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Axisymmetric bending of a circular plate with stiff edge on a soft FGM layer

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Abstract. A circular plate with constant thickness, finite radius and stiff edge lying on an elastic half-space is considered. The half-space consists of a soft functionally graded (FGM) layer with arbitrary varying elastic properties and a homogeneous elastic substrate. The plate bends under the action of arbitrary axisymmetric distributed load and response from the elastic half-space. A semi-analytical solution for the problem effective in whole range of geometric (relative layer thickness) and mechanical (elastic properties of coating and substrate, stiffness of the plate) properties is constructed using the bilateral asymptotic method (Aizikovich *et al*. 2009). Approximated analytical expressions for the contact stresses and deflections of the plate are provided. Numerical results showing the qualitative dependence of the solution from the initial parameters of the problem are obtained with high precision.

Keywords: plate bending; circular plate; Kirchhoff plate; axisymmetric problem; functionally graded; soft layer; elastic layer; analytic method

1. Introduction

The problem of plate bending on an isotropic homogeneous elastic foundation was considered by Gorbunov-Posadov (1940). The solution was constructed by providing the contact stresses in the form of a power series with subsequent determination of the coefficients of expansion from infinite system of algebraic equations. The similar problem was also solved using collocation method (Shatskih 1972, Aleksandrov and Salamatova 2009) and approach based on approximations of the integral equations kernels by orthogonal polynomials (Aleksandrov *et al.* 1973, Bosakov 2008). The convergence of the solution to the exact one was not investigated. Singular and regular asymptotic methods were used to construct solutions effective for big or

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small values of the geometric parameter of the problem (Aleksandrov 1973, Aleksandrov and Salamatova 2009, Aleksandrov and Solodovnik 1974). Most of the known solutions are applicable only for rigid plates. And very few, particularly those described in (Aleksandrov 1973, Bosakov 2008, Aleksandrov and Solodovnik 1974) are applicable either for flexible or rigid plates. There are a number of recent investigations in this area (Silva *et al.* 2001, Guler 2008, Pak *et al*. 2008, Selvadurai *et al*. 2010). Woodward and Kashtalyan (2011, 2012) considered only specific harmonic loading of a rectangular plate that makes it possible to construct an analytical solution of the three-dimensional problem.

Many modern researches focus on elasticity problems for a functionally graded materials and coatings (Guler 2008, Guler *et al.* 2012, Liu *et al.* 2008). Often some special assumptions about the variation of elastic properties in the layer are accepted to make it possible obtaining analytical solutions of the corresponding differential equations. Let us also mention the analysis of deformations of plates and shells made of functionally graded materials within the first-order shear deformable plate theory (Altenbach and Eremeyev 2008, 2009, Arefi and Allam 2015) and higher-order plates and shells models (Arciniega and Reddy 2007). Plasticity and impact analysis of circular plates (Babaei *et al.* 2015 and others) are also of high practical interest.

In this work we construct an approximated analytical solution for problem of bending of a plate in unified form, applicable for any values of geometric and mechanical properties. The problem considered in the paper is of practical interest in modeling of the interaction of thin-walled elements of designs, the foundations of buildings, tank bottoms, sunk wells, etc. as well as in modeling nano- or macro- sized thin films. Our approach based on the bilateral asymptotic method (Aizikovich *et al.* 2009) of solving a certain type of dual integral equations, in particular, allows to consider arbitrary type of variation of elastic moduli in depth, any thickness of the coating and value of the plate stiffness.

2. Mathematical formulation of the problem

Circular plate of radius R and constant thickness h lying on the boundary of an elastic half-space, consisting of inhomogeneous layer (coating) with thickness H and homogeneous half-space (substrate). We use a cylindrical coordinate system r, φ, z, where z axis is perpendicular to the surface of the coating and passes through the center of the plate. Plate is bent under the action of an axisymmetric distributed load $p^*(r)$ and response from the elastic layer.

Young's modulus E and Poisson's ratio ν of the foundation vary with depth according to the following

$$\mathrm{E}(z),\nu(z)=\begin{cases}E_1(z),\nu_1(z),-H\le z\le 0\\ E_2,\nu_2, \quad -\infty<z<-H\end{cases}, E_2,\nu_2=\text{const} \tag{1}$$

where $E_1(z)$, $\nu_1(z)$ are arbitrary continuously differentiable functions. Hereafter, indexes $_1$ and $_2$ correspond to the coating and to the substrate, respectively.

The layer and the substrate are assumed to be glued without sliding

$$z=-H:\tau_{zr}^1=\tau_{zr}^2,\ \sigma_z^1=\sigma_z^2,\ w^1=w^2,\ \ u^1=u^2 \tag{2}$$

Outside of the punch, the surface is traction-free

$$z=0:\ \tau_{zr}^{1}=0,\ \begin{cases}\sigma_z^1=0, & r>R\\ w^1=-w^*(r), & r\le R\end{cases} \tag{3}$$

The stresses and the displacements vanish at $r\to\infty$ and $z\to\infty$.

The quantities of primary interest are the contact stresses under the plate $q^*(r)=\sigma_z\big|_{z=0}$, the deflections of the plate $w^*(r)$, radial and tangential torques M_r, M_φ.

We consider two types of the boundary conditions on the edges of the plate

a) Plate with a stiff edge: $$\left.\frac{\partial w^*}{\partial r}\right|_{r=R}=0,\ P_r=-D\frac{\partial}{\partial r}\left(\Delta w^*\right)\Big|_{r=R}=0 \tag{4}$$

First boundary condition describe quite stiff contours of the plate. It may arise in the calculation of tank bottoms, sunk wells, etc. (Tseitlin 1969, Selvadurai *et al.* 2010).

b) Free edge: $$M_r=-D\left(\frac{\partial^2 w^*}{\partial r^2}+\frac{\nu_{\text{plate}}}{r}\frac{\partial w^*}{\partial r}\right)\Bigg|_{r=R}=0,\quad P_r=-D\frac{\partial}{\partial r}\left(\Delta w^*\right)\Big|_{r=R}=0 \tag{5}$$

Such boundary conditions arise in modeling foundations, free-lying plates, etc.

