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RELSYS: A computer program for structural system reliability

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Abstract. Most reliability-based analyses focus on the reliability of the individual components of a structure. There are many advantages to examining the components in combination as an entire structural system. This paper illustrates an algorithm used in the computer program RELSYS (RELiability of SYStems) which computes the system reliability of any structure which can be modeled as a series-parallel combination of its components. A first-order method is used to initially compute the reliability of each individual component. The system reliability is computed by successively reducing the series and parallel systems until the system has been simplified to a single equivalent component. Equivalent alpha vectors are used to account for the correlation between failure modes during the system reduction process.

Key words: correlation; optimization; random variables; sensitivity; structural reliability; system reliability.

1. Introduction

Over the past several decades, the concepts and methods of structural reliability have developed rapidly and become more widely understood and accepted. Many reliability-based applications are transitioning from hypothetical structures using fictitious data to real-world applications which require the most realistic data possible. Most reliability-based analyses focus on the reliability of the individual components of a structure. There are many advantages to examine the components in combination as an entire structural system. In some cases, the reliability of every component may be satisfactory, but the reliability of the system may become dangerously low. In other cases when the system is highly redundant, the reliability of individual components may be quite low while the system reliability remains high. A system approach to reliability considers both the reliability of components and their relationship and importance to the entire system.

Reliability-based methods have only become practical with the rapid improvements in digital computers. As reliability methods gain greater acceptance, ther is a greater need for software that will provide fast computations. This paper illustrates an algorithm used in the computer program RELSYS, RELiability of SYStems (Estes and Frangopol 1997) which computes the system reliability of any structure which can be modeled as a series-parallel

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combination of its components. The first step is to compute the reliability of each individual component. First-order reliability methods (FORM) are widely accepted and used. The process for transforming all random variables to the uncorrelated standard normal space and finding the shortest distance from the origin to the failure surface is illustrated using an example. Sensitivity with respect to changes in the random variables is provided.

A structural system is modeled as a series-parallel combination of its components. The system reliability is computed by successively reducing the series and parallel systems until the system has been simplified to a single equivalent component. Series systems are solved by taking the average of the Ditlevsen's bounds (1979) and the multi-normal integral is evaluated to compute the reliability of parallel systems. Equivalent alpha vectors are used to account for the correlation between failure modes during the system reduction process. Again, the algorithm is illustrated using a numerical example. The strengths and weaknesses of RELSYS are discussed.

2. Component reliability

A reliability problem may be a function of many random variables. The generalized structural reliability problem is formulated in terms of a vector of basic random variables of the structural system, $X=(X_1, X_2, ..., X_n)^T$, where $X_1, X_2, ..., X_n$ are basic random variables that may describe loads, structural component dimensions, material characteristics, and section properties. A limit state function, $g(X)=0$, describes the performance of a structural component in terms of the basic random variables, X, and defines the failure surface which separates the survival region from the failure region. These regions are as follows:

$$g(X) > 0 \text{ defines the survival or safe region} \tag{1}$$

$$g(X) \leq 0 \text{ defines the failure region} \tag{2}$$

If the joint probability distribution function of the design variables $X_1, X_2, ..., X_n$ is $f_{X_1, X_2, ..., X_n}(x_1, x_2, ..., x_n)$, the probability of the failure state is

$$P_f = \int \cdots \int_{g(X) \leq 0} f_{X_1, X_2, X_n}(x_1, x_2, \ldots, x_n) dx_1 dx_2 \ldots dx_n \tag{3}$$

which may be written as

$$P_f = \int_{g(X) \leq 0} f_X(\boldsymbol{x}) d\boldsymbol{x} \tag{4}$$

Eq. (4) represents the n-fold integral of $f_X(\boldsymbol{x})$ over the failure region $g(X) \leq 0$. This convolution integral can easily become too complex to solve directly. Approximate techniques were developed to approximate this integral. This study uses a first-order reliability method to search for the shortest distance between the origin and the limit state surface in the reduced space of standard normal variates.

3. First order reliability method

In the First Order Reliability Method (FORM), the limit state function at any point is approximated by a first-order Taylor series expansion about that point. This creates a straight

line which is tangent to the limit state function at the point of interest. The minimum distance β from the origin of the reduced space to the tangent plane of the failure surface at point $\boldsymbol{u}^*$ is:

$$\beta = \frac{\mu_g}{\sigma_g} = \frac{-\sum_{i=1}^{n} u_i^* \left(\frac{\partial g}{\partial U_i^*} \right)}{\sqrt{\sum_{i=1}^{n} \left(\frac{\partial g}{\partial U_i^*} \right)^2}} \tag{5}$$

where $\boldsymbol{u}^* = (u_1^*, u_2^*, \ldots, u_n^*)^T$ is the most probable failure point, the derivatives $(\partial g / \partial U_i^*)$ are evaluated at this same point, and μ_g and σ_g are the mean value and the standard deviation of $g(\boldsymbol{X})$, respectively (Shinozuka 1983, Ang and Tang 1984).

Based on the user's guess for the most probable point of failure, the method involves an iterative search which relies on the gradients of the limit state function at the point of failure and the direction cosines to find this minimum distance (Shinozuka 1983). The detailed theory behind the FORM approach is presented in Ang and Tang (1984). This iterative process is best illustrated by an example.

Given the limit state equation $g(\boldsymbol{X})=g(X_1, X_2)=2X_1^2-2X_2=0$, the reliability index can be calculated by finding the minimum distance from the origin in standard normal space to the failure surface. The limit state equation $g(\boldsymbol{X})=0$ is defined in the original space which will be denoted as the x-space. The minimum distance in question is found in the standard normal or reduced space, hereafter referred to as the u-space. The variable X_1 is normally distributed with a mean value $\mu_{X_1}=2.0$ and a standard deviation $\sigma_{X_1}=0.2$ which can be expressed as $X_1=N$ [2.0, 0.2]. The parameters for the normally distributed variable X_2 are N[3.0, 0.3]. With these uncorrelated, normally distributed variables, the transformation equation from the original space x to the reduced space u is:

$$U_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}} \qquad i = 1, 2, \ldots, n \tag{6}$$

The limit state equation $g(\boldsymbol{X})=0$ could be transformed directly to the u-space using Eq. (6) to obtain

$$g(\boldsymbol{U})=0.133U_1^2+2.67U_1-U_2+3.33=0 \tag{7}$$

The algorithm is structured, however, so that the transformation from the x-space to the u-space is never explicitly made. This allows for computer code where the problem and the solution are provided in the original x-space and the user is not aware of the existence of the u-space. The algorithm uses the Direct Derivation approach developed by Lee (1994) which was originally incorporated into the program RELTRAN (Lee *et al.* 1993). The method simultaneously iterates to find the optimal search direction and ensures that the failure point is on the failure surface. The method also allows the user to express the limit state equation in its original x-space form.

Using the stated example problem, this iterative FORM approach proceeds as follows:

· *Step 1*: For each random variable, make an initial guess of the most probable point of failure in the original space. Usually the mean value is a reasonable starting guess. For purposes of illustrating the convergence process, this example will make an intentionally bad initial guess. Therefore, for the first iteration, let $x_1^* = 6.0$ and $x_2^* = 0.5$.

