that $n_1=m_3$=1. We determine the constants a_1 and b_1 (a_3 and b_3) from the way that the ratio $E^{(2)}(r)/E^{(1)}$ (the ratio $\rho^{(2)}(r)/\rho^{(1)}$) increases from

Table 5 Convergence of the numerical results with respect to the number M of the sublayers obtained for the natural frequencies $\Omega = \omega a\sqrt{\rho^{(1)} / \mu^{(1)}}$ obtained for the three-layered hollow sphere with middle layer made of FGM

n	M	Torsional vibration			Spheroidal vibration		
		1	2	3	1	2	3
4	1	7.8615	21.260	29.047	12.169	13.407	24.602
	3	7.3165	19.867	28.758	11.792	12.458	23.200
	5	7.2465	19.788	28.653	11.769	12.311	23.099
	7	7.2255	19.766	28.623	11.760	12.269	23.068
	9	7.2175	19.757	28.611	11.756	12.252	23.054
	11	7.2125	19.752	28.605	11.753	12.243	23.047
	13	7.2105	19.750	28.601	11.752	12.238	23.043
	15	7.2085	19.748	28.599	11.751	12.235	23.040
	17	7.2075	19.747	28.597	11.750	12.233	23.038
	19	7.2065	19.746	28.596	11.750	12.231	23.037
	21	7.2065	19.746	28.596	11.750	12.230	23.036
5	1	9.2245	23.653	31.491	14.032	15.088	26.784
	3	8.6275	21.815	30.948	13.414	14.016	25.327
	5	8.5425	21.709	30.831	13.418	13.820	25.152
	7	8.5175	21.680	30.797	13.419	13.758	25.106
	9	8.5065	21.669	30.783	13.419	13.732	25.087
	11	8.5005	21.663	30.776	13.419	13.718	25.077
	13	8.4975	21.659	30.772	13.419	13.710	25.071
	15	8.4955	21.657	30.769	13.419	13.705	25.068
	17	8.4945	21.656	30.768	13.419	13.702	25.065
	19	8.4935	21.655	30.767	13.419	13.700	25.063
	21	8.4925	21.654	30.766	13.419	13.698	25.062

$(E^{(2)}(r)/E^{(1)})\big|_{r=0.9a} = 3$ (from $(\rho^{(2)}(r)/\rho^{(1)})\big|_{r=0.9a} = 3$) until $(E^{(2)}(r)/E^{(1)})\big|_{r=0.4a} = 7$ (until $(\rho^{(2)}(r)/\rho^{(1)})\big|_{r=0.4a} = 7$). Thus, for determination of the constants a_k and b_k (k=1,3) we obtain the following expressions

$$E^{(2)}(r) = E_0^{(2)}(1+(a_1 r + b_1)^{n_1})$$
$$\rho^{(2)}(r) = \rho_0^{(2)}(1+(a_3 r + b_3)^{n_3})$$
$$a_k = 2\left[(2)^{1/n_k} - (6)^{1/n_k}\right]a$$
$$b_k = \left[1.8\times(6)^{1/n_k} - 0.8\times(2)^{1/n_k}\right]a \qquad (23)$$

Thus, according to the selected relation (21), the changing (i.e., the increasing) character of the modulus of elasticity (or of the material density) is determined through the constant n_1 (n_3). For illustration of this dependence the graphs of the function $E^{(2)}(r)/E^{(1)}$ ($\rho^{(2)}(r)/\rho^{(1)}$) constructed for various values of n_1 (n_3) are given in Fig. 6, according to which, the values of the integrals

$$S_E = \int_{0.4}^{0.9} E^{(2)}(r)/E^{(1)}dr \text{ and } S_\rho = \int_{0.4}^{0.9} \rho^{(2)}(r)/\rho^{(1)}dr \qquad (24)$$

increase with a decrease of the constants $n_1(\geq 0)$ and $n_3(\geq 0)$, respectively. Note that the cases n_1 (n_3)=0.2, 0.5, 1.0, 1.5, 2.0 and 3.0, correspond to $S_E(S\rho)$=3.1672, 2.8333, 2.4999, 2.1666 and 1.9999, respectively. Thus, it follows from the foregoing discussions that the influence of the change character of the FGM in the inward radial direction can also be estimated through the values of S_E and $S\rho$.

