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Bending of an isotropic non-classical thin rectangular plate

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Abstract. This study investigates the bending of an isotropic thin rectangular plate in finite deformation. Employing hyperelastic material of John's type, a non-classical model which generalizes the famous Kirchhoff's plate equation is obtained. Exact solution for deflection of the plate under sinusoidal loads is obtained. Finally, it is shown that the non-classical plate under consideration can be used as a replacement for Kirchhoff's plate on an elastic foundation.

Keywords: bending; non-classical; thin plate

1. Introduction

A thin plate is a structural member having the middle surface in the form of a plane and whose thickness is sufficiently small compared with its other two dimensions. Plate theories take advantage of this disparity in length scale to reduce the full three-dimensional elasticity problem to a two-dimensional problem such that the resulting theory can be used to calculate the deformations and stresses in a plate subjected to loads (Ventsel and Krauthammer 2001). The load-carrying action of plates is similar, to certain extent, to that of beams or cables; thus, plates can be approximated by the gridwork of an infinite number of beams or by the network of an infinite number of cables, depending on the flexural rigidity of the structures. Plates have numerous applications and the increasing use of thin plates structures in many branches of technology such as civil, mechanical, aeronautical, marine, and chemical has prompted intensive research in the fields of engineering, physics, and applied mathematics (Fadodun 2014). In the context of classical elasticity, Kirchhoff's plate model governs the deflection of an isotropic linear elastic thin plate which has been the basis for analysis of thin plate structures in various areas of engineering (Lychev *et al*. 2011, An *et al*. 2015, Imrak and Fetvaci, 2009, Wu *et al*. 2007, Zhong *et al*. 2013, Imrak and Gerdemeli, 2007, Lie *et al*. 2009, Batista, 2010, Zhang *et al*. 2014). However, the hypotheses on which Kirchhoff's model relies limit its range of applicability. Numerous researchers have attempted to refine Kirchhoff's theory and such attempts continue to this day. Reissner and Mindlin made the most important advance in this direction (Ventsel and Krauthammer 2001). In the present work, we consider the deflection of an isotropic thin plate in finite deformation with a view to refine the Kirchhoff's plate model. The plate under consideration is assumed to be made of John's semi-linear hyperelastic material (Akinola 2001). The obtained bending model which governs the deflection of non-classical plate under consideration generalizes the famous Kirchhoff's plate equation. This plate exhibits in-plate harmonic forces whereas the classical model fails to apprehend this phenomenon. Exact solution for deflection of the plate under sinusoidal load is obtained. This paper is organized as follow: section two presents the three-dimensional formulation; section three details the reduction to an equivalent two-dimensional plate equation; section four highlights the moments and stresses within the plates; section five gives the exact solution for deflection of the plate under sinusoidal load while section six concludes the study.

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2. Three-dimensional governing equation

Let Ω be a subset of $\Re^3$ occupied by an isotropic hyperelastic thin plate. Further, assume the deformation function $\vec{\varphi} = \varphi_m$, m=1,2,3 of Ω from reference configuration Ω_0 onto current configuration Ω by the action of load $\vec{g}$ is such that

$$\varphi_\alpha = x_\alpha - x_3 \frac{\partial}{\partial x_\alpha} w(x_1, x_2), \quad \alpha = 1,2$$
$$\varphi_3 = cx_3 + w(x_1, x_2) \qquad (1)$$

where w=$w(x_1,x_2)$ is the deflection (or transverse displacement) of the plate, (x_1,x_2,x_3) is the material coordinates in the reference configuration Ω_0, and $c \in \Re$. The energy function W for the hyperelastic plate in consideration is (Akinola 2011)

$$W = \mu S_1(\tilde{U} - \tilde{E})^2 + \frac{1}{2}\lambda S_1^2(\tilde{U} - \tilde{E}), \qquad (2)$$

where $\nabla\vec{\varphi} = \tilde{U}\tilde{O}^D$ is the gradient of deformation, $\tilde{O}^D$ is the rotation tensor associated with the deformation $\nabla\vec{\varphi}$, $\tilde{U}$ is the left stretch symmetric second rank tensor such

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that $\tilde{U}^2 = \nabla\vec{\varphi}\nabla\vec{\varphi}^T$, $\nabla\vec{\varphi}^T$ is the transpose of $\nabla\vec{\varphi}$, $\tilde{E}$ is the unit second rank tensor, and $S_1(\tilde{A}) = \tilde{E}\cdot\cdot\tilde{A} = tr(\tilde{A})$ is the trace of any second rank tensor $\tilde{A}$.

