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Post-buckling of cylindrical shells with spiral stiffeners under elastic foundation

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Abstract. In this paper, an analytical method for the Post-buckling response of cylindrical shells with spiral stiffeners surrounded by an elastic medium subjected to external pressure is presented. The proposed model is based on two parameters elastic foundation Winkler and Pasternak. The material properties of the shell and stiffeners are assumed to be continuously graded in the thickness direction. According to the Von Karman nonlinear equations and the classical plate theory of shells, strain-displacement relations are obtained. The smeared stiffeners technique and Galerkin method is used to solve the nonlinear problem. To valid the formulations, comparisons are made with the available solutions for nonlinear static buckling of stiffened homogeneous and un-stiffened FGM cylindrical shells. The obtained results show the elastic foundation Winkler on the response of buckling is more effective than the elastic foundation Pasternak. Also the ceramic shells buckling strength higher than the metal shells and minimum critical buckling load is occurred, when both of the stiffeners have angle of thirty degrees.

Keywords: FGM cylindrical shells; nonlinear analysis; spiral stiffener; post-buckling

1. Introduction

The eccentrically stiffened FGM cylindrical shells have more application in modern Engineering.

In many application, the shell under pressure and may be buckling. Therefore, research on nonlinear stability of these structures has been of interest to scientists. In fact, used to stiffeners with low weight to support the structures for bearing Load. Study on nonlinear behavior of these structures is important of the practical.

Van der Neut (1947) showed the importance of stiffeners in the buckling of isotropic cylindrical shell under axial load. A careful analysis of post-buckling behavior of eccentrically stiffened FGM thin circular cylindrical shells is surrounded by an elastic foundation and external pressure was presented by Shaterzadeh and Foroutan (2015). Dung and Nga (2013) studied the Post-buckling of eccentrically stiffened FGM cylindrical shells with elastic foundation under uniform external pressure. Shen, Zhou *et al*. (1993) studied the Buckling and post-buckling behavior of complete and incomplete eccentrically stiffened cylindrical shell under external pressure and axial compression by using boundary layer theory. The mechanical buckling of

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cylindrical shells with functional elastic Pasternak was presented by Bagherizadeh, Kiani *et al*. (2011). Li and Shen (2008) analyzed the three-dimensional post-buckling of composite cylindrical shell under external and axial pressure in the thermal environment. Sadeghifar, Bagheri *et al*. (2011) studied the buckling of laminated cylindrical shell with non-uniform stringer stiffeners based on Love's first order theory. Jiang, Wang *et al*. (2008) studied the buckling of eccentrically stiffened circular cylindrical panels under uniform axial compression by second order differential element method. Bich, Nam *et al*. (2011) analyzed the Nonlinear buckling of eccentrically stiffened functionally graded plates and shallow shells. Post-buckling of shear deformable FGM cylindrical shells surrounded by an elastic medium was presented by Shen (2009). Boroujerdy, Naj *et al*. (2014) studied the buckling of heated temperature dependent FGM cylindrical shell surrounded by elastic medium. Post-buckling of internal pressure loaded FGM cylindrical shells surrounded by an elastic medium was analyzed by Shen, Yang *et al*. (2010). Bagherizadeh, Kiani *et al*. (2012) analyzed the thermal buckling of functionally graded cylindrical shells on elastic foundation. Fan, Chen *et al*. (2015) studied the buckling of axial compressed cylindrical shells with stepwise variable thickness. Buckling analysis of filament wound composite cylindrical shell for considering the filament undulation and crossover presented by Guo, Han *et al*. (2015).

A review of studies shows that the studies on the analytical solution post-buckling of FGM cylindrical shells with spiral stiffeners with elastic foundation have not been done. In this paper, the Post-buckling analysis of FGM cylindrical shells with spiral stiffeners with elastic foundation under uniform external pressure studied. Suppose that stiffened FGM thin circular cylindrical shell is simply supported and subjected to uniformly distributed pressure. The material properties of the shell and stiffeners are assumed to be continuously graded in the thickness direction. The nonlinear equations using the classical plate theory, smeared stiffeners technique and Galerkin method, is obtained. The aim of the study is the finding the best arrangement of stiffeners for achieving the maximum of strength of buckling.

2. Formulation

2.1 FGM power law properties

In this paper, the structure is made of functionally graded materials that varying continuously through the thickness direction of shell. The inside and outside surfaces are ceramic and metal, respectively. Also the stiffeners attached to inside of the shell skin. The volume-fraction to be given by a power law (Shen 2003)

$$\begin{aligned} V_c &= V_c(z) = \left(\frac{2z+h}{2h}\right)^k \\ V_m &= V_m(z) = 1 - V_c(z) \end{aligned} \tag{1}$$

which h is the thickness of shell, $k \geq 0$ is the volume-fraction index, z is the thickness coordinate, footnotes c and m shows ceramic and metal respectively.

Effective properties ($\Pr_{eff}$) of FGM shells by linear combination law is as follows (Sofiyev 2011)

$$\Pr_{eff} = \Pr_m(z) V_m(z) + \Pr_c(z) V_c(z) \tag{2}$$

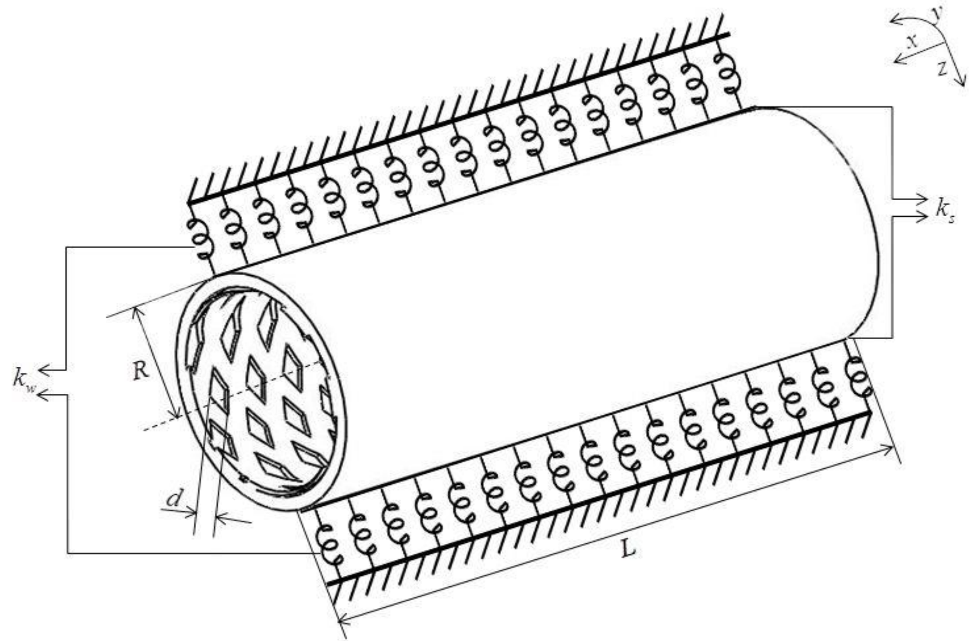


Fig. 1 Configuration of stiffened cylindrical shell surrounded with foundation

According to the mentioned law, The Young's modulus of the shell and stiffeners can be expressed in the following form

$$
\begin{aligned}
&E(z) = E_m V_m + E_c V_c = E_m + (E_c - E_m)\left(\frac{2z+h}{2h}\right)^k, \; -\frac{h}{2} \le z \le \frac{h}{2} \\
&E_s = E_c + (E_m - E_c)\left(\frac{2z-h}{2h_s}\right)^{k_2}, \; \frac{h}{2} \le z \le \frac{h}{2} + h_s
\end{aligned}
\tag{3}
$$

which E_m, E_c are the Young's modulus of the metal and ceramic, respectively, E_s is the Young's modulus of stiffeners, $k_2 \geq 0$ is the volume-fraction of stiffeners.

