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Planar plastic flow of polymers near very rough walls

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Abstract. The main objective of the present paper is to investigate, by means of a boundary value problem permitting a semi-analytic solution, qualitative behaviour of solutions for two pressure-dependent yield criteria used for plastically incompressible polymers. The study mainly focuses on the regime of friction (sticking and sliding). It is shown that the existence of the solution satisfying the regime of sticking depends on other boundary conditions. In particular, there is such a class of boundary conditions depending on the yield criterion adopted that the regime of sliding is required for the existence of the solution independently of the friction law.

Keywords: strength-differential effect; friction; sticking; sliding; polymers

1. Introduction

There are a great number of experimental observations demonstrating that skin layers are generated in the vicinity of frictional interfaces in injection molding of polymers. A review of these experimental observations is provided in Pantani *et al*. (2005). Temperature is one of the main contributory mechanisms responsible for skin layer generation (Viana 2004). In turn, friction is responsible for the temperature field in the vicinity of friction surfaces. In many cases, the effect of friction on temperature is investigated assuming a friction law in terms of stress. For example, Amontons's law has been adopted in Heise (2016). However, the friction laws in terms of stress are only applicable in the regime of sliding whereas typical friction laws include two regimes, sliding and sticking. Moreover, the relative velocity between the workpiece and tool vanishes in the regime of sticking and no heat is generated at the interface. Therefore, it is of importance to study the qualitative behaviour of solutions in the vicinity of frictional interfaces. In particular, the constitutive equations that are usually adopted for metallic materials may or may not be compatible with the regime of sticking (Alexandrov and Harris, 2006, Alexandrov and Mishuris, 2009). The present paper extends these results to constitutive equations that can be used for describing flow of polymers.

A class of polymers can be treated as rigid plastic materials under certain conditions (Harren

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1995, 1997). In general, the yield criterion of isotropic materials may include three stress invariants. However, in most cases yield criteria for polymers are assumed to be independent of the cubic invariant of the stress tensor (Harren 1995, 1997, Deshpande and Fleck 2001, Mills 2010). On the other hand, a sensible yield criterion must depend of the linear invariant of the stress tensor (Raghava *et al*. 1976). The associated flow rule is often adopted for polymers (Harren 1995, 1997, Deshpande and Fleck 2001). However, for several polymers the volume change is similar to that observed in metals (Spitzig and Richmond 1979). It is evident that the associated flow rule used in conjunction with a pressure-dependent yield criterion is not valid for such polymers. The present paper deals with such polymers assuming that elastic strains can be neglected. The plastic potential function is taken in the form of Mises. Then, the boundary value problem formulated in Alexandrov and Harris (2006) is solved to study solution behaviour in the vicinity of the friction surface for the two yield criteria proposed in Raghava *et al*. (1976), Spitzig and Richmond (1979).

2. Material models

The yield criterion for isotropic materials can always be represented as a function of the three stress invariants, I_1, I_2 and I_3, where

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3, \quad I_2 = \sqrt{\frac{3}{2}\left(s_1^2 + s_2^2 + s_3^2\right)}, \quad I_3 = \sqrt[3]{\frac{9}{2}\left(s_1^3 + s_2^3 + s_3^3\right)},$$
$$s_1 = \sigma_1 - \frac{I_1}{3}, \quad s_2 = \sigma_2 - \frac{I_1}{3}, \quad s_3 = \sigma_3 - \frac{I_1}{3}. \tag{1}$$