We use the following notations: ν_{plate} is the Poisson's ratio of the plate, $\Delta=\dfrac{d^2}{dr^2}+\dfrac{1}{r}\dfrac{d}{dr}$ is the Laplace operator. Second boundary condition in (4), (5) means absence of a shear force P_r.

3. Solution of the problem

3.1 Dual integral equation

To reduce the problem to the solution of the dual integral equation we use the classical approach based on the Hankel's integral transformations technique

$$w^1(r,z)=\int_0^\infty W_1(\gamma,z)\mathrm{J}_0(\gamma r)\gamma d\gamma\,, \tag{4}$$

$$Q^*(\gamma)=\int_0^R q^*(\rho)\mathrm{J}_0(\gamma\rho)\rho d\rho\,,\ \ q^*(r)=\int_0^\infty \gamma Q^*(\gamma)\mathrm{J}_0(r\gamma)d\gamma \tag{5}$$

Then Eq. (3) leads to the following

$$\int_0^\infty W_1\ (\gamma,0)\gamma\,\mathrm{J}_0(\gamma r)d\gamma=-w^*(r),\ r\le R \tag{6}$$

Let us introduce the following notations

$$W_1^*(\gamma,z)=-\Theta\frac{\gamma W_1(\gamma,z)}{Q^*(\gamma)}\,,\ \ \Theta=\frac{E(0)}{2(1-\nu^2(0))} \tag{7}$$

Using Eqs. (7) we rewrite equation Eq. (6) in the form

$$\int_0^\infty W_1^*(\gamma,0)Q^*(\gamma)\mathrm{J}_0(\gamma r)d\gamma = \Theta w^*(r),\ r \le R \tag{8}$$

Substituting Eqs. (5) into Eq. (8) we get

$$\int_0^R q^*(\rho)\rho d\rho \int_0^\infty W_1^*(\gamma,0)\mathrm{J}_0(\gamma r)\mathrm{J}_0(\gamma\rho)d\gamma = \Theta w^*(r),\ r \le R \tag{9}$$

Let us introduce the dimensionless variables and functions:

$$\gamma H = u,\ \lambda = H/R,\ r' = r/R,\ \rho' = \rho/R,\ W_1^*(u/H,0) = L(u),\ q^*(r) = q(r')DR^{-3},$$
$$z' = z/H,\ w^*(r) = w(r')R,\ p^*(r) = p(r')DR^{-3},\ \alpha = u\lambda^{-1},\ s = \Theta R^3/D$$

where D is the cylindrical stiffness of the plate, parameter s is the dimensionless bending stiffness of the plate. Function $L(u)$ is the kernel transform of the integral equation, independent of the applied loading $p^*(r)$ and characterizes the compliance of the elastic foundation. The kernel transform $L(u)$ is equal to that appearing in the contact problem of the indentation of a rigid stamp (Aizikovich and Aleksandrov 1984).

Omitting the primes in Eq. (9) we get the Fredholm integral equation of the first kind over the function $q(\rho)$

$$\int_0^1 q(\rho)\rho d\rho \int_0^\infty L(\lambda\alpha)\mathrm{J}_0(\alpha r)\mathrm{J}_0(\alpha\rho)d\alpha = sw(r),\ r \le 1 \tag{10}$$

Eq. (10) is equivalent to the following dual integral equation

$$\begin{cases} \displaystyle\int_0^\infty Q(\alpha)L(\alpha\lambda)\mathrm{J}_0(\alpha r)d\alpha = sw(r),\ 0 \le r \le 1 \\ \displaystyle\int_0^\infty Q(\alpha)\mathrm{J}_0(\alpha r)d\alpha = 0,\ r > 1 \end{cases} \tag{11}$$

According to the Kirchhoff's plate model the deflection of the plate $w(r)$ has to satisfy the differential equation of bending of the plate

$$\mathbf{L_0}w(r) = p(r) - q(r),\ \mathbf{L_0} = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)^2 0 \le r \le 1, \tag{12}$$

3.2 Solution of the problem

In the present paper we construct the solution only for a case of the plate with a stiff edge (see Eqs. (4)).

We represent the deflection function in terms of series with respect to eigenfunctions of oscillations of a circular plate with free edges (similar to Tseitlin 1969)

$$w(r) = \sum_{m=0}^\infty w_m\varphi_m(r), \quad w_m = \int_0^1 w(\rho)\varphi_m(\rho)\rho d\rho, \quad \varphi_m(r) = A_m\mathrm{J}_0(k_m r) \tag{13}$$

where k_m are the roots of the equation $J_1(k_m)=0$, and $A_m = \sqrt{2}/\mathrm{J}_0(k_m), m = 0,1,2,\ldots$.

