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Lateral buckling of reinfroced concrete beams without lateral support

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Abstract. Reinforced concrete beams possess variable flexural and torsional stiffnesses due to formation of cracks in the tension area along the beam. In order to check the stability of the beam, it is thus more appropriate to divide the beam into a finite number of segments for which mean stiffnesses and also bending moments are calculated. The stability analysis is further simplified, by using these mean values for each segment. In this paper, an algorithm for calculating the critical lateral buckling slenderness ratio for a definite load level, in a reinforced concrete beam without lateral support at the flanges, is presented. By using this ratio, the lateral buckling safety level of a slender beam may be checked or estimated.

Key words: reinforced concrete beams; slenderness; lateral buckling.

1. Introduction

A reinforced concrete beam shown in Fig. 1 is considered. This beam is assumed to be subjected to arbitrary loads acting in the *yz* plane, which is the plane of major flexural rigidity. This beam may buckle laterally at a certain critical value of the load. This lateral buckling is of importance in the design of beams without lateral support, provided that the flexural rigidity of the beam in the plane of bending is large in comparison with the lateral bending rigidity (Timoshenko and Gere 1988).

Appropriate design aids for the design of slender concete compression members have been developed in the past (Aydın 1990). In contrast to this for checking the lateral stability of slender reinforced concrete beams of such practically and generally aids are not sufficiently available.

The lateral buckling problem becomes important especially in steel structures. Reinforced concrete beams are generally casted monolithically with slabs. In this case, distorsion of the beam is prevented. Also the widths of the reinforced concrete beams should be selected wider than steel beams due to structural difficulties. Increasing the beam width results in an increase of the lateral flexural rigidity; thus, the lateral buckling possibility is decreased. However, the lateral buckling state is a question of the precast concrete structures. Critical situations may arise during the transportation of components or erection of precast concrete structures before adequate lateral restraint to components is provided (Park and Pauley 1975).

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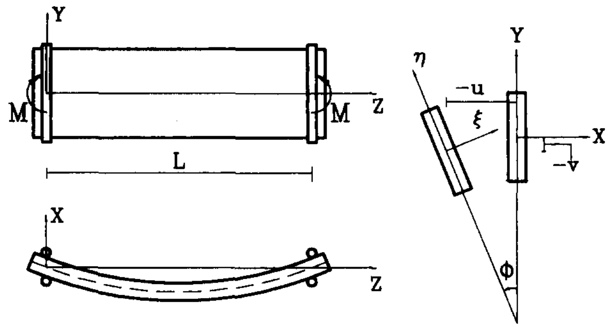

Fig. 1 The curvatures of the deflected axis of the beam.

2. Assumptions

The following assumptions are made:

1) The loads are acting along the centroidal axis of the beam,

2) The beam is subjected to loads acting in the *yz* plane, which is the plane of major flexural rigidity (Fig. 1),

3) The end rotations of the beam with respect to the *z* axis are prevented by some constraint,

4) Serviceability limit state is current. For this state, the mean strength values of concrete and the steel are considered since the real flexural and cracking states should be checked.

3. Models for concrete and steel

The stress-strain properties of concrete should be the same curve as obtained experimentally from uniaxially loaded specimens. However, in designs, some analytical models can be used. The most realistic curve to be used is the one presented by CEN (European Committee for Standardization – EC2 1992) (Fig. 2a). This analytical model gives more accurrate results especially in deformation calculations than other models (Litzner 1993).

The analytical expression for the model is given by

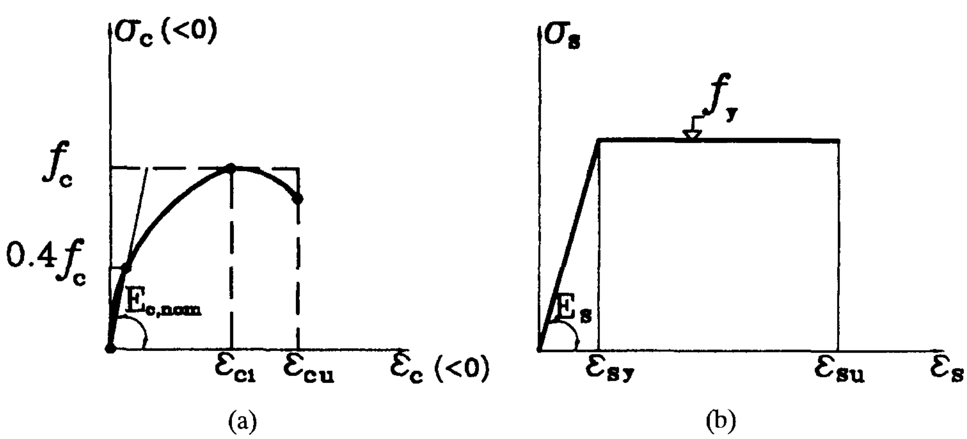

Fig. 2 (a) Assumed concete stress-strain curve; (b) Assumed steel stress-strain relation.