· *Step 2*: Calculate the gradient with respect to each random variable $(\partial g/\partial X_i^*)$ based on the initial guess value. The gradient will establish the search direction for this iteration

$$\partial g/\partial X_1 = 4X_1, \text{ then } \partial g/\partial X_1^* = 4x_1^* = 4(6) = 24$$
$$\partial g/\partial X_2 = -2, \text{ then } \partial g/\partial X_2^* = -2$$

· *Step 3*: Calculate the gradients in the reduced space.

$$\partial g/\partial U_i = \partial g/\partial X_i(\partial X_i/\partial U_i), \text{ where } X_i = \sigma_{X_i} U_i + \mu_{X_i} \text{ and } \partial X_i/\partial U_i = \sigma_{X_i}.$$

Therefore,

$$\partial g/\partial U_i = \partial g/\partial X_i(\sigma_{X_i}) \text{ and } \partial g/\partial U_i^* = \partial g/\partial X_i^*(\sigma_{X_i})$$

Consequently,

$$\partial g/\partial U_1^* = \partial g/\partial X_1^*(\sigma_{X_1}) = 24(0.2) = 4.8$$
$$\partial g/\partial U_2^* = \partial g/\partial X_2^*(\sigma_{X_2}) = -2.0(0.3) = -0.6$$

· *Step 4*: Compute $\Delta\beta$ based on Taylor series expansion about $\boldsymbol{U}^*$.

$$\Delta\beta = \frac{g(\boldsymbol{X}^*)}{\sqrt{\sum_i \left(\frac{\partial g}{\partial U_i^*}\right)^2}}$$

$$g(\boldsymbol{X}^*) = 2x_1^{*2} - 2x_2^* = 2.0(6.0)^2 - 2.0(0.5) = 71$$

$$\sqrt{\sum_i \left(\frac{\partial g}{\partial U_1^*}\right)^2} = \sqrt{\left(\frac{\partial g}{\partial U_1^*}\right)^2 + \left(\frac{\partial g}{\partial U_2^*}\right)^2} = \sqrt{4.8^2 + (-0.6)^2} = 4.837$$

$$\Delta\beta = 71/4.837 = 14.68$$

· *Step 5*: Compute the updated reliability index.

$$\beta_{new} = \beta_{old} + \Delta\beta = 0.0 + 14.68 = 14.68$$

· *Step 6*: Compute the direction cosines.

$$\alpha_{g,U_i} = \frac{\partial g/\partial U_i^*}{\sqrt{\sum_{i=1}^{n} \left(\frac{\partial g}{\partial U_i^*}\right)^2}}$$

Therefore, for the first iteration the direction cosines are

$$\alpha_{1,1} = \frac{\partial g/\partial U_i^*}{\sqrt{\left(\frac{\partial g}{\partial U_1^*}\right)^2 + \left(\frac{\partial g}{\partial U_2^*}\right)^2}} = 4.8/4.837 = .9923$$

$$\alpha_{1,2} = -0.6/4.837 = -.1240$$

· *Step 7*: Compute the new failure point in the reduced space.

$$u_{i,new}^* = -\alpha_{g,U_i}\beta$$

$$u_{1,new}^* = -\alpha_{1,1}\beta = -.9923(14.86) = -14.68$$

$$u_{2,new}^* = -\alpha_{1,2}\beta = -(-.1240)(14.86) = 1.82$$

· *Step 8*: Compute the new failure point in the original space.

$$x_{1,new}^* = \mu_{X_1} + u_{1,new}^*\sigma_{X_1} = 2 + (-14.68)(0.2) = -.9128$$

$$x_{2,new}^* = \mu_{X_2} + u_{1,new}^*\sigma_{X_2} = 3 + 1.82(0.3) = 3.546$$

· *Step 9*: Let ε=.00001. Substitute the coordinates of new failure point $(x_{1,new}^*, x_{2,new}^*)$ into the performance function $g(\boldsymbol{X})$. If $|g(\boldsymbol{X}^*)| < \varepsilon$, then the iteration is complete and the solution has converged. If not, perform another iteration using the new failure point $(x_{1,new}^*, x_{2,new}^*)$.

In this case, the solution did not converge and ten more iterations were necessary. On the eleventh iteration, the value for $|g(\boldsymbol{X}^*)|$ was less than 0.00001 and the solution was sufficiently precise. Table 1 shows the values for these iterations. Figs. 1 and 2 show the iterations graphically in the reduced and original spaces, respectively. After the eighth iteration, the changes were too small to be shown on the graphs. The final reliability index was β=1.2317. The program RELSYS uses this method to compute reliability indices.

When there is correlation between the random variables, an additional transformation matrix using the eigenvalues and eigenvectors of the correlation matrix is used to decouple

Table 1 Iterative FORM solution for $g(\boldsymbol{X})=2X_1^2-2X_2=0$ where $X_1=N[2.0, 0.2]$ and $X_2=N[3.0, 0.3]$

Iteration	Step 2		Step 3		Step 4	Step 5	Step 6			Step 7		Step 8	
Number	$\partial g/\partial X_1^*$	$\partial g/\partial X_2^*$	$\partial g/\partial U_1^*$	$\partial g/\partial U_2^*$	$g(\boldsymbol{X}^*)$	$\Delta\beta$	β_{new}	$\alpha_{1,1}$	$\alpha_{1,2}$	u_1^*	u_2^*	x_1^*	x_2^*
1	24.0	-2.0	4.80	-0.60	71.0	14.68	14.68	0.9923	-0.1240	-14.56	1.82	-0.91	3.55
2	-3.65	-2.0	-0.73	-0.60	-5.426	-5.74	8.94	-0.7727	-0.6348	6.90	5.67	3.38	4.70
3	13.52	-2.0	2.70	-0.60	13.458	4.85	13.79	0.9763	-0.2166	-13.46	2.99	-0.69	3.90
4	-2.77	-2.0	-0.55	-0.60	-6.831	-8.36	5.43	-0.6788	-0.7343	3.69	3.99	2.74	4.20
5	10.95	-2.0	2.19	-0.60	6.597	2.91	8.34	0.9645	-0.2642	-8.04	2.20	0.39	3.66
6	1.57	-2.0	0.31	-0.60	-7.016	-10.37	-2.03	0.4626	-0.8866	0.94	-1.80	2.19	2.46
7	8.75	-2.0	1.75	-0.60	4.649	2.52	0.49	0.9459	-0.3243	-0.46	0.16	1.91	3.05
8	7.63	-2.0	1.53	-0.60	1.187	0.72	1.21	0.9307	-0.3658	-1.13	0.44	1.77	3.13
9	7.10	-2.0	1.42	-0.60	0.035	0.02	1.23	0.9211	-0.3892	-1.13	0.48	1.77	3.14
10	7.09	-2.0	1.42	-0.60	-6E-0.4	0.00	1.23	0.9210	-0.3896	-1.13	0.48	1.77	3.14
11	7.09	-2.0	1.42	-0.60	-1E-0.7	0.00	1.23	0.9210	-0.3896	-1.13	0.48	1.77	3.14

