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On the attenuation of the axisymmetric longitudinal waves propagating in the bi-layered hollow cylinder made of viscoelastic materials

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Abstract. The paper studies the attenuation of the axisymmetric longitudinal waves propagating in the bi-layered hollow cylinder made of linear viscoelastic materials. Investigations are made by utilizing the exact equations of motion of the theory of viscoelasticity. The dispersion equation is obtained for an arbitrary type of hereditary operator of the materials of the constituents and a solution algorithm is developed for obtaining numerical results on the attenuation of the waves under consideration. Specific numerical results are presented and discussed for the case where the viscoelasticity of the materials is described through fractional-exponential operators by Rabotnov. In particular, how the rheological parameters influence the attenuation of the axisymmetric longitudinal waves propagating in the cylinder under consideration, is established.

Keywords: wave attenuation; wave dispersion; viscoelastic material; characteristic creep time; long-term elastic constants

1. Introduction

The success of many applications of guided waves for nondestructive testing of polymer tubes, which are used in the infrastructure of many industries such as gas, oil, and water transport also depends on knowledge of the rules of the attenuation of these waves. Moreover, knowledge of the rules of the attenuation of guided waves in the elements of constructions made of viscoelastic materials can be used for weakening of the vibrations or waves propagating therein, which are caused by earthquakes and other sources of the waves. At the same time, the mentioned knowledge, as noted in the papers by Benjamin, David *et al*. (2016), Yasar, Royston *et al*. (2013a), Yasar, Klatt *et al*. (2013b), can be employed for the nondestructive monitoring the growth of an engineering tissue which can be used for corresponding medical goals. These and other type applications field require to the fundamental studies of the dynamical problems related element of constructions made of viscoelastic materials. These studies are also required for correct application of ultrasonic guided wave (UGW) defect detection methods, the present level of which is discussed in the papers by Lowe, Sanderson *et al*. (2015), Lowe, Sanderson *et al*. (2016) and in the thesis by Yucel (2015) and other ones listed therein.

In the paper by Lowe, Sanderson *et al*. (2015) a high-sensitivity focusing technique, i.e., the hybrid active focusing technique was presented. This technique is based on a combination of numerical simulations with active focusing and time reversal concept. It is shown that this method permits detection of defects the size of which $<0.5\lambda$, where λ is the length of the propagated waves. Investigations are made by employing the $T(0,1)$ (torsional) and $F(i,2)$ ($i=1,2,\ldots$) (flexural) wave modes.

The paper by Low, Sanderson *et al*. (2016) studies the potential of the fundamental mode of the axisymmetric longitudinal wave, i.e., the $L(0,1)$ wave mode for UGW inspection of cylindrical structures. The studies are made experimentally and it is established that the flaw sensitivity of the $L(0,1)$ wave mode can enhance the flaw sensitivity approximately 2.5 times compared to the $T(0,1)$ wave mode and approximately 5 times compared to the $L(0,2)$ wave mode. Note that in this paper, the experiments are made within the 20-100 kHz range, according to which, it is assumed that the wave attenuation is considerably low and the influence of this attenuation on the considered wave propagation is negligible.

The advantages of the high sensitivities of the longitudinal ultrasonic guided waves under long range defect detection were also established and employed in the thesis by Yucel (2015). Moreover, in this work a detailed review of related investigations and employed experimental techniques are also discussed.

As noted in the aforementioned works, the application of the UGW for defect inspection has high sensitivity in the cases where attenuation of the waves is very low. Consequently, the application of the URG for defect inspection requires theoretical knowledge of the influence of the material viscosity on the wave attenuation and dependence of this attenuation on the wave frequencies. This knowledge can be applied with the corresponding investigations on problems related to wave propagation in

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structural elements made of viscoelastic materials. Note that the theoretical studies carried out in the present paper are also among these investigations.

In connection with these, it can be noted that investigations of the rule of attenuation of the axisymmetric longitudinal waves propagating in the bi-layered circular hollow cylinder which is the subject of the study of the present paper, has not only theoretical, but also practical significance.

Now we consider a brief review of related investigations and begin this review with the papers by Weiss (1959) and Tamm and Weiss (1961) which regard the Lamb wave propagation in an isotropic viscoelastic layer with stress-free surfaces. As noted above, in these papers it is assumed that the elastic constants are complex and frequency-independent. Coquin (1964) proposed an approximate method for investigation of the Lamb wave propagation in a plate made from viscoelastic materials with small losses and frequency-dependent elastic moduli. The influence of low-compressibility materials with real Poisson's ratio and frequency dependent complex shear moduli on the propagation of Lamb waves was investigated by Chervinko and Shevchenkov (1986). Lamb wave propagation in elastic plates coated with viscoelastic materials was also studied in the paper by Simonetti (2004) and the results of this paper were also detailed in the monograph by Rose (2004).

Barshinger and Rose (2004) investigated axisymmetric longitudinal guided wave dispersion and attenuation in a metal elastic hollow cylinder coated with a polymer viscoelastic layer. The viscoelasticity of the coated layer was taken into consideration through attenuation coefficients of the longitudinal and shear waves in the corresponding viscoelastic materials. These coefficients are determined experimentally for the frequencies in the order 1-5 MHz and are used for determination of the corresponding complex moduli. Consequently, using these complex moduli, wave dispersion and attenuation dispersion in the bi-layered hollow cylinder were investigated.

There are also investigations, such as Kirby, Zlatev *et al*. (2012, 2013), in which the influence of the viscoelasticity of the coatings of the pipes on the scattering of the longitudinal and torsional waves. Note that in these papers the viscoelasticity of the coating is modeled as in the paper by Barshinger and Rose (2004).

The paper by Leonov, Michael *et al*. (2015) has experimental studies on the attenuation of the torsional and longitudinal wave propagation in pipes buried in sand and it is assumed that this attenuation appears as a result of the energy leakage into the embedding soil. Moreover, in the paper by Jiangong (2011), the dispersion of the viscoelastic SH waves in functionally graded material and laminated plates is studied within the scope of the Kelvin-Voigt model and it is established that the wave attenuation depends not only on the rheological parameters but also on the elastic constants.

The Kelvin-Voigt model is also used in the papers by Manconi and Sorokin (2013), Bartoli, Marzani *et al*. (2006), Mace and Manconi (2008), Manconi and Mace (2009), Mazotti, Marzani *et al*. (2012), Hernando Quintanilla, Fan *et al*. (2015) for investigation of the wave propagation in plates and rods made of viscoelastic materials. Moreover, in these works the hysteretic model, i.e. the model based on the frequency independent complex potential is also examined.

Attempts to use very real and complicated models under investigation of wave propagation and attenuation problems were made in the papers by Meral, Royston *et al*. (2009, 2010). In these papers, the fractional order Voigt (or Kelvin-Voigt) model is used for investigation of 2D dynamic problems for viscoelastic materials. Note that the fractional order Voigt is obtained from the classical Voigt model by replacing the ordinary derivative $\partial/\partial t$, with respect to time, with the fractional order derivative $\partial^{\alpha}/\partial t^{\alpha}$ in the Weyl sense. In this way, a new rheological parameter is introduced into the model through which the description of the experimental data is improved. At the same time, in the paper by Meral, Royston *et al*. (2010), with utilizing the fractional order Voigt model, Lamb wave propagation and attenuation were studied theoretically and verified experimentally for a tissue mimicking phantom material. Moreover, in this paper it was established that successful selection of the rheological parameter α allows for improvement in matching theory to experiment.

This completes the review of the related works from which it follows that the investigations on the dispersion of guided waves in the plates or cylinders made from viscoelastic materials were carried out mainly in the following cases: i) the complex modulus of viscoelastic materials is taken as frequency independent (the hysteretic model); ii) the viscoelasticity of the materials is described by the simplest models such as the classical Kelvin-Voigt or simplest fractional Kelvin-Voigt models; and iii) the expression for the complex elasticity modulus is obtained experimentally for concrete polymer materials. Consequently, in the theoretical investigations reviewed above wave dispersion with given a priori non-dispersive or dispersive attenuation was made for the simplest viscoelastic models and the few numerical results obtained in these works cannot illustrate the character of the influence of the rheological parameters of the viscoelastic materials on the wave dispersion and wave attenuation.

The first attempt on the application of a more real and complicated viscoelastic model, such as fractional exponential operator proposed by Rabotnov (1980), for investigation of wave dispersion was made by Akbarov and Kepceler (2015) in which the torsional wave dispersion also with given a priori attenuation in the sandwich hollow cylinder made from linear viscoelastic materials, was studied. Note that the mentioned fractional exponential operators allow us to describe, with the very high accuracy required, the initial parts of the experimentally constructed creep and relaxation graphs and their asymptotic values. Moreover, these operators can be employed successfully to describe various polymer materials and epoxy-based composites with continuous fibers and layers. At the same time, these operators have many simple rules for complicated mathematical transformations, for example, the Fourier and Laplace transformations which were also used in the paper by Akbarov (2014), Akbarov and Kepceler (2015). The results obtained in these papers were also detailed in the monograph by Akbarov (2015). We note that

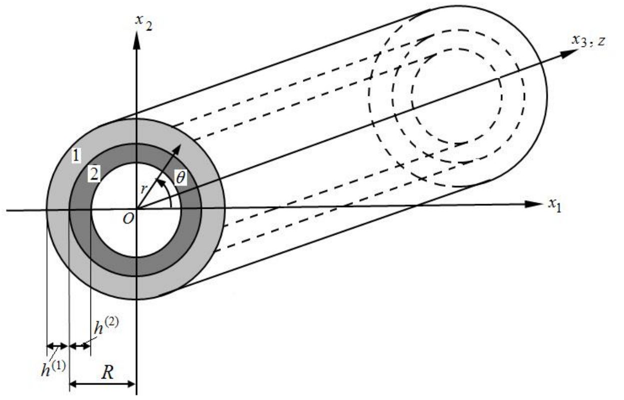

Fig. 1 The geometry of the bi-layered circular hollow cylinder

in this paper the forced vibration of the system consisting of viscoelastic covering layer and viscoelastic half-space was investigated.

Nevertheless, in the paper by Akbarov and Kepceler (2015), dispersion of torsional waves in the viscoelastic hollow sandwich cylinder is studied for the selected wave attenuation rule. However, up to now there has not been any investigations carried out utilizing the fractional exponential operators by Rabotnov (1980) nor has the dispersive attenuation of viscoelastic guided waves for the selected possible dispersion curves of the same viscoelastic waves, been studied. The subject of the present paper relates, namely, to this question. More precisely, the main goal of the present paper is the theoretical investigation of the possible dispersive attenuation of the longitudinal axisymmetric guided waves propagating in the bi-layered circular hollow cylinder for the first fundamental mode, i.e., for the $L(0,1)$ mode in the cases where the constitutive relations for the cylinders' materials are described through the fractional exponential operator by Rabotnov (1980). Moreover, the investigations carried out in the present paper also contain a study of the influence of the rheological parameters of the cylinders' materials on these attenuations. It should be noted that the theoretical investigations carried out in the present paper are the first attempt in this field which is made by utilizing the fractional exponential operators by Rabotnov (1980). The application fields of the obtained theoretical results will be discussed below.

2. Formulation of the problem

Consider the bi-layered hollow circular cylinder (see Fig. 1) and assume that the radius of the outer circle of the inner hollow cylinder is R and the thickness of the inner and outer cylinders is $h^{(2)}$ and $h^{(1)}$, respectively. The values related to the inner and external hollow cylinders will be denoted by the upper indices (2) and (1), respectively.

Assume that the materials of the constituents are isotropic, homogeneous and hereditary-viscoelastic. We use the cylindrical system of coordinates $Or\theta z$ (Fig. 1) for determination of the position of the points of the system under consideration. Moreover, we assume that the cylinders have infinite length in the direction of the Oz axis.

Let us investigate the axisymmetric longitudinal wave propagation along the Oz axis in the considered cylinder with the use of the following field equations and relations.

First we write the equations of motion for the axisymmetric case with respect to the Oz axis (see, for instance Eringen and Suhubi 1975)

$$\frac{\partial T_{rr}^{(n)}}{\partial r}+\frac{\partial T_{rz}^{(n)}}{\partial z}+\frac{1}{r}(T_{rr}^{(n)}-T_{\theta\theta}^{(n)})=\rho^{(n)}\frac{\partial^2 u_r^{(n)}}{\partial t^2},$$

$$\frac{\partial T_{rz}^{(n)}}{\partial r}+\frac{\partial T_{zz}^{(n)}}{\partial z}+\frac{1}{r}T_{rz}^{(n)}=\rho^{(n)}\frac{\partial^2 u_z^{(n)}}{\partial t^2}. \tag{1}$$

In (1) the following notation is used: $T_{rr}^{(n)}$, $T_{\theta\theta}^{(n)}$ and $T_{zz}^{(n)}$ are radial, circumferential and axial normal stresses, respectively, $T_{rz}^{(n)}$ is the shear stress, and $u_r^{(n)}$ and $u_z^{(n)}$ are radial and axial components of the displacement vector in the $n-th$ layer material. We recall that here and below, the case where n=1 (n=2) relates to the outer (inner) layer. This means that for the case where n=1 ((n=2) the equations in (1) are satisfied within the scope of the outer (inner) layer. Moreover, in (1) $\rho^{(n)}$ denotes the density of the n-th layer material.

