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Exact deformation of an infinite rectangular plate with an arbitrarily located circular hole under in-plane loadings

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Abstract. Exact solutions for stresses, strains, and displacements of a perforated rectangular plate by an arbitrarily located circular hole subjected to both linearly varying in-plane normal stresses on the two opposite edges and in-plane shear stresses are investigated using the Airy stress function. The hoop stress occurring at the edge of the non-central circular hole are computed and plotted. Stress concentration factors (the maximum non-dimensional hoop stresses) depending on the location and size of the non-central circular hole and the loading condition are tabularized.

Keywords: perforated plate; arbitrarily located circular hole; hoop stress; stress concentration factor; airy stress function; in-plane linearly varying normal stresses; in-plane shear stresses

1. Introduction

Numerous researchers have investigated the mechanical behaviors of perforated plates, with main concerns being classified into four categories; stress concentration (Savin 1961, Muskhelishvili 1963, Miyata 1970, Timoshenko and Goodier 1970, Peterson 1974, Iwaki and Miyao 1980, Theocaris and Petrou 1987, Mal and Singh 1991, Wanlin 1993, Fu 1996, Radi 2001, Yang and He 2002, Zhang *et al.* 2002, She and Guo 2007, Li *et al.* 2008, Yang *et al.* 2008, Yu *et al.* 2008, Kang 2014, Woo *et al.* 2014), vibration, buckling, and fatigue. The various methods have been used to study them. The finite element method (FEM) is the most widely used for this perforated plate problems. Diverse methods other than FEM have been used like the complex variable method, three-dimensional stress analysis, the Ritz method, the boundary element method, the differential quadrature element method, semi-analytical solution method, experimental method, conjugate load/displacement method, and Galerkin averaging method. Most of the shapes of perforated holes have three types of circular, elliptical, and rectangular cutout. Most of the previous researchers have generated approximate solutions, and have dealt with perforated plates subjected to uni-axial or bi-axial uniform tension or compression at the most.

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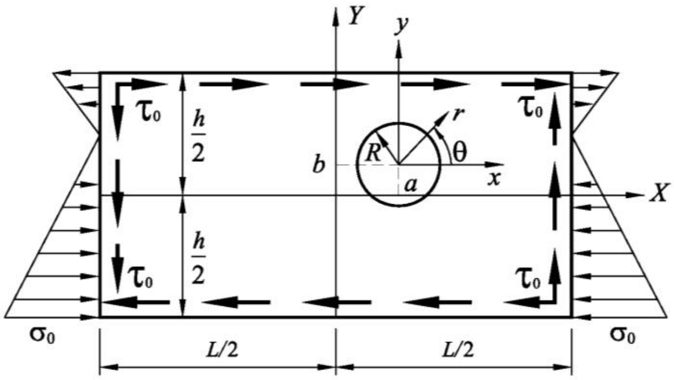

Fig. 1 A perforated rectangular plate by an arbitrarily located circular hole loaded by both linearly varying in-plane normal stresses on two opposite edges and in-plane shear loading τ_0, and the rectangular coordinates (x,y) and (X,Y) and the polar coordinates (r, θ)

In the present study, exact solutions for stresses, strains, and displacements of a perforated rectangular plate by an arbitrarily located circular hole subjected to both linearly varying in-plane normal stresses on two opposite edges and in-plane shear stresses are investigated using the Airy stress function. The present method of analysis is much simpler than the methods used by previous researchers, but it produces an exact solution which is its great strength. The hoop stresses (i.e., circumferential normal stresses) occurring at the edge of the non-central circular hole are computed and plotted. Stress concentration factors (the maximum non-dimensional hoop stresses) depending on the location and size of the non-central circular hole and the loading condition are tabularized.

2. Airy stress function

Fig. 1 shows a perforated rectangular plate of lateral dimensions $L\times h$ by an arbitrarily located circular hole of radius of R under both linearly varying in-plane normal stresses on two opposite edges at $X=-L/2$ and $L/2$ and in-plane shear stresses τ_0. The origin of the rectangular coordinate system (X,Y) is located at the center of the rectangular plate. The origins of the other rectangular coordinates (x,y) and the polar coordinates (r, θ) coincide with the center of the non-central circular hole. The center of the non-central circular hole is located at $(X,Y)=(a,b)$. The axes of x and y are parallel with X and Y axes, respectively. The plate is assumed to be very large compared with the circular hole.

First of all, considering a rectangular plate with no hole subjected to both linearly varying in-plane normal stresses on two opposite edges and in-plane shear stresses τ_0, the stress components through the plate neglecting body forces are

$$\sigma_{XX}^0=\frac{\partial^2\phi^0}{\partial Y^2}=\frac{\sigma_0(1+\alpha)}{h}Y-\frac{\sigma_0}{2}(1-\alpha)$$

$$\sigma_{XY}^0=-\frac{\partial^2\phi^0}{\partial X\partial Y}=\tau_0$$

$$\sigma^0_{YY} = \frac{\partial^2 \phi^0}{\partial X^2} = 0 \tag{1}$$

where ϕ^0 is a fundamental Airy stress function, σ_0 is the intensity of compressive stress at $Y=-h/2$, and α is a numerical loading factor. By changing α, we can obtain various particular cases. For example, by taking $\alpha=-1$ we have the case of uniformly distributed compressive force. When $\alpha=0$, the compressive force varies linearly from $-\sigma_0$ at $Y=-h/2$ to zero at $Y=+h/2$. For $\alpha=1$ we obtain the case of pure in-plane bending. With other α in the range $-1<\alpha<1$, we have a combination of bending and compression. Examples of these cases are shown in Fig. 2. For $\alpha<-1$ or $\alpha>1$ the problems arising are identical with ones having $-1<\alpha<1$. The fundamental Airy stress function ϕ^0 satisfies the governing equation $\nabla^4\phi^0=\nabla^2(\nabla^2\phi^0)=0$ with no body forces, where the Laplacian operator ∇^2 is expressed as