Due to the linearity of the problem the contact stresses $q(r)$ and it's Hankel transform $Q(\alpha)$ can be represented as

$$q(r) = \sum_{m=0}^{\infty} w_m q_m(r),\ Q(\alpha) = \sum_{m=0}^{\infty} w_m Q_m(\alpha)\ \ 0 \le r \le 1, \tag{14}$$

Substituting Eqs. (13) and (14) into Eq. (11) we get the dual integral equation over the function $Q_m(\alpha)$

$$\begin{cases} \int\limits_0^{\infty} Q_m(\alpha) L(\lambda\alpha)\, \mathrm{J}_0(r\alpha) d\alpha = s\varphi_m(r), & r \le 1 \\ \int\limits_0^{\infty} Q_m(\alpha)\, \mathrm{J}_0(r\alpha) \alpha d\alpha = 0, & r > 1 \end{cases} \tag{15}$$

Let us apply to the first equation in Eq. (15) the integral operator $U_1^t \varphi = \dfrac{d}{dt} \int\limits_0^t \dfrac{r\varphi(r)dr}{\sqrt{t^2 - r^2}}$ while to the second equation in Eq. (15) we will apply the integral operator $U_2^t \varphi = \int\limits_t^{\infty} \dfrac{r\varphi(r)dr}{\sqrt{r^2 - t^2}}$. As the result of that we obtain

$$\begin{cases} \int\limits_0^{\infty} Q_m(\alpha) L(\lambda\alpha) \cos(t\alpha) d\alpha = g_m(t), & t \le 1 \\ \int\limits_0^{\infty} Q_m(\alpha) \cos(t\alpha) d\alpha = 0, & t > 1 \end{cases} \tag{16}$$

We used the following notation

$$g_m(t) = sU_1^t \varphi_m = s\frac{d}{dt} \int\limits_0^t \frac{r\varphi_m(r)dr}{\sqrt{t^2 - r^2}} = sA_m \cos(k_m t) \tag{17}$$

The kernel transform $L(\alpha)$ depends on the properties of the nonhomogeneous materials. In (Aizikovich and Aleksandrov 1982) it was shown that $L(\alpha)$ possesses the following properties

$$\begin{aligned} L(\gamma) &= A + B|\gamma| + C\gamma^2 + O(|\gamma|^3), \gamma \to 0 \\ L(\gamma) &= 1 + D|\gamma|^{-1} + E\gamma^{-2} + O(|\gamma|^{-3}), \gamma \to \infty \end{aligned} \tag{18}$$

where A, B, C and D are certain constants which values depend on the material properties. It was shown (Aizikovich and Vasiliev 2013, Vasiliev *et al.* 2014) that the kernel transform $L(\alpha)$ can be precisely approximated by the expression

$$\begin{aligned} & L(\alpha\lambda) \approx L_N(\alpha\lambda) = \frac{P_1(\alpha^2\lambda^2)}{P_2(\alpha^2\lambda^2)} \\ & P_1(\alpha^2\lambda^2) = \prod_{j=1}^{N} (\alpha^2\lambda^2 + a_j^2), \quad P_2(\alpha^2\lambda^2) = \prod_{j=1}^{N} (\alpha^2\lambda^2 + b_j^2) \end{aligned} \tag{19}$$

where a_j and b_j are certain constants obtained by approximation of the kernel transform $L(\alpha)$. A detailed description of the process of determining coefficients a_j, b_j (j=1...N) is described in (Aizikovich and Vasiliev 2013).

It was also shown in (Aizikovich *et al.* 2009) that the solution of approximate dual integral equation resulting from replacing transform $L(\alpha)$ in Eq. (16) by its approximation $L_N(\alpha)$ is asymptotically exact for both thin and thick coatings, i.e., for $\lambda\to 0$ and $\lambda\to\infty$.

Let us introduce the functions

$$d_m(t) = \frac{1}{sA_m}\int_0^\infty Q_m(\alpha)\cos(t\alpha)d\alpha \tag{20}$$

then using operational analysis (Aleksandrov 1973) and Eqs. (19), (20), we represent first equation in Eq. (16) in operator form

$$P_1(-\lambda^2 D)d_m(t) = P_2(-\lambda^2 D)g_m(t), \quad D = \frac{d^2}{dt^2}, \quad t\in[0,1] \tag{21}$$

The solution of the differential equation (21) for d_m has the form

$$d_m(t) = \sum_{j=1}^{N}\vartheta(C_j^m, D_j^m, a_j\lambda^{-1}t) + L_N^{-1}(\lambda k_m)\cos(k_m t) \tag{22}$$

where coefficients C_i^m and D_i^m are arbitrary constants, $\vartheta(x,y,z) = x\,\mathrm{ch}(z) + y\,\mathrm{sh}(z)$.

Using Eqs. (20) and (22) Eq. (16) can be rewritten as follows

$$\begin{cases} \dfrac{1}{sA_m}\displaystyle\int_0^\infty Q_m(\gamma)\cos(t\alpha)d\alpha = \sum_{j=1}^{N}\vartheta(C_j^m, D_j^m, a_j\lambda^{-1}t) + L_N^{-1}(\lambda k_m)\cos(k_m t), \quad 0\le t\le 1 \\ \displaystyle\int_0^\infty Q_m(\gamma)\cos(t\alpha)d\alpha = 0, \quad t>1 \end{cases} \tag{23}$$

Inverting the Fourier's transform in Eq. (23) we obtain

$$Q_m(\alpha) = \frac{2sA_m}{\pi}\left(\sum_{j=1}^{N}\left[\frac{\vartheta(C_j^m, D_j^m, \frac{a_j}{\lambda})\alpha\sin\alpha + \frac{a_i}{\lambda}\vartheta(D_j^m, C_j^m, \frac{a_j}{\lambda})\cos\alpha}{\alpha^2 + A_i^2\lambda^{-2}} - \frac{a_j\lambda^{-1}D_j^m}{\alpha^2 + a_j^2\lambda^{-2}}\right] + \right. \\ \left. + L_N^{-1}(\lambda k_m)\frac{\alpha\sin(\alpha)\cos(k_m) - k_m\sin(k_m)\cos(\alpha)}{\alpha^2 - k_m^2}\right) \tag{24}$$