Thus, after the foregoing preparation procedures we consider the results given in Tables 6 and 7 which show the influence of the change of the modulus of elasticity and of the material densities, respectively, under various n_1 (=n_3) on the values of the dimensionless natural frequency Ω (22) obtained for the torsional and spheroidal vibration modes.

These results are presented for the first, second, third, fourth and fifth harmonics and for the first three roots and, under obtaining the results given in Table 6 (in Table 7), it is assumed that η_3=0 (η_1=0). These results show that an

Table 6 The influence of an increase of the modulus of elasticity in the inward radial direction under constant material densities on the values of the natural frequencies $\Omega = \omega a\sqrt{\rho^{(1)} / \mu^{(1)}}$ obtained for the three-layered hollow sphere with middle layer made of FGM

n	n_1	Torsional vibration			Spheroidal vibration		
		1	2	3	1	2	3
1	0	9.8395	16.394	23.116	5.7685	13.074	13.988
	0.2	13.139	20.889	28.781	7.7565	16.560	18.713
	0.5	12.544	20.167	27.847	7.2605	15.977	17.737
	1	11.855	19.362	26.784	6.7635	15.333	16.598
	1.5	11.413	18.852	26.111	6.4845	14.953	15.872
	2	11.113	18.497	25.653	6.3125	14.704	15.392
	3	10.740	18.032	25.074	6.1195	14.378	14.838
2	0	3.4875	11.966	18.337	4.8905	7.6995	14.819
	0.2	4.1935	15.864	23.488	6.5185	9.4485	18.028
	0.5	3.9875	15.104	22.671	6.2695	8.8815	17.367
	1	3.7935	14.247	21.738	5.9655	8.3935	16.623
	1.5	3.6935	13.712	21.132	5.7595	8.1675	16.166
	2	3.6355	13.356	20.707	5.6145	8.0475	15.867
	3	3.5765	12.925	20.145	5.4295	7.9325	15.512
3	0	5.2115	13.889	20.189	6.9545	9.4945	17.512
	0.2	6.1125	18.419	26.163	9.6955	11.303	21.073
	0.5	5.8225	17.469	25.200	9.2385	10.692	20.204
	1	5.5575	16.423	24.085	8.6765	10.188	19.324
	1.5	5.4285	15.786	23.355	8.3045	9.9635	18.837
	2	5.3575	15.371	22.842	8.0505	9.8475	18.547
	3	5.2905	14.878	22.172	7.7345	9.7335	18.238
4	0	6.6795	15.757	21.946	8.5615	11.279	19.133
	0.2	7.6515	20.816	28.712	12.002	12.979	23.755
	0.5	7.3135	19.671	27.586	11.384	12.335	22.636
	1	7.0135	18.444	26.273	10.625	11.814	21.576
	1.5	6.8745	17.716	25.415	10.126	11.602	21.020
	2	6.8035	17.250	24.816	9.7895	11.502	20.696
	3	6.7385	16.712	24.043	9.3825	11.415	20.346
5	0	8.0095	17.619	23.703	10.051	13.035	20.554
	0.2	8.9905	23.079	31.182	13.816	14.668	26.008
	0.5	8.6275	21.749	29.878	13.089	13.984	24.714
	1	8.3155	20.360	28.359	12.186	13.457	23.506
	1.5	8.1775	19.557	27.374	11.600	13.262	22.876
	2	8.1095	19.056	26.695	11.217	13.179	22.501
	3	8.0515	18.493	25.833	10.773	13.115	22.072