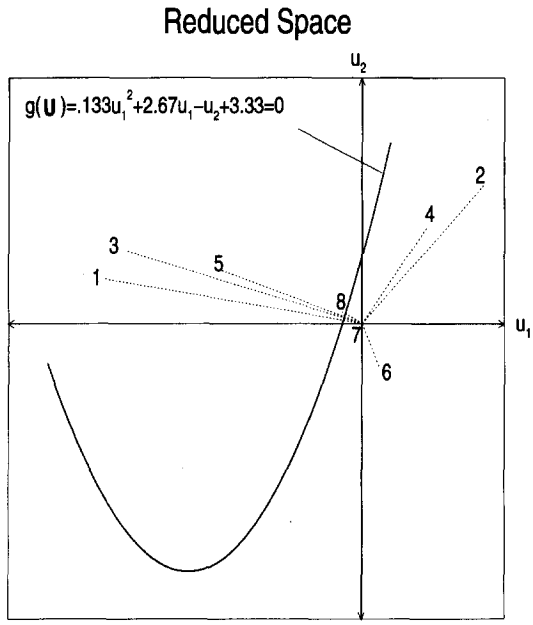


Fig. 1 Results of FORM iterations in the reduced space

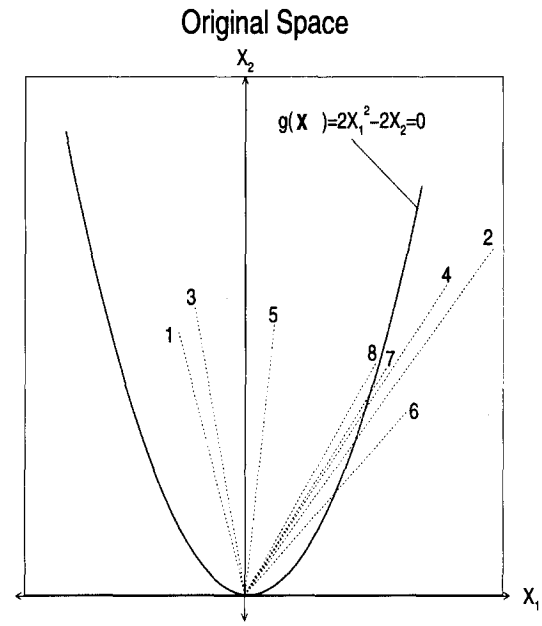


Fig. 2 Results of FORM iterations in the original space

the correlation. When the random variables have non-normal distributions, an equivalent normal distribution is created. The equivalent normal standard deviation $\sigma_{X_i}^N$ is computed (Ang and Tang 1984) as

$$\sigma_{X_i}^N = \frac{\phi\{\Phi^{-1}[F_{X_i}(x_i^*)]\}}{f_{X_i}(x_i^*)} \tag{8}$$

where $F_{X_i}(x_i^*)$ and $f_{X_i}(x_i^*)$ are the cumulative distribution function (CDF) and probability density function (PDF), respectively, of the non-normal distribution evaluated at the current point of failure $\boldsymbol{x}_i^*$. The symbols $\Phi(\cdot)$ and $\phi(\cdot)$ represent the CDF and PDF of the standard normal distribution, respectively. The equivalent normal mean value $\mu_{X_i}^N$ is

$$\mu_{X_i}^N = x_i^* - \sigma_{X_i}^N \Phi^{-1}[F_{X_i}(x_i^*)] \tag{9}$$

When the distribution is non-normal and the variables are correlated, equivalent correlation coefficients developed by Der Kiureghian and Liu (1986) are used for a variety of distributions.

After the reliability of a structural component has been computed, it is useful to examine the sensitivity of the reliability with respect to the random variables which contribute to the uncertainty in the problem (Frangopol 1985, Karamchandani and Cornell 1992). Hohenbichler and Rackwitz (1986) showed that the direction cosines were good approximate measures of the stochastic importance of a random variable. The sensitivity of the reliability index with respect to the mean value of random variable X_j in component a is

$$sens_{\mu_j} = \alpha_{aj} \tag{10}$$

where α_{aj} is the direction cosine with respect to the failure point of random variable X_j in the uncorrelated reduced space. These direction cosines were computed in Step 6 of the iterative solution process shown earlier. The sensitivity with respect to the standard deviation of random variable X_j is

$$sens_{\sigma_j} = -\beta\alpha_{aj}^2 \tag{10}$$

These sensitivity values are used in RELSYS and were demonstrated by Estes (1997) to be valid.

4. System reliability

A general system can be modeled as any combination of series and parallel systems. Consider a series system consisting of y parallel systems. Each parallel system a has z_a components. Then, the probability of failure is given as:

$$P_f = P\left(\bigcup_{a=1}^{y}\bigcap_{b=1}^{z_a}\{g_{ab}(\boldsymbol{X}) \leq 0\}\right) \tag{12}$$

It is often difficult to determine whether a subscript in an equation is referring to a component or a random variable - particularly regarding correlation coefficients and direction cosines. The convention used in this study is to reserve subscripts (i, j, k, ..., n) for random variables. The remaining subscripts (a, b, c, ..., h) and (o, ..., z) are used for components.

The reliability of a series system and a parallel system can be solved separately using the reliabilities and the direction cosines at the point of failure of the individual components. The approach for a complex system will be to sequentially break the system down into simpler equivalent subsystems. The example series-parallel system in Fig. 3 illustrates how the equivalent components are created and the system is simplified until a single system reliability index can be calculated. The original problem consists of six limit state equations $g_1(\boldsymbol{X})=0$ through $g_6(\boldsymbol{X})=0$ and the reliability index of each component, β_1, β_2, ..., β_6 can be calculated using the FORM described earlier.

Using the direction cosines associated with the most probable points of failure for each limit state, the system correlation matrix which provides the correlation between the performance functions $g_1(\boldsymbol{X})$, ..., $g_6(\boldsymbol{X})$ is calculated (Ang and Tang 1984). As shown in Fig. 4, the correlation coefficient between any two performance functions $g_a(\boldsymbol{X})$ and $g_b(\boldsymbol{X})$, ρ_{ab}, is equal to the cosine of the angle between the two reliability vectors associated with $g_a(\boldsymbol{X})$ and $g_b(\boldsymbol{X})$. This can be calculated using the direction cosines, as follows (Ang and Tang 1984):

$$\rho_{g_a,g_b} = \frac{Cov(g_a,g_b)}{\sigma_{g_a}\sigma_{g_b}} = \sum_{k=1}^{n}\alpha_{ak}^{*}\alpha_{bk}^{*} \tag{13}$$

where $Cov(g_a,g_b)$ is the covariance between $g_a(\boldsymbol{X})$ and $g_b(\boldsymbol{X})$, n is the number of random variables associated with $g_a(\boldsymbol{X})$ and $g_b(\boldsymbol{X})$, and α_{ak}^{*} and α_{bk}^{*} are the direction cosines of the tangent planes at the most probable failure point.