Inverting the Hankel transform in Eq. (24) and using Parseval's identity gives

$$q_0(r) = \frac{2\sqrt{2}}{\pi}s\left[\frac{1}{L_N(0)\sqrt{1-r^2}} + \sum_{j=1}^{N}\Psi(r, a_j\lambda^{-1}, C_j^0, D_j^0)\right], \\ q_m(r) = \frac{2}{\pi}A_m s\left[\frac{\Phi(r, ik_m)}{L_N(\lambda k_m)} + \sum_{j=1}^{N}\Psi(r, a_j\lambda^{-1}, C_j^m, D_j^m)\right], \; m = 1,2,\ldots \tag{25}$$

where i is the imaginary unit, Ψ and Φ are given by expressions

$$\Psi(r,A,C,D)=\frac{\vartheta(C,D,A)}{\sqrt{1-r^2}}-CA\int_r^1\frac{\mathrm{sh}(At)dt}{\sqrt{t^2-r^2}}-AD\int_r^1\frac{\mathrm{ch}(At)dt}{\sqrt{t^2-r^2}}$$

$$\Phi(r,A)=\frac{\mathrm{ch}\,A}{\sqrt{1-r^2}}-A\int_r^1\frac{\mathrm{sh}\,Atdt}{\sqrt{t^2-r^2}}$$

To determine constants C_i, D_i we substitute Eq. (25) into Eq. (16). The set of constants $C_i^m(m=0,1,2,\ldots;i=1..N)$ is determined from the system of linear algebraic equations below, while $D_i^m=0(\forall m\forall i)$.

$$\begin{aligned}&\sum_{j=1}^{N}C_j^0\eta(a_j\lambda^{-1},b_k\lambda^{-1})+L_N^{-1}(0)\lambda b_k^{-1}=0,\quad k=1,2,\ldots,N\\&\sum_{j=1}^{N}C_j^m\eta(a_j\lambda^{-1},b_k\lambda^{-1})+\frac{\eta(ik_m,b_k\lambda^{-1})}{L_N(\lambda k_m)}=0,\ k=1,2,\ldots,N,\quad m=1,2,\ldots\end{aligned}\tag{26}$$

where

$$\eta(x,y)=\frac{x\,\mathrm{sh}\,x+y\,\mathrm{ch}\,x}{y^2-x^2}$$

Contact stresses $q_m(r)$ and pressure applied to the plate $p(r)$ can be represented as a following series

$$q_m(r)=\sum_{j=0}^{\infty}y_j^m\varphi_j(r),\quad y_j^m=\int_0^1 q_m^N(\rho)\varphi_j(\rho)\rho d\rho\tag{27}$$

$$p(r)=\sum_{m=0}^{\infty}p_m\varphi_m(r),\quad p_m=\int_0^1 p(\rho)\varphi_m(\rho)\rho d\rho\tag{28}$$

Substituting Eqs. (13), (14), (27) and (28) into Eq. (12) we get the infinite system of a linear algebraic equations over w_m

$$w_m+k_m^{-4}\sum_{j=0}^{\infty}w_j\mathrm{E}_j^m=p_mk_m^{-4}\quad m=0,1,2,\ldots;\tag{29}$$

where

$$\mathrm{E}_j^m=2\pi^{-1}A_jsA_m\left(L_N^{-1}(\lambda k_j)\xi(k_j,k_m)+\sum_{n=0}^{N}C_n^m\zeta(a_n\lambda^{-1},k_m)\right),\quad j=1,2,\ldots,m=0,1,2,\ldots$$

$$\xi(x,y)=\frac{\cos x\sin y}{y}+\frac{x}{2y}\left(\frac{\sin(x-y)}{x-y}-\frac{\sin(x+y)}{x+y}\right)$$

$$\zeta(x,y)=\frac{x\,\mathrm{sh}\,x\cos y+y\sin y\,\mathrm{ch}\,x}{x^2+y^2}$$

In particular,

$$\mathrm{E}_0^m = 2\pi^{-1}\sqrt{2}sA_m\left[L_N^{-1}(0)k_m^{-1}\sin k_m + \sum_{n=1}^{N} C_n^0 \zeta(a_n\lambda^{-1}, k_m)\right],$$

$$\mathrm{E}_0^0 = \pi^{-1}4s\left[L_N^{-1}(0) + \lambda\sum_{n=1}^{N} C_n^0 a_n^{-1} sh(a_n\lambda^{-1})\right],$$

$$\mathrm{E}_j^0 = 2\pi^{-1}\sqrt{2}A_j s\left[L_N^{-1}(\lambda k_j)k_j^{-1}\sin k_j + \lambda\sum_{n=1}^{N} C_n^j a_n^{-1} sh(a_n\lambda^{-1})\right]$$

Using reduction method to the infinite system (29) we get the finite system

$$w_m k_m^4 + \sum_{j=0}^{K} w_j \mathrm{E}_j^m = p_m, \quad m = 0, 1, 2, \ldots, K \tag{30}$$

After determining parameters w_m (m=0,1,...,K) for a fixed value K and substituting them in Eq. (14) we finally get the contact stresses $q(r)$ and from Eq. (13) we get the deflections of the plate $w(r)$.

Radial torque M_r and tangential torque M_φ of the plate can be represented in the following expressions over the deflections

$$M_r = \left(-\frac{D}{R^2}\right)\left(\frac{d^2w}{dr^2} + \frac{\nu_{\text{plate}}}{r}\frac{dw}{dr}\right), \; M_\varphi = \left(-\frac{D}{R^2}\right)\left(\nu_{\text{plate}}\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) \tag{31}$$

where D is the is cylindrical stiffness of the plate.