Let the geometry of deformation be the gradient of deformation $\nabla\vec{\varphi}$, we take the Frechet derivative of the energy function Eq. (2) with respect to the geometry of deformation $\nabla\vec{\varphi}$ and obtain the first Piola-Kirchhoff stress tensor $\tilde{P}$ to which it is energy conjugate

$$\tilde{P} = \frac{\partial W}{\partial \nabla\vec{\varphi}} = W_{\nabla\vec{\varphi}}. \tag{3}$$

Substituting Eq. (2) into Eq. (3) gives the constitutive relation

$$\tilde{P} = 2\mu\nabla\vec{\varphi} + (\lambda S_1(\tilde{U} - \tilde{E}) - 2\mu)\tilde{O}^D. \tag{4}$$

Using Eq. (4), the three-dimensional state equation for the problem under consideration is

$$\begin{cases} \nabla\cdot\tilde{P} + \vec{f} = \vec{0} \\ \tilde{P} = 2\mu\nabla\vec{\varphi} + (\lambda S_1(\tilde{U} - \tilde{E}) - 2\mu)\tilde{O}^D \\ \vec{g}d\Sigma = \vec{N}\cdot\tilde{P}d\Sigma_0, \end{cases} \tag{5}$$

where $d\Sigma$ is the element of the boundary in the current configuration Ω on which the force $\vec{g}$ acts, $d\Sigma_0$ is the element of the boundary in the reference configuration Ω_0, $\vec{f}$ is the body force, and $\vec{N}$, $\vec{n}$ are the orientation outward normal unit vectors on Σ, Σ_0 respectively.

3. An equivalent two-dimensional plate equation

By definition, the gradient of deformation is

$$\nabla\vec{\varphi} = \begin{pmatrix} \frac{\partial}{\partial x_1}\varphi_1 & \frac{\partial}{\partial x_2}\varphi_1 & \frac{\partial}{\partial x_3}\varphi_1 \\ \frac{\partial}{\partial x_1}\varphi_2 & \frac{\partial}{\partial x_2}\varphi_2 & \frac{\partial}{\partial x_3}\varphi_2 \\ \frac{\partial}{\partial x_1}\varphi_3 & \frac{\partial}{\partial x_2}\varphi_3 & \frac{\partial}{\partial x_3}\varphi_3 \end{pmatrix}. \tag{6}$$

Substituting Eq. (1) into Eq. (6) yields

$$\nabla\vec{\varphi} = \begin{pmatrix} 1 - x_3\frac{\partial^2}{\partial x_1^2}w & -x_3\frac{\partial^2}{\partial x_1\partial x_2}w & -\frac{\partial}{\partial x_1}w \\ -x_3\frac{\partial^2}{\partial x_1\partial x_2}w & 1 - x_3\frac{\partial^2}{\partial x_2^2}w & -\frac{\partial}{\partial x_2}w \\ \frac{\partial}{\partial x_1}w & \frac{\partial}{\partial x_2}w & c \end{pmatrix} \tag{7}$$

Employing the polar decomposition $\nabla\vec{\varphi} = \tilde{U}\cdot\tilde{O}^D$ and the relation $\tilde{U}^2 = \nabla\vec{\varphi}\cdot\nabla\vec{\varphi}^T$, where $\cdot$ is the usual scalar product give

$$\tilde{O}^D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8a}$$

$$\tilde{U} = \begin{pmatrix} x_3\frac{\partial^2}{\partial x_1^2}w - 1 & x_3\frac{\partial^2}{\partial x_1\partial x_2}w & -\frac{\partial}{\partial x_1}w \\ x_3\frac{\partial^2}{\partial x_1\partial x_2}w & x_3\frac{\partial^2}{\partial x_2^2}w - 1 & -\frac{\partial}{\partial x_2}w \\ \frac{\partial}{\partial x_1}w & \frac{\partial}{\partial x_2}w & c \end{pmatrix}. \tag{8b}$$