FGM cylindrical thin shell is assumed with length L, radius R, which is surrounded by an elastic foundation. Material properties of stiffeners are assumed FGM (Fig. 1). Original coordinates, x,y,z are in the axial, circumferential, and inward radial directions respectively.

The strains across the shell thickness at a distance z from the mid-surface are represented by

$$
\varepsilon_x = \varepsilon_x^0 - z\,\kappa_x, \; \varepsilon_y = \varepsilon_y^0 - z\,\kappa_y, \; \gamma_{xy} = \gamma_{xy}^0 - z\,\kappa_{xy} \tag{4}
$$

where $\varepsilon_x^0, \varepsilon_y^0$ are normal strains, γ_{xy}^0 is the shear strain at the mid-surface, $\chi_x, \chi_y, \chi_{xy}$ are the change of curvatures and twist of shell.

2.2 Displacement-strain-stress relations

According to the von Karman nonlinear strain-displacement relations (Brush and Almroth

1975) the strain components at the mid-surface of cylindrical shells as

$$\begin{aligned} \varepsilon_x^0 &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \varepsilon_y^0 &= \frac{\partial v}{\partial y} - \frac{w}{R} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \gamma_{xy}^0 &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y},\ \kappa_x = \frac{\partial^2 w}{\partial x^2},\ \kappa_y = \frac{\partial^2 w}{\partial y^2},\ \kappa_{xy} = 2\frac{\partial^2 w}{\partial x \partial y} \end{aligned} \tag{5}$$

where $u=u(x,y)$, $v=v(x,y)$, $w=w(x,y)$ are displacements along x,y,z axes, respectively.

According to Eq. (5), compatibility equation be expressed in the following form

$$\frac{\partial^2 \varepsilon_x^0}{\partial y^2} + \frac{\partial^2 \varepsilon_y^0}{\partial x^2} - \frac{\partial^2 \gamma_{xy}^0}{\partial x \partial y} = -\frac{1}{R}\frac{\partial^2 w}{\partial x^2} + \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \tag{6}$$

The stress-strain relationship for FGM cylindrical shells can be written as follows

$$\begin{aligned} \sigma_x^{sh} &= \frac{E(z)}{1-\nu^2}\left(\varepsilon_x + \nu\varepsilon_y\right) \\ \sigma_y^{sh} &= \frac{E(z)}{1-\nu^2}\left(\varepsilon_y + \nu\varepsilon_x\right) \\ \tau_{xy}^{sh} &= \frac{E(z)}{2(1+\nu)}\gamma_{xy} \end{aligned} \tag{7}$$

the Poisson’s ratio (v) is assumed to be constant, $\sigma_x^{sh}, \sigma_y^{sh}$ normal stress in x,y coordinates, respectively, τ_{xy}^{sh} is shear stress on the un-stiffened shell.

By rotation of the strain tensor, the stress-strain relations of the spiral stiffeners are obtained. With the transformation of strains from the xy -axis to the $1'2'$ -axis and $1''2''$ -axis (Fig. 2), Eqs.

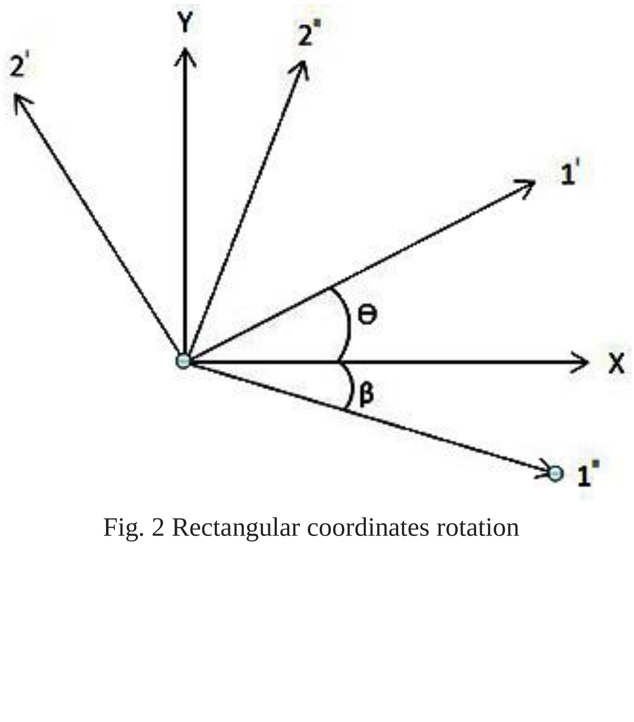


Fig. 2 Rectangular coordinates rotation

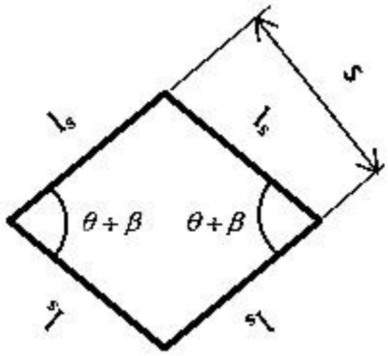


Fig. 3 View a rhombic stiffener grid

(8) and (9) can be made (Yen 1979).

$$\begin{aligned} \varepsilon_1^{'} &= \varepsilon_x \cos^2\theta + 2\gamma_{xy}\sin\theta\cos\theta + \varepsilon_y\sin^2\theta \\ \varepsilon_2^{'} &= \varepsilon_x \sin^2\theta - 2\gamma_{xy}\sin\theta\cos\theta + \varepsilon_y\cos^2\theta \end{aligned} \tag{8}$$

$$\begin{aligned} \varepsilon_1^{''} &= \varepsilon_x \cos^2\beta - 2\gamma_{xy}\sin\beta\cos\beta + \varepsilon_y\sin^2\beta \\ \varepsilon_2^{''} &= \varepsilon_x \sin^2\beta + 2\gamma_{xy}\sin\beta\cos\beta + \varepsilon_y\cos^2\beta \end{aligned} \tag{9}$$

According to the uniaxial Hooke's law

$$\varepsilon_1^{'} = \frac{P'}{hdE_s},\ \varepsilon_1^{''} = \frac{P''}{hdE_s} \tag{10}$$

where d is the width of stiffeners, p', p'' are stiffener loads in the $1'2'$ -axis and $1''2''$ -axis, respectively and θ, β are the angle of the stiffeners.

According to Fig. 3, the length of the stiffener grid is

$$l_s = \frac{s}{\sin(\theta+\beta)} \tag{11}$$

where s is the stiffener spacing.