Using Eqs. (13) and (31) we get the expressions for the torques

$$\begin{aligned} M_r &= \frac{D}{R}\sum_{m=0}^{K} w_m A_m k_m^2\left[\frac{\nu_{\text{plate}}-1}{k_m r} J_1(k_m r) + J_0(k_m r)\right], \\ M_\varphi &= \frac{D}{R}\sum_{m=0}^{K} w_m A_m k_m^2\left[\frac{1-\nu_{\text{plate}}}{k_m r} J_1(k_m r) + \nu_{\text{plate}} J_0(k_m r)\right], \end{aligned} \tag{32}$$

Let us consider the case of the plate with free edge defined by Eq. (5). The solution of this problem is presented in the paper (Aizikovich *et al.* 2011), but it contains a misprint in expressions (2.15), (2.16). Here we present the corrected expressions:

$$q_0^N(r) = \frac{2\sqrt{2}}{\pi}s\left[\frac{1}{L_N(0)\sqrt{1-r^2}} + \sum_{j=1}^{N} C_j^m \Phi(r, a_j\lambda^{-1})\right] \tag{33}$$

$$q_m^N(r) = \frac{2}{\pi}A_m s\left[\frac{\Phi(r, ik_m)}{L_N(\lambda k_m)} - \frac{\mathrm{J}_1(k_m)}{\mathrm{I}_1(k_m)}\frac{\Phi(r, k_m)}{L_N(i\lambda k_m)} + \sum_{j=1}^{N} C_j^m \Phi(r, a_j\lambda^{-1})\right], \; m = 1, 2, \ldots \tag{34}$$

The way of determining the parameters C_j^m are similar to that we used in the present paper: we

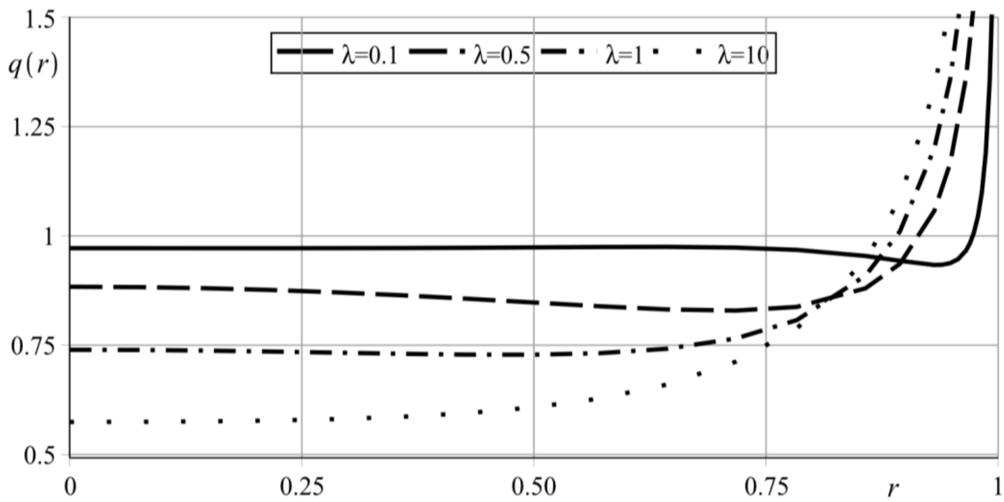

Fig. 1 Graphs of pressure distribution $q(r)$ versus r. The graphs are presented for soft coatings with β=100, plate of intermediate flexibility s=5 and some large and intermediate values of coating thickness λ

substitute Eqs. (33) and (34) into dual integral equation (Eq. (16)) and get the system of linear algebraic equations. Values of parameters A_m and k_m corresponding to the plate with free edge are presented in (Tseitlin 1969).

4. Numerical results

4.1 Soft homogeneous layer

Let us consider the bending of the plate lying on a layer with constant elastic moduli (E_1=*const*, ν_1=*const*) under the action of a uniformly distributed unit load: $p(r)=1$, $r<1$. We assume that the layer is much softer than the substrate ($E_2 << E_1(H)$) and use parameter $\beta=E_2/E_1(-H)$ to characterize softness of the layer. Let us consider case of the coating much softer than the substrate with β=100. The numerical results are provided for a case of the plate with a stiff edge (see Eqs. (4)).

According to (Gorbunov-Posadov 1940) the plate assumed to be flexible (or of infinite radius) if s>10 and stiff if s<1. Numerical examples below are provided for three values of parameter s: s=100, s=0.01 and s=5.

Fig. 1 contains graphs of the contact pressure under the plate of intermediate flexibility (s=5) lying on coatings of large or intermediate thickness λ. It is seen that the pressure on thick coatings (for instance, λ=10) are minimal under the center of the plate (r=0) and monotonically increases when r approaches the edge of the plate. For intermediate thickness of the coating ($0.1<\lambda<1$) the pressure under the center increases while the minimum value is reached near the edge of the plate (r=0.7..0.9).

The pressure distribution for small values of relative layer thickness λ has complicated nonmonotonic behavior (see Fig. 2): near the edges of the contact region one can observe an increase and decrease in pressure, i.e., pressure spikes. With decrease in the relative thickness of the coating λ these pressure maxima and minima become higher, occupy less space, and move