$$\sigma_c = f_c (k\eta - \eta^2)/(1+(k-2)\eta) \tag{1}$$

where

f_c = strenght of concrete

$\eta = \varepsilon_c / \varepsilon_{c1}$ and $\varepsilon_{c1} = -0.0022$ (strain of the peak compressive stress f_c).

$k = (1.1\ E_{c,nom}) \cdot \varepsilon_{c1} / f_c$

$E_{c,nom} = 9.5\,(f_{ck}+8)^{1/3}$ ($E_{c,nom}$ and f_{ck} in N/mm^2)

$E_{c,nom}$ = secant modulus of elasticity

f_{ck} = characteristic compressive strength of concrete

Steel $\sigma_s - \varepsilon_s$ relation is assumed to behave bilinearly, as shown in Fig. 2(b). The modulus of elasticity of steel E_s is 2.10^5 N/mm^2.

4. Lateral buckling of a beam

If a beam is subjected to pure bending at both ends as shown in Fig. 1. The angle of twist ϕ can be represented in the form (Timoshenko and Gere 1988):

$$\phi = A_1 \cdot \cos Cz + A_2 \cdot \sin Cz \tag{2}$$

where

$$C^2 = \frac{M^2}{(GJ)\cdot(EI_\eta)} \tag{3}$$

(EI_η) and (GJ) are the flexural stiffness about the η axis and the torsional stiffness of the cross section, respectively.

The constants of integration A_1 and A_2 are determined from the boundary conditions.

5. Constant bending moment, curvature and stiffnesses in reinforced concrete beam

Eqs. (2) are valid along the beam for which the bending moment and stiffnesses are constant. In fact the bending moment along the beam is variable. In addition to this, the stiffnesses are also variable due to presence of cracks. In this case, in order to apply Eqs. (2), the beam is divided into finite segments.

The procedure for evaluating the bending moment and stiffness that are assumed to be constant on each segment is as follows:

5.1. Conversion of the bending moment diagram to the constant moment values

As an example, the simple supported beam subjected to a uniformly distributed load is divided into 10 segments of equal length. Then, since the strain energy for the varying and constant bending moments is equal for each corresponding segment, κ coefficients shown in Table 1 can be obtained from this equivalence (Fig. 3).

By using the bending moment for a uniformly distributed load,

$$M = \frac{qL^2}{2}\left[\frac{z}{L} - \frac{z^2}{L^2}\right] \tag{4}$$

Table 1 Bending moment conversion coefficients κ for a uniformly distributed load

Segment #	1	2	3	4	5
κ Eq. (8)	0.214	0.513	0.749	0.907	0.987
					Centerline

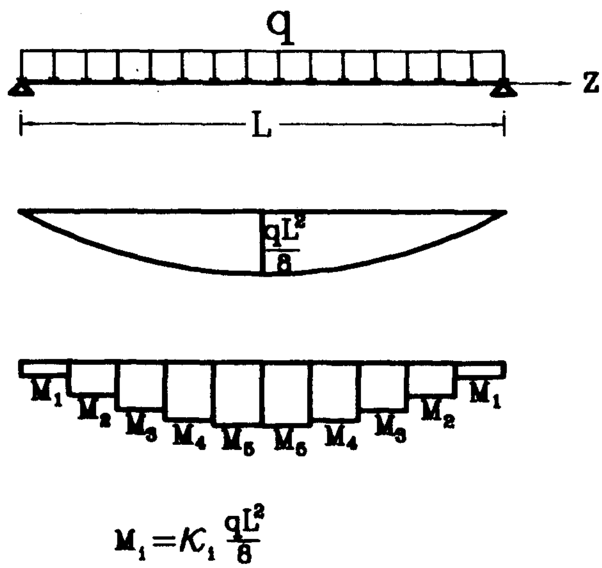

Fig. 3 Conversion of the bending moment diagram to constant moment values of a simple supported beam subjected to a uniformly distributed load.

