Structural Engineering and Mechanics, *Vol. 59, No. 1 (2016) 1-14*
DOI: http://dx.doi.org/10.12989/sem.2016.59.1.001

Nonlinear vibration analysis of a type of tapered cantilever beams by using an analytical approximate method

Weipeng Sun[1a], Youhong Sun[2b], Yongping Yu[*2] and Shaopeng Zheng[2c]

[1]*Department of Mechanics and Engineering Science, School of Mathematics, Jilin University, Changchun 130012, PR China*
[2]*College of Construction Engineering, Jilin University, Changchun 130026, P.R. China*

(Received January 5, 2016, Revised February 19, 2016, Accepted February 23, 2016)

Abstract. In this paper, an alternative analytical method is presented to evaluate the nonlinear vibration behavior of single and double tapered cantilever beams. The admissible lateral displacement function satisfying the geometric boundary conditions of a single or double tapered cantilever beam is derived by using Rayleigh-Ritz method. Based on the Lagrange method and the Newton Harmonic Balance (NHB) method, analytical approximate solutions in closed and explicit form are obtained. These approximate solutions show excellent agreement with those of numeric method for small as well as large amplitude. Moreover, due to brevity of expressions, the present analytical approximate solutions are convenient to investigate effects of various parameters on the large amplitude vibration response of tapered beams.

Keywords: Newton Harmonic Balance method; analytical approximation; nonlinear vibration; tapered beam

1. Introduction

Non-uniform structural members with variable cross section or material properties like rods, beams, plates and shells are commonly used in various aeronautical, civil and mechanical engineering fields, such as helicopter rotor blades, airplane wings, wind turbine blades; non-prismatic pylons of cable-stayed bridges, offshore vertical risers and structure piles, oil platform supports, oil-loading terminals, tower structures and moving arms (Swaddiwudhipong and Liu 1996, Swaddiwudhipong and Liu 1997, Wu and Hsieh 2000, Chen and Liu 2006, Yardimoglu 2006, Gunda *et al*. 2007, Pradhan and Sarkar 2009, Attarnejad *et al*. 2011, Shahba *et al*. 2011, Saboori and Khalili 2012, Bambill *et al*. 2013, He *et al*. 2013, Rajasekaran 2013, Rajasekaran 2013, Baghani *et al*. 2014, Fang and Zhou 2015, Mao 2015). Nowadays, micro- and nano-sized tapered structures and devices such as biosensors, atomic force microscope, microactuators, energy harvesting, and nanoprobes have been widely used in micro-electro-mechanical (MEMS)

∗Corresponding author, Associate Professor, E-mail: yuyongping@jlu.edu.cn
[a]Professor, E-mail: sunwp@jlu.edu.cn
[b]Professor, E-mail: syh@jlu.edu.cn
[c]Ph.D., E-mail: zhengsp0428@126.com

http://www.techno-press.org/?journal=sem&subpage=8 ISSN: 1225-4568 (Print), 1598-6217 (Online)

and nano-electro-mechanical systems (NEMS) (Liu *et al*. 2003, Sadeghi 2012, Akgoz and Civalek 2013, Mohammadimehr *et al*. 2015, Sadeghi 2015). It is significant that predicting and determining of their static and dynamic characteristics for design and analysis of tapered structures. For this purpose, the mechanical analyses of non-uniform beams/columns have been investigated by analytical and approximate numerical methods.

Various types of non-linearity that may arise in beam vibrations has been investigated. Georgian (1965) has proposed the linear vibration frequencies of truncated, tapered cantilever wedges and cones with a free end for vibration problems of turbine and compressor blades with variable cross-sections, and validated his analytical results with experimental ones. For free non-linear oscillations of an initially straight, uniform elastic bar with free-clamped or free-free end conditions, Wagner (1965) has obtained approximate solutions, via combining Hamilton's principle, Bubnov's method and Atkinson's superposition method. Based on an iterative numerical scheme to obtain results for tapered beams with rectangular and circular cross sections, Rao and Rao (1988) presented a simple formulation for the large amplitude free vibrations of tapered beams. More recently, Dugush and Eisenberg (2002), Shahba and Rajasekaran (2012), Bambill *et al*. (2013), Rajasekaran (2013) investigated the vibrations of non-uniform beams under moving loads, curved Timoshenko beams, rotating Timoshenko beams, helicopter blade modeled as Tapered beams of functionally graded materials, respectively.

Exact solutions were hardly obtained for most cases of tapering. However, for some special cases, some special functions such as Bessel or hypergeometric ones are used to obtained solutions (Abrate 1995, Auciello and Nole 1998, Raj and Sujith 2005). Yet, exact solutions are restricted to a few simple cases that can hardly be applied to more realistic geometries, material properties, boundary conditions or loading. Therefore, approximate method is the other choice, such as the Rayleigh quotient or the Ritz method (Sato 1980; Auciello and Nole 1998), the Galerkin-like reduction (Abdel-Jaber *et al*. 2008, Karimpour *et al*. 2012), the method of solving numerically an integral equation (Sakiyama 1985), the analog equation method (Katsikadelis and Tsiatas 2004). In addition, the Poincaré-Lindstedt method (Lenci *et al*. 2013), the multiple time scale method (Clementi *et al*. 2015), and the harmonic balance and the time transformation methods (Abdel-Jaber *et al*. 2008) are also applied to present analytical approximate solutions for nonlinear oscillations of tapered beams.