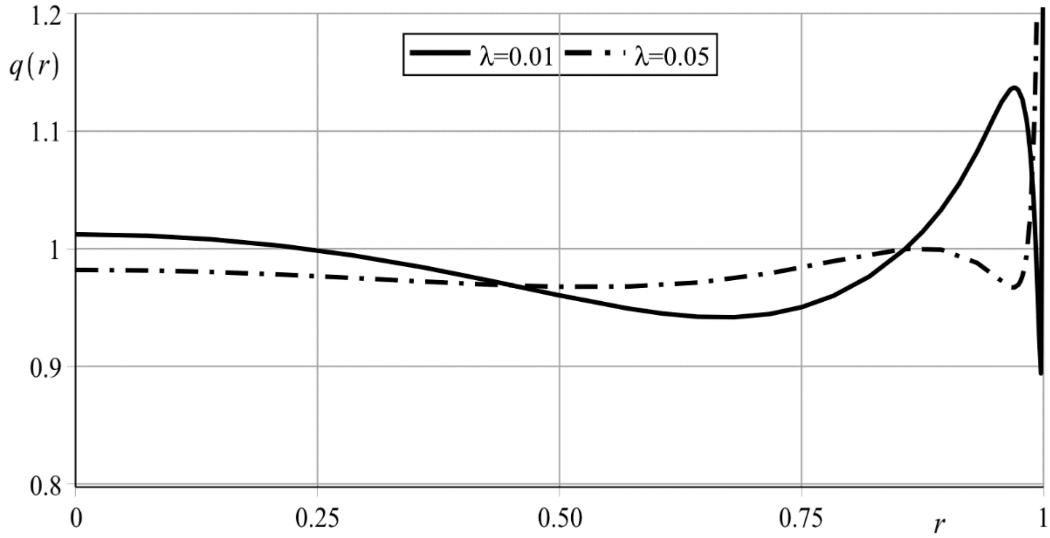

Fig. 2 Graphs of pressure distribution $q(r)$ versus r. The graphs are presented for soft coatings with β=100, plate of intermediate flexibility s=5 and some small coating thicknesses λ

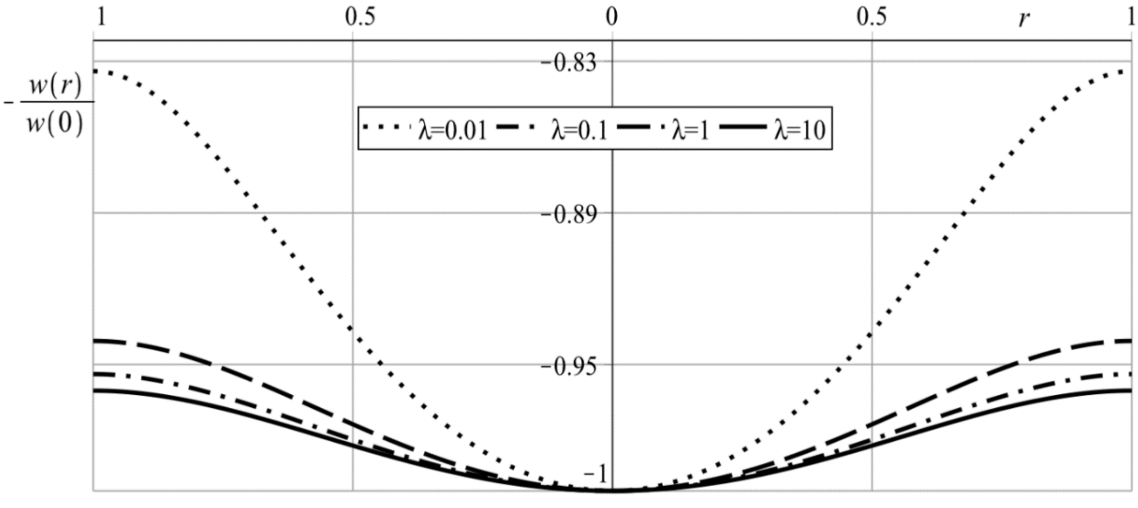

Fig. 3 Graphs of relative deflections of the plate $w_{rel}(r)$ versus r. The graphs are presented for soft coatings with β=100, plate of intermediate flexibility s=5.

toward the end points of the contacts. Relative deflections ($w_{rel}(r)=w(r)/w(0)$) of the plate corresponding to the pressure provided on Figs. 1 and 2 are illustrated on the Fig. 3.

For flexible plates the pressure under the center increases while near the edge its values sufficiently decrease (see Fig. 4), for stiff plates the opposite situation is observed (see Fig. 5).

To illustrate how the layer thickness influences the pressure we provide a dependence of the values $q(0)$ and $q(0.5)$ from the relative layer thickness λ (see Fig. 6). The graphs are provided for β=2, 5, 10, 100 and s=5. The pressure has local maximum for λ=(0.1..0.5). For λ>4 the pressure almost doesn't depend on the value of β. For λ<0.005 the pressure practically does not change with

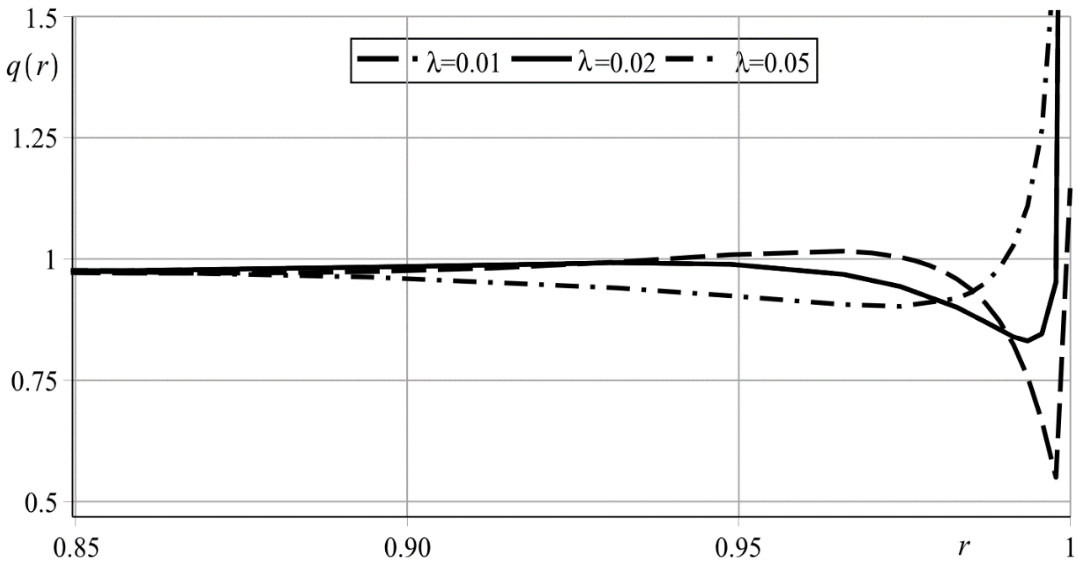

Fig. 4 Graphs of pressure distribution $q(r)$ versus r. The graphs are presented for soft coatings with β=100, flexible plate with s=100 and some small coating thicknesses λ