Substituting Eqs. (7)-(8) into Eq. (4) and invoking the hypothesis of zero stress along the transverse axis of the plate, the components P_{mn}, m,n=1.2.3 of first Piola-Kirchhoff's tensor $\tilde{P}$ are

$$P_{11} = -2x_3\mu\frac{\partial^2}{\partial x_1^2}w - \frac{\mu\lambda}{(2\mu+\lambda)}x_3\nabla^2 w + K, \tag{9a}$$

$$P_{21} = -2x_3\mu\frac{\partial^2}{\partial x_1\partial x_2}w,\ P_{31} = 2\mu\frac{\partial}{\partial x_1}w, \tag{9b}$$

$$P_{22} = -2x_3\mu\frac{\partial^2}{\partial x_2^2}w - \frac{\mu\lambda}{(2\mu+\lambda)}x_3\nabla^2 w + K, \tag{10a}$$

$$P_{21} = -2x_3\mu\frac{\partial^2}{\partial x_1\partial x_2}w,\ P_{13} = -2\mu\frac{\partial}{\partial x_1}w, \tag{10b}$$

$$P_{32} = -P_{23} = 2\mu\frac{\partial}{\partial x_2}w,\ P_{33} = 0, \tag{11}$$

where $K = \frac{8\lambda\mu}{(2\mu+\lambda)} + 4\mu$ is a constant and $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the scalar Laplacian operator in x_1,x_2.

In the absence of the body force ($\vec{f} = \vec{0}$) and setting $\vec{g} = -g\vec{e}_3$, $\vec{e}_3$ being the unit vector along the transverse axis x_3, the components form of Eq. (5) are

$$\frac{\partial}{\partial x_1}P_{11} + \frac{\partial}{\partial x_2}P_{21} + \frac{\partial}{\partial x_3}P_{31} = 0, \tag{12}$$

$$\frac{\partial}{\partial x_1}P_{12} + \frac{\partial}{\partial x_2}P_{22} + \frac{\partial}{\partial x_3}P_{32} = 0, \tag{13}$$

$$\frac{\partial}{\partial x_1}P_{13} + \frac{\partial}{\partial x_2}P_{23} + \frac{\partial}{\partial x_3}P_{33} = g. \tag{14}$$

Substituting the stress components P_{mn} in Eqs. (9)-(11) into Eqs. (12)-(14) gives

$$D^*x_3\left(\frac{\partial^3}{\partial x_1^3}w(x_1,x_2) + \frac{\partial^3}{\partial x_1\partial x_2^2}w(x_1,x_2)\right) = 0, \tag{15}$$

$$D^*x_3\left(\frac{\partial^3}{\partial x_1^2\partial x_2}w(x_1,x_2) + \frac{\partial^3}{\partial x_2^3}w(x_1,x_2)\right) = 0, \tag{16}$$

$$2\mu\left(\frac{\partial^2}{\partial x_1^2}w(x_1,x_2) + \frac{\partial^2}{\partial x_2^2}w(x_1,x_2)\right) = -g, \tag{17}$$

where $D^* = \dfrac{4\mu\lambda}{(2\mu+\lambda)}$.

Multiplying Eq. (15) by x_3; integrate the resulting equation with respect to x_3 in the limit $-\frac{h}{2}$ to $\frac{h}{2}$; and differentiate the result with respect to x_1 yield

$$D\left(\frac{\partial^4}{\partial x_1^4}w(x_1,x_2)+\frac{\partial^4}{\partial x_1^2\partial x_2^2}w(x_1,x_2)\right)=0, \quad (18)$$

In the same fashion, Eq. (16) leads to

$$D\left(\frac{\partial^4}{\partial x_1^4}w(x_1,x_2)+\frac{\partial^4}{\partial x_1^2\partial x_2^2}w(x_1,x_2)\right)=0. \quad (19)$$

The integration of Eq. (17) with respect to x_3 in the limit $-\frac{h}{2}$ to $\frac{h}{2}$ gives

$$-2\mu h\left(\frac{\partial^2}{\partial x_1^2}w(x_1,x_2)+\frac{\partial^2}{\partial x_2^2}w(x_1,x_2)\right)=p, \quad (20)$$

where $p=\int_{-\frac{h}{2}}^{\frac{h}{2}} g\,dx_3$, $D=\dfrac{D^*h^3}{12}$, and h is the thickness of the plate.

Addition of Eqs. (18)-(20) gives

$$D\nabla^4 w(x_1,x_2)-2\mu h\nabla^2 w(x_1,x_2)=p, \quad (21)$$

where $\nabla^4=\nabla^2\nabla^2$, $\nabla^2=\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}$, and D is the flexural rigidity of the plate.