$$\nabla^2 = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \tag{2}$$

and ∇^4 is the bi-harmonic differential operator defined by $\nabla^2(\nabla^2)$ and becomes

$$\nabla^4 = \nabla^2(\nabla^2) = \frac{\partial^4}{\partial X^4} + 2\frac{\partial^4}{\partial X^2 \partial Y^2} + \frac{\partial^4}{\partial Y^4} \tag{3}$$

in rectangular coordinates (X,Y). From the relation of the Airy stress function and stress components in rectangular coordinates in Eq. (1), the fundamental Airy function ϕ^0 can be assumed as

$$\phi^0 = \frac{\sigma_0(1+\alpha)}{6h}Y^3 - \frac{\sigma_0}{4}(1-\alpha)Y^2 - \tau_0 XY + AX + BY + C \tag{4}$$

where A, B, and C are arbitrary integration constants. Since the relations of $X=x+b$ and $Y=y+b$, Eq. (4) becomes

$$\phi^0 = \frac{\sigma_0(1+\alpha)}{6h}(y+b)^3 - \frac{\sigma_0}{4}(1-\alpha)(y+b)^2 - \tau_0(x+a)(y+b) + A(x+a) + B(y+b) + C \tag{5}$$

A linear function of x or y and a constant in the Airy stress function in rectangular coordinates are trivial terms which do not give rise to any stresses and strains (Fu 1996). Dropping the trivial terms in Eq. (5), the fundamental Airy stress function ϕ^0 becomes

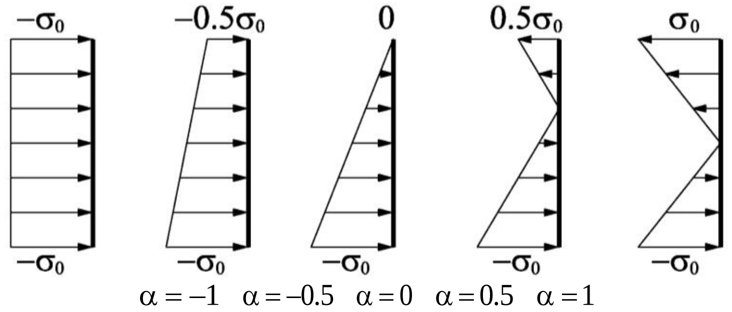

Fig. 2 Examples of in-plane normal stresses σ_{xx} along the edge $X=-L/2$

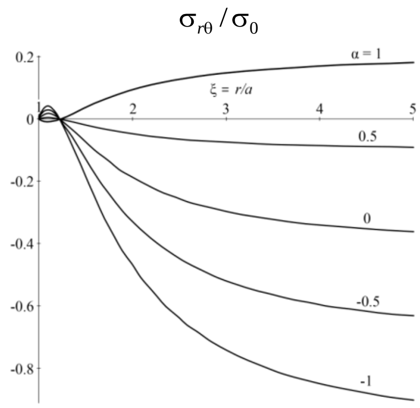

Fig. 3 Plot of non-dimensional shear stress $\sigma_{r\theta}/\sigma_0$ for β=0.1, γ=0.1, τ_0/σ_0=1, and θ=0

$$\phi^0 = \frac{(1+\alpha)\sigma_0}{6h} y^3 + \frac{\sigma_0}{4}\left[\alpha - 1 + \frac{2(1+\alpha)b}{h}\right] y^2 - \tau_0 xy \tag{6}$$

Using the relation of

$$x = r\cos\theta, \quad y = r\sin\theta \tag{7}$$

and the multiple angle formulas

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \tag{8}$$

$$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4} \tag{9}$$

The fundamental Airy stress function ϕ^0 in Eq. (6) can be transformed into the bi-harmonic functions in polar coordinates (r, θ) as below

$$\begin{aligned}\phi^0 = \frac{\sigma_0}{4}\Bigg[\frac{(1+\alpha)}{2h} r^3 \sin\theta - \frac{(1+\alpha)}{6h} r^3 \sin 3\theta + \left\{\frac{1-\alpha}{2} - \frac{b(1+\alpha)}{h}\right\} r^2 \cos 2\theta \\ - \left\{\frac{1-\alpha}{2} - \frac{b(1+\alpha)}{h}\right\} r^2 \Bigg] - \frac{\tau_0}{2} r^2 \sin 2\theta\end{aligned} \tag{10}$$

which satisfies the governing equation $\nabla^4\phi^0=\nabla^2(\nabla^2\phi^0)=0$, where ∇^2 is

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \tag{11}$$

and $\nabla^4=\nabla^2(\nabla^2)$ is expressed as

$$\nabla^4 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) \tag{12}$$

in polar coordinates. From the following relations between stresses and the Airy stress function in

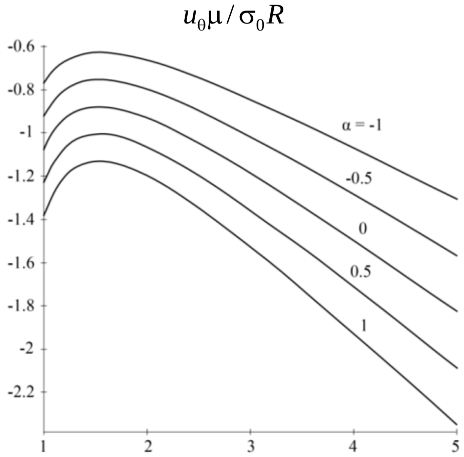

Fig. 4 Plot of non-dimensional circumferential displacement $u_\theta\mu/\sigma_0 R$ for β=−0.1, γ=0.1, τ_0/σ_0=1, ν=0.3 and θ=90°