This paper is focus on the frequency response curves of a single or double beam undergoing nonlinear oscillations determined analytically by the Newton Harmonic Balance (NHB) method (Wu *et al*. 2006), which provides approximate, but accurate results. It is different from other literatures that the admissible lateral displacement function satisfying the geometric boundary conditions are presented by Rayleigh-Ritz method. It is simpler in expression than that of Bessel functions. The accuracy of the present analytical approximate solutions has been illustrated by comparing with numeric results and the results from the Classic Harmonic Balance Method (CHB method). The main result of this work is that the nonlinear frequency can be investigated by simple formulas with respect to amplitude.

2. Mathematical model

A schematic of the tapered beam is shown in Fig. 1. The transverse deflection v and the axial shortening u due to bending deformation are along the vertical axis y and the beam neutral axis x, respectively. The elastic modulus E, density ρ, and the length L_1 of the beam are constants;

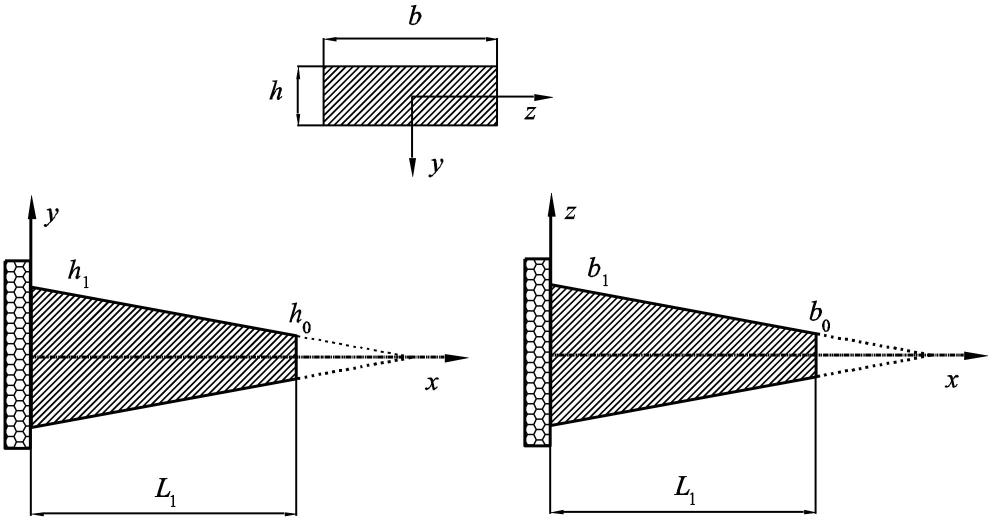

Fig. 1 A schematic for the tapered beam

however, width b and thickness h of the beam vary linearly along the beam neutral axis. The cross-sectional area and moment of inertia at the large end are $A_1=b_1h_1$ and $I_1=b_1h_1^3/12$, respectively, where b_1 and h_1 are the width and the thickness, respectively. While the ones at the small end are $A_0=b_0h_0$ and $I_0=b_0h_0^3/12$, respectively. The thickness of the beam is assumed to be small, compared to the length of the beam, so that the effects of rotary inertia and shear deformation can be ignored. The beam transverse vibration can be considered to be purely planar and the amplitude of vibration may reach large values.

Lagrange method is used to construct the nonlinear vibration equation. Firstly, using Rayleigh-Ritz method (Shames 1985) to obtain an approximate mode $\tilde{\phi}(s)$, and then transverse deflection $v(s,t)$ could be expressed as

$$v(s,t)=\tilde{\phi}(s)q(t) \tag{1}$$

where $q(t)$ is an unknown time modulation of the assumed deflection mode $\tilde{\phi}(s)$. Secondly, based on the assumed mode method discretizing the continuous Lagrangian function, the nonlinear control equation could be expressed as

$$\beta_0\ddot{q}+\beta_1(q^2\ddot{q}+q\dot{q}^2)+\Theta(\beta_2q+2\beta_3q^3)=0\,. \tag{2}$$

Where β_0, β_1, β_2, β_3, are listed in Appendix. For the detail of the derivation about Eq. (2), the readers are kindly advised to see the Appendix.

Next, a new variable is introduced

$$\tau=t\sqrt{\Omega\Theta}\,. \tag{3}$$

Then, the non-dimensional control equation could be written as

$$\Omega\cdot f(q'',q',q)+g(q)=0 \tag{4}$$

where

$$f\left(q'',q',q\right)=q''+\varepsilon_1\left(q^2q''+qq'^2\right) \tag{5}$$

$$g\left(q\right)=\varepsilon_2 q+\varepsilon_3 q^3 . \tag{6}$$

The corresponding non-dimensional parameters are

$$\varepsilon_1=\frac{\beta_1}{\beta_0},\varepsilon_2=\frac{\beta_2}{\beta_0},\varepsilon_3=\frac{2\beta_3}{\beta_0} \tag{7}$$

and " ' " denotes differentiation with respect to τ. The new independent variable is chosen in a way such that the solution of Eq. (4) is a periodic function of τ of period 2π. The corresponding dimensionless frequency of nonlinear oscillation is given by $\omega=\sqrt{\Omega}$. From Eq. (4), the present approximate dimensionless linear frequency $\omega_{La}=\sqrt{\Omega_{La}}=\sqrt{\varepsilon_2}$ and the corresponding fundamental frequency parameter $\varpi_{La}=\sqrt{\Omega_{La}\Theta}=\omega_{La}\sqrt{\Theta}=\sqrt{\Theta\varepsilon_2}$ could be obtained. Similarly, the exact fundamental frequency parameter is $\varpi_L=\sqrt{\Omega_L\Theta}=\omega_L\sqrt{\Theta}$.