Table 7 The influence of an increase of the material densities in the inward radial direction under constant modulus of elasticity on the values of the natural frequencies $\Omega = \omega a \sqrt{\rho^{(1)} / \mu^{(1)}}$ obtained for the three-layered hollow sphere with middle layer made of FGM

n	n_3	Torsional vibration			Spheroidal vibration		
		1	2	3	1	2	3
1	0	2.3505	4.8685	6.3655	1.9935	3.8275	4.2485
	0.2	2.3155	3.8145	5.4335	1.6445	3.2095	3.3985
	0.5	2.3625	3.9455	5.5945	1.7315	3.3405	3.5375
	1	2.3935	4.0735	5.7545	1.8165	3.4635	3.6975
	1.5	2.3975	4.1565	5.8615	1.8625	3.5335	3.7965
	2	2.3915	4.2265	5.9455	1.8885	3.5825	3.8605
	3	2.3755	4.3495	6.0675	1.9175	3.6505	3.9355
2	0	1.6705	3.1715	5.3125	1.5115	2.7835	4.1015
	0.2	1.2175	2.9255	4.2225	1.1325	2.2965	3.4265
	0.5	1.3355	2.9925	4.3615	1.2025	2.4175	3.5865
	1	1.4595	3.0455	4.5035	1.2795	2.5365	3.7475
	1.5	1.5305	3.0645	4.5975	1.3295	2.6035	3.8355
	2	1.5735	3.0705	4.6765	1.3635	2.6435	3.8885
	3	1.6185	3.0755	4.8125	1.4055	2.6855	3.9495
3	0	2.6055	4.0995	5.8865	2.4285	3.6725	4.8555
	0.2	1.8975	3.5725	4.7645	1.7845	3.0665	4.0195
	0.5	2.0875	3.6755	4.9115	1.9115	3.2055	4.2095
	1	2.2865	3.7655	5.0665	2.0535	3.3375	4.4085
	1.5	2.4015	3.8065	5.1715	2.1435	3.4105	4.5195
	2	2.4695	3.8315	5.2595	2.2035	3.4565	4.5875
	3	2.5385	3.8655	5.4105	2.2765	3.5075	4.6655
4	0	3.4565	5.0625	6.5325	3.2295	4.6685	5.7415
	0.2	2.5185	4.1975	5.3805	2.3635	3.8145	4.7715
	0.5	2.7755	4.3495	5.5355	2.5495	3.9925	4.9665
	1	3.0455	4.4895	5.7035	2.7565	4.1605	5.1745
	1.5	3.2015	4.5625	5.8185	2.8865	4.2535	5.2985
	2	3.2935	4.6115	5.9155	2.9715	4.3115	5.3785
	3	3.3835	4.6825	6.0805	3.0705	4.3825	5.4765
5	0	4.2725	6.0335	7.2195	3.9775	5.7285	6.6315
	0.2	3.1125	4.7975	6.0235	2.9225	4.4875	5.5545
	0.5	2.7755	4.3495	5.5355	3.1675	4.7165	5.7655
	1	3.7745	5.2035	6.3765	3.4385	4.9415	5.9825
	1.5	3.9715	5.3155	6.5015	3.6045	5.0725	6.1135
	2	4.0855	5.3915	6.6065	3.7105	5.1585	6.2025
	3	4.1945	5.5065	6.7825	3.8275	5.2675	6.3185

Table 8 The influence of a simultaneous increase of the material densities and modulus of elasticity in the inward radial direction on the values of the natural frequencies $\Omega = \omega a \sqrt{\rho^{(1)} / \mu^{(1)}}$ obtained for the three-layered hollow sphere with middle layer made of FGM