polar coordinates, stress components through the rectangular plate with no hole subjected to both linearly varying in-plane normal stresses on two opposite edges and in-plane shear stresses can be calculated as below

$$\sigma_{rr}^0 = \frac{1}{r}\frac{\partial \phi^0}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi^0}{\partial \theta^2}$$

$$= \frac{\sigma_0}{4}\left[\frac{1+\alpha}{h} r\sin\theta + \frac{1+\alpha}{h} r\sin 3\theta - \left\{1-\alpha-\frac{2b(1+\alpha)}{h}\right\}\cos 2\theta - 1 + \alpha + \frac{2b(1+\alpha)}{h}\right] + \tau_0 \sin 2\theta \quad (13)$$

$$\sigma_{r\theta}^0 = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi^0}{\partial \theta}\right)$$

$$= \frac{\sigma_0}{4}\left[-\frac{1+\alpha}{h} r\cos\theta + \frac{1+\alpha}{h} r\cos 3\theta + \left\{1-\alpha-\frac{2b(1+\alpha)}{h}\right\}\sin 2\theta\right] + \tau_0 \cos 2\theta \quad (14)$$

$$\sigma_{\theta\theta}^0 = \frac{\partial^2 \phi^0}{\partial r^2}$$

$$= \frac{\sigma_0}{4}\left[\frac{3(1+\alpha)}{h} r\sin\theta - \frac{1+\alpha}{h} r\sin 3\theta + \left\{1-\alpha-\frac{2b(1+\alpha)}{h}\right\}\cos 2\theta - 1 + \alpha + \frac{2b(1+\alpha)}{h}\right] - \tau_0 \sin 2\theta \quad (15)$$

Let us return to the original problem of a perforated rectangular plate by an arbitrarily located circular hole under both linearly varying in-plane loading and in-plane shear loading. The total Airy function ϕ becomes

$$\phi = \phi^0 + \phi^* \quad (16)$$

where ϕ^* is an Airy stress function to cancel unwanted traction due to ϕ^0 at the edge of the non-central circular hole on r=R. The normal and shear stresses at the edge of the circular hole on r=R

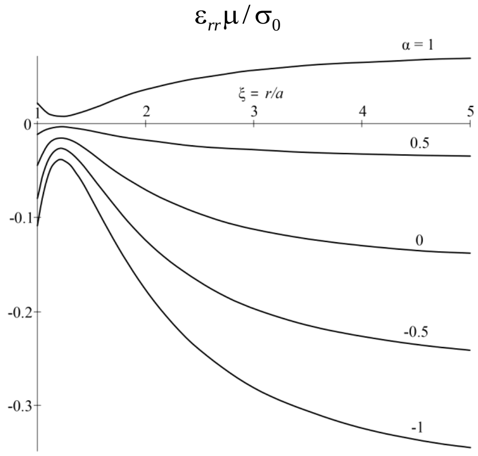

Fig. 5 Plot of radial strain $\varepsilon_{rr}\mu/\sigma_0$ for β=0.1, γ=0.1, τ_0/σ_0=1, v=0.3 and θ=0°

must be free as below

$$\sigma_{rr}\Big|_{r=R} = [\sigma^0_{rr} + \sigma^*_{rr}]_{r=R} = 0 \tag{17}$$

$$\sigma_{r\theta}\Big|_{r=R} = [\sigma^0_{r\theta} + \sigma^*_{r\theta}]_{r=R} = 0 \tag{18}$$

Therefore, σ^*_{rr} and $\sigma^*_{r\theta}$ on $r=R$ must have terms of sin θ, sin 2θ, sin 3θ, cos 2θ, or a constant and have cos θ, cos 2θ, cos 3θ, or sin 2θ, respectively, in order to eliminate the stresses on $r=R$ due to ϕ^0 in Eqs. (13) and (14). Tables 1 and 2 show the potential candidates of the bi-harmonic functions for the present problem from the tables by Dundurs (Fu 1996), which contain stresses and displacements of certain bi-harmonic functions in polar coordinates. However, the terms of r^2sin 2θ, r^3sin θ, r^3sin 3θ, r^2cos 2θ, and r^2 in the fundamental Airy stress function ϕ^0 of Eq. (10) must be excluded in ϕ^* in order not to disturb the traction in Eq. (1) at infinity. The terms of $r$ln $r$sinθ, r^2 ln r, and $r\ \theta$ cos θ give rise to muti-valued displacements u_r and/or u_θ, in the directions of r and θ, respectively. Singularity at infinity occurs in stresses and displacements because of the terms of r^4sin 2θ, r^4cos 2θ, and r^5sin 3θ. Omitting these inadequate terms, the total Airy stress function ϕ in Eq. (16) becomes

$$\phi = \frac{\sigma_0}{4}\left[\frac{(1+\alpha)}{2h}r^3\sin\theta - \frac{(1+\alpha)}{6h}r^3\sin 3\theta + \left\{\frac{1-\alpha}{2} - \frac{b(1+\alpha)}{h}\right\}r^2\cos 2\theta - \left\{\frac{1-\alpha}{2} - \frac{b(1+\alpha)}{h}\right\}r^2\right.$$
$$\left. + C_1R^2\ln r + C_2\frac{R^3\sin\theta}{r} + C_3\frac{R^4\cos 2\theta}{r^2} + C_4R^2\cos 2\theta + C_5\frac{R^5\sin 3\theta}{r^3} + C_6\frac{R^3\sin 3\theta}{r}\right]$$
$$-\frac{\tau_0}{2}\left(r^2\sin 2\theta + C_7R^2\sin 2\theta + C_8R^4\frac{\sin 2\theta}{r^2}\right) \tag{19}$$