The strain energy due to varying bending moment

$$U = \int_{z_1}^{z_2} \frac{M^2}{2(EI_x)}\, dz = \frac{q^2 L^5}{8(EI_x)} \left[\frac{z_2^3 - z_1^3}{3L^3} + \frac{z_2^5 - z_1^5}{5L^5} - \frac{z_2^4 - z_1^4}{2L^4} \right] \tag{5}$$

Strain energy due to constant moment M

$$U = \int_{z_1}^{z_2} \frac{M^2}{2(EI_x)}\, dz = \frac{M^2}{(2EI_x)} (z_2 - z_1) \tag{6}$$

By equating Eqs. (5) and (6) and rearranging, we obtain

$$M = \kappa \frac{qL^2}{8} \tag{7}$$

$$\kappa = \left[\frac{16}{\bar{z}_2 - \bar{z}_1} \left[\frac{1}{3}\left(\bar{z}_2^3 - \bar{z}_1^3\right) + \frac{1}{5}\left(\bar{z}_2^5 - \bar{z}_1^5\right) - \frac{1}{2}\left(\bar{z}_2^4 - \bar{z}_1^4\right) \right] \right]^{1/2} ; \bar{z} = \frac{z}{L} \tag{8}$$

5.2. Moment curvature relationship for reinforced concrete cross section

The value of the concrete compressive force F_c and its acting distance from the extreme com-

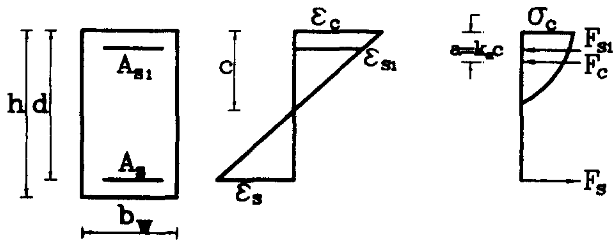

Fig. 4 Reinforced concrete cross section, dimensions, stress and strain distribution and the acting internal forces.

pression fiber may be defined from the $\sigma_c - \varepsilon_c$ relationship Eq. (1) shown in Fig. 2(a) as follows (Grabner 1989)(Fig. 4):

$$F_c = 0.85\ \alpha_v\ f_{cm}\ b_w\ c \tag{9}$$

$$\alpha_v = \frac{k}{k-2}\left(1-\frac{\ln N}{N-1}\right)-\frac{s_1}{\eta}\ ;\ k_a = 1-\frac{1}{\alpha_v \cdot \eta^2}\left(ks_1-\frac{s_2}{(k-2)^4}\right) \tag{10a}$$

Here;

$$N = 1+(k-2)\eta\ ;\quad \eta=\varepsilon_c/\varepsilon_{c1}$$

$$s_1 = \frac{1}{(k-2)^3}\left(0.5N^2-2N+\ln N+1.5\right)$$

$$s_2 = \frac{1}{3}N^3-1.5N^2+3N-\ln N-\frac{11}{6}$$

Eq. (10a) will be transformed into a quadratic equation for k=2. In this case;

$$\alpha_v = \frac{\eta(3-\eta)}{3}\ ;\quad k_a = \frac{4-\eta}{4(3-\eta)} \tag{10b}$$

The steel force is:

$$F_s = A_s \cdot \sigma_s\ ;\quad \sigma_s \le f_{ym} \tag{11}$$

The coefficient 0.85 in Eq. (9) shows the effect of the long term loads. In a cross section shown

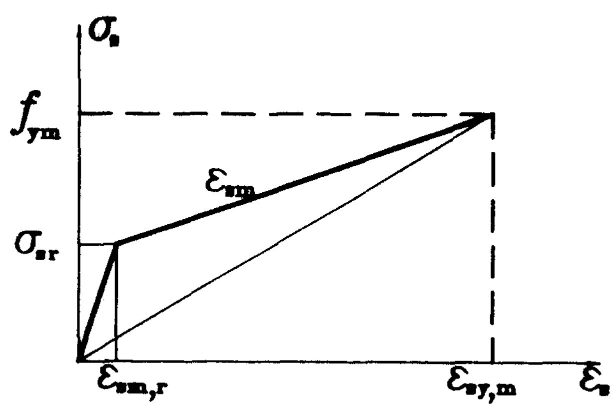

Fig. 5 Accepted mean value of steel strain for consideration of the tension effect between the cracks.