Introducing the standard relation

$$\mu=\frac{E}{2(1+\upsilon)}, \quad \lambda=\frac{E\upsilon}{(1+\upsilon)(1-2\upsilon)}$$

in the above equation yields

$$D\nabla^4 w(x_1,x_2)-\frac{Eh}{(1+\upsilon)}\nabla^2 w(x_1,x_2)=p,$$

$$D=\frac{D^*h^3}{12}=\frac{\mu(\mu+\lambda)h^3}{3(2\mu+\lambda)}=\frac{Eh^3}{12(1-\upsilon^2)}, \quad (22)$$

where E and υ are the Young's modulus and Poisson's ratio of the material of the plate respectively.

Remark 1. Eq. (22) is the two-dimensional model which governs the bending of an isotropic non-classical plate under consideration. Clearly, this equation generalizes the famous Kirchhoff's plate equation.

Remark 2. The existence of harmonic term $\dfrac{Eh}{2(1+\upsilon)}\nabla^2 w(x_1,x_2)$ on the left hand side of Eq. (22) shows that the non-classical plate under consideration exhibits harmonic forces within its planes. Meanwhile, the classical plate model fails to apprehend this phenomenon.

4. Moments and stresses within the plate

The generalized stresses within the plate are the bending moments M_{11}, M_{22}, twisting M_{12}, M_{21}, and the stresses $Z_{\alpha 3}, Z_{3\alpha}$, α=1,2.

These quantities are defined as

$$M_{\alpha\beta}=\int_{-\frac{h}{2}}^{\frac{h}{2}} x_3 P_{\alpha\beta}\,dx_3$$

$$=-\frac{h^3}{12}\left(2\mu\partial_{\alpha\beta}w(x_1,x_2)+\frac{2\mu\lambda}{2\mu+\lambda}\delta_\beta^\alpha\nabla^2 w(x_1,x_2)\right) \quad (23)$$

$$Z_{\alpha 3}=-Z_{3\alpha}=\int_{-\frac{h}{2}}^{\frac{h}{2}} P_{\alpha 3}\,dx_3=-2\mu h\partial_\alpha w(x_1,x_2) \quad (24)$$

where $\delta_\beta^\alpha=\begin{cases}1, when\alpha=\beta\\ 0, when\alpha\neq\beta\end{cases}$.

Using Eqs. (9)-(11), we have

$$M_{11}=-D\left(\frac{\partial^2}{\partial x_1^2}w(x_1,x_2)+\upsilon\frac{\partial^2}{\partial x_2^2}w(x_1,x_2)\right), \quad (25)$$

$$M_{12}=M_{21}=-D(1-\upsilon)\frac{\partial^2}{\partial x_1\partial x_2}w(x_1,x_2), \quad (26)$$

$$M_{22}=-D\left(\frac{\partial^2}{\partial x_2^2}w(x_1,x_2)+\upsilon\frac{\partial^2}{\partial x_1^2}w(x_1,x_2)\right), \quad (27)$$

$$Z_{13}=-Z_{31}=-\frac{E}{(1+\upsilon)}\frac{\partial}{\partial x_1}w(x_1,x_2), \quad (28)$$

and

$$Z_{23}=-Z_{32}=-\frac{E}{(1+\upsilon)}\frac{\partial}{\partial x_2}w(x_1,x_2). \quad (29)$$

5. Exact solution for deflection of thin plate under sinusoidal loads

Consider an hyperelastic rectangular thin plate of sides a and b, simply supported on all edges and subjected to a sinusoidal load

$$p(x_1,x_2)=p_0\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{b}\right), \quad p_0\neq 0.$$

The origin of the plate is assumed to be at any of its corner. Then, one solves the boundary value problem

$$D\nabla^4 w(x_1,x_2)-\frac{Eh}{(1+\upsilon)}\nabla^2 w(x_1,x_2)=p, \quad (30)$$

$$w=0 \text{ and } \frac{\partial^2 w}{\partial x_1^2}=0 \text{ at } x_1=0 \text{ and } x_1=a, \quad (31)$$

$w = 0$ and $\frac{\partial^2 w}{\partial x_2^2} = 0$ at $x_2 = 0$ and $x_2 = b$. (32)

The solution of Eq. (30) in view of the boundary conditions Eqs. (31)-(32) is

$$w(x_1, x_2) = \frac{p_0 \sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{b}\right)}{\left(D\pi^4 (\frac{1}{a^2} + \frac{1}{b^2})^2 + \frac{Eh\pi^2}{(1+\upsilon)}(\frac{1}{a^2} + \frac{1}{b^2})\right)} \quad (33)$$