where C_1~C_8 are arbitrary integration constants to be determined by traction boundary conditions. In order to make the constants C_1~C_8 dimensionless, they are multiplied by R^2, ...,R^5. Applying the stress free boundary conditions at the edge of the circular hole on $r=R$

$$\sigma_{rr}\Big|_{r=R} = \left[\frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right]_{r=R} = 0$$

$$\sigma_{r\theta}\Big|_{r=R} = \left[-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)\right]_{r=R} = 0 \tag{20}$$

and using the property of identity, the unknown constants C_1~C_8 are computed as

$$C_1 = -C_4 = 1-\alpha-2\beta(1+\alpha)\,,\ \ C_2 = C_6 = \frac{\gamma(1+\alpha)}{2}\,,$$

$$C_3 = \frac{C_1}{2}\,,\ \ C_5 = -\frac{\gamma(1+\alpha)}{3}\,,\ \ C_7 = -2\,,\ \ C_8 = 1 \tag{21}$$

where

$$\beta = \frac{b}{h}\,,\ \ \gamma = \frac{R}{h} \tag{22}$$

It is observed from the total Airy stress function ϕ in Eq. (19) that the solutions are not depending on a distance a in the X-direction between the two centers of the rectangular plate and the non-central circular hole.

3. Stresses

Using the relations between the stresses and the Airy stress function in Eqs. (13)-(15) in polar coordinates, the stress components can be expressed as below

$$\sigma_{rr} = \frac{\sigma_0}{4}\left[\gamma(1+\alpha)\left(\xi-\frac{1}{\xi^3}\right)\sin\theta + \gamma(1+\alpha)\left(\xi-\frac{5}{\xi^3}+\frac{4}{\xi^5}\right)\sin 3\theta - \{1-\alpha-2\beta(1+\alpha)\}\left(1-\frac{1}{\xi^2}\right)\right.$$
$$\left. -\{1-\alpha-2\beta(1+\alpha)\}\left(1-\frac{4}{\xi^2}+\frac{3}{\xi^4}\right)\cos 2\theta\right] + \tau_0\left(1-\frac{4}{\xi^2}+\frac{3}{\xi^4}\right)\sin 2\theta \tag{23}$$

$$\sigma_{r\theta} = \frac{\sigma_0}{4}\left[\{1-\alpha-2\beta(1+\alpha)\}\left(1+\frac{2}{\xi^2}-\frac{3}{\xi^4}\right)\sin 2\theta - \gamma(1+\alpha)\left(\xi-\frac{1}{\xi^3}\right)\cos\theta\right.$$
$$\left. +\gamma(1+\alpha)\left(\xi+\frac{3}{\xi^3}-\frac{4}{\xi^5}\right)\cos 3\theta\right] + \tau_0\left(1+\frac{2}{\xi^2}-\frac{3}{\xi^4}\right)\cos 2\theta \tag{24}$$

$$\sigma_{\theta\theta} = \frac{\sigma_0}{4}\left[\gamma(1+\alpha)\left(3\xi+\frac{1}{\xi^3}\right)\sin\theta - \gamma(1+\alpha)\left(\xi-\frac{1}{\xi^3}+\frac{4}{\xi^5}\right)\sin 3\theta\right.$$
$$\left. +\{1-\alpha-2\beta(1+\alpha)\}\left(1+\frac{3}{\xi^4}\right)\cos 2\theta - \{1-\alpha-2\beta(1+\alpha)\}\left(1+\frac{1}{\xi^2}\right)\right] - \tau_0\left(1+\frac{3}{\xi^4}\right)\sin 2\theta \tag{25}$$

where ξ is the non-dimensional radial coordinate defined by r/R. As a numerical example, Fig. 3 shows non-dimensional shear stress $\sigma_{r\theta}/\sigma_0$ for α=1, β=0.1, γ=0.1 and τ_0/σ_0=1 on θ=0° with α=−1, −0.5, 0, 0.5, and 1.

The circumferential stress $\sigma_{\theta\theta}$ on the edge of the non-central circular hole (r=R), which is called the hoop stress σ_{Hoop}, becomes

$$\sigma_{\mathrm{Hoop}} = \sigma_{\theta\theta}\Big|_{r=R} = \sigma_0\left[\gamma(1+\alpha)(\sin\theta - \sin 3\theta) + \{1-\alpha-2\beta(1+\alpha)\}\cos 2\theta + \beta(1+\alpha) - \frac{1-\alpha}{2}\right] - 4\tau_0 \sin 2\theta \tag{26}$$

The limiting case of the plate with no hole ($R\rightarrow 0$, $\beta\rightarrow 0$, $\gamma\rightarrow 0$), the stress components become as below

$$\sigma_{rr} = -\frac{\sigma_0}{4}(1-\alpha)(\cos 2\theta + 1) + \tau_0 \sin 2\theta$$

$$\sigma_{r\theta} = \frac{\sigma_0}{4}(1-\alpha)\sin 2\theta + \tau_0 \cos 2\theta$$

$$\sigma_{\theta\theta} = \frac{\sigma_0}{4}(1-\alpha)(\cos\ 2\theta - 1) - \tau_0 \sin 2\theta \tag{27}$$

Timoshenko and Goodier (1970) analyzed the exact stresses in a rectangular plate with a central circular hole subjected to uni-axial uniform compression (α=−1, β=0, τ_0=0). Substituting α=−1, β=0 and τ_0= into the stress components in Eqs. (23)-(25) results in

$$\sigma_{rr} = -\frac{\sigma_0}{2}\left[1 - \frac{1}{\xi^2} + \left(1 - \frac{4}{\xi^2} + \frac{3}{\xi^4}\right)\cos 2\theta\right]$$

$$\sigma_{r\theta} = \frac{\sigma_0}{2}\left(1 + \frac{2}{\xi^2} - \frac{3}{\xi^4}\right)\sin 2\theta$$

$$\sigma_{\theta\theta} = -\frac{\sigma_0}{2}\left[1 + \frac{1}{\xi^2} - \left(1 + \frac{3}{\xi^4}\right)\cos 2\theta\right] \tag{28}$$

which are exactly same with those by Timoshenko and Goodier (1970).