The stress-strain relations for FGM spiral stiffeners a

$$\begin{aligned} \sigma_x^s &= \frac{P^{'}\cos\theta}{l_s h(\sin\theta+\sin\beta)} + \frac{P^{''}\cos\beta}{l_s h(\sin\theta+\sin\beta)} \\ &= \frac{P^{'}\cos\theta + P^{''}\cos\beta}{l_s h(\sin\theta+\sin\beta)} = \frac{h_s dE_s}{l_s h_s}\frac{\left(\varepsilon_1^{'}\cos\theta+\varepsilon_1^{''}\cos\beta\right)}{(\sin\theta+\sin\beta)} \\ &= \frac{h_s dE_s}{sh_s}\frac{\left(\varepsilon_1^{'}\cos\theta+\varepsilon_1^{''}\cos\beta\right)}{(\sin\theta+\sin\beta)}\sin(\theta+\beta) \\ &= Z_1\left[\varepsilon_x\left(\cos^3\theta+\cos^3\beta\right) + 2\gamma_{xy}\left(\sin\theta\cos^2\theta-\sin\beta\cos^2\beta\right) + \varepsilon_y\left(\sin^2\theta\cos\theta+\sin^2\beta\cos\beta\right)\right] \end{aligned} \tag{12}$$

where

$$Z_1 = \frac{h_s d E_s}{s h_s} \frac{\sin(\theta+\beta)}{(\sin\theta+\sin\beta)} \tag{13}$$

similarly

$$\begin{aligned}
\sigma_y^s &= \frac{h_s d E_s}{s h_s} \frac{(\varepsilon_1^{'}\sin\theta+\varepsilon_1^{''}\sin\beta)}{(\cos\theta+\cos\beta)}\sin(\theta+\beta) \\
&= Z_2\left[\varepsilon_x\left(\sin\theta\cos^2\theta+\sin\beta\cos^2\beta\right)+2\gamma_{xy}\left(\sin^2\theta\cos\theta-\sin^2\beta\cos\beta\right)+\varepsilon_y\left(\sin^3\theta+\sin^3\beta\right)\right] \\
\tau_{xy}^s &= \frac{h_s d E_s}{2 s h_s}\left(\varepsilon_1^{'}-\varepsilon_1^{''}\right)\sin(\theta+\beta) \\
&= Z_3\left[\varepsilon_x\left(\cos^2\theta-\cos^2\beta\right)+2\gamma_{xy}\left(\sin\theta\cos\theta+\sin\beta\cos\beta\right)+\varepsilon_y\left(\sin^2\theta-\sin^2\beta\right)\right]
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
Z_2 &= \frac{h_s d E_s}{s h_s} \frac{\sin(\theta+\beta)}{(\cos\theta+\cos\beta)} \\
Z_3 &= \frac{h_s d E_s}{2 s h_s}\sin(\theta+\beta)
\end{aligned} \tag{15}$$

σ_x^s, σ_y^s is the normal stress of stiffeners. τ_{xy}^s, h_s are shear stress and thickness of the stiffeners, respectively. s is length of the stiffener grid, θ, β are the spiral angle of stiffeners. To consider the effect of stiffeners on the shell used the smeared stiffeners technique. By integrating the stress-strain equations and calculating the resultant forces and moments for stiffened FGM cylindrical shells will be (Najafizadeh *et al*. 2009, Shen 1998)

$$\begin{aligned}
N_x &= B_{11}\varepsilon_x^0 + B_{12}\varepsilon_y^0 - B_{14}\kappa_x - B_{15}\kappa_y \\
N_y &= B_{21}\varepsilon_x^0 + B_{22}\varepsilon_y^0 - B_{24}\kappa_x - B_{25}\kappa_y \\
N_{xy} &= B_{33}\gamma_{xy}^0 - 2B_{36}\kappa_{xy}
\end{aligned} \tag{16}$$

$$\begin{aligned}
M_x &= B_{14}\varepsilon_x^0 + B_{15}\varepsilon_y^0 - B_{41}\kappa_x - B_{42}\kappa_y \\
M_y &= B_{24}\varepsilon_x^0 + B_{25}\varepsilon_y^0 - B_{51}\kappa_x - B_{52}\kappa_y \\
M_{xy} &= B_{36}\gamma_{xy}^0 - 2B_{63}\kappa_{xy}
\end{aligned} \tag{17}$$

B_{ij} are components of the extensional, bending and coupling stiffeners of FGM cylindrical shell. N_x, N_y, N_{xy} are in-plane normal force and shearing force intensities, respectively. M_x, M_y, M_{xy} are bending moment and twisting moment intensities, respectively.

$$\begin{aligned}
B_{11} &= \frac{E_1}{1-\nu^2} + Z_1 E_{1s}\left(\cos^3\theta+\cos^3\beta\right),\ B_{12} = \frac{E_1\nu}{1-\nu^2} + Z_1 E_{1s}\left(\sin^2\theta\cos\theta+\sin^2\beta\cos\beta\right) \\
B_{14} &= \frac{E_2}{1-\nu^2} + Z_1 E_{2s}\left(\cos^3\theta+\cos^3\beta\right),\ B_{15} = \frac{E_2\nu}{1-\nu^2} + Z_1 E_{2s}\left(\sin^2\theta\cos\theta+\sin^2\beta\cos\beta\right) \\
B_{21} &= \frac{E_1\nu}{1-\nu^2} + Z_2 E_{1s}\left(\sin\theta\cos^2\theta+\sin\beta\cos^2\beta\right),\ B_{22} = \frac{E_1}{1-\nu^2} + Z_2 E_{1s}\left(\sin^3\theta+\sin^3\beta\right)
\end{aligned}$$

$$B_{24}=\frac{E_2\nu}{1-\nu^2}+Z_2E_{2s}\left(\sin\theta\cos^2\theta+\sin\beta\cos^2\beta\right),\ B_{25}=\frac{E_2}{1-\nu^2}+Z_2E_{2s}\left(\sin^3\theta+\sin^3\beta\right)$$
$$B_{33}=\frac{E_1}{2(1+\nu)}+2Z_3E_{1s}\left(\sin\theta\cos\theta+\sin\beta\cos\beta\right),\ B_{36}=\frac{E_2}{2(1+\nu)}+2Z_3E_{2s}\left(\sin\theta\cos\theta+\sin\beta\cos\beta\right)$$
$$B_{41}=\frac{E_3}{1-\nu^2}+Z_1E_{3s}\left(\cos^3\theta+\cos^3\beta\right),\ B_{42}=\frac{E_3\nu}{1-\nu^2}+Z_1E_{3s}\left(\sin^2\theta\cos\theta+\sin^2\beta\cos\beta\right)$$
$$\begin{aligned}&B_{51}=\frac{E_3\nu}{1-\nu^2}+Z_2E_{3s}\left(\sin\theta\cos^2\theta+\sin\beta\cos^2\beta\right),\ B_{55}=\frac{E_3}{1-\nu^2}+Z_2E_{3s}\left(\sin^3\theta+\sin^3\beta\right)\\&B_{63}=\frac{E_3}{2(1+\nu)}+2Z_3E_{3s}\left(\sin\theta\cos\theta+\sin\beta\cos\beta\right)\end{aligned}\tag{18}$$

where

$$\begin{aligned}E_1&=\int_{-h/2}^{h/2}E_{sh}(z)\,dz=\left(E_m+\frac{E_c-E_m}{k+1}\right)h\\E_2&=\int_{-h/2}^{h/2}zE_{sh}(z)\,dz=\frac{(E_c-E_m)kh^2}{2(k+1)(k+2)}\\E_3&=\int_{-h/2}^{h/2}z^2E_{sh}(z)\,dz=\left[\frac{E_m}{12}+(E_c-E_m)\left(\frac{1}{k+3}-\frac{1}{k+2}+\frac{1}{4k+4}\right)\right]h^3\\E_{1s}&=\int_{h/2}^{h/2+h_s}E_s(z)\,dz=\left(E_c+\frac{E_m-E_c}{k_2+1}\right)h_s\\E_{2s}&=\int_{h/2}^{h/2+h_s}zE_s(z)\,dz=\frac{E_c}{2}hh_s\left(\frac{h_s}{h}+1\right)+(E_m-E_c)hh_s\left(\frac{1}{k_2+2}\frac{h_s}{h}+\frac{1}{2k_2+2}\right)\\E_{3s}&=\int_{h/2}^{h/2+h_s}z^2E_s(z)\,dz=\frac{E_c}{3}h_s^3\left(\frac{3}{4}\frac{h^2}{h_s^2}+\frac{3}{2}\frac{h}{h_s}+1\right)+(E_m-E_c)h_s^3\left(\frac{1}{k_2+3}+\frac{1}{k_2+2}\frac{h}{h_s}+\frac{1}{4(k_2+1)}\frac{h^2}{h_s^2}\right)\end{aligned}\tag{19}$$