3. Methods of solution

In this section, Eq. (4) will be solved by using NHB method (Wu *et al*. 2006). The initial conditions are taken as

$$q\left(0\right)=a\ ,\quad q'\left(0\right)=0 \tag{8}$$

where a is the amplitude of the motion. Here, both periodic solution $q(\tau)$ nd frequency $\omega=\sqrt{\Omega}$ depend on a. The periodic solution $q(\tau)$ can be represented by a Fourier series containing only odd multiples of τ, i.e.

$$q(\tau)=\sum_{j=1}^{\infty}z_j\cos[(2j-1)\tau] . \tag{9}$$

Following the single term HB (Harmonic Balance) approximation, set

$$q_1(\tau)=a\cos\tau \tag{10}$$

which satisfies the initial conditions in Eq. (8). Substituting Eq. (10) into Eq. (4), expanding the resulting expression in a trigonometric series, and setting the coefficient of $\cos\tau$ to vanish, yields

$$16a+12a^3\varepsilon_2+\left(-16a-8a^3\varepsilon_1-6a^5\varepsilon_3\right)\Omega=0 \tag{11}$$

which can be solved for Ω as function of a as

$$\Omega_1\left(a\right)=\frac{4\varepsilon_2+3a^2\varepsilon_3}{4+2a^2\varepsilon_1} . \tag{12}$$

The first approximation to the dimensionless frequency of the nonlinear oscillator is

$$\omega_1(a)=\sqrt{\Omega_1(a)}=\sqrt{\frac{4\varepsilon_2+3a^2\varepsilon_3}{4+2a^2\varepsilon_1}} \tag{13}$$

and the corresponding approximate periodic solution is

$$q_1(\tau)=a\cos\tau\,,\quad \tau=\omega_1(a)\sqrt{\Theta}t\,. \tag{14}$$

Initial approximations $q_1(\tau)$ and $\Omega_1(a)$ to Eqs. (4) and (8) have been obtained. Next, the combination of Newton's method and the HB method is formulated to solve Eqs. (4) and (8). The first step is the Newton-linearization procedure. The periodic solution and the square of frequency of Eq. (4) can be expressed as

$$q=q_1+\Delta q_1\,,\quad \Omega=\Omega_1+\Delta\Omega_1\,. \tag{15}$$

Substituting Eq. (15) into Eqs. (4) and (8), and linearizing about the correction terms Δq_1 and $\Delta\Omega_1$ lead to

$$\begin{aligned}&\left(\Omega_1+\Delta\Omega_1\right)f\left(q_1'',q_1',q_1\right)+g\left(q_1\right)+g_q\left(q_1\right)\cdot\Delta q_1\\&+\Omega_1\cdot\left[f_{q''}\left(q_1'',q_1',q_1\right)\Delta q_1''+f_{q'}\left(q_1'',q_1',q_1\right)\Delta q_1'+f_q\left(q_1'',q_1',q_1\right)\Delta q_1\right]\end{aligned} \tag{16}$$

$$\Delta q_1(0)=0\,,\,\Delta q_1'(0)=0\,. \tag{17}$$

Here, Δq_1 is a periodic function of τ of period 2π, and both Δq_1 and $\Delta\Omega_1$ are yet undetermined. The resulting linearized equations (16) and (17) in Δq_1 and $\Delta\Omega_1$ will be solved by the HB method.

The second approximate solution to Eq. (4) could be developed by setting Δq_1 in Eq. (16) as

$$\Delta q_1(\tau)=z_1(\cos\tau-\cos 3\tau) \tag{18}$$

which satisfies initial condition in Eq. (17) at the outset. Substituting Eqs. (10) and (18) into Eq. (16), expanding the resulting expression in a trigonometric series and setting the coefficients of $\cos\tau$ and $\cos 3\tau$ to zeros, respectively, yield

$$4a\varepsilon_2+3\varepsilon_3a^3-\left(4a+2\varepsilon_1a^3\right)\left(\Omega_1+\Delta\Omega_1\right)+\left(4\varepsilon_2+6\varepsilon_3a^2-4\Omega_1\right)z_1=0 \tag{19}$$

$$\varepsilon_3a^3-2\varepsilon_1a^3\left(\Omega_1+\Delta\Omega_1\right)+\left[-4\varepsilon_2-3\varepsilon_3a^2+\Omega_1\left(36+14\varepsilon_1a^2\right)\right]z_1=0\,. \tag{20}$$

Solving Eqs. (19) and (20) for z_1 and $\Delta\Omega_1$ gives

$$\Delta\Omega_1(a)=\frac{a^4\left(2\varepsilon_1\varepsilon_2-\varepsilon_3+\varepsilon_1\varepsilon_3a^2\right)\left(2\varepsilon_1\varepsilon_2+3\varepsilon_3+3\varepsilon_1\varepsilon_3a^2\right)}{\left(2+\varepsilon_1a^2\right)\left[64\varepsilon_2+\left(56\varepsilon_1\varepsilon_2+48\varepsilon_3\right)a^2+\left(10\varepsilon_1^2\varepsilon_2+39\varepsilon_1\varepsilon_3\right)a^4+6\varepsilon_1^2\varepsilon_3a^6\right]} \tag{21}$$

$$z_1(a)=\frac{\left(2+\varepsilon_1a^2\right)\left(2\varepsilon_1\varepsilon_2-\varepsilon_3+\varepsilon_1\varepsilon_3a^2\right)a^3}{64\varepsilon_2+\left(56\varepsilon_1\varepsilon_2+48\varepsilon_3\right)a^2+\left(10\varepsilon_1^2\varepsilon_2+39\varepsilon_1\varepsilon_3\right)a^4+6\varepsilon_1^2\varepsilon_3a^6}\,. \tag{22}$$