Using Eqs. (25)-(29), the corresponding moments and stresses are

$$M_{11} = D\left(\frac{1}{a^2} + \frac{\upsilon}{b^2}\right)G^0 p_0 \sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{b}\right), \quad (34)$$

$$M_{22} = D\left(\frac{\upsilon}{a^2} + \frac{1}{b^2}\right)G^0 p_0 \sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{b}\right), \quad (35)$$

$$M_{12} = -D\pi^2 (1-\upsilon)G^0 \frac{p_0}{ab}\cos\left(\frac{\pi x_1}{a}\right)\cos\left(\frac{\pi x_2}{b}\right), \quad (36)$$

$$Z_{13} = -\frac{E\pi}{a(1+\upsilon)}G^0 p_0 \cos\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{b}\right), \quad (37)$$

$$Z_{23} = -\frac{E\pi}{b(1+\upsilon)}G^0 p_0 \sin\left(\frac{\pi x_1}{a}\right)\cos\left(\frac{\pi x_2}{b}\right), \quad (38)$$

where $G^0 = \left(D\pi^4\left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2 + \frac{Eh\pi^2}{(1+\upsilon)}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\right)^{-1}$.

Meanwhile the governing differential equation of Kirchhoff's plate on an elastic foundation in classical elasticity is given by Winkler's model

$$D\nabla^4 w(x_1, x_2) + kw(x_1, x_2) = p, \quad (39)$$

where k is the foundation modulus.

The solution of Eq. (39) subject to the boundary conditions Eqs. (31)-(32) coincides completely with solution Eq. (33) when one sets the foundation modulus

$$k = \frac{Eh\pi^2}{(1+\upsilon)}\left(\frac{1}{a^2} + \frac{1}{b^2}\right).$$

6. Conclusions

The obtained bending model for non-classical plate considered generalizes the famous Kirchhoff's plate equation. This plate exhibits in-plane harmonic forces. Exact solution for deflection of the plate under sinusoidal loads is obtained. Finally, it is shown that the non-classical plate in this study can be used as a replacement for Kirchhoff's plate on an elastic foundation when the foundation modulus k is chosen correctly.

References

Akinola, A. (2001), "An application of nonlinear fundamental problems of a transversely isotropic layer in finite deformation", *Int. J. Nonlin. Mech.*, **91**(3), 307-321.

An, C., Gu, J. and Su, J. (2015), "Exact solution of bending of clamped orthotropic rectangular thin plates", *J. Braz. Soc. Mech. Sci. Eng*., **38**(2), 601-607.

Batista, M. (2010), "New analytic solution for bending problem of uniformly loaded rectangular plate supported on corner points", *IES J. Part A: Civil Struct. Eng.*, **3**(2), 462-474.

Fadodun, O.O. (2014), "Two-dimensional theory for a transversely isotropic thin plate in nonlinear elasticity", Ph.D. dissertation, Obafemi Awolowo University, Ile-Ife, Nigeria.

Imrak, E. and Fetvaci, C. (2009), "An exact solution of a clamped rectangular plate under uniform Load", *Appl. Math. Sci.*, **1**(43), 2129-2137.

Imrak, E. and Gerdemali, I. (2009), "The deflection solution of a clamped rectangular thin plate carrying uniformly load", *Mech. Bas. Des. Struct. Mach.*, **37**, 462-474

Lie, R., Zhong, Y. and Liu, Y. (2009), "On finite integral transform method for exact bending solutions of fully clamped orthtropic rectangular thin plates", *Appl. Math. Lett.*, **22**, 1821-1827.

Lychev, S.A., Lycheva, T.N. and Manzhirov, A.V. (2011), "Unsteady vibration of a growing circular plate", *Mech. Solid.*, **46**(2), 325-333.

Ventsel, E. and Krauthammer, T. (2001), *Thin plate and shell theory, analysis and application*, Marce Dekker, Inc., New York and Basel NY, USA.

Wu, H.J., Liu, A.Q. and Chen, H.L. (2007), "Exact solution for free-vibration analysis of rectangular plates using Bessel functions", *J. Appl. Mech.*, **74**, 1247-1251.

Zhang, C.C., Zhu, H.H., Shi, B. and Mei, G.X. (2014), "Bending of a rectangular plate resting on a fractionalized Zener foundation", *Struct. Eng. Mech.*, **52**(6), 1069-1084.

Zhong, Y., Zhao, X. and Liu, H. (2013), "Vibration of plate on foundation with four edges free by finite cosine integral transform method", *Latin Am. J. Solid. Struct.*, **11**(5), 854-862.

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