Mal and Singh (1991) presented the exact stresses in a perforated rectangular plate by a central hole under in-plane pure shear (σ_0=0, β=0). Substituting σ_0=0 and β=0 into Eqs. (3.1)-(3.3), the stress components become

$$\sigma_{rr} = \tau_0\left(1 - \frac{4}{\xi^2} + \frac{3}{\xi^4}\right)\sin 2\theta$$

$$\sigma_{r\theta} = \tau_0\left(1 + \frac{2}{\xi^2} - \frac{3}{\xi^4}\right)\cos 2\theta$$

$$\sigma_{\theta\theta} = -\tau_0\left(1 + \frac{3}{\xi^4}\right)\sin 2\theta \tag{29}$$

Table 1 Stresses of potential candidates of bi-harmonic functions ϕ

ϕ	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$
r^2	2	0	2
$\ln r$	$1/r^2$	0	$-1/r^2$
$r^2 \ln r$	$2\ln r + 1$	0	$2\ln r + 3$
$r^3 \sin\theta$	$2r\sin\theta$	$-2r\cos\theta$	$6r\sin\theta$
$r\theta\cos\theta$	$-2\sin\theta/r$	0	0
$r\ln r\sin\theta$	$\sin\theta/r$	$-\cos\theta/r$	$\sin\theta/r$
$\sin\theta/r$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$	$2\sin\theta/r^3$
$r^2\cos 2\theta$	$-2\cos 2\theta$	$2\sin 2\theta$	$2\cos 2\theta$
$r^4\cos 2\theta$	0	$6r^2\sin 2\theta$	$12r^2\cos 2\theta$
$\cos 2\theta/r^2$	$-6\cos 2\theta/r^4$	$-6\sin 2\theta/r^4$	$6\cos 2\theta/r^4$
$\cos 2\theta$	$-4\cos 2\theta/r^2$	$-2\sin 2\theta/r^2$	0
$r^3\sin 3\theta$	$-6r\sin 3\theta$	$-6r\cos 3\theta$	$6r\sin 3\theta$
$r^5\sin 3\theta$	$-4r^3\sin 3\theta$	$-12r^3\cos 3\theta$	$20r^3\sin 3\theta$
$\sin 3\theta/r^3$	$-12\sin 3\theta/r^5$	$12\cos 3\theta/r^5$	$12\sin 3\theta/r^5$
$\sin 3\theta/r$	$-10\sin 3\theta/r^3$	$6\cos 3\theta/r^3$	$2\sin 3\theta/r^3$
$r^2\sin 2\theta$	$-2\sin 2\theta$	$-2\cos 2\theta$	$2\sin 2\theta$
$\sin 2\theta$	$-4\sin 2\theta/r^2$	$2\cos 2\theta/r^2$	0
$r^4\sin 2\theta$	0	$-6r^2\cos 2\theta$	$12r^2\sin 2\theta$
$\sin 2\theta/r^2$	$-6\sin 2\theta/r^4$	$6\cos 2\theta/r^4$	$6\sin 2\theta/r^4$

which exactly coincide with those by Mal and Singh (1991).

Woo *et al.* (2014) obtained the exact solutions for stresses of a perforated rectangular plate by a central circular (β=0) hole subjected to in-plane bending moment (α=1, τ_0=0) on two opposite edges. The stress components in Eqs. (3.1)-(3.3) becomes

$$\sigma_{rr} = \frac{\gamma\sigma_0}{2}\left[\left(\xi - \frac{1}{\xi^3}\right)\sin\theta + \left(\xi - \frac{5}{\xi^3} + \frac{4}{\xi^5}\right)\sin 3\theta\right]$$

$$\sigma_{r\theta} = \frac{\gamma\sigma_0}{2}\left[-\left(\xi - \frac{1}{\xi^3}\right)\cos\theta + \left(\xi + \frac{3}{\xi^3} - \frac{4}{\xi^5}\right)\cos 3\theta\right]$$

$$\sigma_{\theta\theta} = \frac{\gamma\sigma_0}{2}\left[\left(3\xi + \frac{1}{\xi^3}\right)\sin\theta - \left(\xi - \frac{1}{\xi^3} + \frac{4}{\xi^5}\right)\sin 3\theta\right] \tag{30}$$

which are exactly same with those by Woo *et al* (2014).

4. Displacements

Using Table 2 by Dundurs (Fu 1996), the displacement components u_r and u_θ can be easily