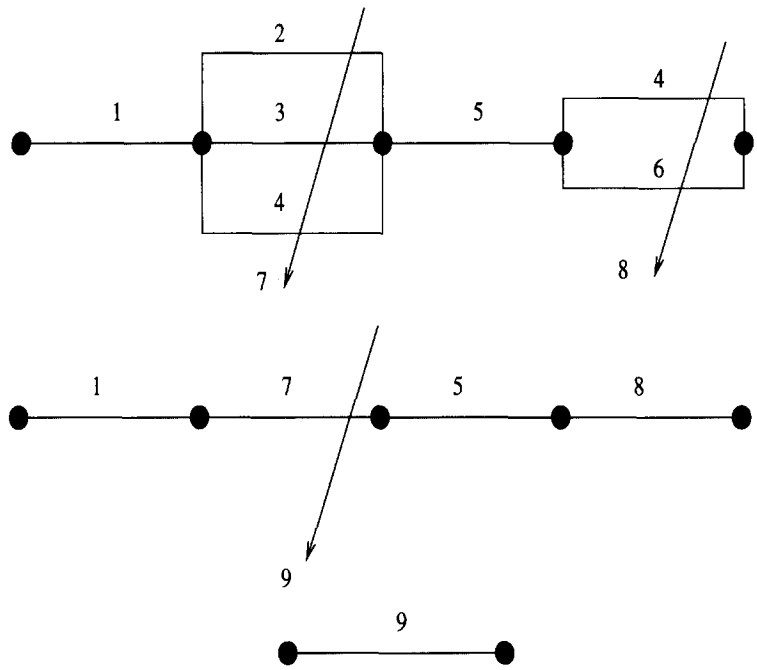


Fig. 3 Reduction of a series-parallel system

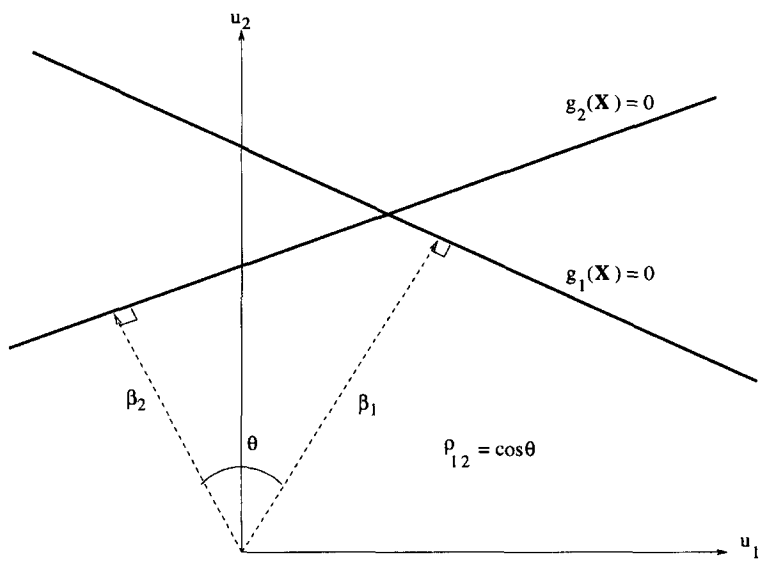


Fig. 4 Correlation between two failure modes

Consider the series-parallel system shown in Fig. 3. It consists of a series of four subsystems with two subsystems in parallel. The first step is to reduce the two parallel subsystems in Fig. 3 to equivalent components. By numerically integrating the trivariate normal distribution, the probability of failure of the parallel subsystem comprised of $g_2(\boldsymbol{X})$, $g_3(\boldsymbol{X})$, and $g_4(\boldsymbol{X})$ is found and an equivalent component $g_7(\boldsymbol{X})$ is created as follows:

$$P(F_7) = P(F_2 \cap F_3 \cap F_4) = \int_{\beta_2}^{\infty} \int_{\beta_3}^{\infty} \int_{\beta_4}^{\infty} f_3(\beta_a, \beta_b, \beta_c; [\rho_{sys_{a,b,c}}]) d\beta_c \, d\beta_b \, d\beta_a \tag{14}$$

where F_i is the failure event associated with component i and

$$f_3(\beta_a, \beta_b, \beta_c; [\rho_{sys_{a,b,c}}]) = \frac{1}{(2\pi)^{3/2} \sqrt{det\,[\rho_{sys_{a,b,c}}]}} e^{[-1/2\{\beta_a, \beta_b, \beta_c\}[\rho_{sys_{a,b,c}}]^{-1}\{\beta_a, \beta_b, \beta_c\}^T]} \tag{15}$$

in which $[\rho_{sys_{a,b,c}}]$ is the correlation matrix of $g_2(\boldsymbol{X})$, $g_3(\boldsymbol{X})$, and $g_4(\boldsymbol{X})$. Similarly, the bivariate normal distribution

$$P(F_8) = P(F_4 \cap F_6) = \int_{\beta_4}^{\infty} \int_{\beta_6}^{\infty} \frac{1}{2\pi\sqrt{1 - \rho_{sys_{ab}}^2}} e^{-1/2(1 - \rho_{sys_{ab}}{}^2)(\beta_a{}^2 + \beta_b{}^2 - 2\rho_{sys_{ab}}\beta_a\beta_b)} d\beta_a \, d\beta_b \tag{16}$$

is used to find the probability of failure of the two-member parallel subsystem comprised of $g_4(\boldsymbol{X})$ and $g_6(\boldsymbol{X})$, and create the equivalent component $g_8(\boldsymbol{X})$. The notations a, b, c used in Eqs. (14) and (15) are for the performance functions associated with components 2, 3, and 4 in Fig. 3, and the notations a and b in Eq. (16) are for the performance functions associated with components 4 and 6 in Fig. 3.

For parallel systems with four members or more, the number of computations to numerically integrate the multinormal distribution becomes too large and the Hohenbichler approximation is used (Hohenbichler and Rackwitz 1983, Sørensen 1987).

As shown in Fig. 3, the intermediate result is a series system of four components whose performance functions are $g_1(\boldsymbol{X})$=$g(1)$, $g_7(\boldsymbol{X})$=$g(7)$, $g_5(\boldsymbol{X})$=$g(5)$, and $g_8(\boldsymbol{X})$=$g(8)$. The probability of failure of the equivalent single component with the performance function $g_9(\boldsymbol{X})$=$g(9)$ is the average of the Ditlevsen's bounds (1979) for this series system. The lower bound can be written as:

$$P_{f_{lower}} = P(F_1) + \sum_{a=2}^{z} max\,[P(F_a) - \sum_{b=1}^{a-1} (P(F_a \cap F_b); 0] \tag{17}$$

where z is the total number of potential failure modes, F_a is the failure event associated with mode a, and F_1 is assumed to be the dominant failure mode. The lower bound is based on considering only the individual mode probabilities, $P(F_a)$, and all possible combinations of joint probabilities involving two failure modes, $P(F_a \cap F_b)$. By neglecting joint probabilities involving three (i.e., $P(F_a \cap F_b \cap F_c)$) or more modes, which are more difficult to calculate, a lower bound is assured. The upper bound for the probability of the system failure can be written as

$$P_{f_{upper}} = \sum_{a=1}^{z} P(F_a) - \sum_{a=2, b<a}^{z} max\,[P(F_b \cap F_a)] \tag{18}$$

The upper bound is computed by selectively excluding certain two-member joint probabilities. The joint probabilities are computed using Eq. (16). The best bounds are obtained by rank ordering the failure events, (F_1, ..., F_z), from the event associated with the highest probability of occurrence to lowest (Melchers 1987).