Sort by Eq. (16) in terms of the strain as follows

$$\begin{aligned}\varepsilon_x^0&=A_{22}^*N_x-A_{12}^*N_y+B_{11}^*\kappa_x+B_{12}^*\kappa_y\\\varepsilon_y^0&=A_{11}^*N_y-A_{21}^*N_x+B_{21}^*\kappa_x+B_{22}^*\kappa_y\\\gamma_{xy}^0&=A_{33}^*+2B_{36}^*\kappa_{xy}\end{aligned}\tag{20}$$

where

$$\begin{aligned}&\Delta=A_{11}A_{22}-A_{12}A_{21},\ A_{22}^*=\frac{A_{22}}{\Delta},\ A_{12}^*=\frac{A_{12}}{\Delta}\\&A_{11}^*=\frac{A_{11}}{\Delta},\ A_{21}^*=\frac{A_{21}}{\Delta},\ A_{33}^*=\frac{1}{A_{33}},\ B_{36}^*=\frac{A_{36}}{A_{33}}\\&B_{11}^*=A_{22}^*A_{14}-A_{12}^*A_{24},\ B_{12}^*=A_{22}^*A_{15}-A_{12}^*A_{25}\\&B_{21}^*=A_{11}^*A_{24}-A_{21}^*A_{14},\ B_{22}^*=A_{11}^*A_{25}-A_{21}^*A_{15}\end{aligned}\tag{21}$$

Substituting Eq. (20) in Eq. (17) can be written

$$\begin{aligned}M_x&=B_{11}^{**}N_x+B_{21}^{**}N_y-D_{11}^*\kappa_x-D_{12}^*\kappa_y\\M_y&=B_{12}^{**}N_x+B_{22}^{**}N_y-D_{21}^*\kappa_x-D_{22}^*\kappa_y\\M_{xy}&=B_{36}^*N_{xy}-2D_{36}^*\kappa_{xy}\end{aligned}\tag{22}$$

where

$$\begin{aligned}
&B_{11}^{**}=A_{22}^{*}A_{14}-A_{21}^{*}A_{15},\ B_{21}^{**}=A_{11}^{*}A_{15}-A_{12}^{*}A_{14}\\
&B_{12}^{**}=A_{22}^{*}A_{24}-A_{21}^{*}A_{25},\ B_{22}^{**}=A_{11}^{*}A_{25}-A_{12}^{*}A_{24}\\
&D_{11}^{*}=B_{11}^{*}A_{14}-B_{21}^{*}A_{15}-A_{41},\ D_{12}^{*}=B_{12}^{*}A_{14}-B_{22}^{*}A_{15}-A_{42}\\
&D_{21}^{*}=B_{11}^{*}A_{24}-B_{21}^{*}A_{25}-A_{51},\ D_{22}^{*}=B_{12}^{*}A_{24}-B_{22}^{*}A_{25}-A_{52}\\
&D_{36}^{*}=A_{36}A_{36}^{*}-A_{63}
\end{aligned}\tag{23}$$

Non-linear equations thin circular cylindrical shell based on the classical shell theory follow as (Darabi, Darvizeh *et al*. 2008, Sofiyev and Schnack 2004, Ghiasian, Kiani *et al*. 2013)

$$\begin{aligned}
&\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=0\\
&\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}=0\\
&\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\,\partial y}+\frac{\partial^2 M_y}{\partial y^2}+N_x\frac{\partial^2 w}{\partial x^2}+2N_{xy}\frac{\partial^2 w}{\partial x\,\partial y}+N_y\frac{\partial^2 w}{\partial y^2}+\frac{1}{R}N_y+q_0-k_w w+k_s\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=0
\end{aligned}\tag{24}$$

In Eq. (24) k_w is Winkler foundation modulus, k_s is shear stiffness layer based on Pasternak, q_0 is external pressure.

According to the first two of Eq. (24), a stress function F may be defined as

$$N_x=\frac{\partial^2 F}{\partial y^2},\ N_y=\frac{\partial^2 F}{\partial x^2},\ N_{xy}=-\frac{\partial^2 F}{\partial x\,\partial y}\tag{25}$$

substituting Eq. (20) in to Eq. (6) and Eq. (22) in to the third of Eq. (24) and according to the Eq. (5) and (25)

$$\begin{aligned}
&A_{11}^{*}\frac{\partial^4 F}{\partial x^4}+\left(A_{33}^{*}-A_{12}^{*}+A_{21}^{*}\right)\frac{\partial^4 F}{\partial x^2\partial y^2}+A_{22}^{*}\frac{\partial^4 F}{\partial y^4}+B_{21}^{*}\frac{\partial^4 w}{\partial x^4}+\left(B_{11}^{*}+B_{22}^{*}-2B_{36}^{*}\right)\frac{\partial^4 w}{\partial x^2\partial y^2}\\
&\quad+B_{12}^{*}\frac{\partial^4 w}{\partial y^4}+\frac{1}{R}\frac{\partial^2 w}{\partial x^2}-\left[\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]=0
\end{aligned}\tag{26}$$

$$\begin{aligned}
&D_{11}^{*}\frac{\partial^4 w}{\partial x^4}+\left(D_{12}^{*}+D_{21}^{*}+4D_{36}^{*}\right)\frac{\partial^4 w}{\partial x^2\partial y^2}+D_{22}^{*}\frac{\partial^4 w}{\partial y^4}-B_{21}^{**}\frac{\partial^4 F}{\partial x^4}-\left(B_{11}^{**}+B_{22}^{**}-2B_{36}^{*}\right)\frac{\partial^4 F}{\partial x^2\partial y^2}-B_{12}^{**}\frac{\partial^4 F}{\partial y^4}\\
&\quad-\frac{1}{R}\frac{\partial^2 F}{\partial x^2}-\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2}+2\frac{\partial^2 F}{\partial x\,\partial y}\frac{\partial^2 w}{\partial x\,\partial y}-\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-q_0+k_w w-k_s\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=0
\end{aligned}\tag{27}$$

Eqs. (26) and (27) are a non-linear equation system in terms of two unknown parameter F and w.