Table 2 Displacements of potential candidates of bi-harmonic functions ϕ

ϕ	$2\mu u_r$	$2\mu u_\theta$
r^2	$(\kappa-1)r$	0
$\ln r$	$-1/r$	0
$r^2\ln r$	$(\kappa-1)r\ln r-r$	$(\kappa+1)r\theta$
$r^3\sin\theta$	$(\kappa-2)r^2\sin\theta$	$-(\kappa+2)r^2\cos\theta$
$r\theta\cos\theta$	$[(\kappa-1)\theta\cos\theta-(\kappa+1)\ln r\sin\theta+\sin\theta]/2$	$[-(\kappa-1)\theta\sin\theta-(\kappa+1)\ln r\cos\theta-\cos\theta]/2$
$r\ln r\sin\theta$	$[-(\kappa+1)\theta\cos\theta+(\kappa-1)\ln r\sin\theta-\sin\theta]/2$	$[(\kappa+1)\theta\sin\theta+(\kappa-1)\ln r\cos\theta+\cos\theta]/2$
$\sin\theta/r$	$\sin\theta/r^2$	$-\cos\theta/r^2$
$r^2\cos2\theta$	$-2r\cos2\theta$	$2r\sin2\theta$
$r^4\cos2\theta$	$-(3-\kappa)r^3\cos2\theta$	$(3+\kappa)r^3\sin2\theta$
$\cos2\theta/r^2$	$2\cos2\theta/r^3$	$2\sin2\theta/r^3$
$\cos2\theta$	$(\kappa+1)\cos2\theta/r$	$-(\kappa-1)\sin2\theta/r$
$r^3\sin3\theta$	$-3r^2\sin2\theta$	$-3r^2\cos2\theta$
$r^5\sin3\theta$	$-(4-\kappa)r^4\sin3\theta$	$-(4+\kappa)r^4\cos3\theta$
$\sin3\theta/r^3$	$3\sin3\theta/r^4$	$-3\cos3\theta/r^4$
$\sin3\theta/r$	$(2+\kappa)\sin3\theta/r^2$	$-(2-\kappa)\cos3\theta/r^2$
$r^2\sin2\theta$	$-2r\sin2\theta$	$-2r\cos2\theta$
$\sin 2\theta$	$(\kappa+1)\sin2\theta/r$	$(\kappa-1)\cos2\theta/r$
$r^4\sin2\theta$	$-(3-\kappa)r^3\sin2\theta$	$-(3+\kappa)r^3\cos2\theta$
$\sin2\theta/r^2$	$2\sin2\theta/r^3$	$-2\cos2\theta/r^3$

obtained as selecting and summing displacement corresponding to each term of the total Airy stress functions ϕ in Eq. (2.19) as below

$$u_r=\frac{\sigma_0R}{8\mu}\left[\frac{\gamma(1+\alpha)}{2}\left\{(\kappa-2)\xi^2+\frac{1}{\xi^2}\right\}\sin\theta+\gamma(1+\alpha)\left(\frac{\xi^2}{2}-\frac{1}{\xi^4}+\frac{2+\kappa}{2\xi^2}\right)\sin3\theta\right.$$
$$\left.-\{1-\alpha-2\beta(1+\alpha)\}\left(\xi-\frac{1}{\xi^3}+\frac{\kappa+1}{\xi}\right)\cos2\theta-\{1-\alpha-2\beta(1+\alpha)\}\left(\frac{\kappa-1}{2}\xi+\frac{1}{\xi}\right)\right]$$
$$+\frac{\tau_0R}{2\mu}\left(\xi+\frac{\kappa+1}{\xi}-\frac{1}{\xi^3}\right)\sin2\theta \tag{31}$$

$$u_\theta=\frac{\sigma_0R}{8\mu}\left[\{1-\alpha-2\beta(1+\alpha)\}\left(\xi+\frac{1}{\xi^3}+\frac{\kappa-1}{\xi}\right)\sin2\theta-\frac{\gamma(1+\alpha)}{2}\left\{(\kappa-2)\xi^2+\frac{1}{\xi^2}\right\}\cos\theta\right.$$
$$\left.+\gamma(1+\alpha)\left(\frac{\xi^2}{2}+\frac{1}{\xi^4}-\frac{2-\kappa}{2\xi^2}\right)\cos3\theta\right]+\frac{\tau_0R}{2\mu}\left(\xi+\frac{\kappa-1}{\xi}+\frac{1}{\xi^3}\right)\cos2\theta \tag{32}$$

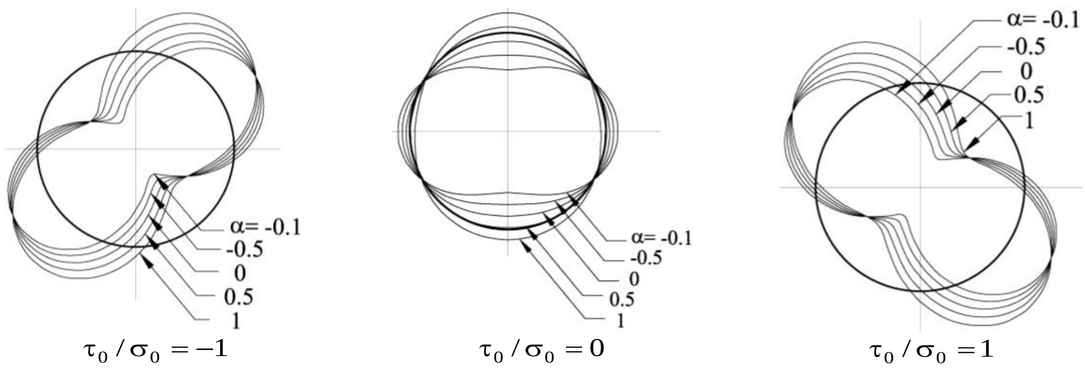

Fig. 6 The non-dimensional hoop stress $\sigma_{\theta\theta}/\sigma_0$ at $r=R$ for b/h=1/5, R/h=1/10, and τ_0/σ_0=−1,0,1, 0, 1