Using Eq. (21) results in the second approximations to frequency and periodic solution as

$$\omega_2(a)=\sqrt{\frac{128\varepsilon_2^2+\left(48\varepsilon_1\varepsilon_2^2+192\varepsilon_2\varepsilon_3\right)a^2+\left(70\varepsilon_1\varepsilon_2\varepsilon_3+69\varepsilon_3^2\right)a^4+24\varepsilon_1\varepsilon_3^2a^6}{128\varepsilon_2+2\left(56\varepsilon_1\varepsilon_2+48\varepsilon_3\right)a^2+2\left(10\varepsilon_1^2\varepsilon_2+39\varepsilon_1\varepsilon_3\right)a^4+12\varepsilon_1^2\varepsilon_3a^6}} \tag{23}$$

and

$$q_2(t) = q_1(\tau) + \Delta q_1(\tau) = a\cos\tau + (\cos\tau - \cos 3\tau)\, z_1(a), \quad \tau = \omega_2(a)\sqrt{\Theta} t \ . \tag{24}$$

It should be clear how the procedure works for constructing further analytical approximate solutions. For brevity, further higher order analytical approximation is omitted. Nevertheless, the procedure can be carried out recursively to desired order. In the next section, one could find that present formulas are capable of providing excellent analytical approximate representations to frequencies of the nonlinear oscillator in Eqs. (4) and (8).

Furthermore, the formulas of the Classic Harmonic Balance Method (CHB method) with two terms in Eqs. (4) and (8) are listed as

$$y = \frac{4\varepsilon_2 a + 3\varepsilon_3 a^3 - 4a\omega_{\mathrm{H}}^2 - 2\varepsilon_1\omega_{\mathrm{H}}^2 a^3 + \left(9\varepsilon_3 a - 14\varepsilon_1\omega_{\mathrm{H}}^2 a\right) y^2 + \left(6\varepsilon_3 - 16\varepsilon_1\omega_{\mathrm{H}}^2\right) y^3}{4\omega_{\mathrm{H}}^2 - 4\varepsilon_2 - 6\varepsilon_3 a^2} \tag{25}$$

$$\omega_{\mathrm{H}}^2 = \frac{4\varepsilon_2 y - \varepsilon_3 a^3 + 3\varepsilon_3 y a^2 + 9\varepsilon_3 a y^2 + 8\varepsilon_3 y^3}{2\left(18y - \varepsilon_1 a^3 + 7\varepsilon_1 y a^2 + 17\varepsilon_1 a y^2 + 18\varepsilon_1 y^3\right)} \tag{26}$$

$$q_{\mathrm{H}}(t) = a\cos\tau + y(\cos\tau - \cos 3\tau), \quad \tau = \omega_{\mathrm{H}}(a)\sqrt{\Theta} t \ . \tag{27}$$

However, Eqs. (25) and (26) should be solved numerically for a given amplitude a, using an iterative technique. Namely, the formulas of the CHB method with two terms could only obtain numerical solutions to Eqs. (4) and (8), instead of analytical ones. Unlike the CHB method, the present method carries out linearization of the governing differential equation prior to harmonic balancing. Simple linear algebraic equations are constructed by the present approach, instead of nonlinear algebraic equations without analytical solution. Therefore, accurate higher-order approximate analytical expressions for period and periodic solution could be established.

4. Results and discussion

In this section, accuracy of the proposed analytical approximations will be illustrated by comparing with the exact (numerical) solution obtained by the improved shooting method (Yu *et al*. 2012) and the results of the CHB method. For a double tapered beam and wedge-type beam (single taper), the present result of linear frequency ω_{La}, the analytical and experimental ones ω_{LGA} and ω_{LGE} from Georgian (1965), and the result ω_{LRR} obtained by Rao and Rao (1988) are firstly listed in Tables 1 and 2, respectively. From Tables 1 and 2, excellent agreements of the present results with those obtained by experiment method can be observed. Therefore, the present mode $\tilde{\phi}(s) = L_1\phi(\xi)$, where $\phi(\xi)$ is given by Eq. (A.6), is exact enough to calculate the fundamental frequency of a single or double tapered beam.

Let ω_N represent the nonlinear vibration frequency of the beam. Accuracy of the proposed analytical approximate nonlinear frequency is presented by comparing with the exact (numerical) frequency ω_{e} obtained by the improved shooting method (Yu *et al*. 2012) and the frequency ω_H from the CHB method, where ω_1, ω_2 represent the first and second analytical approximate nonlinear frequencies obtained by present method. For taper ratios α=0.1,0.3,0.5, the nonlinear frequencies of a single taper ‘‘wedge-shaped beam’’ and a double tapered beam ω_e, ω_1, ω_2 with

Table 1 The smallest natural frequency parameters ω_L from linear free vibration analysis of a tapered cantilever wedge with a free end

α	ω_{La}	ω_{LGA}	ω_{LGE}	ω_{LRR}
1.000	3.521	3.516	3.55	3.516
0.800	3.610	3.608	——	3.608
0.797	3.611	——	3.65	3.610
0.600	3.737	3.737	——	3.737
0.592	3.743	——	3.82	3.743
0.407	3.926	——	3.99	3.926
0.400	3.934	3.934	——	3.934
0.206	4.278	——	4.31	4.277
0.200	4.293	4.292	——	4.292

Table 2 The smallest natural frequency parameters ω_L from linear free vibration analysis of a tapered cantilever cone with a free end

α	ω_{La}	ω_{LGA}	ω_{LGE}	ω_{LRR}
1.000	3.521	3.516	3.59	3.516
0.803	3.850	——	3.88	3.849
0.601	4.316	——	4.41	4.316
0.500	4.625	4.625	——	4.625
0.411	4.963	——	4.96	4.962
0.333	5.329	5.289	——	5.328
0.250	5.825	5.85	——	5.823
0.207	6.142	——	6.13	6.140
0.100	7.209	7.201	——	7.205