In these equations, σ_1, σ_2 and σ_3 are the principal stresses and s_1, s_2 and s_3 are the principal deviator stresses. A linear combination of the three stress invariants has been adopted as the yield criterion for polyethylene and polycarbonate in Spitzig and Richmond (1979). Then, it has been shown experimentally that the term involving I_3 is insignificant for polyethylene but is significant for polycarbonate. Thus the yield criterion for polyethylene is

$$I_2 + c_1 I_1 = c \tag{2}$$

where c and c_1 are material constants. Macroscopic yielding of polyvinylchloride and polycarbonate has been studied experimentally in Raghava *et al*. (1973). It has been found that the yield criterion for these materials can be represented as

$$I_2^2 + \left(C - T\right) I_1 = CT \tag{3}$$

where C and T represent the absolute values of the compressive and tensile yield strengths respectively. It is assumed that both C and T are constant. In what follows, the yield criteria (2) and (3) will be used. It has been demonstrated in both, Raghava *et al*. (1973), Spitzig and Richmond (1979), that the materials considered are practically incompressible. The associated flow rule in conjunction with Eqs. (2) and (3) does not reflect this material property. Therefore, the plastic potential function of Mises is adopted in the present paper. In this case the flow rule is

$$\xi_1 = \lambda s_1, \quad \xi_2 = \lambda s_2, \quad \xi_3 = \lambda s_3. \tag{4}$$

Here ξ_1, ξ_2 and ξ_3 are the principal strain rates and λ is a non-negative multiplier. The models are co-axial (i.e., the principal stress and strain rate directions coincide).

3. Statement of the boundary value problem

The boundary value problem that consists of a planar deformation comprising the simultaneous shearing and expansion/contraction of a hollow cylindrical specimen of material is an ideal benchmark problem for understanding qualitative features of solution behaviour in the vicinity of frictional interfaces (Alexandrov and Harris 2006, Alexandrov and Mishuris 2009). Therefore, in the present paper this boundary value problem is used in conjunction with the models introduced in Section 2.

Consider an infinite circular hollow cylinder of internal radius a_0 and external radius b_0. It is convenient to introduce a cylindrical coordinate system (r,θ,z) with its z-axis coinciding with the axis of symmetry of the cylinder. Then, the internal surface of the cylinder is given by the equation $r=a_0$ and the external surface by the equation $r=b_0$. Both surfaces are rough. In particular, it is assumed that the regime of sticking occurs at these surfaces, unless such a solution does not exist. Let u_r and u_θ denote the radial and circumferential velocities respectively. The internal radius is fixed against rotation. Therefore

$$u_\theta = 0 \tag{5}$$

for $r=a_0$, unless the regime of sliding occurs at the internal surface. The rate of expansion/contraction of the internal surface is denoted by U. Then,

$$u_r = U \tag{6}$$

for $r=a_0$. It is evident that the cylinder expands if $U>0$ and contracts if $U<0$. The final velocity boundary condition is

$$u_\theta = V \tag{7}$$

for $r=b_0$. The stress boundary condition is

$$\sigma_{rr} = -p < 0 \tag{8}$$

for $r=a_0$. Here σ_{rr} is the radial stress ($\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ will stand for the circumferential and shear stresses respectively). p involved in Eq. (8) is given.

The constitutive equations should be supplemented with the equilibrium equations. The prescribed boundary conditions dictate that the solution is independent of both θ and z. Therefore, the equilibrium equations reduce to

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \quad \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} = 0\,. \tag{9}$$

In the case of the material models under consideration, it is sufficient to find an instantaneous solution.

4. General solution

Under plane strain deformation, $\xi_3=0$, it follows from Eq. (4) that

$$s_3 = 0\,. \tag{10}$$

Then, it can be found from the identity $s_1+s_2+s_3=0$ that

$$s_1 = -s_2 . \tag{11}$$

It is possible to assume, without a loss of generality, that $\sigma_1>\sigma_2$. Then, $s_1>s_2$ and, as follows from Eq. (11), $s_1>0$. In this case, substituting Eqs. (10) and (11) into Eq. (1) leads to

$$I_2 = \sqrt{3}s_1, \quad I_2 = -\sqrt{3}s_2 . \tag{12}$$

Let φ be the inclination of the principal stress axis corresponding to the principal stress σ_1 to the r-axis, measured anti-clockwise. Then, the transformation equations for stress components result in

$$\sigma_{rr} = \frac{I_1}{3} + \frac{(\sigma_1 - \sigma_2)}{2}\cos 2\varphi, \quad \sigma_{\theta\theta} = \frac{I_1}{3} - \frac{(\sigma_1 - \sigma_2)}{2}\cos 2\varphi, \quad \sigma_{r\theta} = \frac{(\sigma_1 - \sigma_2)}{2}\sin 2\varphi . \tag{13}$$