n	n_1 (= n_3)	Torsional vibration			Spheroidal vibration		
		1	2	3	1	2	3
1	0	5.6135	10.197	15.165	3.9415	8.0285	10.299
	0.2	6.8745	10.943	14.636	4.4365	9.2905	10.267
	0.5	6.7225	10.732	14.627	4.4215	9.2395	10.279
	1	6.4715	10.568	14.709	4.3615	9.0905	10.236
	1.5	6.2735	10.551	14.794	4.2925	8.9895	10.144
	2	6.1315	10.587	14.850	4.2335	8.9265	10.055
	3	5.9575	10.677	14.907	4.1475	8.8345	9.9515
2	0	3.0815	8.2625	11.234	3.6355	5.8875	8.4225
	0.2	3.1775	8.6735	12.115	3.4535	6.0775	9.5945
	0.5	3.2095	8.4945	11.922	3.5795	6.0555	9.5105
	1	3.2215	8.2505	11.798	3.7025	5.9915	9.3565
	1.5	3.2125	8.1065	11.792	3.7635	5.9335	9.2355
	2	3.1985	8.0305	11.816	3.7915	5.8925	9.1415
	3	3.1725	7.9815	11.854	3.8065	5.8495	9.0025
3	0	4.6705	10.433	12.993	5.2995	8.0635	10.436
	0.2	4.8615	10.360	13.602	5.2935	8.1915	11.241
	0.5	4.8855	10.153	13.457	5.4955	8.1575	11.112
	1	4.8735	9.9355	13.390	5.6735	8.0835	10.931
	1.5	4.8425	9.8575	13.396	5.7365	8.0345	10.803
	2	4.8135	9.8505	13.406	5.7455	8.0125	10.712
	3	4.7695	9.9145	13.395	5.7035	8.0105	10.594
4	0	6.0865	12.005	15.214	6.6115	9.9435	13.356
	0.2	6.3545	11.835	15.071	6.7725	10.202	13.372
	0.5	6.3585	11.615	15.018	7.0125	10.138	13.210
	1	6.3165	11.443	15.031	7.1945	10.033	13.039
	1.5	6.2655	11.427	15.062	7.2275	9.9785	12.957
	2	6.2245	11.474	15.074	7.1975	9.9595	12.923
	3	6.1725	11.608	15.057	7.0925	9.9615	12.926
5	0	7.4145	13.370	17.154	7.8665	11.726	15.613
	0.2	7.7515	13.166	16.413	8.1315	12.094	15.137
	0.5	7.7305	12.956	16.471	8.3895	11.990	14.970
	1	7.6545	12.844	16.566	8.5515	11.850	14.842
	1.5	7.5875	12.884	16.626	8.5425	11.787	14.827
	2	7.5395	12.967	16.652	8.4755	11.768	14.869
	3	7.4845	13.129	16.662	8.3245	11.770	15.012

increase in the values of the modulus of elasticity in the inward radial direction causes an increase in the values of the natural frequency Ω. According to Table 7, except in some particular cases, a similar conclusion also occurs for the influence of the increase of the material density in the inward direction on the natural frequencies, i.e. the increase of the material density causes a decrease in the values of the natural frequency. However, this conclusion is violated in first harmonics for the first roots in the torsional mode.

Comparison of the results obtained for the various values of n_1 and n_3 shows that the foregoing effects of the FGM of the middle layer on the natural frequencies become more considerable with the parameters S_E and $S\rho$ in (24) (or with decreasing of the constants $n_1(\geq 0)$ and $n_3(\geq 0)$). Thus, the parameters S_E and $S\rho$ can be used as global characteristics of the FGM determined through the relations in (21).

Consider also the numerical results which are obtained in the case where the modulus of elasticity and the density of the FGM of the middle layer increase simultaneously in the inward radial direction. Assume that the ratio $E^{(2)}(r)/E^{(1)}$ increases from $(E^{(2)}(r)/E^{(1)})\big|_{r=0.9a} = 3$ until $(E^{(2)}(r)/E^{(1)})\big|_{r=0.4a} = 7$, and the ratio $\rho^{(2)}(r)/\rho^{(1)}$ increases from $\rho^{(2)}(r)/\rho^{(1)}\big|_{r=0.9a} = 2$ until $(\rho^{(2)}(r)/\rho^{(1)})\big|_{r=0.4a} = 5$ and $\rho^{(3)}(r)/\rho^{(1)}$=7. The values of the other parameters remain as above. Analyze the values of the natural frequencies obtained for this case which are given in Table 8. As in the foregoing tables, these results are obtained for the first, second, third, fourth and fifth harmonics and for each