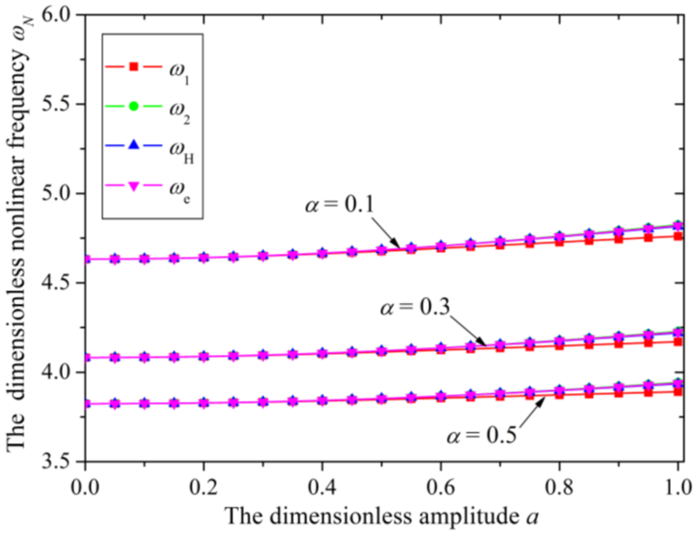

Fig. 2 Comparison of the approximate and exact dimensionless nonlinear frequencies for a single tapered beam (α=0.1, 0.3, 0.5)

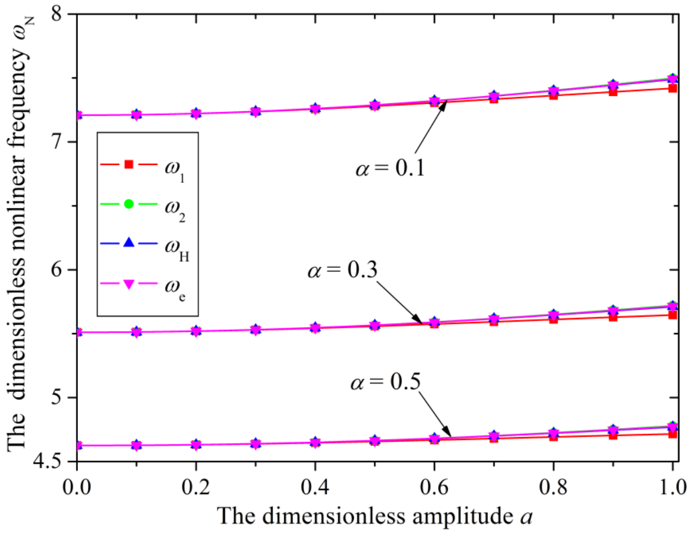

Fig. 3 Comparison of the approximate and exact dimensionless nonlinear frequencies for a double tapered beam (α=0.1, 0.3, 0.5)

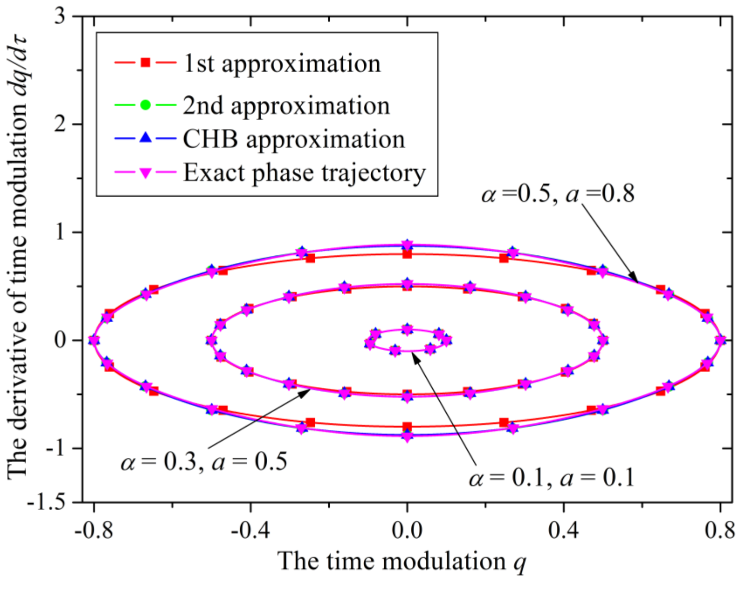

Fig. 4 Comparison of approximate phase trajectories with exact ones for a single tapered beam (α=0.1, a=0.1, α=0.3, a=0.5 and α=0.5, a=0.8)

respect to the amplitude a are shown in Figs. 2 and 3, respectively.

According to Figs. 2 and 3, the considered example exhibits a softening nonlinear behaviour. From Figs. 2 and 3, it also can be concluded that Eqs. (13) and (23) can provide excellent approximate frequencies for oscillation amplitude a<0.5, but Eq. (13) is not very accurate when a>0.5. Especially, for a single taper "wedge-shaped beam" and α=0.1, the relative error of ω_2 to ω_e