Since $\sigma_1-\sigma_1=s_1-s_2$, Eqs. (12) and (13) combine to give

$$\sigma_{rr} = \frac{I_1}{3} + \frac{I_2}{\sqrt{3}}\cos 2\varphi, \quad \sigma_{\theta\theta} = \frac{I_1}{3} - \frac{I_2}{\sqrt{3}}\cos 2\varphi, \quad \sigma_{r\theta} = \frac{I_2}{\sqrt{3}}\sin 2\varphi . \tag{14}$$

Without a loss of generality, it is possible to assume that

$$\sigma_{r\theta} > 0 . \tag{15}$$

Note that this inequality dictates that $V>0$ in Eq. (7). In the case of the expansion of the cylinder $\sigma_{\theta\theta}>\sigma_{rr}$ and in the case of the contraction of the cylinder $\sigma_{\theta\theta}<\sigma_{rr}$. Therefore, it follows from Eqs. (14) and (15) that

$$\frac{\pi}{4} \le \varphi \le \frac{\pi}{2} \tag{16}$$

in the case of expansion of the cylinder and

$$0 \le \varphi \le \frac{\pi}{4} \tag{17}$$

in the case of contraction of the cylinder.

It is seen from Eq. (4) that the material is incompressible,

$$\xi_1 + \xi_2 + \xi_3 = 0 . \tag{18}$$

Under plane strain deformation, $\xi_3=0$, this equation becomes $\xi_1+\xi_2=0$ or, in the cylindrical system of coordinates

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} = 0 . \tag{19}$$

It has been taken into account here that the solution is independent of both θ and z. Since the models under consideration are co-axial, it follows from Eqs. (13) and (18) that the strain rate components in the cylindrical coordinate system are

$$\xi_{rr} = \frac{(\xi_1 - \xi_2)}{2}\cos 2\varphi, \quad \xi_{\theta\theta} = -\frac{(\xi_1 - \xi_2)}{2}\cos 2\varphi, \quad \xi_{r\theta} = \frac{(\xi_1 - \xi_2)}{2}\sin 2\varphi .$$

Eliminating $\xi_1 - \xi_2$ between these equations leads to $2\xi_{r\theta} = (\xi_{rr} - \xi_{\theta\theta})\tan 2\varphi$ or

$$\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} = \left(\frac{\partial u_r}{\partial r} - \frac{u_r}{r}\right)\tan 2\varphi . \tag{20}$$

It has been taken here into account that

$$\xi_{rr} = \frac{\partial u_r}{\partial r}, \ \xi_{\theta\theta} = \frac{u_r}{r} \ \text{and} \ \xi_{r\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

in the case under consideration.

The solutions to Eqs. $(9)^2$ and (19) are independent of the yield criterion chosen. In particular

$$\frac{\sigma_{r\theta}}{\sigma_0} = \frac{r_0^2}{r^2} \tag{21}$$

where r_0 is a constant of integration and σ_0 is a constant introduced for further convenience. Also

$$\frac{u_r}{U} = \frac{a_0}{r} . \tag{22}$$

This solution of Eq. (19) satisfies the boundary condition (6). Eqs. (14) and (21) combine to give

$$\frac{I_2}{\sqrt{3}\sigma_0}\sin 2\varphi = \frac{r_0^2}{r^2} . \tag{23}$$

Eliminating u_r in Eq. (20) by means of Eq. (22) yields

$$\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{2a_0 U \tan 2\varphi}{r^2} = 0 . \tag{24}$$

In what follows, the dimensionless radii ρ and b defined as

$$\rho = \frac{r}{a_0}, \quad b = \frac{b_0}{a_0} \tag{25}$$

will be used.