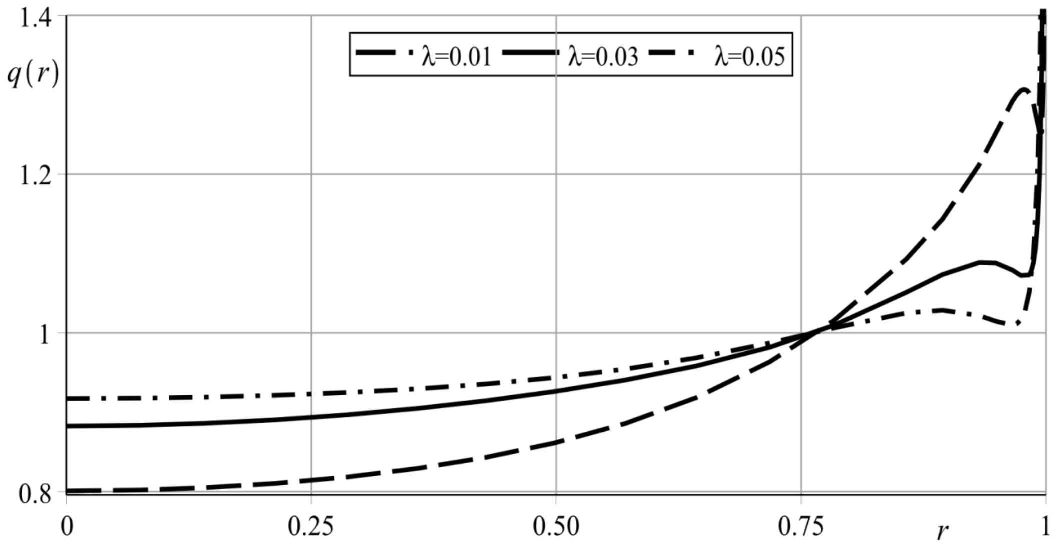

Fig. 5 Graphs of pressure distribution $q(r)$ versus r. The graphs are presented for soft coatings with β=100, stiff plate with s=0.01 and some small coating thicknesses λ

decreasing λ (the coating is so thin that practically has no effect on the pressure redistribution).

The pressure for the plates with free and stiff edges (see Eqs. (5) and (4)) are illustrated in Fig. 7. It is seen that the pressure for the plate with free boundaries are greater in the neibourhood of r=0 and smaller near its edge than for the plate with stiff edge. The qualitative differences were not found.

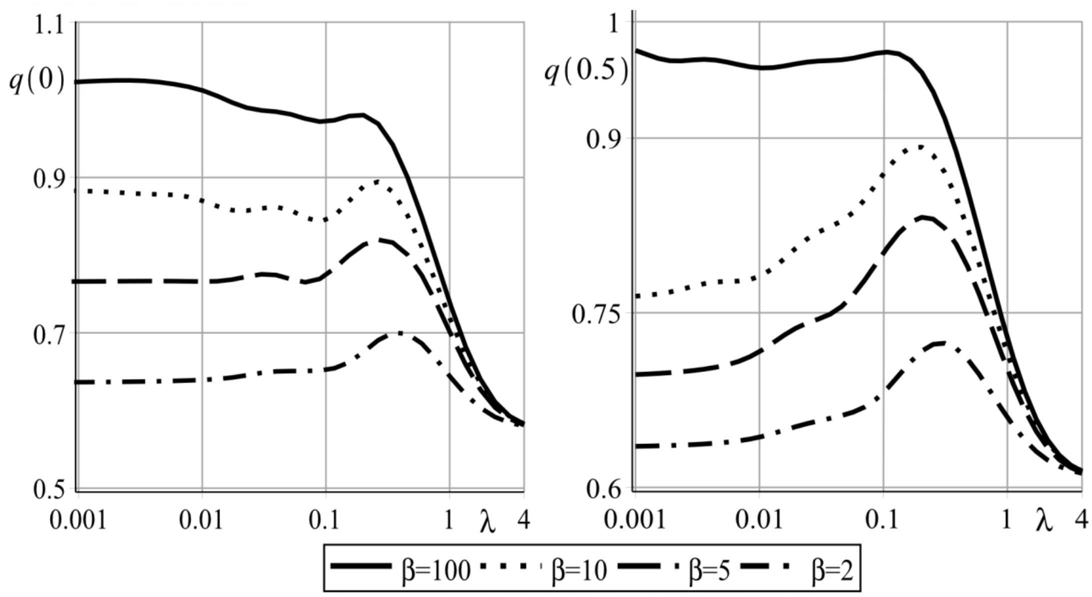

Fig. 6 Graphs of pressure $q(0)$ and $q(0.5)$ versus λ. The graphs are presented for soft coatings with β=2, 5, 10, 100, plate of intermediate flexibility s=5