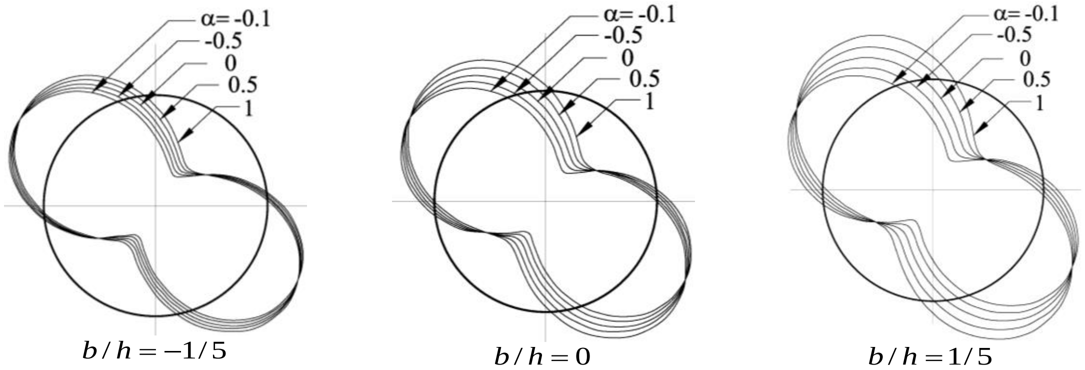

Fig. 7 The non-dimensional hoop stress $\sigma_{\theta\theta}/\sigma_0$ at $r=R$ for R/h=1/10, τ_0/σ_0=1, and b/h=−1/5,0,1/5

where μ is shear modulus, κ is a secondary elastic constant defined as

$$\kappa = \frac{3-\nu}{1+\nu} \tag{33}$$

As a numerical example, Fig. 4 shows non-dimensional circumferential displacement $u_\theta\mu/\sigma_0 R$ for β=−0.1, γ=0.1, τ_0/σ_0=1, and ν=0.3 on θ=90° with α=−1, – 0.5, 0, 0.5, and 1.

The circumferential displacement u_θ on $r=R$, which is called the hoop displacement, becomes

$$u_\theta\big|_{r=R} = \frac{\sigma_0 R}{16\mu}[\gamma(1+\alpha)(1-\kappa)\cos\theta + \gamma(1+\alpha)(1+\kappa)\cos 3\theta - 2(1+\kappa)\{2\beta(1+\alpha)+\alpha-1)\}\sin 2\theta]$$

$$+\frac{\tau_0 R}{2\mu}(1+\kappa)\cos 2\theta \tag{34}$$

The limiting case of the plate with no hole ($R\rightarrow 0$, $\beta\rightarrow 0$, $\gamma\rightarrow 0$), the displacements become as below

$$u_r = -\frac{\sigma_0 r}{8\mu}(1-\alpha)\left(\cos 2\theta + \frac{\kappa-1}{2}\right) + \frac{\tau_0}{2\mu} r\sin 2\theta \tag{35}$$

Table 3 The stress concentration factors $(\sigma_{\theta\theta})_{\max}/\sigma_0$ at $r=R$

$\frac{\tau_0}{\sigma_0}$	$\frac{R}{h}$	$\frac{b}{h}$	α				
			-1	-0.5	0	0.5	1
-1	0.05	-0.2	-5.472	-5.213	-4.966	-4.732	-4.513
		0	-5.472	-5.037	-4.643	-4.297	$\pm$4.003
		0.2	-5.472	-4.869	-4.357	4.055	4.513
	0.2	-0.2	-5.472	-5.265	-5.060	-4.857	-4.657
		0	-5.472	-5.083	-4.714	-4.367	$\pm$4.046
		0.2	-5.472	-4.910	-4.403	4.088	4.657
0	0.05	-0.2	-3	-2.6	-2.2	-1.8	-1.4
		0	-3	-2.3	-1.6	-0.9	$\pm$0.2
		0.2	-3	-2	-1	0.3	1.4
	0.2	-0.2	-3	-2.75	-2.5	-2.25	-2
		0	-3	-2.45	-1.9	-1.35	$\pm$0.8
		0.2	-3	-2.15	-1.3	0.75	2
1	0.05	-0.2	-5.472	-5.213	-4.966	-4.732	-4.413
		0	-5.472	-5.037	-4.643	-4.297	$\pm$4.003
		0.2	-5.472	-4.869	-4.357	4.055	4.513
	0.2	-0.2	-5.472	-5.265	-5.060	-4.857	-4.657
		0	-5.472	-5.083	-4.714	-4.367	$\pm$4.046
		0.2	-5.472	-4.910	-4.403	4.088	4.657

$$u_\theta = \frac{\sigma_0 r}{8\mu}(1-\alpha)\sin 2\theta + \frac{\tau_0}{2\mu} r\cos 2\theta \tag{36}$$

5. Strains

Substituting Eqs. (31) and (32) into the well-known relations of strain-displacement in polar coordinates

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{r\theta} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right], \quad \varepsilon_{\theta\theta} = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) \tag{37}$$

the strain components can be calculated as below

$$\varepsilon_{rr} = \frac{\sigma_0}{8\mu}\left[\gamma(1+\alpha)\left\{(\kappa-2)\xi - \frac{1}{\xi^3}\right\}\sin\theta + \gamma(1+\alpha)\left(\xi + \frac{4}{\xi^5} - \frac{2+\kappa}{\xi^3}\right)\sin 3\theta\right.$$
$$\left. - \{1-\alpha-2\beta(1+\alpha)\}\left(1+\frac{3}{\xi^4} - \frac{1+\kappa}{\xi^2}\right)\cos 2\theta + \{1-\alpha-2\beta(1+\alpha)\}\left(\frac{1}{\xi^2} - \frac{\kappa-1}{2}\right)\right]$$

$$+\frac{\tau_0}{2\mu}\left[1-(1+\kappa)\frac{1}{\xi^2}+\frac{3}{\xi^4}\right]\sin 2\theta \tag{38}$$

$$\varepsilon_{r\theta}=\frac{\sigma_0}{8\mu}\left[\frac{\gamma(1+\alpha)}{\xi^3}\cos\theta+\gamma(1+\alpha)\left(\xi+\frac{3}{\xi^3}-\frac{4}{\xi^5}\right)\cos 3\theta+\{1-\alpha-2\beta(1+\alpha)\}\left(1+\frac{2}{\xi^2}-\frac{3}{\xi^4}\right)\sin 2\theta\right]$$

$$+\frac{\tau_0}{2\mu}\left[1+\frac{2}{\xi^2}-\frac{3}{\xi^4}\right]\cos 2\theta \tag{39}$$