5. Solution for the yield criterion (2)

It is convenient to put $\sigma_0 = c$. Then, using Eq. (25) it is possible to transform Eq. (23) to

$$\frac{I_2}{\sqrt{3}c}\sin 2\varphi = \frac{r_0^2}{a_0^2 \rho^2} . \tag{26}$$

Eliminating I_1 in Eq.(14) by means of Eq. (2) yields

$$\sigma_{rr} = \frac{c}{3c_1} + \frac{I_2}{3}\left(\sqrt{3}\cos 2\varphi - \frac{1}{c_1}\right), \quad \sigma_{\theta\theta} = \frac{c}{3c_1} - \frac{I_2}{3}\left(\sqrt{3}\cos 2\varphi + \frac{1}{c_1}\right). \tag{27}$$

Substituting these expressions for σ_{rr} and $\sigma_{\theta\theta}$ into Eq. $(9)^1$ and using Eq. (25) result in

$$\frac{\partial\left(I_2\cos 2\varphi\right)}{\partial\rho}-\frac{1}{\sqrt{3}c_1}\frac{\partial I_2}{\partial\rho}+\frac{2I_2\cos 2\varphi}{\rho}=0\,. \tag{28}$$

Using the identity

$$\frac{\partial\left(\rho^2 I_2\cos 2\varphi\right)}{\partial\rho}=2\rho I_2\cos 2\varphi+\rho^2\frac{\partial\left(I_2\cos 2\varphi\right)}{\partial\rho}$$

Eq. (28) can be transformed to

$$\frac{\partial\left(\rho^2 I_2\cos 2\varphi\right)}{\partial\rho}-\frac{\rho^2}{\sqrt{3}c_1}\frac{\partial I_2}{\partial\rho}=0.$$

Eliminating I_2 in this equation by means of Eq. (23) and using Eq. (25) yield

$$\frac{d\varphi}{d\rho}=\frac{\sin 2\varphi}{\rho\left(\sqrt{3}c_1-\cos 2\varphi\right)}\,. \tag{29}$$

This equation can be immediately integrated to give

$$\rho=\sqrt{\frac{\sin 2\varphi_a}{\sin 2\varphi}}\left(\frac{\cot\varphi_a}{\cot\varphi}\right)^m,\quad m=\frac{\sqrt{3}c_1}{2}\,. \tag{30}$$

Here φ_a is the value of φ at ρ=1. Using Eqs. (25), (29) and (30) it is possible to rewrite Eq. (24) as

$$\frac{\partial w}{\partial\varphi}+\frac{2\left(2m-\cos 2\varphi\right)\tan 2\varphi}{\sin 2\varphi_a}\left(\frac{\cot\varphi}{\cot\varphi_a}\right)^{2m}=0 \tag{31}$$

where

$$w=\frac{u_\theta}{U\rho}\,. \tag{32}$$

Using this definition for w, the definition for φ_a and Eq. (25) the solution to Eq. (31) satisfying the boundary condition (5) can be written as

$$w=2\int\limits_{\varphi_a}^{\varphi}\frac{\left(\cos 2\chi-2m\right)\tan 2\chi}{\sin 2\varphi_a}\left(\frac{\cot\chi}{\cot\varphi_a}\right)^{2m}d\chi \tag{33}$$

where χ is a dummy variable of integration. Then, the boundary condition (7) leads to

$$\frac{V}{Ub}=2\int\limits_{\varphi_a}^{\varphi_b}\frac{\left(\cos 2\varphi-2m\right)\tan 2\varphi}{\sin 2\varphi_a}\left(\frac{\cot\varphi}{\cot\varphi_a}\right)^{2m}d\varphi\,. \tag{34}$$