Consider the constitutive relations for the linear viscoelastic materials which, according to the Volterra principle (see, for instance, Christenson 2010, Fung 1965, Rabotnov 1980), are obtained from the elasticity relations (i.e., from Hook's law) by replacing the elastic constants with the corresponding operators. As a result of this replacement, the following constitutive relations are obtained for the not aging viscoelastic material

$$T_{(ii)}^{(n)}=\lambda^{(n)*}\theta^{(n)}+2\mu^{(n)*}\varepsilon_{(ii)}^{(n)},\ (ii)=rr,zz,\theta\theta,$$

$$T_{rz}^{(n)}=2\mu^{(n)*}\varepsilon_{rz}^{(n)},\ \theta^{(n)}=\varepsilon_{rr}^{(n)}+\varepsilon_{\theta\theta}^{(n)}+\varepsilon_{zz}^{(n)}, \tag{2}$$

where $\lambda^{(n)*}$ and $\mu^{(n)*}$ are the following viscoelastic operators

$$\begin{Bmatrix}\lambda^{(n)*}\\ \mu^{(n)*}\end{Bmatrix}\varphi(t)=\begin{Bmatrix}\lambda_0^{(n)}\\ \mu_0^{(n)}\end{Bmatrix}\varphi(t)+\int_0^t\begin{Bmatrix}\lambda_1^{(n)}\\ \mu_1^{(n)}\end{Bmatrix}(t-\tau)\varphi(\tau)d\tau\ . \tag{3}$$

In Eq. (3), $\lambda_0^{(n)}$ and $\mu_0^{(n)}$ are the instantaneous values of Lame's constants as $t\to 0$, and $\lambda_1^{(n)}(t)$ and $\mu_1^{(n)}(t)$ are the corresponding kernel functions describing the hereditary properties of the materials of the constituents. Note that the form (3) for the operators in (2) is obtained from the Boltzmann superposition principle (see, for instance, Christenson (2010), Fung (1965) and others listed therein), according to which, the stress in the viscoelastic material is presented as a superposition of multiplying the relaxation function to deformation increments, i.e., if for simplicity we consider the one axial stress-strain state, then the Boltzmann superposition principle means that $\sigma=\int_0^t J(t,\tau)d\varepsilon(\tau)$, where σ and ε are the axial stress and strain, respectively, and J is the relaxation function. If the relaxation function J is presented as $J=J(t,\tau)$, then the material is called an aging one, however, if the relaxation function J is presented as $J=J(\tau-t)$, then the material is called a not aging one. Consequently, for the not aging viscoelastic one, the Boltzmann superposition principle is expressed as $\sigma=\int_0^t J(t-\tau)d\varepsilon(\tau)$ and employing the partial integration procedure to this integral we obtain $\sigma=\varepsilon(t)J(0)-\int_0^t (dJ(\xi)/d\xi)\big|_{\xi=t-\tau}\,\varepsilon(\tau)d\tau$ and introducing the notation $E=J(0)$ (where E is the modulus of elasticity) and $K(t-\tau)=-(dJ(\xi)/d\xi)\big|_{\xi=t-\tau}$ we obtain the expression $\sigma=\varepsilon(t)E+\int_0^t K(t-\tau)\varepsilon(\tau)d\tau$, the form of which coincides with the form of the operators. So with the corresponding generalization of the expression $\sigma=\varepsilon(t)E+\int_0^t K(t-\tau)\varepsilon(\tau)d\tau$, the constitutive relations given in (2) and (3) are obtained. Note that the kernel function $K(t)$ as well as the $\lambda_1^{(n)}(t)$ and $\mu_1^{(n)}(t)$ kernel functions in (3) are determined experimentally and under this determination, various types of models are employed, such as the simplest Kelvin-Voigt, fractional Kelvin-Voigt and more complicated models, one of which is the fractional exponential operator by Rabotnov (1980) which will be discussed and used below.

For completeness of the foregoing field equations, it is necessary to add to these equations the following strain-displacement relations

$$\varepsilon_{rr}^{(n)}=\frac{\partial u_r^{(n)}}{\partial r},\ \varepsilon_{rz}^{(n)}=\frac{1}{2}\left(\frac{\partial u_r^{(n)}}{\partial z}+\frac{\partial u_z^{(n)}}{\partial r}\right),\ \varepsilon_{\theta\theta}^{(n)}=\frac{u_r^{(n)}}{r},$$

$$\varepsilon_{zz}^{(n)}=\frac{\partial u_z^{(n)}}{\partial z}, \tag{4}$$

where $\varepsilon_{rr}^{(n)}$, $\varepsilon_{\theta\theta}^{(n)}$ and $\varepsilon_{zz}^{(n)}$ are radial, circumferential and axial normal deformations, respectively, and $\varepsilon_{rz}^{(n)}$ is the shear deformation.

This completes the consideration of the complete system of field equation given in (1)-(4).

Consider also formulation of the boundary and contact conditions. According to Fig. 1 we can write the boundary and contact conditions:

$$T_{rr}^{(2)}\Big|_{r=R(1-h^{(2)}/R)}=0,\ T_{rz}^{(2)}\Big|_{r=R(1-h^{(2)}/R)}=0,$$

$$T_{rr}^{(2)}\Big|_{r=R}=T_{rr}^{(1)}\Big|_{r=R},\ T_{rz}^{(2)}\Big|_{r=R}=T_{rz}^{(1)}\Big|_{r=R},$$

$$u_r^{(2)}\Big|_{r=R}=u_r^{(1)}\Big|_{r=R},\ u_z^{(2)}\Big|_{r=R}=u_z^{(1)}\Big|_{r=R},$$

$$T_{rr}^{(1)}\Big|_{r=R(1+h^{(1)}/R)}=0,\ T_{rz}^{(1)}\Big|_{r=R(1+h^{(1)}/R)}=0. \tag{5}$$

This completes the formulation of the problem on the axisymmetric longitudinal wave dispersion in the bi-layered hollow cylinder made of viscoelastic materials with arbitrary kernel functions $\lambda_1^{(n)}(t)$ and $\mu_1^{(n)}(t)$ which enter the constitutive relations (2) and (3).

3. Method of solution

As we consider the time-harmonic wave propagation in the Oz axis direction, we can represent all the sought values as follows

$$u_r^{(n)}=v_r^{(n)}(r)e^{i(kz-\omega t)},\ u_z^{(n)}=v_z^{(n)}(r)e^{i(kz-\omega t)},$$

$$\theta^{(n)}=\upsilon^{(n)}(r)e^{i(kz-\omega t)},\ \varepsilon_{(ii)}^{(n)}=\gamma_{(ii)}^{(n)}(r)e^{i(kz-\omega t)},$$

$$(ii)=rr;\theta\theta;zz;rz \tag{6}$$

where k is the wave number and ω is the circular frequency, and

$$\gamma_{rr}^{(n)}=\frac{dv_r^{(n)}(r)}{dr},\ \gamma_{\theta\theta}^{(n)}=\frac{v_r^{(n)}(r)}{r},\ \gamma_{zz}^{(n)}=\frac{dv_z^{(n)}(r)}{dz},$$

$$\gamma_{rz}^{(n)}=\frac{1}{2}\left(\frac{dv_r^{(n)}(r)}{dz}+\frac{dv_z^{(n)}(r)}{dr}\right), \tag{7}$$

Using the relation

$$\int_0^t f_1(t-\tau)f_2(\tau)d\tau\approx\int_{-\infty}^t f_1(t-\tau)f_2(\tau)d\tau, \tag{8}$$

and taking the relations (6)-(8) into account in Eqs. (2) and (3), we can write the following relations:

$$T_{(ii)}^{(n)}=\lambda_0^{(n)}\vartheta^{(n)}(r)e^{i(kz-\omega t)}+e^{ikz}\vartheta^{(n)}(r)\int_{-\infty}^t\lambda_1^{(n)}(t-\tau)e^{-i\omega\tau}d\tau+$$

$$2\mu_0^{(n)}\gamma_{(ii)}^{(n)}(r)e^{i(kz-\omega t)}+e^{ikz}\gamma_{(ii)}^{(n)}(r)\int_{-\infty}^t\mu_1^{(n)}(t-\tau)e^{-i\omega\tau}d\tau. \tag{9}$$

Employing the transformation $t-\tau=s$, the following manipulations can be made for the integrals which enter into Eq. (9)

$$\int_{-\infty}^t\begin{Bmatrix}\lambda_1^{(n)}\\ \mu_1^{(n)}\end{Bmatrix}(t-\tau)e^{-i\omega\tau}d\tau=-\int_{\infty}^0\begin{Bmatrix}\lambda_1^{(n)}\\ \mu_1^{(n)}\end{Bmatrix}(s)e^{-i\omega t}e^{i\omega s}ds=$$

$$e^{-i\omega t}\int_0^\infty\begin{Bmatrix}\lambda_1^{(n)}\\ \mu_1^{(n)}\end{Bmatrix}(s)e^{i\omega s}ds=e^{-i\omega t}\left(\begin{Bmatrix}\lambda_{1c}^{(n)}\\ \mu_{1c}^{(n)}\end{Bmatrix}+i\begin{Bmatrix}\lambda_{1s}^{(n)}\\ \mu_{1s}^{(n)}\end{Bmatrix}\right), \tag{10}$$

where

$$\begin{Bmatrix}\lambda_{1c}^{(n)}\\ \mu_{1c}^{(n)}\end{Bmatrix}=\int_0^\infty\begin{Bmatrix}\lambda_1^{(n)}\\ \mu_1^{(n)}\end{Bmatrix}(s)\cos(\omega s)\,\mathrm{ds},$$

$$\begin{Bmatrix}\lambda_{1s}^{(n)}\\ \mu_{1s}^{(n)}\end{Bmatrix}=\int_0^\infty\begin{Bmatrix}\lambda_1^{(n)}\\ \mu_1^{(n)}\end{Bmatrix}(s)\sin(\omega s)\,\mathrm{ds}. \tag{11}$$

Thus, according to (8)-(11), we can write the following expressions for the stresses

$$T_{(ii)}^{(n)}=\Lambda^{(n)}\vartheta^{(n)}(r)e^{i(kz-\omega t)}+$$

$$2M^{(n)}\gamma_{(ii)}^{(n)}(r)e^{i(kz-\omega t)}=\sigma_{(ii)}^{(n)}(r)e^{i(kz-\omega t)},$$

$$T_{rz}^{(n)}=2M^{(n)}\gamma_{rz}^{(n)}(r)e^{i(kz-\omega t)}=\sigma_{rz}^{(n)}(r)e^{i(kz-\omega t)}, \tag{12}$$

where

$$\Lambda^{(n)} = \lambda_0^{(n)} + \lambda_{1c}^{(n)} + i\lambda_{1s}^{(n)} \ , \ M^{(n)} = \mu_0^{(n)} + \mu_{1c}^{(n)} + i\mu_{1s}^{(n)} . \tag{13}$$

It follows from the Eqs. (12) and (13) that the complete system of field Eqs. (1), (2), (4), (12) and (13) for the viscoelastic system can also be obtained for the purely elastic system by replacing the elastic constants $\lambda_0^{(n)}$ and $\mu_0^{(n)}$ with the complex constants $\Lambda^{(n)}$ and $M^{(n)}$. In other words, the foregoing mathematical calculations confirm the dynamic correspondence principle (see Fung 1965) for the problem under consideration and the solution method used here coincides with this principle.

Thus, substituting the expression (12) into the equation of motion (1) and taking the relation (6) into consideration, we obtain the following equations of motion in terms of the displacement amplitudes

$$m_1^{(n)} \frac{d^2 v_r^{(n)}}{d(kr)^2} + m_2^{(n)} \frac{d}{d(kr)}\left(\frac{v_r^{(n)}}{kr}\right) + i(m_2^{(n)} + m_3^{(n)}) \frac{dv_z^{(n)}}{d(kr)} -$$

$$m_3^{(n)} v_r^{(n)} + \frac{1}{kr}(m_1^{(n)} - m_2^{(n)}) \frac{dv_r^{(n)}}{dr} + (m_2^{(n)} - m_1^{(n)}) \frac{v_r^{(n)}}{(kr)^2} =$$

$$-\frac{\omega^2}{k^2} \rho^{(n)} v_r^{(n)} \ ,$$

$$im_3^{(n)} \frac{dv_r^{(n)}}{d(kr)} + m_3^{(n)} \frac{d^2 v_z^{(n)}}{d(kr)^2} + i\frac{1}{kr} m_3^{(n)} v_r^{(n)} + \frac{1}{kr} m_3^{(n)} \frac{dv_z^{(n)}}{d(kr)} +$$

$$ikm_2^{(n)} \frac{dv_r^{(n)}}{d(kr)} + im_2^{(n)} \frac{v_r^{(n)}}{kr} - m_1^{(n)} v_z^{(n)} = -\frac{\omega^2}{k^2} \rho^{(n)} v_z^{(n)} \ , \tag{14}$$

where

$$m_1^{(n)} = \Lambda^{(n)} + 2M^{(n)} \ , \ m_2^{(n)} = \Lambda^{(n)}, \ m_3^{(n)} = M^{(n)} . \tag{15}$$