Since the reliability of a system is dependent on the correlation of the individual failure

modes of the system, the equivalent correlation must be included. The equivalent correlation coefficients $\rho_{sys\text{-}equiv_{ab}}$ that are associated with the equivalent components created in the system analysis are a function of equivalent alpha vectors of equivalent direction cosines associated with each random variable in the sub-system analysis, as follows:

$$\rho_{sys-equiv_{ab}} = \sum_{k=1}^{n} \alpha_{equiv_{ak}} \alpha_{equiv_{bk}} \tag{19}$$

where $\alpha_{equiv_{ak}}$ is the equivalent direction cosine for equivalent component a with respect to random variable k. Just as shown with step 6 of the iterative FORM procedure, the equivalent direction cosines are a function of the gradients. The gradient under consideration however is the change in the system reliability with respect to each random variable in the system

$$\alpha_{equiv_{ak}} = \frac{\partial(\beta_{sys})/\partial(X_k)}{\sqrt{\sum_{i=1}^{n} \left(\frac{\partial(\beta_{sys})}{\partial(X_i)} \right)^2}} \tag{20}$$

The gradients $\frac{\partial(\beta_{sys})}{\partial(X_i)}$ are computed by using the direction cosines associated with each component α_{ak}^* in a system to individually assess the effect of each random variable on the reliability of the system. The sensitivity of the system reliability with respect to each random variable becomes the equivalent direction cosine for that random variable. Consider random variable X_k for a series or parallel systems with z components. The reliability index associated with an individual component β_a is varied by a small amount (an arbitrary small constant ψ multiplied the direction cosine α_{ak}^*) to create β_{ak}, as follows:

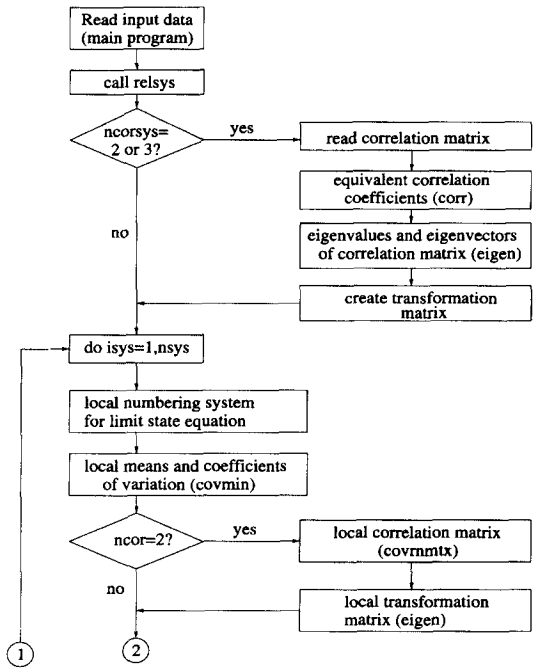
$$\begin{aligned} \beta_{ak} &= \beta_a + \psi(\alpha_{ak}^*) \\ \beta_{bk} &= \beta_b + \psi(\alpha_{bk}^*) \\ &\vdots = \vdots \\ \beta_{zk} &= \beta_z + \psi(\alpha_{zk}^*) \end{aligned} \tag{21}$$

Using these altered reliability indices and the original system correlation matrix ρ_{sys}, the revised system reliability index for the series or parallel system $\beta_{sys_{new(k)}}$ with respect to the random variable X_k is computed. The system gradient with respect to X_k is computed as

$$\frac{\partial(\beta_{sys})}{\partial(X_k)} = \frac{\beta_{sys_{new(k)}} - \beta_{sys}}{\psi} \tag{22}$$

This process is completed with respect to the rest of the random variables in the system from which the remainder of the system gradients are computed. Eq. (20) uses these gradients to create the equivalent alpha vectors and Eq. (19) computes the equivalent correlation coefficients.

The flowchart of the computer program RELSYS is shown in Fig. 5. For additional information the reader is referred to Estes and Frangopol (1997).



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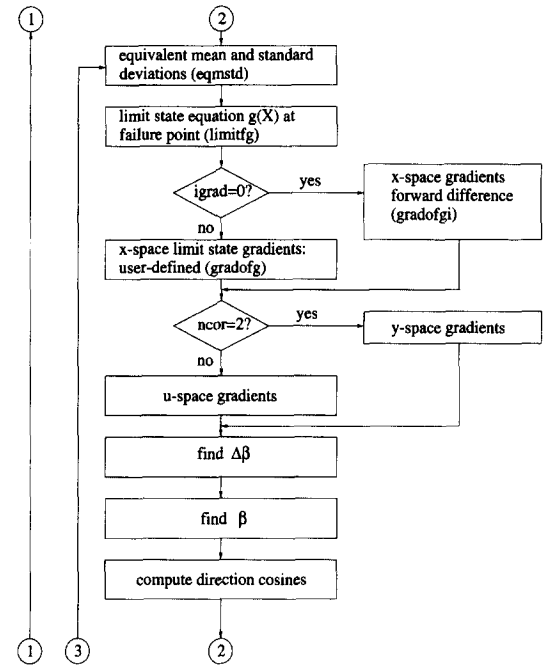
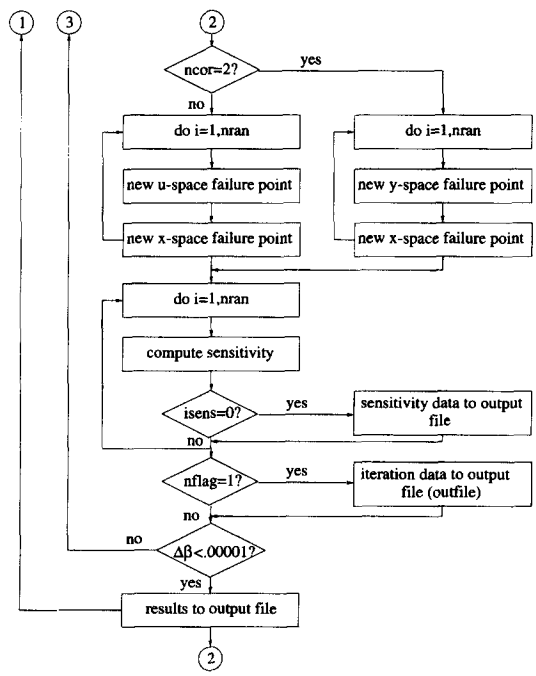
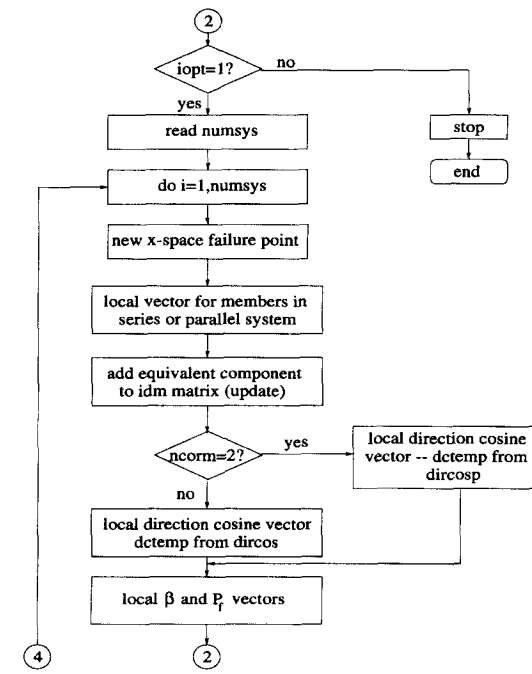