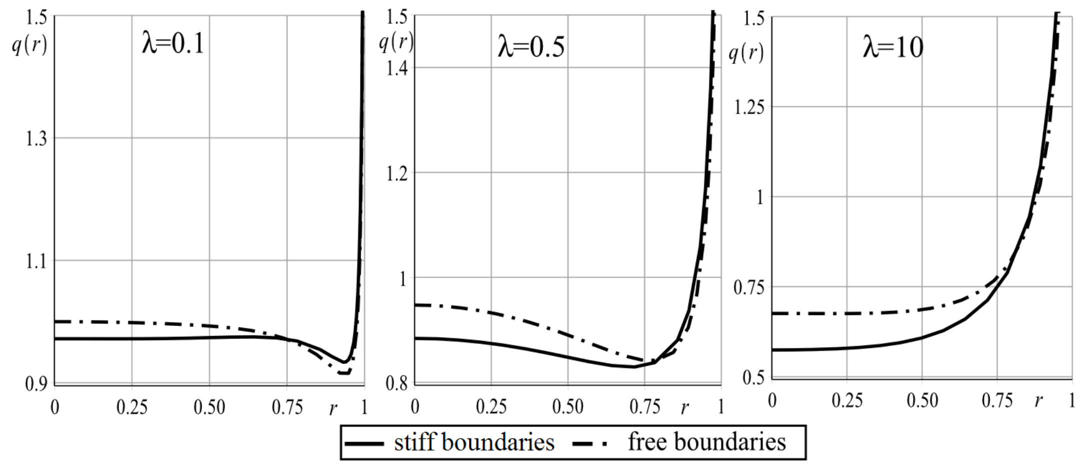

Fig. 7 Graphs of pressure distribution $q(r)$ versus r. The graphs are presented for soft coatings with β=100, plate of intermediate flexibility s=5 and some small coating thicknesses λ

4.2 Soft functionally graded layer

Let the Young's modulus of the coating varies with depth according to one of the following laws:

$$\text{coating 1: } E_1(z) = \frac{4.5}{7} + \frac{2.5}{7}\cos\left(2\pi\frac{z}{H}\right) \qquad \text{coating 2: } E_1(z) = \frac{4.5}{2} - \frac{2.5}{2}\cos\left(2\pi\frac{z}{H}\right)$$

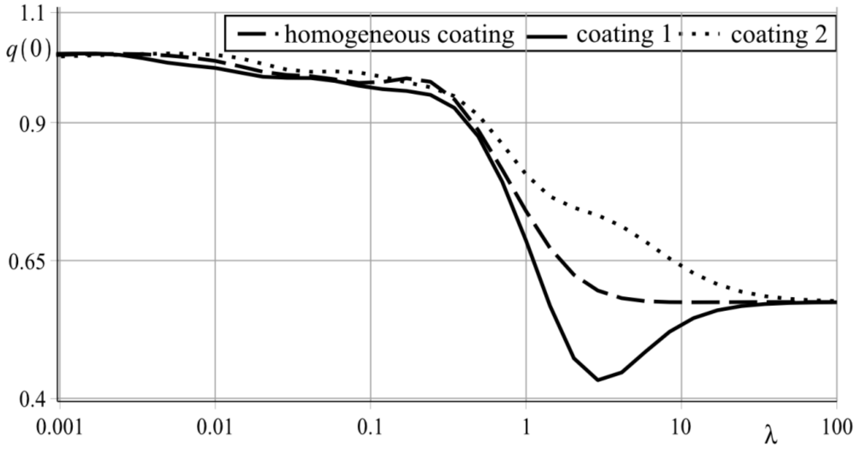

Fig. 8 Graphs of pressure $q(0)$ versus λ. The graphs are presented for soft homogeneous and functionally graded coatings with β=100 and plate of intermediate flexibility s=5

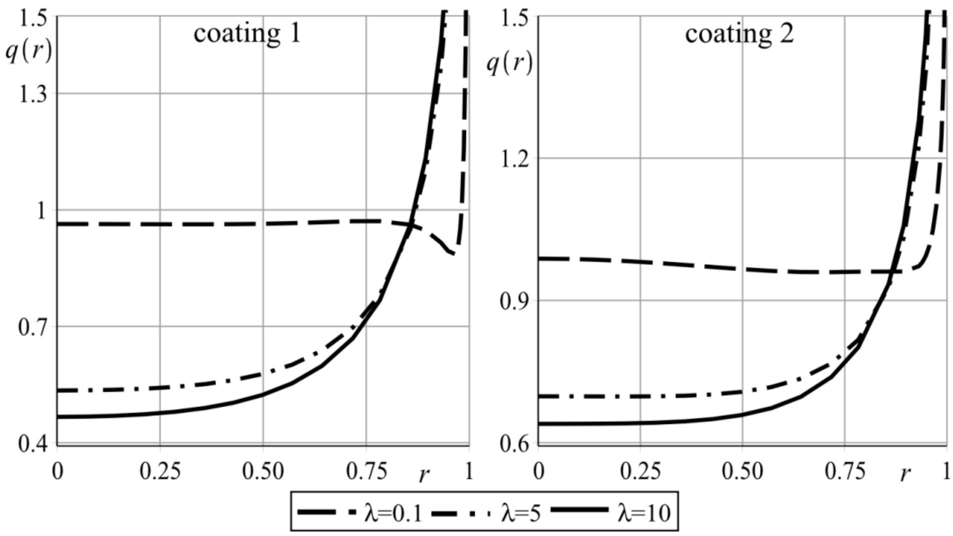

Fig. 9 Graphs of pressure distribution $q(r)$ versus r. The graphs are presented for soft functionally graded coatings with β=100, plate of intermediate flexibility s=5 and some large and intermediate coating thicknesses λ

The pressure under the center of the plate $q(0)$ versus λ and pressure distribution versus r for functionally-graded coatings 1 and 2 and homogeneous coating for β=100 and s=5 are presented in the Figs. 8 and 9. It is seen that the functionally graded properties of the coating sufficiently redistribure the pressure expecially for λ=1..10.

5. Conclusions

Analytical expressions for the contact stresses appearing under the plate and the deflection function are constructed using the bilateral asymptotic method. The method allows to consider the elastic layer lying on a much stiffer substrate. Using approximations for the kernel transform of high accuracy it is possible to obtain a solution of the problem which is applicable for all possible values of λ and any stiffness of the plate. Same method was successfully applied to a wide class of contact problems for materials with functionally-graded coatings (Aizikovich and Aleksandrov 1984, Vasiliev *et al.* 2012, 2014, 2015, 2016, Kudish *et al.* 2016).

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