3. Buckling analysis

Suppose the stiffened FGM cylindrical shell is simply supported. The deflection of cylindrical shells consider the three-term as follows (Huang and Han 2010, Volmir 1972)

$$w = f_0 + f_1 \sin\frac{m\pi x}{L}\sin\frac{ny}{R} + f_2 \sin^2\frac{m\pi x}{L} \tag{28}$$

in which f_0 is pre-buckling uniform unknown amplitude, f_1 is linear unknown amplitude, f_2 is nonlinear unknown amplitude, $\sin\frac{m\pi x}{L}\sin\frac{ny}{R}$, $\sin^2\frac{m\pi x}{L}$, m, n are linear buckling shape, nonlinear buckling shape, the number of half wave and full wave in the axial and circumferential direction, respectively. It should be noted that Eq. (28) does not satisfy the condition of problem. Volmir has stated that cylindrical shells are generally insensitive to this condition (Volmir 1967).

Substituting Eq. (28) in Eq. (26) and solving it, obtained the equation for the unknown function F as follows

$$F = F_1\cos\frac{2m\pi x}{L} + F_2\cos\frac{2ny}{R} - F_3\sin\frac{m\pi x}{L}\sin\frac{ny}{R} + F_4\sin\frac{3m\pi x}{L}\sin\frac{ny}{R} - \sigma_{0y} h\frac{x^2}{2} \tag{29}$$

σ_{0y} is the average circumferential stress and the coefficients F_i as follows

$$\begin{aligned}
F_1 &= \frac{n^2\lambda^2}{32A_{11}^* m^2\pi^2} f_1^2 - \frac{\left(4\lambda L - 16A_{24}^* m^2\pi^2\right)}{32A_{11}^* m^2\pi^2} f_2 \\
F_2 &= \frac{m^2\pi^2}{32A_{22}^* n^2\lambda^2} f_1^2 \\
F_3 &= \frac{B}{A} f_1 + \frac{m^2 n^2\pi^2\lambda^2}{A} f_1 f_2 \\
F_4 &= \frac{m^2 n^2\pi^2\lambda^2}{G} f_1 f_2
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
A &= A_{11}^* m^4\pi^4 + \left(A_{33}^* - A_{12}^* - A_{21}^*\right) m^2 n^2\pi^2\lambda^2 + A_{22}^* n^4\lambda^4 \\
B &= B_{21}^* m^4\pi^4 + \left(B_{11}^* + B_{22}^* - 2B_{36}^*\right) m^2 n^2\pi^2\lambda^2 + B_{12}^* n^4\lambda^4 - \frac{L^2}{R} m^2 n^2 \\
B_1 &= B_{21}^{**} m^4\pi^4 + \left(B_{11}^{**} + B_{22}^{**} - 2B_{36}^*\right) m^2 n^2\pi^2\lambda^2 + B_{12}^{**} n^4\lambda^4 - \frac{L^2}{R} m^2 n^2 \\
D &= D_{11}^* m^4\pi^4 + \left(D_{12}^* + D_{21}^* + 4D_{36}^*\right) m^2 n^2\pi^2\lambda^2 + D_{22}^* n^4\lambda^4 \\
G &= 81A_{11}^* m^4\pi^4 + 9\left(A_{33}^* - A_{12}^* - A_{21}^*\right) m^2 n^2\pi^2\lambda^2 + A_{22}^* n^4\lambda^4 \\
\lambda &= \frac{L}{R}
\end{aligned} \tag{31}$$

Substituting Eq. (28) and (29) in to Eq. (27) and by applying the Galerkin method in the ranges $0\le x\le L$ and $0\le y\le 2\pi R$

$$\sigma_{0y} = \frac{R}{h}\left(q_0 - \frac{1}{2}k_w\left(f_2 - 2f_0\right)\right) \tag{32}$$

$$\left[\frac{m^2n^2\pi^2\lambda^2}{A}+\frac{B_1}{A}m^2n^2\pi^2\lambda^2-\frac{n^2\lambda^2\left(\lambda L-4B_{21}^{*}m^2\pi^2\right)}{4A_{11}^{*}m^2\pi^2}\right]f_1f_2$$
$$+\left(\frac{m^2n^2\pi^2\lambda^2}{A}+\frac{m^2n^2\pi^2\lambda^2}{G}\right)f_1f_2^{\,2}+\left(\frac{m^4\pi^4}{16A_{22}^{*}}+\frac{n^4\lambda^4}{16A_{11}^{*}}\right)f_1^{\,3}+\left(D+\frac{BB_1}{A}\right)f_1 \tag{33}$$
$$-\sigma_{0y}hn^2L^2\lambda^2f_1+L^4k_wf_1+L^2k_sf_1\left[(\lambda n)^2+(m\pi)^2\right]=0$$

$$\left\{\left[4B_{21}^{**}\left(\frac{m\pi}{L}\right)^4-\frac{1}{R}\left(\frac{m\pi}{L}\right)^2\right]\frac{n^2\lambda^2}{32A_{11}^{*}m^2\pi^2}+\frac{1}{2}\frac{B}{A}\left(\frac{m\pi}{L}\right)^2\left(\frac{n}{R}\right)^2\right\}f_1^{\,2}$$
$$+\frac{1}{2}m^2n^2\pi^2\lambda^2\left(\frac{m\pi}{L}\right)^2\left(\frac{n}{R}\right)^2\left(\frac{1}{A}-\frac{1}{G}\right)f_1^{\,2}f_2$$
$$+\left\{4D_{11}^{*}\left(\frac{m\pi}{L}\right)^4-\left[4B_{21}^{**}\left(\frac{m\pi}{L}\right)^4-\frac{1}{R}\left(\frac{m\pi}{L}\right)^2\right]\frac{\left(\lambda L-4B_{21}^{*}m^2\pi^2\right)}{4A_{11}^{*}m^2\pi^2}\right\}f_2 \tag{34}$$
$$+\frac{\sigma_{0y}h}{R}-q_0+k_w\left(\frac{3}{4}f_2+f_0\right)+k_sf_2\left(\frac{m\pi}{L}\right)^2=0$$

In addition, cylindrical shell must be satisfy the circumferential close conditions as

$$\int_0^{2\pi R}\int_0^{L}\frac{\partial v}{\partial y}\mathrm{d}x\mathrm{d}y=\int_0^{2\pi R}\int_0^{L}\left[\varepsilon_y^0+\frac{w}{R}-\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]\mathrm{d}x\mathrm{d}y=0 \tag{35}$$

Using Eqs. (20), (25), (28) and (29), can be written

$$-2A_{11}^{*}\sigma_{0y}h+\frac{1}{R}\left(f_2+2f_0\right)-\frac{1}{4}\left(\frac{n}{R}\right)^2f_1^{\,2}=0 \tag{36}$$

According to Eq. (33)

$$f_1^{\,2}=\frac{\left(a_{11}+a_{12}f_2+a_{13}f_2^{\,2}-\sigma_{0y}h\frac{n^2}{R^2}L^4+L^4k_w+L^2k_s\left((\lambda n)^2+(m\pi)^2\right)\right)}{a_{14}} \tag{37}$$