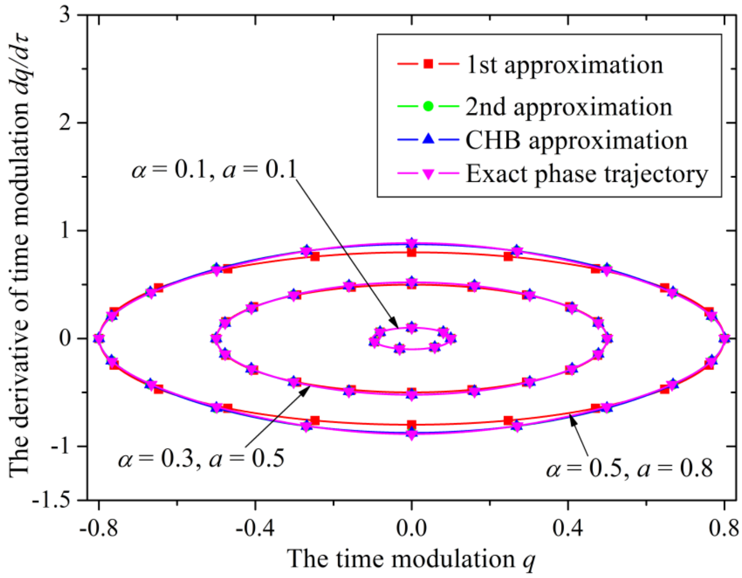

Fig. 5 Comparison of approximate phase trajectories with exact ones for a double tapered beam (α=0.1, a=0.1, α=0.3, a=0.5 and α=0.5, a=0.8)

is only 0.0427% when a=0.8, while the corresponding one for ω_1 is 0.657%. Note that vibration amplitude a=0.8 is a large value, which corresponds to a ratio of tip displacement/length of the beam equal to 0.8. What's more, it is easier and more convenient to investigate vibration behavior of a tapered beam by applying the analytical approximate solutions than numeric method or the CHB method, since the former is an explicit expression in terms of dimensionless oscillation amplitude a.

With α=0.1, a=0.1, α=0.3, a=0.5 and α=0.5, a=0.8, Figs 4 and 5 present comparison of the analytical approximate phase trajectories computed by Eqs. (14) and (24) with the exact one obtained by numerically integrating Eq. (4), for a single taper "wedge-shaped beam" and a double tapered beam, respectively. These figures show that the approximate phase trajectories proposed from Eq. (24) provide excellent approximations to exact ones. Moreover, the result of Eq. (14) also gives generally acceptable approximation to the exact solution when oscillation amplitude is not very large, such as a<0.5.

5. Conclusions

The Lagrangian method and the NHB method have been successfully applied to investigate large amplitude vibration behavior of tapered beams. The novel ideas proposed in this study are briefly summarized here:

- The admissible lateral displacement function satisfying geometrical conditions of tapered a cantilever beam is presented.
- Brief analytical approximate solutions of closed form, for the problem considered, are directly obtained.

- The present analytical approximate solutions are directly validated by comparing with those obtained from numeric method.
- The proposed method in this paper is general and could be directly applied to study large amplitude vibration behavior of circular and rectangular thin plates.

Acknowledgments

The work described in this paper is supported by the National Natural Science Foundation of China (Grant No. 11402095), the Science and Technology Developing Plan Project of Jilin Province (Grant No. 20160520021JH) and Innovative Project of Scientific Forefront and Interdisciplinary of Jilin University (Grant No. 2013ZY14).

References

Abdel-Jaber, M.S., Al-Qaisia, A.A., Abdel-Jaber, M. and Beale, R.G. (2008), "Nonlinear natural frequencies of an elastically restrained tapered beam", *J. Sound Vib.*, **313**(3-5), 772-783.

Abrate, S. (1995), "Vibration of non-uniform rods and beams", *J. Sound Vib.*, **185**(4), 703-716.

Akgoz, B. and Civalek, O. (2013), "Free vibration analysis of axially functionally graded tapered Bernoulli-Euler microbeams based on the modified couple stress theory", *Compos. Struct.*, **98**, 314-322.

Attarnejad, R., Shahba, A. and Eslaminia, M. (2011), "Dynamic basic displacement functions for free vibration analysis of tapered beams", *J. Vib. Control*, **17**(14), 2222-2238.

Auciello, N.M. and Nole, G. (1998), "Vibrations of a cantilever tapered beam with varying section properties and carrying a mass at the free end", *J. Sound Vib.*, **214**(1), 105-119.

Baghani, M., Mazaheri, H. and Salarieh, H. (2014), "Analysis of large amplitude free vibrations of clamped tapered beams on a nonlinear elastic foundation", *Appl. Math. Model.*, **38**(3), 1176-1186.

Bambill, D.V., Rossit, C.A., Rossi, R.E., Felix, D.H. and Ratazzi, A.R. (2013), "Transverse free vibration of non uniform rotating Timoshenko beams with elastically clamped boundary conditions", *Meccanica*, **48**(6), 1289-1311.

Chen, D.W. and Liu, T.L. (2006), "Free and forced vibrations of a tapered cantilever beam carrying multiple point masses", *Struct. Eng. Mech.*, **23**(2), 209-216.

Clementi, F., Demeio, L., Mazzilli, C.E.N. and Lenci, S. (2015), "Nonlinear vibrations of non-uniform beams by the MTS asymptotic expansion method", *Continuum. Mech. Thermodyn.*, **27**(4-5), 703-717.

Dugush, Y.A. and Eisenberger, M. (2002), "Vibrations of non-uniform continuous beams under moving loads", *J. Sound Vib.*, **254**(5), 911-926.

Fang, J. and Zhou, D. (2015), "Free vibration analysis of rotating axially functionally graded-tapered beams using Chebyshev-Ritz method", *Mater. Res. Innov.*, **19**, 1255-1262.

Georgian, J.C. (1965), "Discussion: 'Vibration Frequencies of Tapered Bars and Circular Plates' (Conway, HD, Becker, ECH, and Dubil, JF, 1964, *ASME J. Appl. Mech*., **31**, 329-331)", *J. Appl. Mech.*, **32**(1), 234-235.