According to the foregoing transformations and expressions in (6) and (12), the boundary and contact conditions in (5) can be rewritten as follows

$$\sigma_{rr}^{(2)}\Big|_{r=R(1-h^{(2)}/R)} = 0 \, , \ \sigma_{rz}^{(2)}\Big|_{r=R(1-h^{(2)}/R)} = 0 \ ,$$

$$\sigma_{rr}^{(2)}\Big|_{r=R} = \sigma_{rr}^{(1)}\Big|_{r=R} , \ \sigma_{rz}^{(2)}\Big|_{r=R} = \sigma_{rz}^{(1)}\Big|_{r=R} ,$$

$$v_r^{(2)}\Big|_{r=R} = v_r^{(1)}\Big|_{r=R} , \ v_z^{(2)}\Big|_{r=R} = v_z^{(1)}\Big|_{r=R} ,$$

$$\sigma_{rr}^{(1)}\Big|_{r=R(1+h^{(1)}/R)} = 0 \, , \ \sigma_{rz}^{(1)}\Big|_{r=R(1+h^{(1)}/R)} = 0 . \tag{16}$$

For the solution to Eqs. (14), (6), (7) and (12), with the boundary and contact conditions in (16), according to Guz (2004), we employ the following representation for the displacement amplitudes

$$v_r^{(n)} = -ik\frac{\partial}{\partial r} X^{(n)} ,$$

$$v_z^{(n)} = \frac{1}{m_2^{(n)} + m_3^{(n)}} \left(m_1^{(n)} \Delta_1 - k^2 m_3^{(n)} + \omega^2 \rho^{(n)} \right) X^{(n)} ,$$

$$\Delta_1 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} \ , \tag{17}$$

where $\boldsymbol{X}^{(n)}$ satisfies the equation

$$\left[\left(\Delta_1 - k^2 (\zeta_2^{(n)})^2 \right) \left(\Delta_1 - k^2 (\zeta_3^{(n)})^2 \right) \right] X^{(n)} = 0 . \tag{18}$$

In (18) $\zeta_2^{(n)}$ and $\zeta_3^{(n)}$ are determined from the following equations

$$(\Lambda^{(n)} + 2M^{(n)})M^{(n)} \zeta'^{(n)^4} -$$

$$k^2 \ \zeta'^{(n)\ 2} \left[(\Lambda^{(n)} + 2M^{(n)}) \left(\rho^{(n)} \left(\frac{\omega}{k} \right)^2 - (\Lambda^{(n)} + 2M^{(n)}) \right) + \right.$$

$$\left. M^{(n)} \left(\rho^{(m)} \left(\frac{\omega}{k} \right)^2 - M^{(n)} \right) + \Lambda^{(n)} + M^{(n)} \ ^2 \right] +$$

$$k^4 \left(\rho^{(n)} \left(\frac{\omega}{k} \right)^2 - (\Lambda^{(n)} + 2M^{(n)}) \right) \left(\rho^{(n)} \left(\frac{\omega}{k} \right)^2 - M^{(n)} \right) = 0, \tag{19}$$

where ω/k is the complex phase velocity of the wave propagation.

Thus, we determine the following expression for the function $\boldsymbol{X}^{(n)}$ from Eqs. (18) and (19).

$$X^{(n)} = A_1^n J_0(\zeta_2^{(n)} kr) + A_2^n J_0(\zeta_3^{(n)} kr) + B_1^n Y_0(\zeta_2^{(n)} kr) +$$

$$B_2^n Y_0(\zeta_3^{(n)} kr) \tag{20}$$

where $J_0(x)$ and $Y_0(x)$ are Bessel functions of the first and second kinds with zeroth order, respectively.

Using the expression (20) and Eqs. (17), (13), (6) and (7), we obtain the following dispersion equation from the conditions in (16)

$$\det \left\| \beta_{nm} \right\| = 0 \ , \ n; m = 1, 2, ..., 8 , \tag{21}$$

The explicit expressions of the components of the matrix (β_{nm}) are given in Appendix A through the expressions (A1) and (A2).

Thus, the dispersion equations obtained for the considered wave propagation problems have been derived in the form (21), (A1) and (A2). In the case where $\lambda_{1c}^{(n)} = \lambda_{1s}^{(n)} = \mu_{1c}^{(n)} = \mu_{1s}^{(n)} = 0$ in (13), i.e., in the case where $\Lambda^{(n)} = \lambda_0^{(n)}$ and $M^{(n)} = \mu_0^{(n)}$ the foregoing dispersion equation transforms into the corresponding one obtained for the wave dispersion in the purely elastic case which is detailed for instance in the papers by Akbarov and Guliev (2009), Akbarov and Ipek (2010, 2012), Ipek (2015) and in the monograph by Akbarov (2015) which also contains results related to the dynamics of the complicated medium discussed, for instance, in the paper by Ilhan and Koc (2015).

4. Numerical results and discussions

4.1 Selection of the operators in (3) and dimensionless rheological parameters

According to the well-known physico-mechanical considerations, under time harmonic wave propagation in a viscoelastic material, it is necessary to assume that the wave number k is a complex one and can be presented as follows

$$k = k_1 + ik_2 = k_1(1+i\beta)\,,\ \beta = \frac{k_2}{k_1}\,, \tag{22}$$

where k_2 (or parameter β in (22)), i.e., the imaginary part of the wave number k, defines the attenuation of the wave amplitude under consideration and β is called the coefficient of the attenuation.

We determine the phase velocity of the studied waves through the expression

$$c = \frac{\omega}{k_1} \tag{23}$$

and introduce the notation $c_{20}^{(n)} = \sqrt{\mu_0^{(n)} / \rho^{(n)}}$.

We use below the arguments

$$\frac{c}{c_{20}^{(2)}}\ ,\ k_1 R\,,\ \frac{h^{(1)}}{R}\ \text{and}\ \frac{h^{(2)}}{R}\,. \tag{24}$$

To solve the dispersion Eq. (21) it is necessary to give the explicit expressions for the functions $\mu_1^{(n)}(t)$ and $\lambda_1^{(n)}(t)$ which enter into the operators in order to determine the quantities $\lambda_{1c}^{(n)}$, $\lambda_{1s}^{(n)}$, $\mu_{1c}^{(n)}$ and $\mu_{1s}^{(n)}$ through the expressions in (11). For this purpose, as in the papers by Akbarov (2014) and Akbarov and Kepceler (2015), here we also assume that the viscoelasticity of the materials of the cylinder's layers is described by Rabotnov's (1980) fractional exponential operator, i.e., we assume that

$$\mu^{(n)*}\varphi(t) = \mu_0^{(n)}\left[\varphi(t) - \frac{3\beta_0^{(n)}}{2(1+\nu_0^{(n)})}\varPi_{\alpha^{(n)}}^{(n)*}\left(-\frac{3\beta_0^{(n)}}{2(1+\nu_0^{(n)})} - \beta_\infty^{(n)}\right)\varphi(t)\right],$$

$$\lambda^{(n)*}\varphi(t) = \lambda_0^{(n)}\left[\varphi(t) + \frac{\beta_0^{(n)}}{(1+\nu_0^{(n)})}\varPi_{\alpha^{(n)}}^{(n)*}\left(-\frac{3\beta_0^{(n)}}{2(1+\nu_0^{(n)})} - \beta_\infty^{(n)}\right)\varphi(t)\right],$$

$$E^{(n)*}\varphi(t) = E_0^{(n)}\left[\varphi(t) - \beta_0^{(n)}\varPi_{\alpha^{(n)}}^{(n)*}\left(-\beta_0^{(n)} - \beta_\infty^{(n)}\right)\varphi(t)\right],$$

$$\nu^{(n)*}\varphi(t) = \nu_0^{(n)}\left[\varphi(t) + \frac{1-2\nu_0^{(n)}}{2\nu_0^{(n)}}\beta_0^{(n)}\varPi_{\alpha^{(n)}}^{(n)*}\left(-\beta_0^{(n)} - \beta_\infty^{(n)}\right)\varphi(t)\right], \tag{25}$$

where

$$\varPi_{\alpha^{(n)}}^{(n)*}(x^{(n)})\varphi(t) = \int_0^\infty \varPi_{\alpha^{(n)}}^{(n)}(x^{(n)}, t-\tau)\varphi(\tau)d\tau\,,$$

$$\varPi_{\alpha^{(n)}}^{(n)}(x^{(n)}, t) = t^{-\alpha^{(n)}}\sum_{p=0}^{\infty}\frac{(x^{(n)})^p\, t^{p(1-\alpha^{(n)})}}{\Gamma((1+n)(1-\alpha^{(n)}))}\,,\ 0 \le \alpha^{(n)} < 1. \tag{26}$$

In (26) $\Gamma(x)$ is the gamma function. Moreover, the constants $\alpha^{(n)}$, $\beta_0^{(n)}$ and $\beta_\infty^{(n)}$ in (25) and (26) are the rheological parameters of the n-th layer's material.

As in the papers by Akbarov (2014) and Akbarov and Kepceler (2015), we introduce the parameters

$$d^{(n)} = \frac{\beta_\infty^{(n)}}{\beta_0^{(n)}}\ \text{and}\ Q^{(n)} = \frac{c_{20}^{(n)}}{R(\beta_{01}^{(n)} + \beta_\infty^{(n)})^{\frac{1}{1-\alpha^{(n)}}}}\,, \tag{27}$$

and the long-term values of the elastic constants and characteristic creep time can be estimated, respectively for the n-th material. In this case, the values of the long-term elastic constants and the quantities of $\mu_{1c}^{(n)}$ and $\mu_{1s}^{(n)}$ are determined through the expressions (28) and (29) respectively, as given below

$$E_\infty^{(n)} = \lim_{t\to\infty} E^{(n)*} = E_0^{(n)}\left(1 - \frac{1}{1+d^{(n)}}\right),$$

$$\nu_\infty^{(n)} = \lim_{t\to\infty} \nu^{(n)*} = \nu_0^{(n)}\left(1 + \frac{1-2\nu_0^{(n)}}{2\nu_0^{(n)}}\frac{1}{1+d^{(n)}}\right),$$

$$\mu_\infty^{(n)} = \lim_{t\to\infty} \mu^{(n)*} = \mu_0^{(n)}\left(1 - \frac{3}{2(1+\nu_0^{(n)})}\frac{1}{(3/(2(1-\nu_0^{(n)})) + d^{(n)})}\right), \tag{28}$$

$$\mu_c^{(n)} = \mu_0^{(n)}\left[1 - \frac{3}{2(1+\nu_0^{(n)})}\left(d^{(n)} + \beta_{01}^{(n)}\right)^{-1}\varPi_{\alpha^{(n)}c}^{(n)}(-\beta_{01}^{(n)} - \beta_\infty^{(n)}, k_1 Rc)\right],$$

$$\mu_c^{(n)} = -\mu_0^{(n)}\frac{3}{2(1+\nu_0^{(n)})}\left(d^{(n)} + \beta_{01}^{(n)}\right)^{-1}\varPi_{\alpha^{(n)}s}^{(n)}(-\beta_{01}^{(n)} - \beta_\infty^{(n)}, k_1 Rc), \tag{29}$$

where

$$\beta_{01}^{(n)} = \frac{3\beta_0^{(n)}}{2(1+\nu_0^{(n)})}\,.$$

$$\varPi_{\alpha^{(n)}c}^{(n)}(-\beta_{01}^{(n)} - \beta_\infty^{(n)}, k_1 Rc) = \frac{(\xi^{(n)})^2 + \xi^{(n)}\sin\frac{\pi\alpha^{(n)}}{2}}{(\xi^{(n)})^2 + 2\xi^{(n)}\sin\frac{\pi\alpha^{(n)}}{2} + 1}\,,$$

$$\varPi_{\alpha^{(n)}s}^{(n)}(-\beta_{01}^{(n)} - \beta_\infty^{(n)}, k_1 Rc) = \frac{\xi^{(n)}\cos\frac{\pi\alpha^{(n)}}{2}}{(\xi^{(n)})^2 + 2\xi^{(n)}\sin\frac{\pi\alpha^{(n)}}{2} + 1}\,,$$

$$\xi^{(n)} = (Q^{(n)}\Omega)^{\alpha^{(n)}-1}\,,\ \Omega = k_1 R\frac{c}{c_{20}^{(2)}}\,. \tag{30}$$

It follows from the foregoing expressions (29) and (30) and from the numerical analyses made in the papers by Akbarov (2014), Akbarov and Kepceler (2015) that

$\mu_s^{(n)} / \mu_0^{(n)} \to 0$ as $\xi^{(n)} \to 0$ or as $\xi^{(n)} \to \infty$, but the absolute values of $\mu_c^{(n)}$ ($\Pi_{\alpha^{(n)}c}^{(n)}$) decrease (increase) monotonically with $\xi^{(n)}$ and $\mu_c^{(n)} / \mu_0^{(n)} \to 1$ ($\mu_c^{(n)} / \mu_0^{(n)} \to \mu_\infty^{(n)} / \mu_0^{(n)}$) as $\xi^{(n)} \to 0$ ($\xi^{(n)} \to \infty$). Moreover, according to the expressions (28)-(30), we can write the following limit cases

$$\mu_\infty^{(n)} \to \mu_0^{(n)}, \ \mu_s^{(n)} \to 0 \ \text{ as } d^{(n)} \to \infty,$$

$$\xi^{(n)} \to \infty; \ \Pi_{\alpha^{(n)}c}^{(n)}(-\beta_1^{(n)} - \beta_\infty^{(n)}, k_1 Rc) \to 1;$$

$$\Pi_{\alpha^{(n)}s}^{(n)}(-\beta_1^{(n)} - \beta_\infty^{(n)}, k_1 Rc) \to 0 \qquad (31)$$

as

$$(Q^{(n)}\Omega) \to 0 \ \text{ or as } \ k_1 R \to 0,$$

$$\xi^{(n)} \to 0; \ \Pi_{\alpha^{(n)}c}^{(n)}(-\beta_1^{(n)} - \beta_\infty^{(n)}, k_1 Rc) \to 0;$$

$$\Pi_{\alpha^{(n)}s}^{(n)}(-\beta_1^{(n)} - \beta_\infty^{(n)}, k_1 Rc) \to 0 \qquad (32)$$

as

$$(Q^{(n)}\Omega) \to \infty \ \text{ or as } \ k_1 R \to \infty. \qquad (33)$$

It follows from the relation (32) that in the cases where ($Q^{(n)}$ Ω)<<1, the behavior of the viscoelastic system must be very close to the corresponding purely elastic system with long-term values of the elastic constants. As well, it follows from the relation (33) that in the cases where ($Q^{(n)}$ Ω)>>1, the behavior of the viscoelastic system must be very close to that of the corresponding purely elastic system with instantaneous values of the elastic constants at t=0.