Part 2 of 5



Part 3 of 5



Part 4 of 5

Fig. 5 Flow chart for the program RELSYS

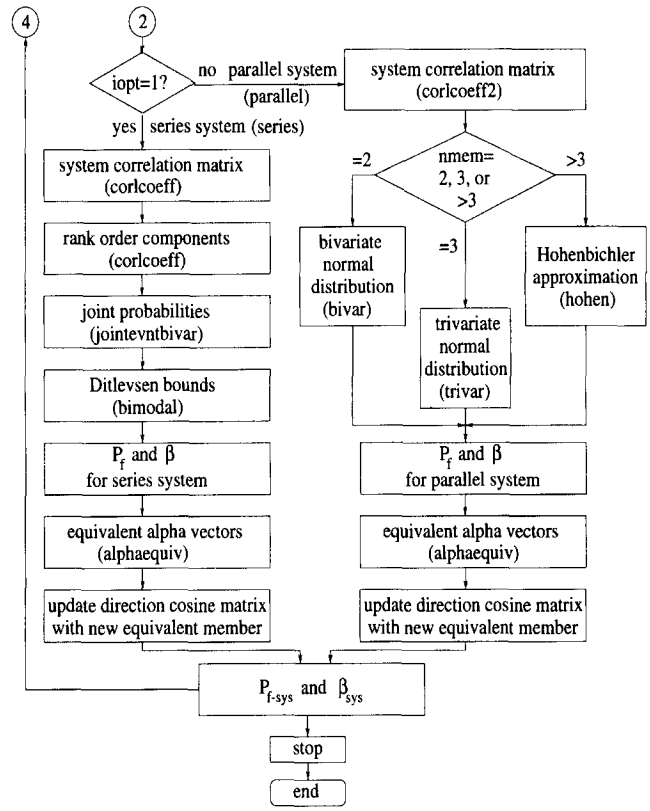


Fig. 5 Part 5 of 5

5. System reliability example

Consider the general series-parallel system shown in Fig. 6. It contains three components 1, 2, and 3 represented by the limit state equations $g(1)$, $g(2)$, and $g(3)$, as follows:

$$g(1) = 2X_1^2 - 2X_2 = 0$$
$$g(2) = X_1 - X_3 = 0$$
$$g(3) = 1.5X_1 + 0.5X_3^2 - X_2 = 0 \tag{23}$$

The reliability associated with the component 1 was computed earlier: β_1=1.2317. The general system is a parallel subsystem of components 1 and 3 in series with component 1, in series with a parallel combination of components 1, 2, and 3. The parameters of random variables X_1, X_2, and X_3 are listed in Table 2 and the direction cosines and reliability associated with each component are shown in Table 3.

Considering the parallel system of components 1 and 3, Eq. (13) uses the direction cosines to compute the system correlation coefficient $\rho_{sys_{13}}$=0.9032. The bivariate normal distribution, Eq. (16), is used to compute the reliability of the parallel system β_{sys}=1.3902 for β_1=1.2317, β_3=1.1479, and $\rho_{sys_{13}}$=0.9032.

Table 2 Random variables X_1, X_2, and X_3

Random variable	Mean value	Standard deviation	Distribution
X_1	2.0	0.2	Normal
X_2	3.0	0.3	Normal
X_3	1.0	0.1	Normal

Table 3 Component reliability results for $g(1)$, $g(2)$, and $g(3)$

Component	β	P_f	P_s=1-P_f	α^*_{a1}	α^*_{a2}	α^*_{a3}
g(1)	1.2317	0.1090	0.8910	0.9210	-0.3896	0.0
g(2)	4.4721	0.3875E-5	1.0	0.8944	0.0	-0.4472
g(3)	1.1479	0.1255	0.8750	0.6892	-0.6892	0.2238

The effect of random variable X_1 on the system reliability is considered first. Setting ψ=0.001, the adjusted component reliabilities based on the sensitivity of X_1 are:

$$\beta_{11} = \beta_1 + \psi(\alpha^*_{11}) = 1.2317 + 0.001(0.9210) = 1.2326$$
$$\beta_{31} = \beta_3 + \psi(\alpha^*_{31}) = 1.1479 + 0.001(0.6892) = 1.1485 \quad (24)$$

Solving the bivariate normal distribution where β_{11}=1.2326, β_{31}=1.1485 and $\rho_{sys_{11,31}}$=0.9033 yields $\beta_{sys_{new(1)}}$=1.3911. Table 4 shows the results for β_{1i}, β_{3i}, and $\beta_{sys_{new(i)}}$ for random variables X_2 and X_3.

The gradients of the system reliability with respect to each random variable are:

$$\frac{\partial(\beta_{sys})}{\partial(X_1)} = \frac{\beta_{sys_{new(1)}} - \beta_{sys}}{\psi} = 0.8344$$
$$\frac{\partial(\beta_{sys})}{\partial(X_2)} = -0.5286$$
$$\frac{\partial(\beta_{sys})}{\partial(X_3)} = 0.0989 \quad (25)$$

Using Eq. (20), the equivalent direction cosines $\alpha_{equiv_{ak}}$ which make up the equivalent alpha vector for equivalent component 4 are

$$\alpha_{equiv_{41}} = \frac{0.8344}{\sqrt{(0.8344)^2 + (-0.5286)^2 + (0.0989)^2}} = \frac{0.8344}{0.9927} = 0.8405$$

Table 4 Adjusted system reliabilities for two-member parallel system based on the sensitivities with respect to random variables X_1, X_2, and X_3

Random Variable	Component 1		Component 3		System
	α^*_{1i}	β_{1i}	α^*_{3i}	β_{3i}	$\beta_{sys_{new(i)}}$
X_1	0.9210	1.2326	0.6892	1.1485	1.3911
X_2	-0.3896	1.2313	-0.6892	1.1472	1.3897
X_3	0.0	1.2317	0.2238	1.1481	1.3903

$$\alpha_{equiv_{42}} = \frac{-0.5286}{0.9927} = -0.5325$$
$$\alpha_{equiv_{43}} = \frac{0.0989}{0.9927} = 0.0996 \quad (26)$$

This equivalent alpha vector can be used to compute the equivalent correlation coefficients between component 4 and any of the other component in the system. The reliabilities of the two components 1, β_1=1.2317, and 3, β_3=1.1479, are relatively close and each have relatively equal contributions to the reliability of the system. The equivalent direction cosines for each random variable in this case are close to the average of the individual direction cosines for each component.