Gunda, J.B., Singh, A.P., Chhabra, P.S. and Ganguli, R. (2007), "Free vibration analysis of rotating tapered blades using Fourier-p superelement", *Struct. Eng. Mech.*, **27**(2), 243-257.

He, P., Liu, Z.S. and Li, C. (2013), "An improved beam element for beams with variable axial parameters", *Shock Vib.*, **20**(4), 601-617.

Karimpour, S., Ganji, S.S., Barari, A., Ibsen, L.B. and Domairry, G. (2012), "Nonlinear vibration of an elastically restrained tapered beam", *Sci. China-Phys. Mech. Astron.*, **55**(10), 1925-1930.

Katsikadelis, J.T. and Tsiatas, G.C. (2004), "Non-linear dynamic analysis of beams with variable stiffness", *J. Sound Vib.*, **270**(4), 847-863.

Lenci, S., Clementi, F. and Mazzilli, C.E.N. (2013), "Simple formulas for the natural frequencies of non-uniform cables and beams", *Int. J. Mech. Sci.*, **77**, 155-163.
Liu, A.Q., Zhang, X.M., Lu, C., Wang, F. and Liu, Z.S. (2003), "Optical and mechanical models for a variable optical attenuator using a micromirror drawbridge", *J. Micromech. Microeng.*, **13**(3), 400-411.
Mao, Q.B. (2015), "AMDM for free vibration analysis of rotating tapered beams", *Struct. Eng. Mech.*, **54**(3), 419-432.
Mohammadimehr, M., Monajemi, A.A. and Moradi, M. (2015), "Vibration analysis of viscoelastic tapered micro-rod based on strain gradient theory resting on visco-pasternak foundation using DQM", *J. Mech. Sci. Technol.*, **29**(6), 2297-2305.
Pradhan, S.C. and Sarkar, A. (2009), "Analyses of tapered fgm beams with nonlocal theory", *Struct. Eng. Mech.*, **32**(6), 811-833.
Raj, A. and Sujith, R.I. (2005), "Closed-form solutions for the free longitudinal vibration of inhomogeneous rods", *J. Sound Vib.*, **283**(3), 1015-1030.
Rajasekaran, S. (2013), "Buckling and vibration of axially functionally graded nonuniform beams using differential transformation based dynamic stiffness approach", *Meccanica*, **48**(5), 1053-1070.
Rajasekaran, S. (2013), "Free vibration of tapered arches made of axially functionally graded materials", *Struct. Eng. Mech.*, **45**(4), 569-594.
Rao, B.N. and Rao, G.V. (1988), "Large amplitude vibrations of a tapered cantilever beam", *J. Sound Vib.*, **127**(1), 173-178.
Saboori, B. and Khalili, S.M.R. (2012), "Free vibration analysis of tapered FRP transmission poles with flexible joint by finite element method", *Struct. Eng. Mech.*, **42**(3), 409-424.
Sadeghi, A. (2012), "The flexural vibration of V shaped atomic force microscope cantilevers by using the Timoshenko beam theory", *ZAMM-Z. Angew. Math. Mech.*, **92**(10), 782-800.
Sadeghi, A. (2015), "A new investigation for double tapered atomic force microscope cantilevers by considering the damping effect", *ZAMM-Z. Angew. Math. Mech.*, **95**(3), 283-296.
Sakiyama, T. (1985), "A method of analyzing the bending vibration of any type of tapered beams", *J. Sound Vib.*, **101**(2), 267-270.
Sato, K. (1980), "Transverse vibrations of linearly tapered beams with ends restrained elastically against rotation subjected to axial force", *Int. J. Mech. Sci.*, **22**(2), 109-115.
Shahba, A., Attarnejad, R. and Hajilar, S. (2011), "Free vibration and stability of axially functionally graded tapered Euler-Bernoulli beams", *Shock Vib.*, **18**(5), 683-696.
Shahba, A. and Rajasekaran, S. (2012), "Free vibration and stability of tapered Euler-Bernoulli beams made of axially functionally graded materials", *Appl. Math. Model.*, **36**(7), 3088-3105.
Shames, I.H. (1985), *Energy and finite element methods in structural mechanics*, CRC Press
Swaddiwudhipong, S. and Liu, Z.S. (1996), "Dynamic response of large strain elasto-plastic plate and shell structures", *Thin Wall. Struct.*, **26**(4), 223-239.
Swaddiwudhipong, S. and Liu, Z.S. (1997), "Response of laminated composite plates and shells", *Compos. Struct.*, **37**(1), 21-32.
Wagner, H. (1965), "Large-amplitude free vibrations of a beam", *J. Appl. Mech.*, **32**(4), 887-892.
Wu, B.S., Sun, W.P. and Lim, C.W. (2006), "An analytical approximate technique for a class of strongly non-linear oscillators", *Int. J. Nonlin. Mech.*, **41**(6), 766-774.
Wu, J.S. and Hsieh, M. (2000), "Free vibration analysis of a non-uniform beam with multiple point masses", *Struct. Eng. Mech.*, **9**(5), 449-467.
Yardimoglu, B. (2006), "Vibration analysis of rotating tapered Timoshenko beams by a new finite element model", *Shock Vib.*, **13**(2), 117-126.
Yu, Y.P., Wu, B.S. and Lim, C.W. (2012), "Numerical and analytical approximations to large post-buckling deformation of MEMS", *Int. J. Mech. Sci.*, **55**(1), 95-103.