Substituting Eq. (33) into Eqs. (36) and (37), we have

$$a_{21}f_1^{\,2}+a_{22}f_1^{\,2}f_2+\left(a_{23}+\frac{1}{4}k_w+\left(\frac{m\pi}{L}\right)^2k_s\right)f_2=0 \tag{38}$$

$$f_0=a_{31}q+a_{32}f_2+a_{33}f_1^{\,2} \tag{39}$$

where

$$a_{11}=D+\frac{BB_1}{A},\ a_{12}=\frac{B_1}{A}m^2n^2\pi^2\lambda^2-\frac{n^2\lambda^2\left(\lambda L-4B_{21}^{*}m^2\pi^2\right)}{4A_{11}^{*}m^2\pi^2}$$
$$a_{13}=\frac{m^2n^2\pi^2\lambda^2}{A}+\frac{m^2n^2\pi^2\lambda^2}{G},\ a_{14}=\frac{m^4\pi^4}{16A_{22}^{*}}+\frac{n^4\lambda^4}{16A_{11}^{*}} \tag{40}$$

$$a_{21}=\left[4B_{21}^{**}\left(\frac{m\pi}{L}\right)^4-\frac{1}{R}\left(\frac{m\pi}{L}\right)^2\right]\frac{n^2\lambda^2}{16A_{11}^{*}m^2\pi^2}+\frac{1}{2}\frac{B}{A}\left(\frac{m\pi}{L}\right)^2\left(\frac{n}{R}\right)^2$$

$$a_{22}=\frac{1}{2}m^2n^2\pi^2\lambda^2\left(\frac{m\pi}{L}\right)^2\left(\frac{n}{R}\right)^2\left(\frac{1}{A}+\frac{1}{G}\right) \tag{41}$$

$$a_{23}=4D_{11}^{*}\left(\frac{m\pi}{L}\right)^4-\left[4B_{21}^{**}\left(\frac{m\pi}{L}\right)^4-\frac{1}{R}\left(\frac{m\pi}{L}\right)^2\right]\frac{\left(\lambda L-4B_{21}^{*}m^2\pi^2\right)}{4A_{11}^{*}m^2\pi^2}$$

$$a_{31}=\frac{R^2A_{11}^{*}}{\left(1+k_{\mathrm{w}}R^2A_{11}^{*}\right)},\ a_{32}=\frac{1}{2},\ a_{33}=\frac{n^2}{8R\left(1+k_{\mathrm{w}}R^2A_{11}^{*}\right)} \tag{42}$$

Substituting Eq. (32) into Eqs. (33) and (39) as follows

$$q=a_{41}+a_{42}f_2+a_{43}f_2^{\,2}+a_{44}f_0-\frac{\left(a_{23}+\frac{1}{4}k_{\mathrm{w}}+\left(\frac{m\pi}{L}\right)^2k_{\mathrm{s}}\right)f_2}{a_{21}T+a_{22}Tf_2} \tag{43}$$

where

$$a_{41}=\frac{a_1+L^4k_{\mathrm{w}}+L^2k_{\mathrm{s}}\left(\left(\lambda n\right)^2+\left(m\pi\right)^2\right)}{Ta_4},\ T=\frac{n^2L^4}{a_4R}$$

$$a_{42}=\frac{1}{T}\left(\frac{Tk_{\mathrm{w}}}{2}+\frac{a_2}{a_4}\right),\ a_{43}=\frac{a_3}{Ta_4},\ a_{44}=k_{\mathrm{w}} \tag{44}$$

Substituting Eq. (32) into Eq.(37) and Eq.(37) into Eqs. (36) and (38), we have

$$f_1^{\,2}=a_{51}+a_{52}f_2+a_{53}f_2^{\,2}+a_{54}q \tag{45}$$

$$f_0=a_{61}q+a_{62}f_2+a_{63}f_2^{\,2}+a_{64} \tag{46}$$

$$q=a_{71}+a_{72}f_2+a_{73}f_2^{\,2}-\frac{\left(a_{23}+\frac{1}{4}k_{\mathrm{w}}+\left(\frac{m\pi}{L}\right)^2k_{\mathrm{s}}\right)f_2}{\left(a_{21}T+a_{22}Tf_2\right)\left(1-a_{54}a_{61}\right)} \tag{47}$$

where

$$a_{51}=\frac{-R^3\left(1+k_{\mathrm{w}}R^2A_{11}^{*}\right)}{R^3\left(1+k_{\mathrm{w}}R^2A_{11}^{*}\right)+8Tn^2}\left(\frac{a_1+L^4k_{\mathrm{w}}+L^2k_{\mathrm{s}}\left(\left(\lambda n\right)^2+\left(m\pi\right)^2\right)}{Ta_4}\right)$$

$$a_{52}=\frac{R^3\left(1+k_{\mathrm{w}}R^2A_{11}^{*}\right)}{R^3\left(1+k_{\mathrm{w}}R^2A_{11}^{*}\right)+8Tn^2}\left(\frac{-2Ra_2-n^2L^4k_{\mathrm{w}}}{2Ra_4}+\frac{T\left(k_{\mathrm{w}}R^2A_{11}^{*}-1\right)}{2\left(1+k_{\mathrm{w}}R^2A_{11}^{*}\right)}\right) \tag{48}$$

$$a_{53}=\frac{-R^3\left(1+k_{\mathrm{w}}R^2A_{11}^{*}\right)}{R^3\left(1+k_{\mathrm{w}}R^2A_{11}^{*}\right)+8Tn^2}\left(\frac{a_3}{a_4}\right)$$

$$a_{54}=\frac{R^3\left(1+k_{\mathrm{w}}R^2A_{11}^{*}\right)}{R^3\left(1+k_{\mathrm{w}}R^2A_{11}^{*}\right)+8Tn^2}\left(\frac{n^2L^4}{Ra_4}-\frac{TR^2A_{11}^{*}}{\left(1+k_{\mathrm{w}}R^2A_{11}^{*}\right)}\right)$$

Table 1 The critical buckling loads (q) for cylindrical shell under external pressure

	Present	Baruch and Singer	Reddy and Stames	Shen
Un-stiffened	103.327 (4)*	102	93.5	100.7
Stringer stiffened (θ=0°, β=0°)	104.494 (4)	103	94.7	102.2
Ring stiffened (θ=90°, β=90°)	379.694 (3)	370	357.5	368.3
Orthogonal stiffened (θ=0°, β=90°)	387.192 (3)	377	365	374.1

*The numbers in the parenthesis denote the buckling modes (n).

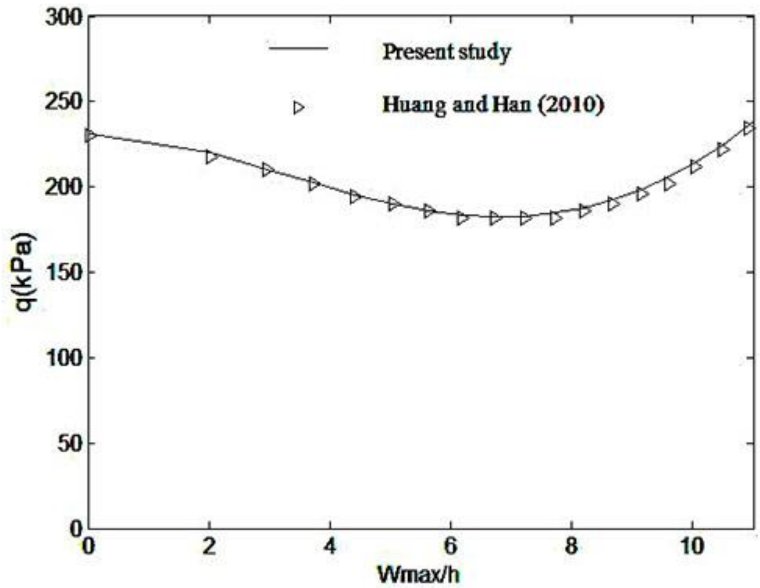


Fig. 4 Post-buckling curve of un-stiffened FGM shells

$$a_{61} = a_{31} + a_{33}a_{54},\ a_{62} = -a_{32} + a_{33}a_{52}$$
$$a_{63} = a_{33}a_{53},\ a_{64} = a_{33}a_{51} \tag{49}$$

$$a_{71} = \frac{a_{41} + a_{44}a_{64}}{1 - a_{44}a_{61}},\ a_{72} = \frac{a_{42} + a_{62}}{1 - a_{44}a_{61}},\ a_{73} = \frac{a_{43} + a_{63}}{1 - a_{44}a_{61}} \tag{50}$$