The parallel system consisting of components 1, 2, and 3 is reduced to equivalent component 5 in a similar manner. Eq. (13) provides the system correlation coefficients $\rho_{sys_{12}}$= 0.8238, $\rho_{sys_{13}}$=0.9032, $\rho_{sys_{23}}$=0.5163 from which the system correlation matrix ρ_{sys} is created. The reliability of the parallel system β_{sys}=4.488 is computed from the trivariate normal distribution, Eq. (14), using ρ_{sys} and β=(β_1, β_2, β_3)^T=(1.2317, 4.4721, 1.1479)^T. Table 5 shows the adjusted reliabilities of the components based on the sensitivities of the random variables and the resulting equivalent alpha vector for the three-member parallel system.

In this three-member system, component 2 has a much higher reliability than the other two members and has the dominant effect on the reliability of the system. The equivalent direction cosines are likewise dominated by the component 2.

The system has been reduced to an equivalent three-member series system of components 1, 4, and 5 (see Fig. 6). The reliabilities and direction cosines for these three components are shown in Table 6. The direction cosines are used to compute the system correlation coefficients using Eq. (13). The component reliabilities and correlation coefficients are substituted into the bivariate normal distribution integral, Eq. (16), to compute all two-event joint probabilities. The upper and lower bimodal bounds are computed using Eqs. (18) and (17), respectively. For the series system of components 1, 4, and 5, the Ditlevsen's bounds were $P_{f_{upper}}$=0.11281 and $P_{f_{lower}}$=0.11281.

Using the average of the bi-modal bounds, the reliability index of the single equivalent component 6 and thus the reliability of the system is β_{sys}=1.2119. The equivalent alpha vector could be computed in the same manner based on the contribution of the random variables to the system reliability, but there is no need since the solution is complete. RELSYS uses this method to break any system which can be modeled as a series-parallel combination of the individual components into successively smaller equivalent systems, until only a single equivalent component is remaining.

Table 5 Adjusted system reliabilities for three-member parallel system based on the sensitivities of the random variables X_1, X_2, and X_3

Random variable	Component 1	Component 2	Component 3	System			
	β_{1i}	β_{2i}	β_{3i}	$\beta_{sys_{new(i)}}$	β_{sys}	$\frac{\partial(\beta_{sys})}{\partial(X_i)}$	$\alpha_{equiv_{5i}}$
X_1	1.2326	4.4730	1.1485	4.4891	4.4882	0.8885	0.9020
X_2	1.2313	4.4721	1.1472	4.4882	4.4882	-0.0174	-0.0177
X_3	1.2317	4.4717	1.1480	4.4878	4.4882	-0.4248	-0.4313

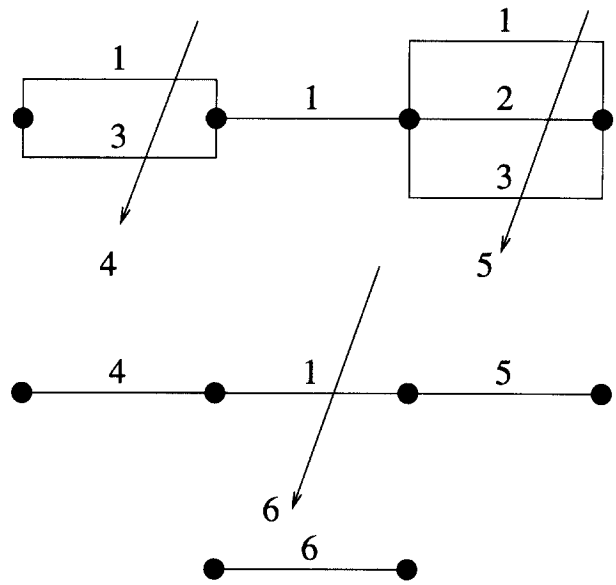


Fig. 6 Reduction for a series-parallel system to equivalent single component: Example problem

Table 6 Results for components 1, 4, and 5

Component	β	P_f	P_s=1-P_f	α^*_{a1}	α^*_{a2}	α^*_{a3}
1	1.2317	0.1090	0.8910	0.9210	-0.3896	0.0
4	1.3902	0.0823	0.9177	0.8405	-0.5325	-0.0996
5	4.4882	0.3598E-05	1.0	0.9020	-0.0177	-0.4313

The approximate system FORM approach described herein is very effective in many cases and provides good results with minimum computational effort. Until such methods were developed, the most popular estimation method for such analysis was Monte Carlo simulation which has some serious limitations, particularly for problems that require millions and sometimes billions of simulations to obtain a good solution. However, powerful adaptive importance sampling procedures significantly reduces the number of samples needed in a Monte Carlo approach and has successfully been used for system problems (e.g., Mori and Ellingwood 1993, Enright and Frangopol 1998a, b).

6. Strengths and limitations of RELSYS

RELSYS has the ability to compute the system reliability of any structure which can be modeled as a combination of series and parallel systems, regardless of the complexity. It is much faster than Monte Carlo simulation and gives reasonable and conservative results for most situations. RELSYS produces highly accurate results for all parallel systems with five or fewer members. The divergence of the Ditlevsen's bounds produced the largest errors when the reliabilities of components in a series system were all equal, but even the worst case produced answers that were within 10% of the Monte Carlo simulation solution (Estes 1997).

In general, RELSYS provides an excellent approximation for most problems with relatively little computational effort.

There are certain limitations to this approach. The FORM methods are approximations and have errors associated with them, especially with problems that are highly non-linear. Sometimes a local minimum is identified by the optimization process and the global minimum distance to the failure surface is missed. This causes the reliability to be over-estimated. This FORM approach uses only the first and second moment of random variables and, therefore, only the mean and standard deviation are considered. Other descriptors of random variables such as skewness are ignored. Creating an equivalent normal distribution from a non-normal distribution can generate errors, especially when that assumption is carried through the entire system analysis.

There is error associated with the bimodal bounds and the numerical integration of the multi-normal integral, especially when there is very high correlation between the components. In fact, any correlation higher than 0.99 is reduced to $\rho_{sys_{ab}}=0.99$ which creates an additional minor error - especially for those failure modes which are perfectly correlated. It has been argued that a joint β-point is a more accurate measure of the reliability of a parallel system than the multi-normal integral solution (Enevoldsen 1991, Enevoldsen and Sϕrensen 1990).