Thus, according to the foregoing discussions, we can conclude that the influence of the viscosity of the viscoelastic materials under consideration on the wave propagation velocity dispersion (i.e., on the dependence between $c / c_{20}^{(2)}$ and k_1R) and on the wave attenuation dispersion (i.e., on the dependence between the attenuation coefficient β (22) and k_1R) can be characterized through the parameters $Q^{(n)}$ and $d^{(n)}$. It must be taken into account that an increase in the values of the parameters $Q^{(n)}$ and $d^{(n)}$ will correspond to a decrease in the viscous part of all the viscoelastic deformations of the constituents. Note that the influence of the other rheological parameter $\alpha^{(n)}$ on the viscous part of the viscoelastic deformations can be taken into account through the parameter $Q^{(n)}$ (27).

Note that the fractional exponential operator given through the expressions in (25) and (26) was proposed for the first time by Rabotnov (1948). This operator has developed since the 1950's and has numerous applications under corresponding theoretical and experimental investigations, a review of which is given in the monograph by Rabotnov (1980) and in the papers by Rossikhin (2010), Bosiakov (2014), Rossikhin and Shitikova (2014) and other ones listed therein. The main advantages of this operator are that it not only allows description, with the very high accuracy required, of the initial parts of the experimentally constructed creep and relaxation graphs and their asymptotic values, but it is also a resolvent one. This means that this operator allows creep experiments to be carried out. It also allows for determination of the rheological constants involving not only the strain-stress relation, but also to simultaneously determine the corresponding constants involving the same stress-strain relation. However, under application of the operators which are not resolvent for construction of the stress-strain relation using the strain-stress relation it is necessary to solve the corresponding integral equations. A more detailed discussion of the properties and advantages of the resolvent operators are given in the monograph by Rabotnov (1980). At the same time, it should be noted that in the related literature the application of other types of fractional order operators (see, for instance, Adolfson, Enelund *et al*. (2005, Sawicki and Padovan 1999 and others listed therein) can also be found.

Up to now, many investigations (see, Rabotnov 1980, Kaminskii and Selivanov 2005, Golub, Fernati *et al*. 2008 and others listed therein) have been made on the experimental validation of the fractional exponential operators in (25) and on determination of the rheological parameters α, β_∞ and β_0 which enter into these operators. For instance, in the paper by Golub, Fernati *et al*. (2008), it was experimentally established that for Polymer concrete, these rheological parameters have the following values: α=0.723778, $\beta_0+\beta_\infty$=0.18875$h^{\alpha-1}$ and β_0=0.02598$h^{\alpha-1}$. Another example on the experimental determination of the values of these parameters for polymethylmethacrylate is given in the paper by Kaminskii and Selivanov (2005) and these values are: α=0.53, $\beta_0+\beta_\infty$=49day$^{1-\alpha}$ and β_0=0.98day$^{1-\alpha}$. Thus, through these operators, the constitutive relations can be described for a very wide range of linear viscoelastic materials with the corresponding values of the rheological parameters α, β_∞ and β_0.

Numerical results, which will be discussed below, are obtained for selected values of the dimensionless parameters $d^{(n)}$ and $Q^{(n)}$, the values of which are determined through the rheological parameters $\alpha^{(n)}$, $\beta_\infty^{(n)}$ and $\beta_0^{(n)}$. According to the foregoing discussions, the proposed investigation approach and obtained concrete numerical results can be employed for each really determined value of these rheological parameters which correspond to the concrete values of the dimensionless rheological parameters $d^{(n)}$ and $Q^{(n)}$.

Moreover, the results which will be discussed in the present paper can be used mainly in the theoretical - orientational sense. For instance, using these results it can be predicted how the increase or decrease of the rheological parameters such as creep time and long-term values of the mechanical properties of some polymer-viscoelastic materials can act (in the qualitative sense) on wave dispersion in the elements of construction made of these materials. Moreover, using these results it can be predicted how the change of the sizes of the elements of construction can act on the influence of the material viscosity on the wave dispersion. At the same time, the results obtained in the present paper allow us to understand the nature of the wave propagation and dispersion processes in the elements of construction made of viscoelastic materials and to control these processes.

This completes consideration of the selection of the dimensionless rheological parameters through which we will study the influence of the viscoelasticity properties of the layers' materials on the axisymmetric longitudinal wave dispersion.

4.2 On the algorithm of the numerical solution of the dispersion equation with respect to the wave attenuation

As the values of the determinant obtained in (21) are complex, the dispersion Eq. (21) can be reduced to the following one

$$\left|\det\left\|\beta_{ij}\right\|\right| = 0\,, \tag{34}$$

where $|\det\|\beta_{ij}\||$ means the modulus of the complex number $\det\|\beta_{ij}\|$. Consequently for construction of the attenuation or dispersion curves it is necessary to solve numerically the Eq. (34) for the selected problem parameters. In this solution procedure, the values of all the problem parameters (except c, k_1R and β) are selected in advance. Consequently, the equation (34) has three unknowns: c, k_1R and β which must be determined from this equation.

Note that in the corresponding purely elastic problems the dispersion equation contains only two unknowns: c and k_1R. The values of c are determined for each possible selected value of k_1R through the solution to this equation. Moreover, in the purely elastic case this solution procedure is carried out by employing the well-known "bi-section" method which is based on the sign change of the dispersion determinant. A more detailed description of the solution algorithm of the dispersion equations related to the purely elastic problems is given in the monograph by Akbarov (2015) and in papers such as Akbarov and Guliev (2009), Akbarov and Ipek (2010, 2012, 2015), Akbarov, Negin *et al*. (2015), Ipek (2015) etc. However, in the case under consideration we have not changed the sign of the dispersion determinant, i.e., $|\det\|\beta_{ij}\||\geq 0$ and this determinant, as noted above, contains three unknowns. Consequently, for the solution to the dispersion Eq. (34) we cannot employ the aforementioned algorithm based on the "bi-section" method. Therefore, for the solution to the dispersion Eq. (34) we use the algorithm which is based on direct calculation of the values of the moduli of the dispersion determinant $|\det\|\beta_{ij}\||$ and determination of the sought roots from the criterion $|\det\|\beta_{ij}\||\leq 10^{-12}$. It should be noted that under employing this algorithm it is necessary to give in advance a certain value of one of the unknowns c, k_1R or β. For instance, in the paper by Barshinger and Rose (2004) the admissible values for the wave propagation velocity c are given in advance and the values of the attenuation coefficient β are determined for each selected value of k_1R. It is also possible to give a value to the attenuation coefficient β and then to determine the phase velocity c for each selected value of k_1R. This latter case was considered in the paper by Akbarov and Kepceler (2015) and the expressions given by Ewing *et al*. (1957) and Kolsky (1963) are used for determination of the attenuation coefficient β through $\operatorname{Im}\mu_1^{(n)}(\omega)$ and $\operatorname{Re}\mu_1^{(n)}(\omega)$. It is evident that this approach can give certain results on the dispersion curves but cannot illustrate sufficiently and correctly the influence of the rheological, mechanical and geometrical parameters on the attenuation dispersion of the waves. In the application sense, knowledge of this influence has great significance. Taking this statement into consideration, in the present paper we aim to determine the dispersion attenuation curves directly from the solution of the dispersion Eq. (34) for selected admissible wave propagation velocities under selected values of the dimensionless wavenumber k_1R and to determine how the mechanical and rheological parameters affect these curves. How we select the admissible wave propagation velocity (or wave dispersion curves) will be discussed below.

4.3 On the low and high wavenumber limit values of the wave propagation velocity

According to the well-known physico-mechanical principle (see, for instance Rabotnov 1980), the effect of the viscosity of the viscoelastic material on its vibration increases with decreasing of the vibration frequency, and this effect decreases with increasing of the vibration frequency. Consequently, using this principle we can predict that if the wave propagation velocity approaches a finite limit in the case where $k_1R\to 0$ (in the case where $k_1R\to\infty$), then this means that along the dispersion curves, the wave frequency ω also approaches zero (infinity). Consequently, in the cases under consideration, a decrease (an increase) in the values of k_1R also means a decrease (an increase) in the values of the wave frequency. Therefore, according to the foregoing physical principle, in the cases where $k_1R\to 0$ (in the cases where $k_1R\to\infty$) the influence of the viscosity of the layers' materials on the dispersion curves, i.e., on the wave propagation velocity must disappear and the dynamic behavior of the viscoelastic system under consideration must approach that of the corresponding purely elastic system with long-term values (with instantaneous values) of the elastic constants. In other words, according to the foregoing physico-mechanical principle, the attenuation coefficient β must satisfy the following conditions.

$$\beta \to 0 \text{ as } k_1R \to 0 \text{ and } \beta \to 0 \text{ as } k_1R \to \infty\,. \tag{35}$$

Taking into consideration the foregoing discussions and those made in subsection 4.1, as well as the expressions obtained in the monograph by Akbarov (2015), for the low wavenumber limit values of the propagation velocity of the axisymmetric longitudinal waves in the bi-layered hollow cylinder, we can write the following low wavenumber limit values for the wave propagation velocity in the bi-layered circular hollow cylinder made of viscoelastic materials.

$$\frac{c}{c_{20}^{(2)}} = \sqrt{\frac{\mu_\infty^{(2)}}{\mu_0^{(2)}}}\left(\frac{e_\infty^{(2)}\eta^{(2)} + e_\infty^{(1)}\eta^{(1)}\,\dfrac{\mu_\infty^{(1)}}{\mu_\infty^{(2)}}}{\eta^{(2)} + \eta^{(1)}\,\dfrac{\rho^{(1)}}{\rho^{(2)}}}\right)^{\frac{1}{2}} \text{ as } k_1R \to 0\,, \tag{36}$$

where

$$e_{\infty}^{(n)}=2\left(1+\frac{\lambda_{\infty}^{(n)}}{2(\lambda_{\infty}^{(n)}+\mu_{\infty}^{(n)})}\right),$$

$$\eta^{(2)}=\frac{2+\dfrac{h^{(2)}}{R}}{\left(1+\dfrac{h^{(1)}}{h^{(2)}}\right)\left(2+\dfrac{h^{(2)}}{R}-\dfrac{h^{(1)}}{R}\right)},$$

$$\eta^{(1)}=\frac{2-\dfrac{h^{(1)}}{R}}{\left(1+\dfrac{h^{(2)}}{h^{(1)}}\right)\left(2+\dfrac{h^{(2)}}{R}-\dfrac{h^{(1)}}{R}\right)}. \tag{37}$$

In a similar manner, according to the foregoing discussions and those in subsection 4.1, as well as the monograph by Akbarov (2015) we can write the following high wavenumber limit values for the case under consideration

$$\frac{c}{c_{20}^{(2)}}=\min\left\{\frac{c_R^{(1)}}{c_{20}^{(2)}},\frac{c_R^{(2)}}{c_{20}^{(2)}},\frac{c_S}{c_{20}^{(2)}}\right\}, \tag{38}$$

where $c_R^{(n)}$ is the Rayleigh wave propagation velocity of the n-th material for the instantaneous values of the elastic constants of this material and c_S is the Stoneley wave propagation velocity of the selected pair of materials of the layers as well as for the instantaneous values of the elastic constants of these materials.

It should be noted that the expressions (35)-(38) occur not only for the fractional exponential operators given in (25) and (26), but also for the arbitrary possible operators describing the viscoelasticity of the layers' materials of the cylinder.

Thus, it follows from the foregoing results and from the expressions in (28) that the low wavenumber limit values of the wave propagation velocity determined by expressions (36) and (37) depend on the rheological parameter d only and do not depend on the rheological parameters Q and α. However, the high wavenumber limit values of the wave propagation velocity determined by the expression (38) do not depend on any of the above-introduced rheological parameters.

4.4 The selection of the admissible dispersion curves and algorithm for determination of the attenuation curves

Numerous numerical results and mechanical considerations show that the dispersion curves obtained for the corresponding purely elastic case with long-term values of the elastic constants and the dispersion curves also obtained for the corresponding purely elastic case with instantaneous values of the elastic constants under satisfaction of certain conditions can be taken as the lower and upper limit cases, respectively, for the dispersion curves obtained for the viscoelastic case. Namely this statement allows us to select admissible dispersion curves (and wave propagation velocity c) for the viscoelastic case and according to these curves (or velocity), to find the attenuation coefficient β from the solution of the dispersion Eq. (34) for each fixed value of the dimensionless wavenumber k_1R.