CC

Appendix

Construction of Mathematical model

The potential energy of the system could be expressed as

$$\mathrm{V} = \frac{E}{2}\int_0^{L_1} I(s)\kappa^2 ds \tag{A.1}$$

and the kinetic energy T of beam can be written as

$$T = \frac{1}{2}\rho\int_0^{L_1} A(s)\left[\dot{u}^2 + \dot{v}^2\right]ds \tag{A.2}$$

where

$$\kappa^2 = \left(\frac{d\theta}{ds}\right)^2 = \frac{\left(d^2v/ds^2\right)^2}{1-\left(dv/ds\right)^2} \approx \left(\frac{d^2v}{ds^2}\right)^2\left[1+\left(\frac{dv}{ds}\right)^2\right] \tag{A.3}$$

and

$$u = s - \int_0^s \cos\theta d\eta = s - \int_0^s \sqrt{1-\left(\frac{dv}{ds}\right)^2}d\eta \approx \frac{1}{2}\int_0^s \left(\frac{dv}{ds}\right)^2 d\eta\,. \tag{A.4}$$

Here u is the axial shortening due to bending deformation. Note that Eqs. (A.3) and (A.4) are obtained by assuming $(dv/ds)^2 \ll 1$. The Lagrangian function of the beam can be expressed as

$$L = T - V\,. \tag{A.5}$$

An assumed single mode of transverse deflection Eq. (1) is used to discretize the continuous Lagrangian function. Let $\xi=s/L_1$, and $\tilde{\phi}(s) = L_1\phi(\xi)$, and $\phi(\xi)$ is the non-dimensional deflection mode and satisfies the condition $\phi(1)=1$. For simplicity, $\phi(\xi)$ which satisfies geometrical boundary (i.e. clamped condition) could be taken as

$$\phi(\xi) = \sum_{i=1}^{4} C_i\phi_i(\xi) = \sum_{i=1}^{4} C_i\left[1-\cos(i\xi)\right]. \tag{A.6}$$

Where C_1−C_4 are arbitrary constants to be determined by Rayleigh-Ritz method (Shames 1985). Using Eqs. (A.1) - (A.6), the Lagrangian function could be expressed as

$$L = \frac{1}{2}\rho\left(\tilde{\beta}_0\dot{q}^2 + \tilde{\beta}_1 q^2\dot{q}^2 - \frac{E}{\rho}\tilde{\beta}_2 q^2 - \frac{E}{\rho}\tilde{\beta}_3 q^4\right) \tag{A.7}$$

where

$$\tilde{\beta}_0 = L_1^3\int_0^1 A(\xi)\phi(\xi)^2 d\xi \tag{A.8}$$

$$\tilde{\beta}_1 = L_1^3 \int_0^1 A(\xi) \left\{ \int_0^{\xi} \left[\frac{d\phi(\chi)}{d\chi} \right]^2 d\chi \right\}^2 d\xi \tag{A.9}$$

$$\tilde{\beta}_2 = \frac{1}{L_1} \int_0^1 I(\xi) \left[\frac{d^2\phi(\xi)}{d\xi^2} \right]^2 d\xi \tag{A.10}$$

$$\tilde{\beta}_3 = \frac{1}{L_1} \int_0^1 I(\xi) \left[\frac{d\phi(\xi)}{d\xi} \right]^2 \left[\frac{d^2\phi(\xi)}{d\xi^2} \right]^2 d\xi \tag{A.11}$$

For a double tapered beam, $A(\xi) = A_1\left[1-(1-\alpha)\xi\right]^2$ and $I(\xi) = I_1\left[1-(1-\alpha)\xi\right]^4$, here $\alpha = b_0/b_1 = h_0/h_1$. While for wedge-type beams (single taper), $A(\xi) = A_1\left[1-(1-\alpha)\xi\right]$ and $I(\xi) = I_1\left[1-(1-\alpha)\xi\right]^3$, where $\alpha = h_0/h_1, b_0 = b_1$.

Applying the Euler-Lagrangian relation to the system Lagrangian function

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{A.12}$$

the nonlinear equation of motion is obtained.

$$\tilde{\beta}_0 \ddot{q} + \tilde{\beta}_1 \left(q^2\ddot{q} + q\dot{q}^2\right) + \frac{E}{\rho}\left(\tilde{\beta}_2 q + 2\tilde{\beta}_3 q^3\right) = 0 . \tag{A.13}$$

Eq. (A.13) could be rewritten as

$$\beta_0 \ddot{q} + \beta_1 \left(q^2\ddot{q} + q\dot{q}^2\right) + \Theta\left(\beta_2 q + 2\beta_3 q^3\right) = 0 \tag{A.14}$$

where

$$\beta_0 = \int_0^1 A^* \phi(\xi)^2 \, d\xi \tag{A.15}$$

$$\beta_1 = \int_0^1 A^* \left\{ \int_0^{\xi} \left[\frac{d\phi(\chi)}{d\chi} \right]^2 d\chi \right\}^2 d\xi \tag{A.16}$$

$$\beta_2 = \int_0^1 I^* \left[\frac{d^2\phi(\xi)}{d\xi^2} \right]^2 d\xi \tag{A.17}$$

$$\beta_3 = \int_0^1 I^* \left[\frac{d\phi(\xi)}{d\xi} \right]^2 \left[\frac{d^2\phi(\xi)}{d\xi^2} \right]^2 d\xi \tag{A.18}$$

$$\Theta = \frac{EI_1}{\rho A_1 L_1^4} . \tag{A.19}$$

For a double tapered beam, $A^* = \left[1-(1-\alpha)\xi\right]^2$ and $I^* = \left[1-(1-\alpha)\xi\right]^4$, here $\alpha = b_0/b_1 = h_0/h_1$, while for wedge-type beams (single taper), $A^* = \left[1-(1-\alpha)\xi\right]$ and $I^* = \left[1-(1-\alpha)\xi\right]^3$, where $\alpha = h_0/h_1, b_0 = b_1$.