There is a potential problem with failure surfaces that are symmetrical or otherwise have multiple points that have the same minimum distance from the origin to the failure surface. The β value would be the same for any of these points but the direction cosines could be very different which may have a profound effect on the system reliability.

7. Highway bridge application

Estes (1997) used this approach and the RELSYS program to develop an optimum repair strategy for the existing Colorado State Highway Bridge E-17-AH located in the metro Denver area and shown in Fig. 7. The series-parallel model for the bridge system is shown in Fig. 8 where the performance functions $g(1)$ through $g(16)$ represent the various failure modes of the bridge to include moment failure of the deck, shear and moment failure in all of the girders, and various failure modes in the pier cap, pier columns, and column footings. The

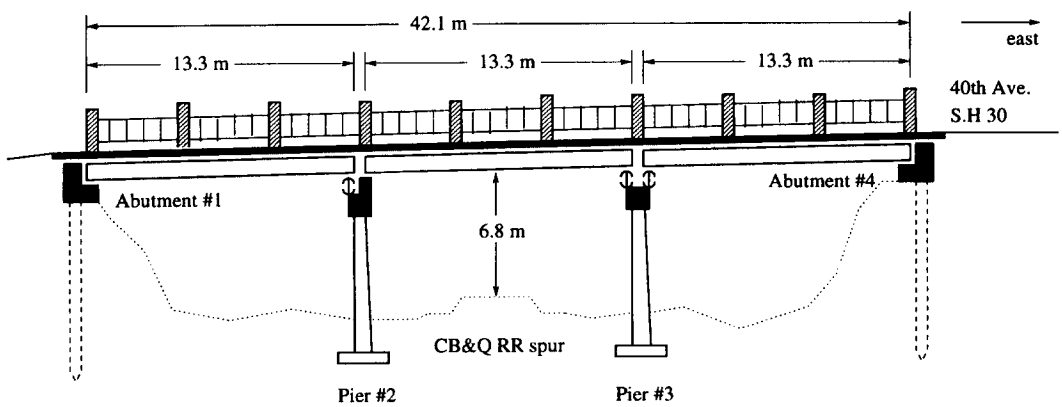


Fig. 7 Profile of Colorado State highway bridge E-17-AH

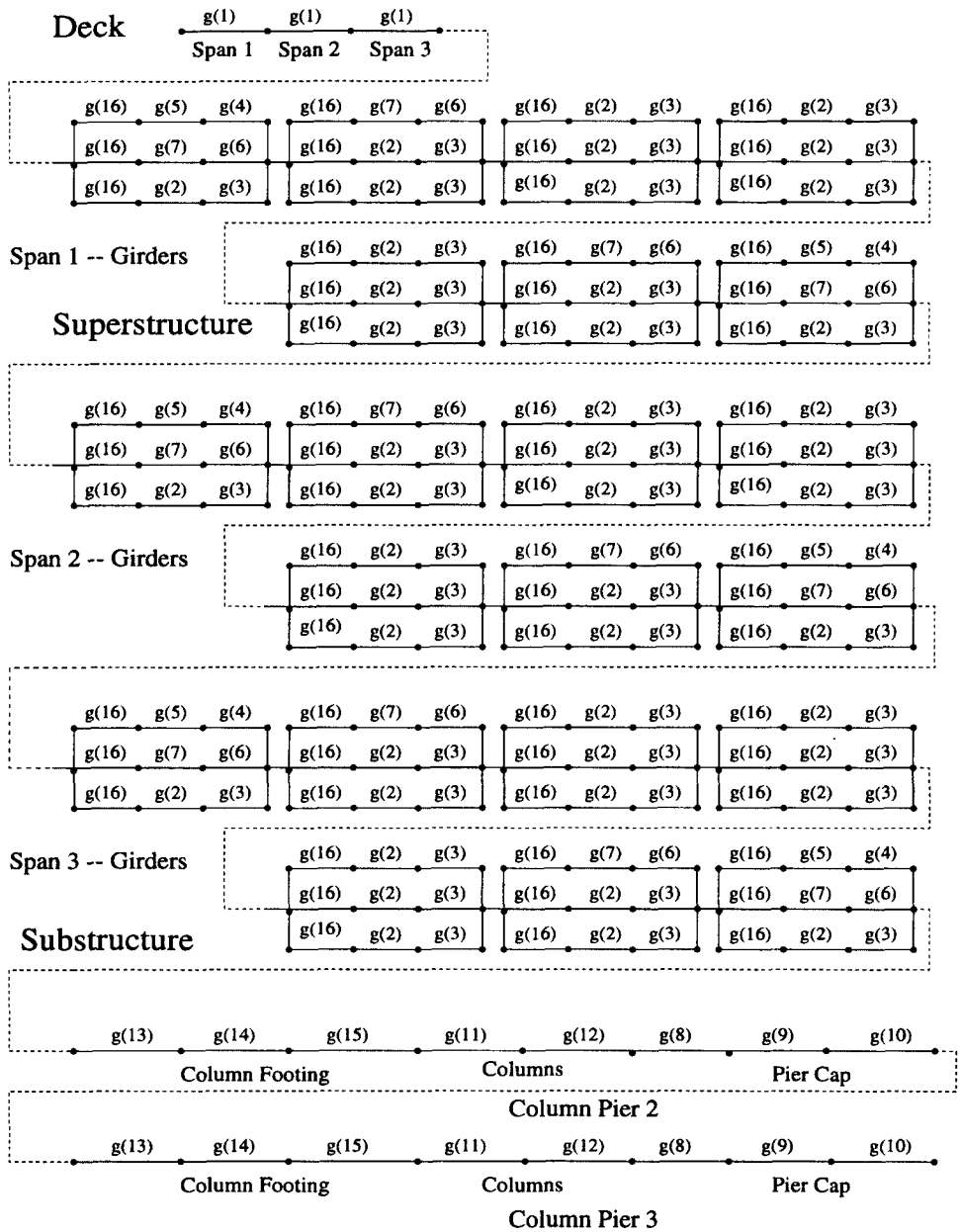


Fig. 8 Series-parallel model for bridge E-17-AH: deck, superstructure, and substructure

bridge system was modeled as a combination of the deck, superstructure, and substructure. It was assumed that the failure of any three adjacent girders was required for the structure to fail. Using deterioration and live load models, the system and component reliabilities were computed at many points in time through the life of the structure. RELSYS was able to provide accurate results in relatively little time for this highway bridge. Some of these results were reported by Estes and Frangopol (1997) and Frangopol and Estes (1997).

8. Conclusions

There has been tremendous progress in applying reliability-based methods to anlyze the safety and performance of structures. In general, structural reliability problems require a large number of complex computations. For this reason, efficient computer programs will continue to be needed to solve these problems. There is a constant trade-off between time consuming computations that provide very precise solutions and simplified methods that allow one to solve a wider variety of problems in less time, but with perhaps lower accuracy.

There remain many advantages to using a system approach to reliability-based problems where the importance of individual components relative to the safety of the overall structure is considered. The algorithm described herein provides quick and reasonable results for many problems where a structure can effectively be modeled as a series-parallel system of its components.

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