For detailed illustration of the foregoing algorithm we consider the case where $d^{(1)}=d^{(2)}=25$, $h^{(1)}/R=h^{(2)}/R=0.1$ and $\mu^{(2)}/\mu^{(1)}=0.5$. Moreover, we assume that $Q^{(1)}=Q^{(2)}$ and consider how we can determine the attenuation curves from the dispersion Eq. (34). First we construct the dispersion curves related to the purely elastic case with long-term and instantaneous values of the elastic constants of the layers of the cylinder. These dispersion curves are illustrated in Fig. 2 with dashed lines. Thus, after construction of the dispersion curves related to the purely elastic cases, according to the discussions made in the foregoing subsections of the present section, we can select the admissible dispersion curves related to the viscoelastic case. As an example, for selection, we can take the dispersion curves shown in Fig. 2 with solid lines which are numbered from the dispersion curve constructed at $t=\infty$ to the dispersion curve constructed at $t=0$.

Thus, after the foregoing preparation, we select values for the dimensionless wavenumber k_1R and the wave propagation velocity $c/c_{20}^{(2)}$. Note that selection of the values for the velocity $c/c_{20}^{(2)}$ is made according to the admissible dispersion curves indicated in Fig. 2. For instance, if as an admissible dispersion curve we take the dispersion curve indicated by number 1 in Fig. 2, then the values of $c/c_{20}^{(2)}$ for the given k_1R are determined from this curve. After this determination, given the values for the rheological parameters $Q^{(1)}$, $Q^{(2)}$, $\alpha^{(1)}$ and $\alpha^{(2)}$ we determine the attenuation β from the dispersion Eq. (34) and as a result of this determination we construct the attenuation curves which will be discussed in the next subsection.

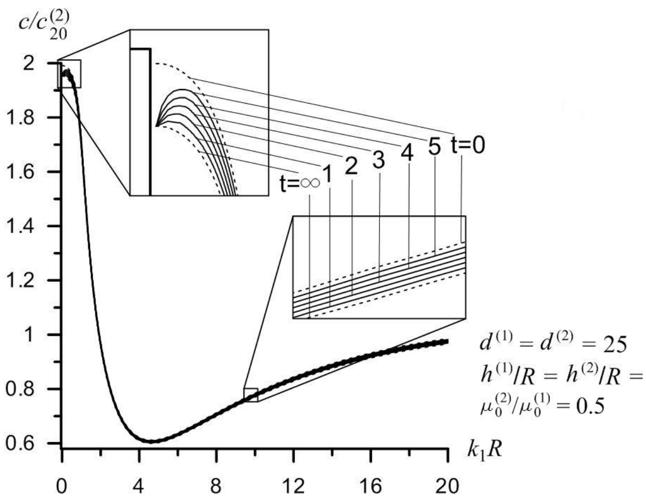

Fig. 2 Selected dispersion curves (solid lines) and dispersion curves (dashed lines) obtained under instantaneous (i.e., under $t=0$) and long-term (i.e., under $t=\infty$) values of the elastic constants for the case where $d^{(1)}=d^{(2)}=25$

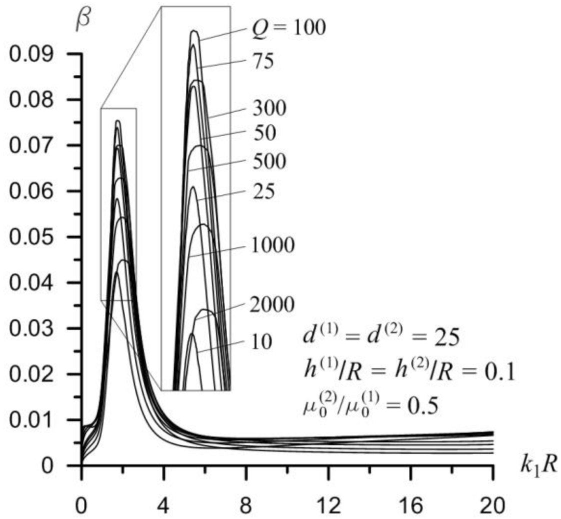

Fig. 3 Attenuation curves obtained for the admissible dispersion curve indicated by number 1 in Fig. 2 for various values of the rheological parameter Q

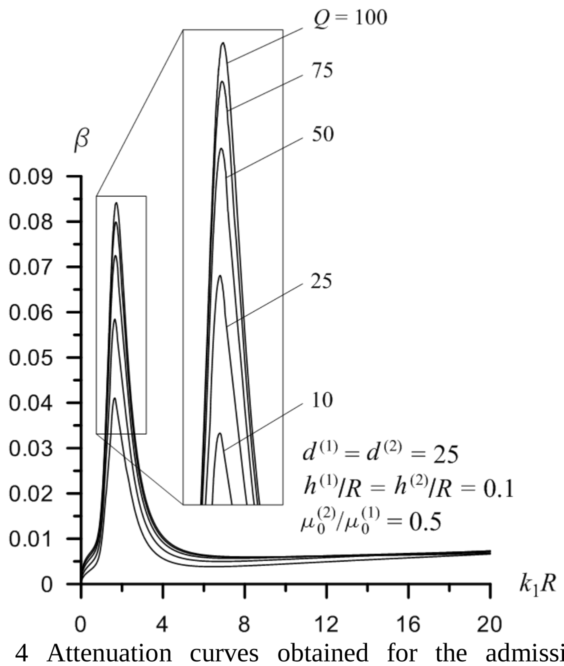

Fig. 4 Attenuation curves obtained for the admissible dispersion curve indicated by number 2 in Fig. 2 for various values of the rheological parameter Q

Under theoretical investigations of the considered type of problems, the selection of the concrete material properties and sizes of the constituents limits the application field of the obtained results. For instance, the results obtained in the case where $\mu_0^{(2)} \big/ \mu_0^{(1)} = 0.5$ relate to any pair of materials, the ratio of the instantaneous modulus elasticity of which is equal to 0.5 (we assume that the Poisson ratios of the materials are equal to each other and to 0.3). Moreover, the results obtained in the case where $h^{(1)}/R=h^{(2)}/R=0.1$ relate simultaneously to all the cases where the thickness of the cylinder layers is equal to each other and the ratio of this thickness to the radius R as shown in Fig. 1 is equal to 0.1. However, in connection with this,

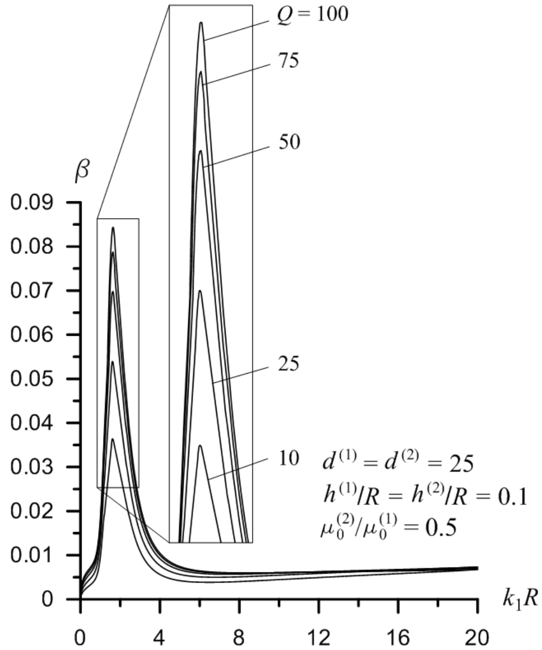

Fig. 5 Attenuation curves obtained for the admissible dispersion curve indicated by number 3 in Fig. 2 for various values of the rheological parameter Q

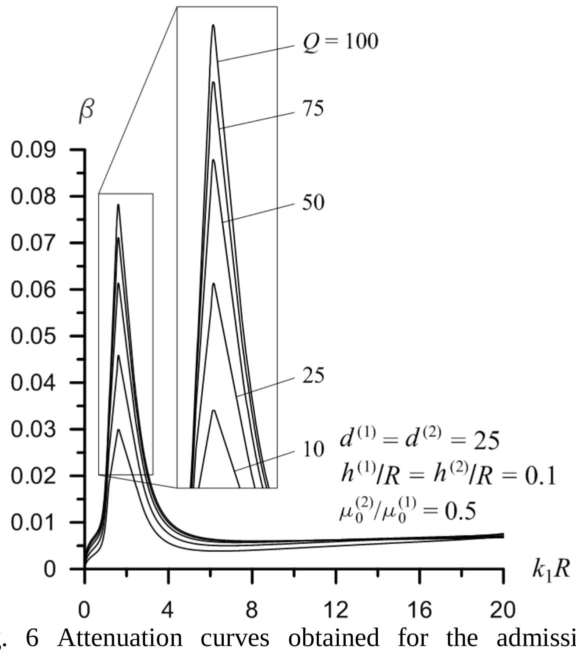

Fig. 6 Attenuation curves obtained for the admissible dispersion curve indicated by number 4 in Fig. 2 for various values of the rheological parameter Q

the concrete values of the material properties and sizes of the constituents are not indicated in Fig. 2 or in other figures given below or in any illustrated obtained numerical results.

4.5 Attenuation curves and their discussions

As above we assume that $\mu^{(2)}/\mu^{(1)}=0.5$, $h^{(1)}/R=h^{(2)}/R=0.1$, $\alpha^{(1)}=\alpha^{(2)}=0.5$ and $Q^{(1)}=Q^{(2)}(=Q)$, and consider the attenuation curves obtained by employing the foregoing solution algorithm. These curves are given in Figs. 3, 4, 5, 6 and 7 which are constructed under various values of the parameter Q for the dispersion curves indicated by numbers 1, 2, 3, 4 and 5 in Fig. 2, respectively.

First of all, we note that throughout the present paper under “wave propagation” we mean “guided wave propagation” only and the attenuation is estimated by the parameter β which is determined through the expression (22). According to the expressions in (6), this parameter characterizes a decrease of the wave amplitude under consideration with the wave propagation distance.

Moreover, we note that the attenuation curves are constructed as follows: first, the wave propagation velocity c is selected from the corresponding admissible dispersion curves given in Fig. 2 for the selected value of k_1R and the unknown β is determined from the solution to the dispersion Eq. (34). Thus, according to this procedure, the function $\beta=\beta(k_1R)$ is determined numerically (or graphically), however, the possible function $c=c(k_1R)$ is given a priori.

The frequency related to the point selected on the arbitrary attenuation curves given in Figs. 3-7 is determined as follows. For simplicity we consider the points on the attenuation curves given in Fig. 3 which relate to the dispersion curve 1 in Fig. 2. Thus, as an example, we consider the point on the attenuation curve obtained in the case where Q=100 under k_1R=0.2. After this selection, we again turn to the dispersion curve 1 in Fig. 2 and from this we find the wave propagation velocity related to k_1R=0.2. Multiplying this velocity with k_1R=0.2 we obtain the frequency related to the selected point on the attenuation curve. It is evident that this frequency also relates to the point on each attenuation curve in Fig. 3 which corresponds to the case where k_1R=0.2. In a similar manner, the frequency related to an arbitrary point on the attenuation curves can be determined.

We recall that the attenuation curves given in Figs. 3-7 are obtained in the case where $d^{(1)}=d^{(2)}=25$. For illustration of the influence of the rheological parameter $d^{(1)}$ ($=d^{(2)}$) on the attenuation curves, these curves are also constructed for the cases where $d^{(1)}=d^{(2)}=35$ and $d^{(1)}=d^{(2)}=50$. The dispersion curves related to the case where $d^{(1)}=d^{(2)}=35$ are given in Fig. 8 and the corresponding attenuation curves are given in Figs. 9, 10, 11, 12 and 13 which are also constructed under various values of the rheological parameter Q for the dispersion curves indicated by the numbers 1, 2, 3, 4 and 5 in Fig. 8, respectively.

However, the dispersion curves related to the case where $d^{(1)}=d^{(2)}=50$ are given in Fig. 14 and the corresponding attenuation curves are given in Figs. 15, 16, 17, 18 and 19 which are also constructed under various values of the rheological parameter Q for the dispersion curves indicated by the numbers 1, 2, 3, 4 and 5 in Fig. 14, respectively.