in Fig. 4, the neutral axis depth c that satisfies the horizontal equilibrium condition for a determined value of ε_c can be defined by using the Eqs. (9) and (11) and employing a trial and error method so that the moment resisting capacity of the cross section is:

$$M = M_c + M_s = F_c(c - a) + F_s(d - c) \tag{12}$$

The curvature of cross section is:

$$\psi = \frac{\varepsilon_s - \varepsilon_c}{d} \tag{13}$$

Curvature obtained from Eq. (13) refers to a cracked section. Tension effects between the cracks contribute to curvature. In different Codes (Specification 1978, "Building" 1989, EC2 1992, CEB-FIB 1990) the consideration of this tension effect has been taken into account and were studied in detail (Ghali 1993, Favre and Charif 1994).

In this paper, the tension effect between the cracks has been considered as shown in Fig. 5.

ε_{sm} = The mean value of steel strain.
$\varepsilon_{sm,r}$ = The mean value of steel strain of the uncracked cross section for the load at first cracking (state 1)
$\varepsilon_{sy,m}$ = The mean value of steel strain for yield stress f_{ym}
σ_s = Steel stress
σ_{sr} = Steel stress at the cracked cross section for the load at the first cracking (state 2)
f_{ym} = The mean value of yield stress of steel

In calculating curvature ψ_m for each segment, instead of strain ε_s the mean value of strain ε_{sm} is used.

$$\psi_m = \frac{\varepsilon_{sm} - \varepsilon_c}{d} \tag{14}$$

5.3. Stiffnesses of reinforced cross section

If the moment expression for concrete Eq. (12) is divided to the mean value of curvature, then the mean value of stiffness for concrete is obtained. Thus:

$$\frac{M_c}{\psi_m} = (EI)_{cm} = E_{cm} \cdot I_{cm} \tag{15}$$

where E_{cm} is the mean value of modulus of elasticity for concrete and it is defined from the Eq. (15), as follows:

$$E_{cm} = \frac{M_c}{\psi_m I_{cm}} \tag{16}$$

The mean value of modules of elasticity E_{cm} obtained of Eq. (16) is assumed to be valid for bending about both x and y axes.

I_{cm} is the mean value of the moment of inertia of the concrete compressive area about the modified neutral axis for a rectangular cross section, I_{cm} is given by:

$$I_{cm} = \frac{b_w C_m^3}{3} \tag{17}$$

Here c_m is the modified neutral axis depth and is calculated from:

$$c_m = \frac{\varepsilon_c}{\psi_m} \tag{18}$$

Once the mean value of the modulus of elasticity for concrete and the neutral axis depth have been calculated, then the stiffnesses are obtained by calculating the torsional stiffness, the modulus of elasticity in shear $G(G \approx 0.40E_{cm})$ and torsional moment of inertia J as that of the gross concrete cross-section. For calculating the flexural stiffness about the y axis, the mean values of flexural stiffnesses for cracked and uncracked section might be taken (Litzner 1993) and it might also be considered that before buckling the η and y axes coincide.

Thus

$$(EI_y) = \frac{1}{2}\left[(E_{cm}I_{cy} + E_s I_{sy}) + (E_{cm}I_{y1})\right] \tag{19}$$

where:

I_{cy} = Moment of inertia of the concrete compressive area about the y axis at the cracked section.
I_{sy} = Moment of inertia of steel about the y axis.
I_{y1} = Moment of inertia about the y axis including the reinforcement at the uncracked section.

If the moment acting at the cross section is less than the cracking moment M_{cr}, then the mean value of modulus of elasticity for concrete becomes

$$E_{cm} = \frac{M_{cr}}{\psi_{cr} I_1} \tag{20}$$

where

I_1 = Moment of inertia of transformed cross section composed of A_c plus A_s

$$\psi_{cr} = \frac{\varepsilon_{sm,r}}{d - c_1} \tag{21}$$

c_1 = Neutral axis depth of transformed cross section.
$\varepsilon_{sm,r}$ = (see Fig. 5)

$$M_{cr} = \frac{f_r \cdot I_1}{h - c_1}; \quad f_r = 0.6(f_{cm} - 8)^{2/3} \tag{22}$$

f_r = Modulus of rupture of concrete in N/mm^2 (EC2 1992)

$$(EI_y) = E_{cm} \cdot I_{y1} \tag{23}$$

(EI_y) and (GJ) are the flexural stiffness and torsional stiffness of the beam about y axis and it is more appropriate to have those values in a dimensionless form as follows;

$$(EI_y) = (\overline{EI}_y)\, b_w^3 h \;\; f_{cm} \tag{24}$$

$$(GJ) = (\overline{GJ})\, b_w^3 h \;\; f_{cm} \tag{25}$$

where

$(\overline{EI}_y)$ and $(\overline{GJ})$ are the dimensionless stiffness coefficients.