If f=$W_{\max}$ then according to Eq. (28) it is obvious that the maximal deflection of the shells is

$$f = f_0 + f_1 + f_2 \tag{51}$$

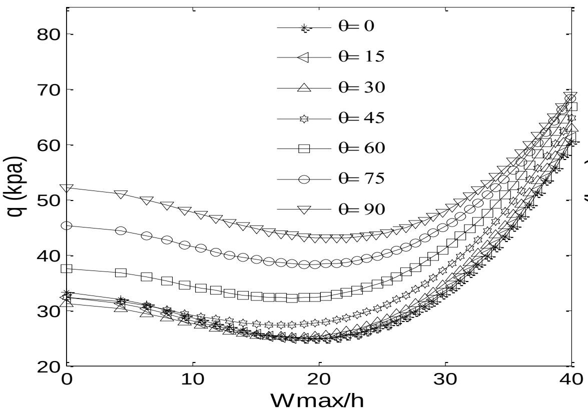
It should be noted the exact solutions despite all benefits include restrictions such as the kind of boundary conditions and geometric shape of structure.

4. Numerical results

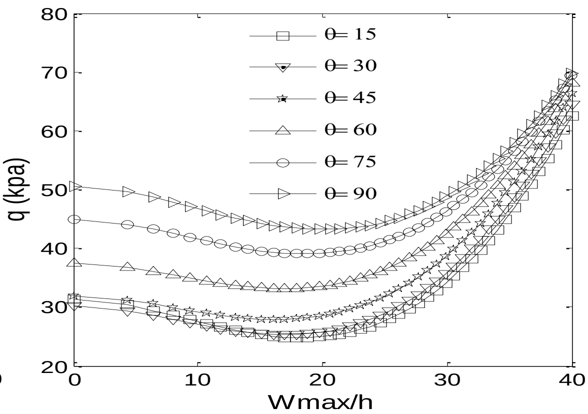
In this section, the stiffened and un-stiffened FGM cylindrical shells by an elastic foundation are considered with R=60.643 mm, L=387.35 mm. The combination of materials consists of Aluminum E_m=70 GPa and Zirconia E_c=380 GPa. The Poisson's ratio is chosen to be 0.3. The height of stiffeners is 0.076 mm and width is 1.27 mm. Each of the stiffener system includes 15 stiffeners distributed regular.

In order to verify the formulation, in Table 1 the critical buckling loads (q) for stiffened and un-stiffened cylindrical shell under external pressure to compared with the results given by Baruch and Singer (1963), Reddy and Stames (1993), Shen (2009). In Fig. 4 post-buckling curve of un-stiffened FGM cylindrical shell compared with the results of the analysis Huang and Han (2010). As can be seen, good agreement is obtained in this comparison.

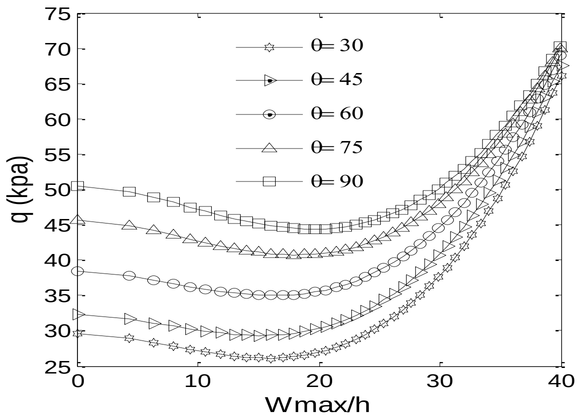
Fig. 5 shows the effects of stiffeners with various angles on the curve of post-buckling of cylindrical shells. In previous works, only stringer and ring stiffeners are investigated that buckling load of stringer stiffeners is lower than ring stiffeners and this subject in the present work is confirmed. According to Fig. 5 for the mid-states have been chosen the stiffeners with various angle that can be observed the effects of them on the post-buckling behavior. As can be seen of Fig. 5 maximum buckling load related to the shell with ring stiffeners and minimum of the buckling load when the angle of both series stiffeners together is 30°.



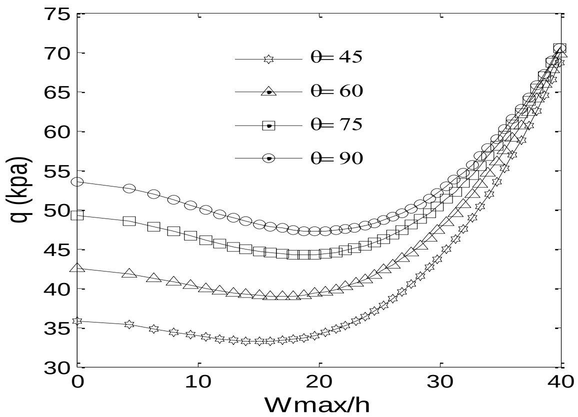
(a) β=0°



(b) β=15°

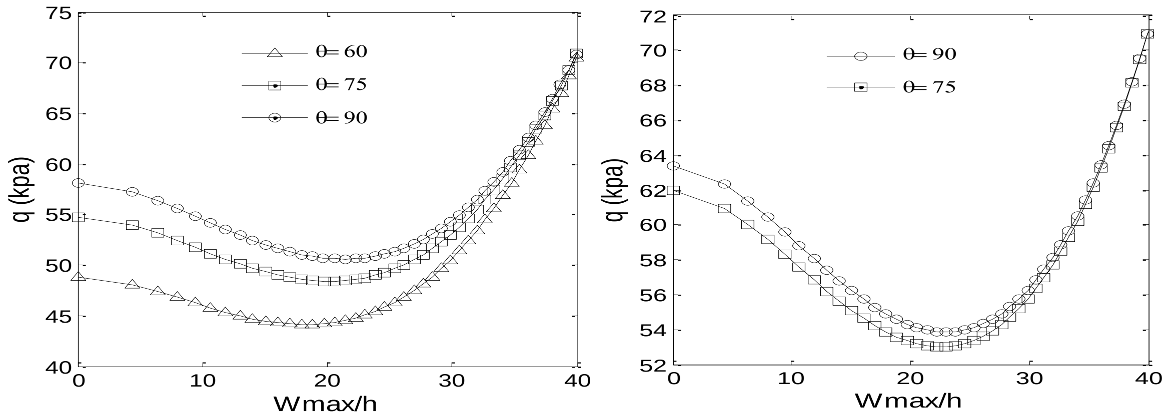


(c) β=30°



(d) β=45°

Fig. 5 Post-buckling curves of cylindrical shells with various angle of stiffeners