Note that in the foregoing graphs for clarity of the illustration, in general, the cases where $10\le Q\le 100$ are considered. However, under obtaining the results given Figs. 3, 9 and 15 it is assumed that $10\le Q\le 2000$ to show the character of the dependence between the maximum of β (denote it by $\max\{\beta\}$) and the parameter Q. According to these results, we can conclude that the dependence between $\max\{\beta\}$ and the rheological parameter Q is non-monotonic. In other words, we can conclude that up to a certain value of the rheological parameter Q (denote it by Q^*) before which, i.e., under $Q\le Q^*$ an increase in the values of Q causes an increase in the values of $\max\{\beta\}$, but after which, i.e., under $Q>Q^*$ an increase in the values of Q causes a decrease in the values of $\max\{\beta\}$. The results given in Figs. 3, 9 and 15 also show that the values of Q^* increase with the rheological parameter $d^{(1)}$ ($=d^{(2)}$)

Also, it follows from the foregoing results that as has been predicted, the limit case $\beta\to 0$ takes place as $k_1R\to 0$. It should be noted that $\beta\to 0$ as $k_1R\to\infty$, although this result does not clearly follow from the foregoing results (because here, for convenience, only the cases where $k_1R\le 20$ are considered). However, a further increase in the values of k_1R causes β to approach zero. The foregoing results also show that the values of the attenuation coefficient β become

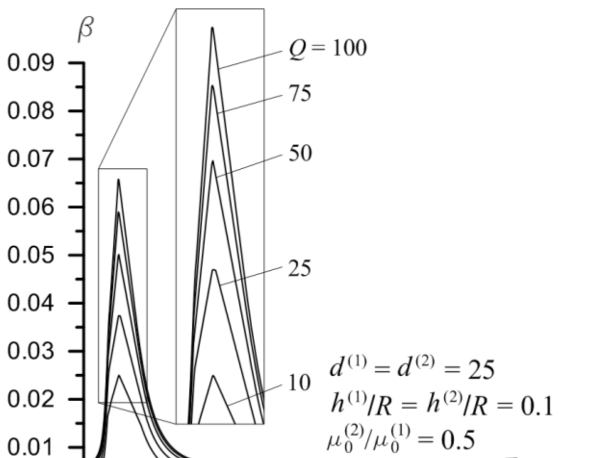

Fig. 7 Attenuation curves obtained for the admissible dispersion curve indicated by number 5 in Fig. 2 for various values of the rheological parameter Q

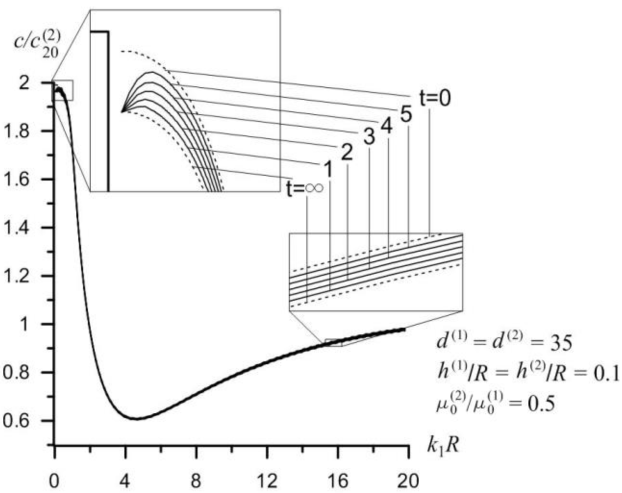

Fig. 8 Selected dispersion curves (solid lines) and dispersion curves (dashed lines) obtained under instantaneous (i.e., under t=0) and long-term (i.e., under $t=\infty$) values of the elastic constants for the case where $d^{(1)}=d^{(2)}=35$

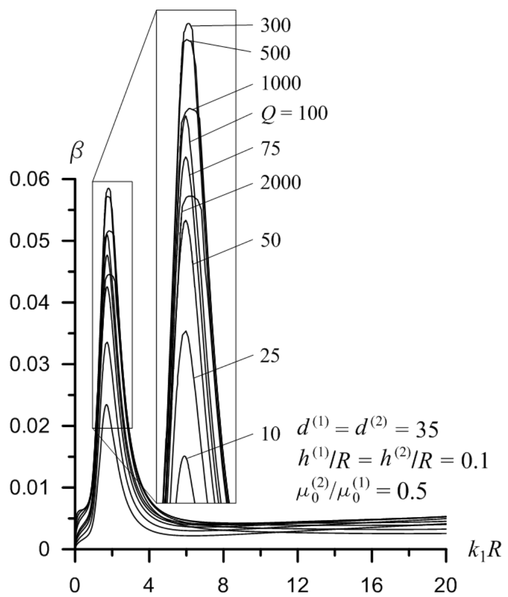

Fig. 9 Attenuation curves obtained for the admissible dispersion curve indicated by number 1 in Fig. 8 for various values of the rheological parameter *Q*

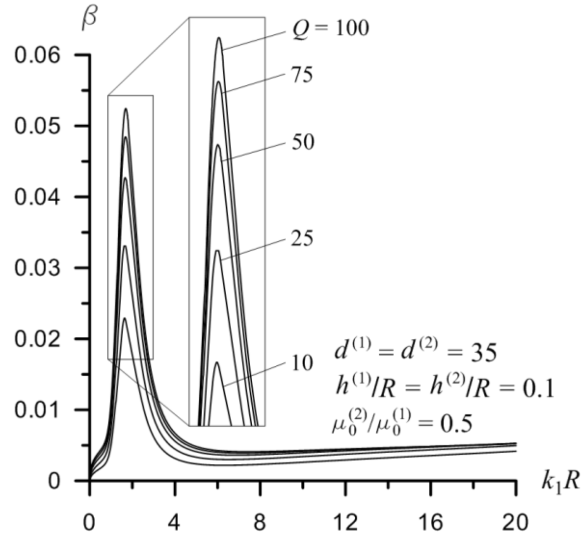

Fig. 10 Attenuation curves obtained for the admissible dispersion curve indicated by number 2 in Fig. 8 for various values of the rheological parameter *Q*

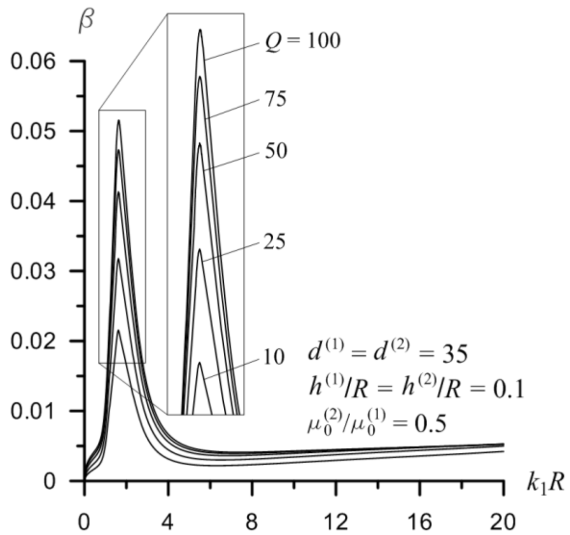

Fig. 11 Attenuation curves obtained for the admissible dispersion curve indicated by number 3 in Fig. 8 for various values of the rheological parameter *Q*

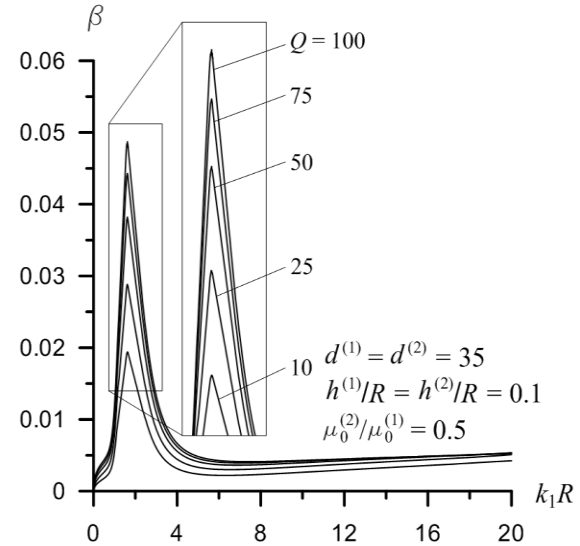

Fig. 12 Attenuation curves obtained for the admissible dispersion curve indicated by number 4 in Fig. 8 for various values of the rheological parameter *Q*

more considerable in the cases where $1.5<k_1R<2.5$. According to the numerical results we can conclude that $\max\{\beta\}$appears in the near vicinity of $k_1R=2$.

Comparison of the corresponding results given in Figs. 4-7, Figs. 9-13 and Figs. 15-19 with each other shows that the approach of these curves to the dispersion curves constructed for the purely elastic cases with instantaneous values of the elastic constants (we call this case the “instantaneous” case and note that in the figures this curve is indicated by t=0) and with long-term values of the elastic constants (we call this case the “long-term” case and note that in the figures this case is indicated by $t=\infty$), the values of $\max\{\beta\}$ decrease and are close to zero. This result agrees with the well-known mechanical consideration, according to which the attenuation coefficient of the purely elastic waves must be equal to zero.

Moreover, comparison of the results given in Figs. 4-7 with the corresponding results given in Figs. 9-13 and comparison of the latter ones with the corresponding results given in Figs. 15-19 show that an increase in the values of the rheological parameter $d^{(1)}$ $(=d^{(2)})$ causes a decrease in the values of $\max\{\beta\}$.

Now we attempt to make a comparison of the attenuation curves obtained in the present investigation and given in Figs. 4-7, 9-13 and 15-19 with the corresponding ones given in the works by Barshinger and Rose (2004), Hernando Quintanilla, Fan *et al*. (2015), Mazotti, Marzani *et al*. (2012). It is evident that this comparison can be made in the qualitative sense only, because the viscoelasticity models and concrete materials selected in these works differ from those selected in the present paper. Consequently,

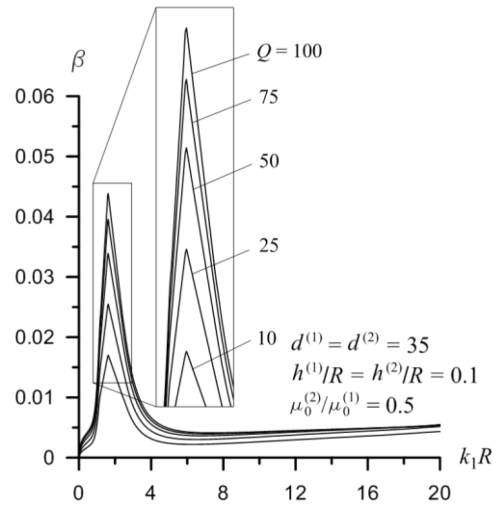

Fig. 13 Attenuation curves obtained for the admissible dispersion curve indicated by number 5 in Fig. 8 for various values of the rheological parameter Q

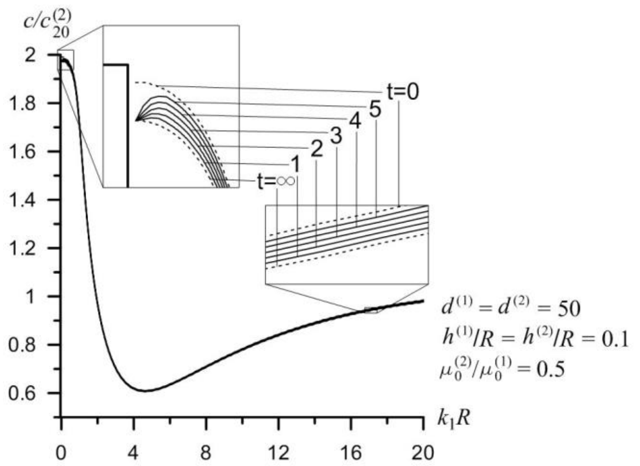

Fig. 14 Selected dispersion curves (solid lines) and dispersion curves (dashed lines) obtained under instantaneous (i.e. under t=0) and long-term (i.e. under t=∞) values of the elastic constants for the case where d(1)=d(2)=50

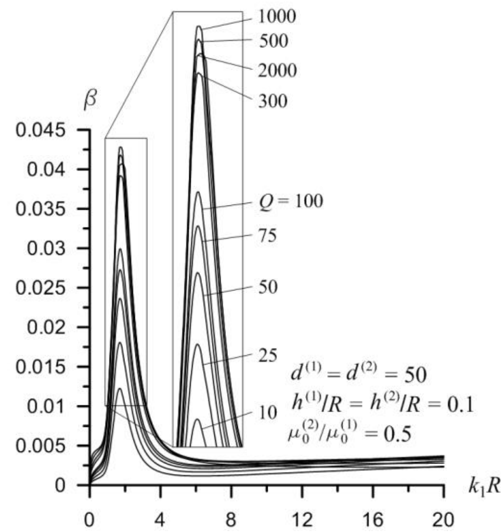

Fig. 15 Attenuation curves obtained for the admissible dispersion curve indicated by number 1 in Fig. 14 for various values of the rheological parameter Q

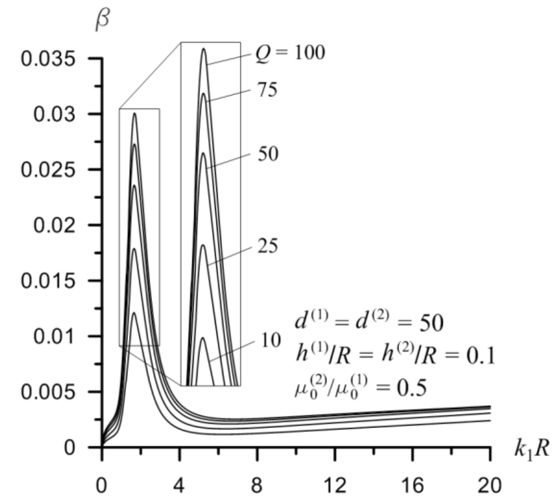

Fig. 16 Attenuation curves obtained for the admissible dispersion curve indicated by number 2 in Fig. 14 for various values of the rheological parameter Q

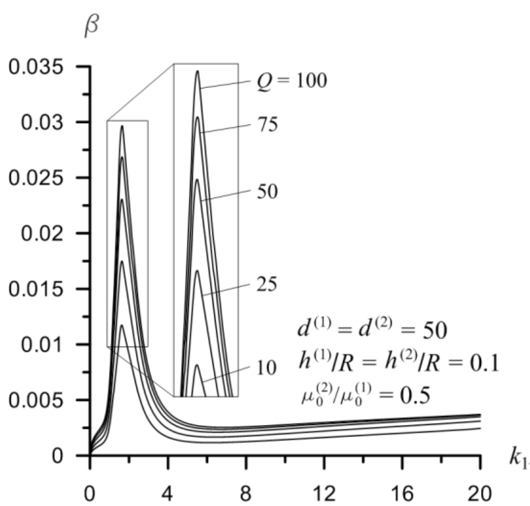

Fig. 17 Attenuation curves obtained for the admissible dispersion curve indicated by number 3 in Fig. 14 for various values of the rheological parameter Q