6. Calculation of critical lateral buckling ratio

Aydın (Aydın 1992) has studied the critical lateral buckling slenderness ratios which cause lateral buckling of beams for different external loadings. In this study, the Hognestad (Hognestad 1951) model is used for the concrete stress strain relationship and the effect of concrete stiffness in the tension area is ignored.

The following algorithm is outlined for a beam of known steel and concrete grades and steel arrangements. A simple supported beam subjected to a uniformly distributed load is taken as an example for explaining the algorithm, but if desired, it can also be applied to different beams of varying cross-sections and loads.

1) The beam is divided into 10 segments.

2) For each segment, coefficient C of Eq. (3) is obtained (Fig. 3 and Eqs. (7) and (19) to (25)).

$$CL = \frac{M}{((GJ)(EI_y))^{1/2}} L = \frac{\kappa}{((\overline{GJ})(\overline{EI}_y))^{1/2}} \cdot \frac{qL^2}{8b_w h^2 f_{cm}} \cdot \frac{Lh}{b_w^2} \tag{26}$$

In the above Eq. (26), the first term is relating to the cross-section properties and the second term to internal and external moments. These both terms are constant. Only the last term Lh/b_w^2 is relating to slenderness of the beam. The ratio of Lh/b_w^2 is defined as the critical slenderness ratio λ_{cr} in lateral buckling cases and it is also the ratio which brings the beam to the buckling state.

3) Eqs. (2) are established by taking the boundary conditions into account. For a simple supported beam, the following conditions may be used: ϕ=0 at z=0 and z=L and at the interfaces of each segment ϕ and its derivative ($d\phi/dz$) are equal. Finally, a set of homogeneous equations are obtained and the resulting equations will be twice the number of segments. If the external loading of the simple supported beam is symmetric, then the number of equations are halved. In this case the boundary condition $d\phi/dz$=0 at $z=L/2$ can be applied. If the beam is a cantilever, then the boundary conditions will be ϕ=0 at z=0 and $d\phi/dz$=0 at z=L.

4) A set of equations that are obtained in step 3 are solved. In solution of these equations, the determinant of the coefficient matrix is set to zero. An iterative procedure that satisfies this condition is applied; that is, an initial guess to the term Lh/b_w^2 of Eq. (26) is assigned and this guess is updated with a small increment until a zero-determinant solution is satisfied.

5) Slenderness of the beam λ is compared with the value λ_{cr} found in step 4. As a result of this comparison the safety coefficient of lateral buckling γ is calculated as follows:

$$\gamma = \frac{\lambda_{cr}}{\lambda} \tag{28}$$

According to this comparison, it might be said: If $\gamma<1$ the beam is unsafe for lateral buckling.

7. Numerical example

To demonstrate the application of this method presented here, the numerical example of Deneke, Holz and Litzner (1985) was used.

Datum for the beam are :

Simple supported beam (end rotations of the beam are prevented with respect to z axis)

Span of the beam L=18 m.
Total uniformly distributed load q=25.14 kN/m.

$f_{cm}^{*} = 50$ N/mm^2 (The mean value of 28 days cube compressive strength)
$f_{cm} = 50/1.18 = 42.37$ N/mm^2 (The mean value of 28 days cube compressive strength converted to cylinder compressive strength),
$f_r = 0.6\ (42.37 - 8)^{2/3} = 6.34$ N/mm^2
$f_{ym} = 42000$ N/mm^2
$A_{s1} = 452$ mm^2 (4ϕ12)
$A_s = 3694$ mm^2 (6ϕ28)

$$maxM = \frac{qL^2}{8} = \frac{25.14 \times 18^2}{8} = 1018.17 \text{ kNm}.$$

Cracking moment of the beam M_{cr}= 550.46 kNm.
A computer program was prepared to calculate the safety coefficient of lateral buckling and the following values of this numerical example are obtained by using this program.
The computed C coefficients are given below.