(e) β=60° (f) β=90°

Fig. 5 Continued

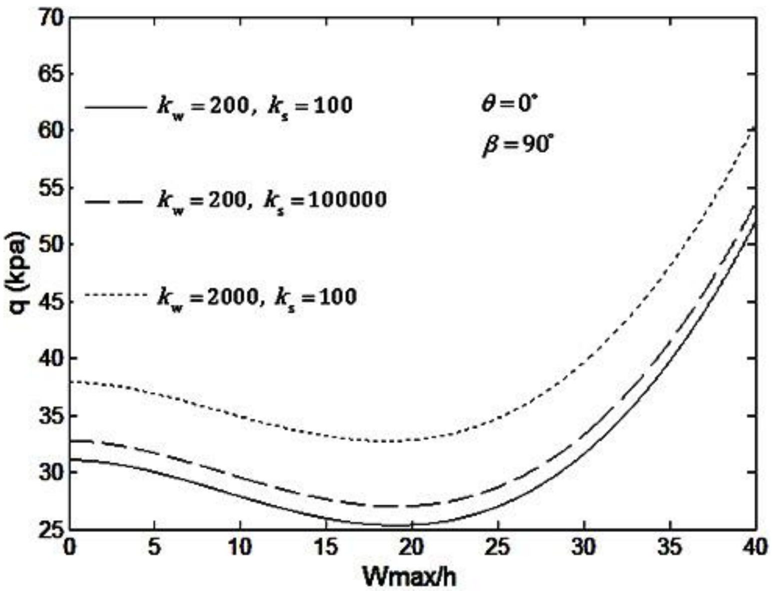


Fig. 6 The effect of elastic foundation Winkler and Pasternak on stiffened FGM cylindrical shells

The obtained results from Fig. 6. show that when the buckling load-bearing capacity of shell increased 6.6 percent, the Pasternak index is about 1000 times increases, but the buckling load-bearing capacity of shell increased 29.4 percent, when the Winkler index is about 10 times increases. This results show the effect of elastic foundation Winkler on the response of buckling is more than the elastic foundation Pasternak.

According to Fig. 7 can be achieved interesting results. Ceramic and metal shells have the highest and lowest resistance to buckling, respectively. The use of steel/ceramic stiffeners leads to increase/decrease the load-bearing capacity of shell than the FGM stiffeners. Therefore, the ceramic cylindrical shell with steel spiral stiffeners is the best choice.

In Fig. 8 the curve of maximum radial deflection along circumferential and along of the stiffened FGM cylindrical shells for different parts of the length and *m*=1 is showed. In Fig. 8 maximum deflection for *x*=200 (the middle of the cylinder), It can be seen. Mode number of buckling is 7 that is clearly identified in Fig. 8.

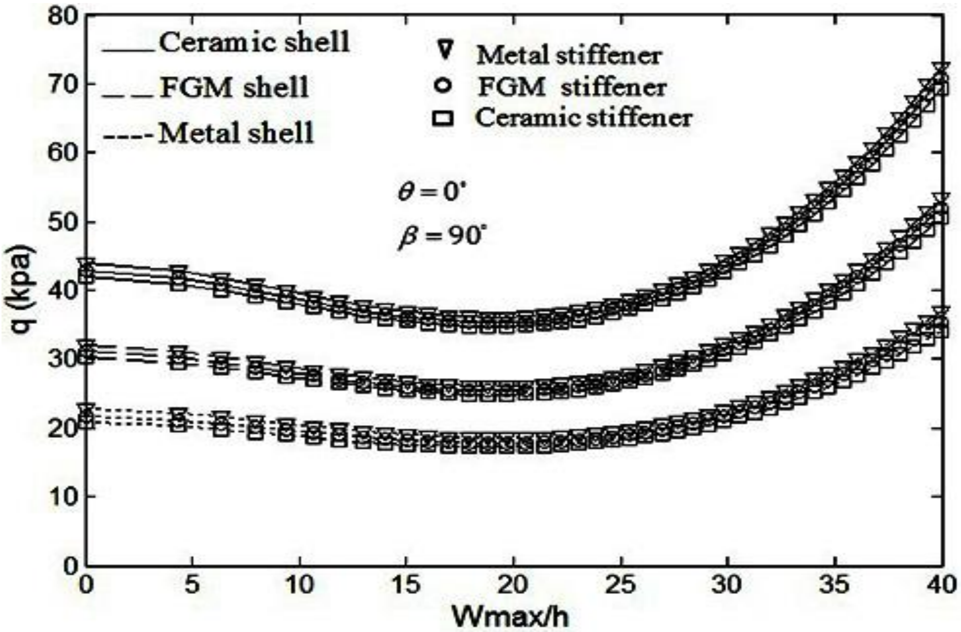


Fig. 7 The effect of material on the load-bearing capacity of shell

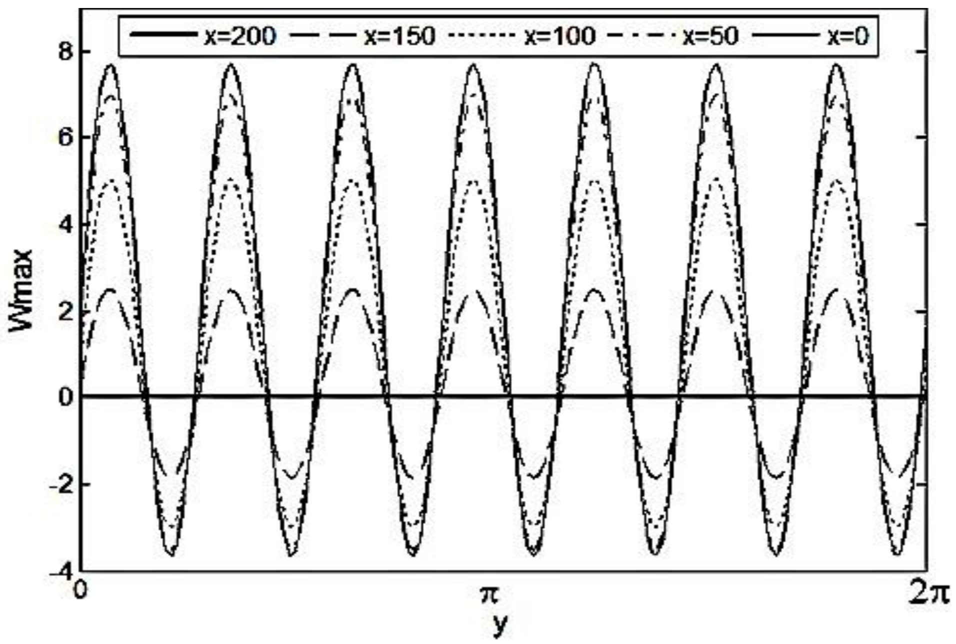


Fig. 8 Maximum radial deflection along circumferential of the cylindrical shells

5. Conclusions

The exact analytical method for FGM cylindrical shells with spiral stiffeners with elastic foundation under uniform external pressure is presented. The proposed model is based on Winkler and Pasternak elastic foundation parameters. According to the Von Karman nonlinear equations and the classical plate theory (CPT) of shells, strain displacement relations are obtained. The smeared stiffeners technique and Galerkin method, used for solving nonlinear problem. With considering three terms approximation for the deflection shape, the relation for non-linear buckling obtained.

Some conclusions are obtained from this study

- When both of the stiffeners have angle of 30° critical buckling load is minimum.
- The effect of elastic foundation Winkler on the response of buckling is more than the elastic foundation Pasternak.
- Ceramic shells have the greatest resistance to buckling load and metal shells have the lowest resistance to buckling load.
- The ceramic cylindrical shells with steel stiffeners is the best choice.
- Maximum deflection is arisen the middle of length of cylindrical.

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