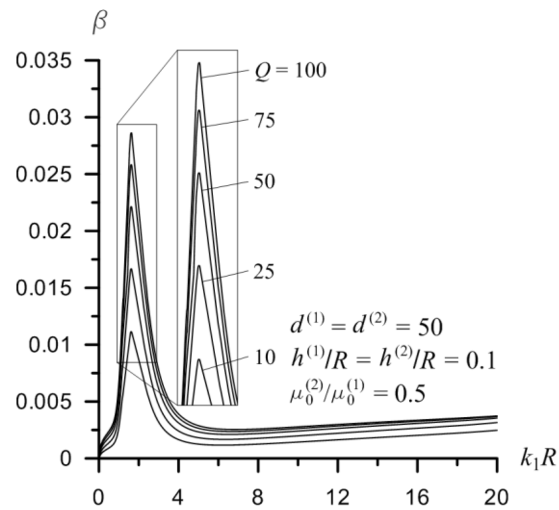

Fig. 18 Attenuation curves obtained for the admissible dispersion curve indicated by number 4 in Fig. 14 for various values of the rheological parameter Q

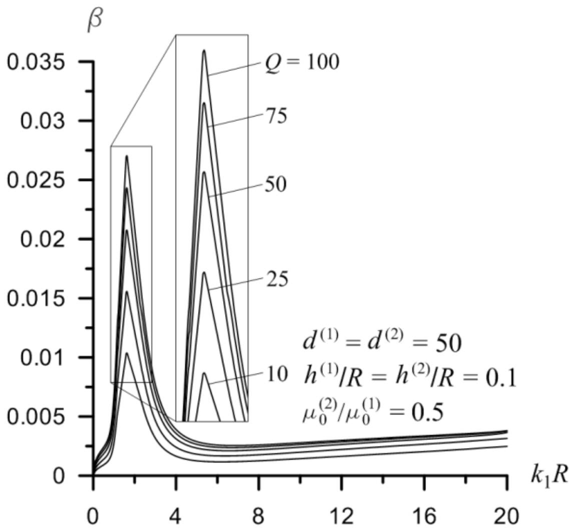

Fig. 19 Attenuation curves obtained for the admissible dispersion curve indicated by number 5 in Fig. 14 for various values of the rheological parameter Q

comparison can be made with respect to the character of the attenuation curves, i.e., with respect to the graphs of the dependence between β and k_1R (or frequency ω). It follows from the Figs. 4-7, 9-13 and 15-19 that each graph obtained for the dependence between β and k_1R (or ω) has a certain maximum under a certain value of β and k_1R (or ω). Observation of the attenuation curves given in the aforementioned works shows that these curves also have such character. Namely, this situation can be taken as validation of the obtained numerical results in the qualitative sense.

We attempt to explain why the aforementioned character of the attenuation curves appears. This explanation can be based on the behavior of the viscoelastic system under lowest and highest frequencies, so that under lower (highest) frequencies the behavior of the viscoelastic system is very near to the corresponding elastic system with long-term (instantaneous) values of the elastic constants. Therefore, under lowest (highest) frequencies or lowest and highest values of k_1R, attenuation must decrease with decreasing (increasing) either the frequency or the values of the dimensionless wavenumber k_1R. Consequently, according to this statement, the attenuation must have its maximum under moderate values of the frequency or of the dimensionless wavenumber k_1R. Namely, this property of the attenuation is a cause of the character of the dependencies between β and k_1R which is observed in all investigations related to this question.

This completes the discussions of the numerical results.

5. Conclusions

Thus, in the present paper an approach is proposed and applied for determination of the longitudinal axisymmetric wave attenuation in the bi-layered hollow cylinder made of viscoelastic materials. This approach is based on selection of the admissible dispersion curves of the considered waves which can propagate in the viscoelastic bi-layered cylinder under consideration. The investigations are made within the scope of the exact equations of motion of the theory of linear viscoelasticity. The fractional exponential operators by Rabotnov (1980) are employed for describing the constitutive relations of the constituents of the cylinder. Dimensionless rheological parameters are introduced and through these parameters the influence of the viscosity of the cylinder's materials on the attenuation curves is studied. The numerical results related to these curves are presented and discussed. According to these results, the following concrete conclusions can be drawn:

- For the considered values of the problem parameters, attenuation of the axisymmetric waves becomes more significant in the cases where $1.5<k_1R<2.5$;
- Maximal values of the attenuation coefficient β appear approximately at $k_1R\approx 2.0$;
- There exists such a value of the rheological parameter Q (denoted by $Q*$) under which the attenuation coefficient max{β} has its maximum, however, in the cases where $Q>Q*$ (in the cases where $Q<Q*$) an increase (a decrease) in the values of Q causes a decrease in max{β};
- The values of $Q*$ increase with the rheological parameter $d^{(1)}$ ($=d^{(2)}$);
- At first max{β} increases as the "distance" between the admissible dispersion curve and the "long-term" dispersion curve grows and this increase continues up to a certain "distance". After this "distance" max{β} decreases with the approaching of the admissible dispersion curve to the "instantaneous" dispersion curve;
- An increase in the values of the rheological parameter $d^{(1)}$ ($=d^{(2)}$) causes a decrease in the values of max{β}:
- The approach proposed in the present paper for the axisymmetric longitudinal waves can be employed for investigation of the attenuation of other types of waves, such as torsional and flexural waves in cylinders, Lamb waves in layers, near-surface waves in the layered half-space etc. made of viscoelastic materials;
- Using the theoretical results obtained by employing the proposed approach, attenuation of the waves through the rheological parameters of the viscoelastic material of the elements of constructions can be controlled;
- The character of the obtained attenuation curves agrees well with the corresponding ones obtained by other researchers and in this way validation of these curves is established in the qualitative sense.

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Appendix A

Here we give the expressions of the components of the matrix (β_{nm}), where n;m=1,2,…,8 the determinant of which enters the dispersion Eq. (21). These expressions are the following ones.

$$\beta_{11}\left(\zeta_2^{(2)},\chi_{2h^{(2)}}^{(2)}\right)=$$

$$(\Lambda^{(2)}(\omega)+M^{(2)}(\omega))\left(-\zeta_2^{(2)2}\,\frac{1}{2}\left(J_2(\chi_{2h^{(2)}}^{(2)})-J_0(\chi_{2h^{(2)}}^{(2)})\right)\right)+$$

$$\frac{M^{(2)}(\omega)}{\eta}\zeta_2^{(2)}J_1(\chi_{2h^{(2)}}^{(2)})+$$

$$\frac{M^{(2)}(\omega)}{\eta}\left(\beta_1^{(2)}\;\zeta_2^{(2)2}\;\left(J_2\;(\zeta_{2h^{(2)}}^{(2)})-J_0\;(\chi_{2h^{(2)}}^{(2)})\right)\right)$$

$$\left.-\frac{\zeta_2^{(2)}}{\eta}J_1\;(\chi_{2h^{(2)}}^{(2)})-\beta_2^{(2)}\;J_0\;(\chi_{2h^{(2)}}^{(2)})\right),$$

$$\beta_{21}\left(\zeta_2^{(2)},\chi_{2h^{(2)}}^{(2)}\right)=-M(\omega)\zeta_2^{(2)}J_1\left(\chi_{2h^{(2)}}^{(2)}\right)+$$

$$\frac{M(\omega)}{4}\left(\beta_1^{(2)}\left(\zeta_2^{(2)3}\left(3J_1(\chi_{2h^{(2)}}^{(2)})-J_3(\chi_{2h^{(2)}}^{(2)})\right)+\frac{\zeta_2^{(2)}}{\eta^2}J_1(\chi_{2h^{(2)}}^{(2)})+\right.\right.$$

$$\left.\frac{\zeta_2^{(2)2}}{2\eta}\left(J_2(\chi_{2h^{(2)}}^{(2)})-J_0(\chi_{2h^{(2)}}^{(2)})\right)+\beta_2^{(2)}\zeta_2^{(2)}J_1\;(\chi_{2h^{(2)}}^{(2)})\right),$$

$$\beta_{13}\left(\zeta_2^{(2)},\chi_{2h^{(2)}}^{(2)}\right)=$$

$$(\Lambda^{(2)}(\omega)+2M^{(2)}(\omega))\left(-\zeta_2^{(2)2}\,\frac{1}{2}\left(Y_2(\chi_{2h^{(2)}}^{(2)})-Y_0(\chi_{2h^{(2)}}^{(2)})\right)\right)+$$

$$\frac{M^{(2)}(\omega)}{\eta}\zeta_2^{(1)}Y_1(\chi_{2h^{(2)}}^{(2)})+\frac{\Lambda^{(2)}(\omega)}{2}\times$$

$$\left(\beta_1^{(2)}\;\zeta_2^{(2)2}\left(Y_2\;(\chi_{2h^{(2)}}^{(2)})-Y_0\;(\chi_{2h^{(2)}}^{(2)})\right)-\frac{\zeta_2^{(2)}}{\eta}Y_1(\chi_{2h^{(2)}}^{(2)})-\beta_2^{(2)}Y_0(\chi_{2h^{(2)}}^{(2)})\right),$$

$$\beta_{23}\left(\zeta_2^{(2)},\chi_{2h^{(2)}}^{(2)}\right)=-M^{(2)}(\omega)\zeta_2^{(2)}Y_1\left(\chi_{2h^{(2)}}^{(2)}\right)+$$

$$\frac{M^{(2)}(\omega)}{4}\left(\beta_1^{(2)}\left(\zeta_2^{(2)3}\left(3Y_1(\chi_{2h^{(2)}}^{(2)})-Y_3\;(\chi_{2h^{(2)}}^{(2)})\right)+\right.\right.$$

$$\frac{\zeta_2^{(2)}}{\eta^2}Y_1\;(\chi_{2h^{(2)}}^{(2)})+$$

$$\left.\frac{\zeta_2^{(2)2}}{2\eta}\left(Y_2\;(\chi_{2h^{(2)}}^{(2)})-Y_0\;(\chi_{2h^{(2)}}^{2})\right)+\beta_2^{(2)}\zeta_2^{(2)}Y_1\;(\chi_{2h^{(2)}}^{(2)})\right),$$

$$\beta_{n2}=\beta_{n1}\left(\zeta_3^{(2)},\chi_{3h^{(2)}}^{(2)}\right),\;\beta_{n4}=\beta_{n3}\left(\zeta_3^{(2)},\chi_{3h^{(2)}}^{(2)}\right),$$

$$\beta_{n5}=\beta_{n6}=\beta_{n7}=\beta_{n8}=0\,,\;n=1,2\,,$$

$$\beta_{31}\;\zeta_2^{(2)},\chi_2^{(2)}=$$

$$(\Lambda^{(2)}(\omega)+2M^{(2)}(\omega))\left(-\zeta_2^{(2)2}\,\frac{1}{2}J_2(\chi_2^{(2)})-J_0(\chi_2^{(2)})\right)+$$

$$\frac{\Lambda^{(2)}(\omega)}{\eta}\zeta_2^{(2)}J_1(\chi_2^{(2)})+\frac{\Lambda^{(2)}(\omega)}{2}\times$$

$$\left(\beta_1^{(2)}\ \zeta_2^{(2)\ 2}\ J_2\ \chi_2^{(2)}-J_0\ \chi_2^{(2)}-\frac{\zeta_2^{(2)}}{\eta}J_1\,\chi_2^{(2)}-\beta_2^{(2)}J_0\,\chi_2^{(2)}\right),$$

$$\beta_{41}(\zeta_2^{(2)},\chi_2^{(2)})=-M^{(2)}(\omega)\zeta_2^{(2)}J_1(\chi_2^{(2)})+$$

$$\frac{M^{(2)}(\omega)}{4}\Bigg(\beta_1^{(2)}\Big(\zeta_2^{(2)^3}\ (3J_1(\chi_2^{(2)})-J_3(\chi_2^{(2)}))+$$

$$\frac{\zeta_2^{(2)}}{\eta^2}J_1\ (\chi_2^{(2)})+$$

$$\frac{\zeta_2^{(2)^2}}{2\eta}\Big(J_2\ (\chi_2^{(2)})-J_0\ (\chi_2^{2}))+\beta_2^{(2)}\zeta_2^{(2)}J_1\ (\chi_2^{(2)})\Big),$$

$$\beta_{51}\left(\zeta_2^{(2)},\chi_2^{(2)}\right)=-\zeta_2^{(2)}J_1\left(\chi_2^{(2)}\right),$$

$$\beta_{61}(\zeta_2^{(2)},\chi_2^{(2)})=\left(-\beta_1^{(2)}\ \zeta_2^{(2)^2}-\beta_2^{(2)}\right)J_0\ (\chi_2^{(2)}),$$

$$\beta_{33}(\zeta_2^{(2)},\chi_2^{(2)})=$$

$$(\Lambda^{(2)}(\omega)+2M^{(2)}(\omega))\left(-\zeta_2^{(2)\ 2}\frac{1}{2}(Y_2(\chi_2^{(2)})-Y_0(\chi_2^{(2)}))\right)+$$

$$\frac{\Lambda^{(2)}(\omega)}{\eta}\zeta_2^{(2)}Y_1(\chi_2^{(2)})+\frac{\Lambda^{(2)}(\omega)}{2}\times$$