$$\varepsilon_{\theta\theta}=\frac{\sigma_0}{8\mu}\left[\gamma(1+\alpha)\left\{(\kappa-2)\xi+\frac{1}{\xi^3}\right\}\sin\theta-\gamma(1+\alpha)\left(\xi+\frac{4}{\xi^5}-\frac{4-\kappa}{\xi^3}\right)\sin 3\theta\right.$$

$$\left.+\{1-\alpha-2\beta(1+\alpha)\}\left(1+\frac{3}{\xi^4}-\frac{3-\kappa}{\xi^2}\right)\cos 2\theta-\{1-\alpha-2\beta(1+\alpha)\}\left(\frac{1}{\xi^2}-\frac{\kappa-1}{2}\right)\right]$$

$$-\frac{\tau_0}{2\mu}\left[1-\frac{3-\kappa}{\xi^2}+\frac{3}{\xi^4}\right]\sin 2\theta \tag{40}$$

As a numerical example, Fig. 5 shows radial strain $\varepsilon_{rr}\mu/\sigma_0$ for β=0.1, γ=0.1, τ_0/σ_0=1, and ν=0.3 on θ=0° with α=−1, – 0.5, 0, 0.5, and 1.

The circumferential stain $\varepsilon_{\theta\theta}$ on the edge of the non-central circular hole (r=R), which is called the hoop strain, becomes

$$\varepsilon_{\theta\theta}\big|_{r=R}=\frac{\sigma_0}{8\mu}\Big[\gamma(1+\alpha)(\kappa-1)\sin\theta-\gamma(1+\alpha)(1+\kappa)\sin 3\theta+\{1-\alpha-2\beta(1+\alpha)\}(1+\kappa)\cos 2\theta$$

$$\left.-\{1-\alpha-2\beta(1+\alpha)\}\left(1-\frac{\kappa-1}{2}\right)\right]-\frac{\tau_0}{2\mu}(1+\kappa)\sin 2\theta \tag{41}$$

The limiting case of the plate with no hole (R→0, β→0, γ→0), the strain components become as below

$$\varepsilon_{rr}=-\frac{\sigma_0}{8\mu}(1-\alpha)\left(\cos 2\theta+\frac{\kappa-1}{2}\right)+\frac{\tau_0}{2\mu}\sin 2\theta \tag{42}$$

$$\varepsilon_{r\theta}=\frac{\sigma_0}{8\mu}(1-\alpha)\sin 2\theta+\frac{\tau_0}{2\mu}\cos 2\theta \tag{43}$$

$$\varepsilon_{\theta\theta}=\frac{\sigma_0}{8\mu}(1-\alpha)\left(\cos 2\theta-\frac{\kappa-1}{2}\right)-\frac{\tau_0}{2\mu}\sin 2\theta \tag{44}$$

6. Stress concentration factors

The circumferential stress $\sigma_{\theta\theta}$ occurring at the edge of the non-central circular hole (r=R) is called the hoop stress $\sigma_{\text{Hoop}} = \sigma_{\theta\theta}\big|_{r=R}$. By means of Eq. (26) the non-dimensional hoop stress becomes

$$\frac{\sigma_{\text{Hoop}}}{\sigma_0} = \left[\gamma(1+\alpha)(\sin\theta - \sin 3\theta) + \{1-\alpha-2\beta(1+\alpha)\}\cos 2\theta + \beta(1+\alpha) - \frac{1-\alpha}{2}\right] - 4\frac{\tau_0}{\sigma_0}\sin 2\theta \quad (45)$$

It is seen that the hoop stress is independent of shear modulus μ and Poisson's ratio v. Also it is not depending on a distance a in the X-direction between the two centers of the rectangular plate and the non-central circular hole. Figs. 6 and 7 show the non-dimensional hoop stresses $\sigma_{\text{Hoop}}/\sigma_0$. The stress concentration factor (SCF) is the non-dimensional maximum hoop stress defined by the ratio between the maximum hoop stress and a nominal stress $(\sigma_{\text{Hoop}})_{\max}/\sigma_0$. Table 3 shows the stress concentration factors varying with α, $\beta(=b/h)$, $\gamma(=R/h)$, and τ_0/σ_0.

7. Conclusions

Exact solutions for stresses, strains, and displacement of a perforated rectangular plate by an arbitrarily located circular hole subjected to both linearly varying in-plane normal stresses on two opposite edges and in-plane shear stresses are investigated using the Airy stress function. The hoop stress occurring at the edge of the non-central circular hole are derived and plotted. The non-dimensional maximum hoop stresses, which is called the stress concentration factor, are also computed.

Considering multi-valuedness and singularity in stresses and displacements at infinity, the Airy stress function $\phi(r, \theta)$ is obtained as selecting and summing proper bi-harmonic functions from the tables presented by Dundurs (Fu 1996), which contain stresses and displacements for certain bi-harmonic functions. The Airy stress function $\phi(r, \theta)$ satisfies the governing equation $\nabla^4\phi=0$ and the stress free boundary conditions at the edge of the non-central circular hole.

The stress components are calculated using the relations of stresses and the Airy stress function, the displacement components can be obtained as selecting and summing displacements corresponding to each term of the Airy stress functions from the table by Dundurs (Fu 1996), and then the strain components are computed using the relation of the strains and displacements. The solutions are not depending upon the location in the X-direction of the center of the non-central circular hole.

The solutions for stress, strain, and displacement components for the present study can be used in cases of a very thick plate or a very long cuboid with a non-central circular cylindrical hole and subjected to both linearly varying normal stresses and shear stresses along the longitudinal direction, which is a kind of plane strain problems. In the plane strain problem, the secondary elastic constant κ in Eq. (33) must be changed to $3-4v$ from $(3-v)/(1+v)$.

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