Segment	1	2	3	4	5
$C(b_w^3\, hf_{cm})$	148.97	68.37	89.06	98.96	102.81

By using these coefficients, the value of $\lambda_{cr} = Lh / b_w^2$ that satisfies zero-determinant of the coefficient matrix of Eqs. (2) is found to be 4944.
Slenderness of the beam

$$\lambda = \frac{Lh}{b_w^2} = \frac{18 \times 1.6}{0.13^2} = 1704$$

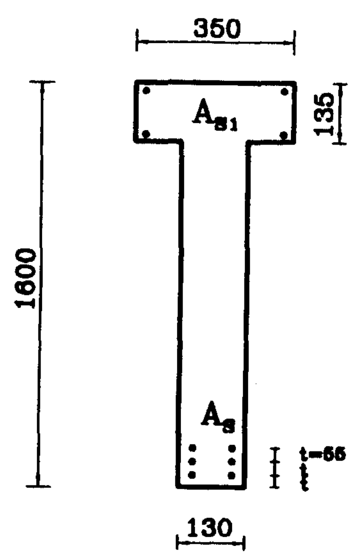

Fig. 6 Reinforcement and dimension of the beam cross-section.

Safety coefficient of lateral buckling $\gamma = \frac{\lambda_{cr}}{\lambda} = \frac{4944}{1704} = 2.9$

If the load coefficient were considered, then the safety coefficient γ would be found lower than the value 2.9

The reported safety coefficient by Deneke, *et al.* (1985) is 2.67

8. Conclusions

Developments in material technologies and production techniques in application brings about the possibility of lateral buckling in structural components. There is not sufficient proposals relating to the verification of lateral buckling safety in reinforced concrete beams. EC2 (EC2 1992) recommends that in the case of slenderness ratios Lh/b_w^2 of greater than 125, a more detailed analysis should be carried out. It is seen that the addmisible lateral buckling slenderness ratio of a beam is rather small. For this reason, methods that are safe and easily applicable should be developed. In this study, a such method is proposed.

The proposed algorithm is a general one and can be applied to any cross sections with different steel arrangements. It can also be applied to beams of varying cross sections and to continuous beams.

The number of segments were varied in the computations. As a result, it was observed that increasing the segments did not result in variation in the safety coefficient. The optimum segment number was found to be 10.

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Notations

The following symbols are used in this paper.

A_s	area of tension steel
b	width of the compression flange
b_w	width of the web of beam
c	distance from extreme compression fiber to neutral axis
d	distance from extreme compression fiber to centroid of bottom steel layer
E_{cm}	the mean value of modulus of elasticity of concrete
E_s	modulus of elasticity of steel
$(EI)_{cm}$	the mean value of flexural stiffness of concrete
(EI_x), (EI_y)	flexural stiffness of the cross section about x and y axes, respectively
(EI_ξ), (EI_η)	flexural stiffnesses of the beam about the ξ and η axes
$(\overline{EI_y})$	dimensionless flexural stiffness coefficient
F_c, F_s	concrete compressive force, steel force respectively
f_{cm}	the mean value of concrete cylinder compressive strength
f_{ym}	the mean value of yield stress of steel
G	modulus of elasticity in shear
GJ	torsional stiffness
$(\overline{GJ})$	dimensionless torsional stiffness coefficient
h	overall thickness of member
I_{cm}	the mean value of moment of inertia of concrete compressive area about the modified neutral axis
I_{cy}	moment of inertia of the concrete compressive area about y axis at the cracked section
I_{sy}	moment of inertia of steel about y the axis
I_1, I_{y1}	moment of inertia of the transformed cross section composed of A_c plus A_s about the x and y axes, respectively
J	torsional moment of inertia
L	span of the beam
M	constant bending moment about the x axis
M_c	moment resistance capacity by the concrete at the cross section
M_{cr}	cracking moment
x, y, z	fixed coordinate axes
ε_c	strain of concrete

ε_s	strain of steel
ε_{cu}	ultimate compressive strain of concrete
ε_{su}	ultimate strain of steel
ϕ	the angle of rotation of the cross section about the z axis
γ	safety coefficient of lateral buckling
κ	ratio of the moment that is accepted as constant at the 1/10 th segments of the beam span to the maximum moment
λ	slenderness of the beam
λ_{cr}	critical lateral buckling slenderness ratio
σ_c	concrete stress
σ_s	steel stress
ψ	curvature
ψ_m	the mean curvature
ψ_{cr}	curvature of the cracking moment