$$\left(\beta_1^{(2)}\ \zeta_2^{(2)\ 2}\ (Y_2\ (\chi_2^{(2)})-Y_0\ (\chi_2^{(2)}))-\frac{\zeta_2^{(2)}}{\eta}Y_1(\chi_2^{(2)})-\beta_2^{(2)}Y_0(\chi_2^{(2)})\right),$$

$$\beta_{43}(\zeta_2^{(2)},\chi_2^{(2)})=-M^{(2)}(\omega)\ \zeta_2^{(2)^2}Y_1(\chi_2^{(2)})+$$

$$\frac{M^{(2)}(\omega)}{4}\Big(\beta_1^{(2)}\Big(\zeta_2^{(2)\ 3}\ (3Y_1(\chi_2^{(2)})-Y_3\ (\chi_2^{(2)}))+$$

$$\frac{\zeta_2^{(2)}}{\eta^2}Y_1\ (\chi_2^{(2)})+$$

$$\frac{\zeta_2^{(2)^2}}{2\eta}\Big(Y_2\ (\chi_2^{(2)})-Y_0\ (\chi_2^{(2)}))+\beta_2^{(2)}\zeta_2^{(2)}Y_1\ (\chi_2^{(2)})\Big),$$

$$\beta_{53}\left(\zeta_2^{(2)},\chi_2^{(2)}\right)=-\zeta_2^{(2)}Y_1\left(\chi_2^{(2)}\right),$$

$$\beta_{63}(\zeta_2^{(2)},\chi_2^{(2)})=\left(-\beta_1^{(2)}\ \zeta_2^{(2)^2}-\beta_2^{(2)}\right)Y_0\ (\chi_2^{(2)}),$$

$$\beta_{n2}=\beta_{n1}\left(\zeta_3^{(2)},\chi_3^{(2)}\right),\ \beta_{n4}=\beta_{n3}\left(\zeta_3^{(2)},\chi_3^{(2)}\right),\ n=3,4,5,6\ ,$$

$$\beta_{35}(\zeta_2^{(2)},\chi_2^{(2)})=$$

$$(\Lambda^{(1)}(\omega)+2M^{(1)}(\omega))\left(-\zeta_2^{(1)2}\ \frac{1}{2}(J_2(\chi_2^{(1)})-J_0(\chi_2^{(1)}))\right)+$$

$$\frac{\Lambda^{(1)}(\omega)}{\eta}\zeta_2^{(1)}J_1(\chi_2^{(1)})+\frac{\Lambda^{(1)}(\omega)}{2}\times$$

$$\left(\beta_1^{(1)}\ \zeta_2^{(1)\ 2}(J_2\ (\chi_2^{(1)})-J_0\ (\chi_2^{(1)}))-\frac{\zeta_2^{(1)}}{\eta}J_1(\chi_2^{(1)})-\beta_2^{(1)}J_0(\chi_2^{(1)})\right),$$

$$\beta_{37}(\zeta_2^{(2)},\chi_2^{(2)})=$$

$$(\Lambda^{(1)}(\omega)+2M^{(1)}(\omega))\left(-\zeta_2^{(1)2}\ \frac{1}{2}(Y_2(\chi_2^{(1)})-Y_0(\chi_2^{(1)}))\right)+$$

$$\frac{\Lambda^{(1)}(\omega)}{\eta}\zeta_2^{(1)}Y_1(\chi_2^{(1)})+\frac{\Lambda^{(1)}(\omega)}{2}\times$$

$$\left(\beta_1^{(1)}\ \zeta_2^{(1)\ 2}(Y_2\ (\chi_2^{(1)})-Y_0\ (\chi_2^{(1)}))-\frac{\zeta_2^{(1)}}{\eta}Y_1(\chi_2^{(1)})-\beta_2^{(1)}Y_0(\chi_2^{(1)})\right),$$

$$\beta_{45}(\zeta_2^{(1)},\chi_2^{(1)})=-M^{(2)}(\omega)\ \zeta_2^{(1)}J_1(\chi_2^{(1)})+$$

$$\frac{M^{(1)}(\omega)}{4}\Bigg(\beta_1^{(1)}\Big(\zeta_2^{(1)^3}\ (3J_1(\chi_2^{(1)})-J_3\ (\chi_2^{(1)}))+$$

$$\frac{\zeta_2^{(1)}}{\eta^2}J_1\ (\chi_2^{(1)})+$$

$$\frac{\zeta_2^{(1)^2}}{2\eta}(J_2\ (\chi_2^{(1)})-J_0\ (\chi_2^{(1)}))+\beta_2^{(2)}\zeta_2^{(1)}Y_1\ (\chi_2^{(1)})\Big),$$

$$\beta_{47}\left(\zeta_2^{(1)},\chi_2^{(1)}\right)=-M^{(1)}(\omega)s_2^{(1)}Y_1\left(\chi_2^{(1)}\right)+\ ,$$

$$\frac{M^{(1)}(\omega)}{4}\Big(\beta_1^{(1)}\Big(\zeta_2^{(1)\ 3}\ \Big(3Y_1(\chi_2^{(1)})-Y_3\ (\chi_2^{(1)})\Big)+$$

$$\frac{\zeta_2^{(1)}}{\eta^2}Y_1\ (\chi_2^{(1)})+\frac{\zeta_2^{(1)2}}{\eta^2}\Big(Y_2\ (\chi_2^{(1)})-Y_0\ (\chi_2^{(1)}))+\beta_2^{(1)}\zeta_2^{(1)}Y_1\ (\chi_2^{(1)})\Big),$$

$$\beta_{36}=\beta_{35}\left(\zeta_3^{(1)},\chi_3^{(1)}\right),\ \beta_{38}=\beta_{37}\left(\zeta_3^{(1)},\chi_3^{(1)}\right),$$

$$\beta_{46}=\beta_{45}\left(\zeta_3^{(1)},\chi_3^{(1)}\right),\ \beta_{48}=\beta_{47}\left(\zeta_3^{(1)},\chi_3^{(1)}\right),$$

$$\beta_{55}\left(\zeta_2^{(1)},\chi_2^{(1)}\right)=-\zeta_2^{(1)}J_1\left(\chi_2^{(1)}\right),$$

$$\beta_{65}(\zeta_2^{(1)},\chi_2^{(1)})=\left(-\beta_1^{(1)}\ \zeta_2^{(1)^2}-\beta_2^{(1)}J_0\ (\chi_2^{(1)})\right),$$

$$\beta_{57}\left(\zeta_2^{(1)},\chi_2^{(1)}\right)=-\zeta_2^{(1)}Y_1\left(\chi_2^{(1)}\right),$$

$$\beta_{67}(\zeta_2^{(1)},\chi_2^{(1)})=\left(-\beta_1^{(1)}\ \zeta_2^{(1)^2}-\beta_2^{(1)}\right)Y_0\ (\chi_2^{(1)}),$$

$$\beta_{56}=\beta_{55}\left(\zeta_3^{(1)},\chi_3^{(1)}\right),\ \beta_{58}=\beta_{57}\left(\zeta_3^{(1)},\chi_3^{(1)}\right),$$

$$\beta_{66}=\beta_{65}\left(\zeta_3^{(1)},\chi_3^{(1)}\right),\ \beta_{68}=\beta_{67}\left(\zeta_3^{(1)},\chi_3^{(1)}\right),$$

$$\beta_{75}(\zeta_2^{(1)},\chi_{2h^{(1)}}^{(1)})=$$

$$(\Lambda^{(1)}(\omega)+2M^{(1)}(\omega))\left(-\zeta_2^{(1)^2}\ \frac{1}{2}\Big(J_2(\chi_{2h^{(1)}}^{(1)})-J_0(\chi_{2h^{(1)}}^{(1)})\Big)\right)+$$

$$\frac{\Lambda^{(1)}(\omega)}{\eta}\zeta_2^{(1)}J_1(\chi_{2h^{(1)}}^{(1)})+\frac{\Lambda^{(1)}(\omega)}{2}\times$$

$$\left(\beta_1^{(1)}\ \zeta_2^{(1)\ 2}\Big(J_2\ (\chi_{2h^{(1)}}^{(2)})-J_0\ (\chi_{2h^{(1)}}^{(1)})\Big)-\frac{s_2^{(1)}}{\eta}J_1(\chi_{2h^{(1)}}^{(1)})-\beta_2^{(1)}J_0(\chi_{2h^{(1)}}^{(1)})\right),$$

$$\beta_{85}\left(\zeta_2^{(1)},\chi_{2h^{(1)}}^{(1)}\right)=-M^{(1)}(\omega)\zeta_2^{(1)}J_1\left(\chi_{2h^{(1)}}^{(1)}\right)+$$

$$\frac{M^{(1)}(\omega)}{4}\Big(\beta_1^{(1)}\Big(\zeta_2^{(1)\ 3}\ \Big(3J_1(\chi_{2h^{(1)}}^{(1)})-J_3\ (\chi_{2h^{(1)}}^{(1)})\Big)+$$

$$\frac{\zeta_2^{(1)}}{\eta^2}J_1(\chi_{2h^{(1)}}^{(1)})+$$

$$\frac{\zeta_2^{(1)^2}}{2\eta}\Big(J_2\ (\chi_{2h^{(1)}}^{(1)})-J_0\ (\chi_{2h^{(1)}}^{(1)})\Big)+\beta_2^{(1)}\zeta_2^{(1)}J_1\ (\chi_{2h^{(1)}}^{(1)})\Big),$$

$$\beta_{77}\ \zeta_2^{(1)},\chi_{2h^{(1)}}^{(1)}\ =$$

$$(\Lambda^{(1)}(\omega)+2M^{(1)}(\omega))\left(-\zeta_2^{(1)^2}\ \frac{1}{2}(Y_2\ (\chi_2^{(1)})-Y_0\ (\chi_{2h^{(1)}}^{(1)}))\right)+$$

$$\frac{\Lambda^{(1)}(\omega)}{\eta}\zeta_2^{(1)}Y_1(\chi_{2h^{(1)}}^{(1)})+\frac{\Lambda^{(1)}(\omega)}{2}\times$$

$$\left(\beta_1^{(2)}\ \zeta_2^{(2)^2}\ \Big(Y_2\ (\chi_{2h^{(2)}}^{(2)})-Y_0\ (\chi_{2h^{(2)}}^{(2)})\Big)-\frac{\zeta_2^{(2)}}{\eta}Y_1(\chi_{2h^{(2)}}^{(2)})-\beta_2^{(2)}Y_0\ (\chi_{2h^{(2)}}^{(2)})\right),$$

$$\beta_{87}\left(\zeta_2^{(1)},\chi_{2h^{(1)}}^{(1)}\right)=-M^{(1)}(\omega)\zeta_2^{(1)}Y_1\left(\chi_{2h^{(1)}}^{(1)}\right)+$$

$$\frac{M^{(1)}(\omega)}{4}\Big(\beta_1^{(1)}\Big(\zeta_2^{(1)^3}\quad\Big(3Y_1(\chi_{2h^{(1)}}^{(1)})-Y_3\ (\chi_{2h^{(1)}}^{(1)})\Big)+$$

$$\frac{\zeta_2^{(1)}}{\eta^2}Y_1(\chi_{2h^{(1)}}^{(1)})+$$

$$\frac{\zeta_2^{(1)}{}^2}{2\eta}\Big(Y_2\ (\chi_{2h^{(1)}}^{(1)})-Y_0\ (\chi_{2h^{(1)}}^{(1)})\Big)+\beta_2^{(1)}\zeta_2^{(1)}Y_1\ (\chi_{2h^{(1)}}^{(1)})\Big),$$

$$\beta_{n6}=\beta_{n5}\left(\zeta_3^{(1)},\chi_{3h^{(1)}}^{(1)}\right),\ \beta_{n8}=\beta_{n7}\left(\zeta_3^{(1)},\chi_{3h^{(1)}}^{(1)}\right),$$

$$\beta_{n1}=\beta_{n2}=\beta_{n3}=\beta_{n4}=0,\ n=7,8. \tag{A1}$$

In relation (A1), $J_n(x)$ and $Y_n(x)$ are Bessel functions of the first and second kinds, respectively. Moreover in (A1) the following notation is used

$$\chi_2^{(n)}=kR\zeta_2^{(n)},\chi_3^{(n)}=kR\zeta_3^{(n)},n=1,2,$$

$$\chi_{2h^{(2)}}^{(2)}=kR\left(1-\frac{h^{(2)}}{R}\right)\zeta_2^{(2)},\chi_{3h^{(2)}}^{(2)}=kR\left(1-\frac{h^{(2)}}{R}\right)\zeta_3^{(2)},$$

$$\chi_{2h^{(1)}}^{(1)}=kR\left(1-\frac{h^{(1)}}{R}\right)\zeta_2^{(1)},\chi_{3h^{(1)}}^{(1)}=kR\left(1-\frac{h^{(1)}}{R}\right)\zeta_3^{(1)},$$

$$\beta_1^{(n)}=\frac{(\Lambda^{(n)}(\omega)+2M^{(n)}(\omega))}{(\Lambda^{(n)}(\omega)+M^{(n)}(\omega))},$$

$$\beta_2^{(n)}=\frac{M^{(n)}(\omega)}{(\Lambda^{(n)}(\omega)+M^{(n)}(\omega))}-\rho^{(n)}\left(\frac{\omega}{k}\right)^2(\Lambda^{(n)}(\omega)+M^{(n)}(\omega))^{-1}. \tag{A2}$$

This completes the consideration of the explicit expressions of the components of the matrix (β_{nm}).