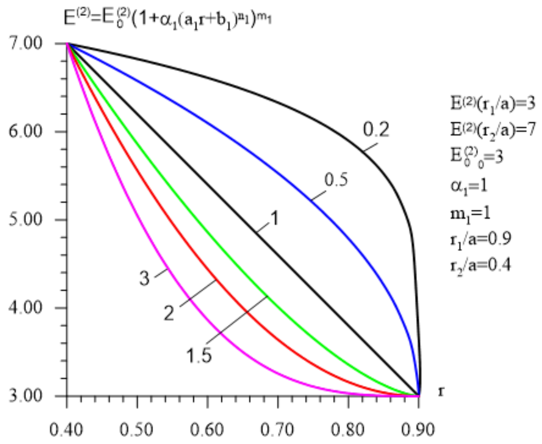


Fig. 6 Distribution of modulus of elasticity in the radial direction for various n_1

harmonic the first three roots are found in the torsional and spheroidal vibration modes. It follows from the data given in Table 8 that for the case under consideration, the natural frequencies increase with simultaneous increases in the parameters S_E and $S\rho$ in (24) (i.e., with simultaneous decreases in the constants $n_1(\geq 0)$ and $n_3(\geq 0)$). However, the magnitude of this increase is significantly less than the corresponding ones observed in Table 6. It is evident that this situation can be explained with the foregoing results, according to which, an increase of the material density in the inward radial direction causes a decrease, but the corresponding increase of the modulus of elasticity causes an increase in the values of the natural frequencies. Consequently, the simultaneous existence of the noted "decrease" and "increase" determines the character of the results given in Table 8.

This completes the consideration and analysis of the numerical results.

5. Conclusions

Thus, in the present paper, the natural vibration of the three-layered hollow sphere with middle layer made of FGM is investigated by employing the exact three-dimensional equations and relations of elastodynamics in spherical coordinates. The perfect contact conditions between the layers are satisfied. The corresponding eigenvalue problem is solved by employing the discrete analytical method described in the previous papers by the authors.

The case where the properties of the FGM change in the radial direction according to the power law, is considered, and numerical results on the dimensionless natural frequency Ω (22) obtained for certain concrete cases of this law are presented for the first, second, third, fourth and fifth harmonics and for the first three roots of each harmonic for the torsional and spheroidal vibration modes. Analysis of these results allows us to make the following main concrete conclusions:

- The results obtained for the three-layered hollow sphere approach the corresponding ones obtained for the corresponding solid sphere;
- The aforementioned approach may be "from below" as well as "from above", in other words, for certain harmonics and for roots of certain order, the values of the natural frequency obtained for the hollow sphere can be greater (or less) than those obtained for the solid sphere;
- It follows from the foregoing two conclusions that it is impossible to make any conclusion on the values of the natural vibration of the hollow sphere using the corresponding results obtained for the solid sphere and this situation illustrates the main significance and necessity of the investigation of the natural vibration of the hollow sphere to which the present work has been devoted.
- At the same time, in the qualitative sense all the conclusions made for the solid sphere in the paper by Akbarov *et al*. (2016) are also confirmed with the results of the present investigation and these conclusions are:

• An increase (a decrease) of the modulus of elasticity of the FGM in the inward radial direction causes an increase (a decrease) in the values of the natural frequency Ω;
• A decrease (an increase) of the density of the FGM in the inward radial direction, in general, causes an increase (a decrease) in the values of the natural frequency Ω;
• The influence of the character of the aforementioned change (increase or decrease) law on the values of Ω can be determined through the parameters S_E and S_ρ (24);
• An increase in the values of S_E, i.e., a decrease in the values of the constant n_1 which enter the relation (23), also causes an increase in the values of Ω;
• A decrease in the values of S_ρ, i.e., an increase in the values of the constant n_3 which enter the relation (23), also, in general, causes an increase in the values of Ω;
•The character of the results obtained in the cases where the modulus of elasticity and the density of the FGM are changed simultaneously, can be explained with the use of the foregoing conclusions.

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