Here φ_b is the value of φ at ρ=b. Therefore, it follows from Eq. (30) that

$$b=\sqrt{\frac{\sin 2\varphi_a}{\sin 2\varphi_b}}\left(\frac{\cot\varphi_a}{\cot\varphi_b}\right)^m. \tag{35}$$

Equations (34) and (35) constitute a system for determining φ_a and φ_b at given values of b, m and V/U. This system may or may not have a solution. Of special interest is the existence of the solution as the ratio $V/|U|$ increases whereas the other parameters are kept constant. In this case, the non-existence of the solution means that the regime of sticking at the friction surface $\rho=1$ is not compatible with other boundary conditions and the only possibility to find a solution is to assume that the regime of sliding occurs at that friction surface.

The integrand in Eq. (34) reduces to the expression $0\cdot\infty$ at $\varphi=0$ and $\varphi=\pi/2$. However, it is evident from Eq. (14) that $\varphi=0$ and $\varphi=\pi/2$ are the special solutions of the boundary value problem corresponding to expansion/contraction with no twist. Therefore, these special cases are not of interest. The integrand in Eq. (34) vanishes at $\varphi=\varphi_{cr}$ where φ_{cr} should be found from the equation

$$\cos 2\varphi_{cr} - 2m = 0 . \tag{36}$$

This equation has no solution if $\cos 2\varphi<0$. It is seen from Eq. (16) that this case corresponds to expansion of the cylinder. In the case of contraction, Eq. (36) has no solution if

$$m > \frac{1}{2} . \tag{37}$$

If Eq. (36) has a solution then $\varphi_{cr}<\pi/4$. Consider the function $f(\varphi)=\sin(2\varphi)\cot^{2m}\varphi$. Differentiating gives

$$\frac{df}{d\varphi} = 2\left(\cos 2\varphi - 2m\right)\cot^{2m}\varphi, \quad \frac{d^2 f}{d\varphi^2} = \frac{2\left[4m\left(2m-\cos 2\varphi\right)-1+\cos 4\varphi\right]\cot^{2m}\varphi}{\sin 2\varphi} . \tag{38}$$

It is seen from Eqs. (36) and (38) that $df/d\varphi=0$ and $d^2f/d\varphi^2<0$ at $\varphi=\varphi_{cr}$. Therefore, the function $f(\varphi)$ attains a maximum at $\varphi=\varphi_{cr}$. Then, it is seen from Eq. (35) that

$$\varphi_{cr} \le \varphi_a < \varphi_b < \frac{\pi}{2} \tag{39}$$

and

$$0 < \varphi_b < \varphi_a \le \varphi_{cr} . \tag{40}$$

Eq. (39) is valid in the case of expansion. The integrand in Eq. (34) is positive in the range of φ given in Eq. (16). This condition and Eq. (39) ensure that U is positive. Assume that $\varphi_a\to\pi/4$. Since

$$\tan 2\varphi = \frac{1}{2}\left(\frac{\pi}{4}-\varphi\right)^{-1} + o\left[\left(\frac{\pi}{4}-\varphi\right)^{-1}\right]$$

as $\varphi\to\pi/4$, the integral involved in Eq. (34) is divergent and $V/U\to\infty$ as $\varphi_a\to\pi/4$. Thus the solution satisfying the regime of sticking at $\rho=1$ always exists.

Equation (40) is valid in the case of contraction. Equation (39) can also be valid in the case of contraction if $\varphi_a<\pi/4$. Assume that Eq. (40) is valid. The integrand in Eq. (34) is positive in the range $0<\varphi<\varphi_{cr}$. This condition and Eq. (40) ensure that U is negative. Since $\varphi_{cr}<\pi/4$, the integrand in Eq. (34) is bounded in the range $0<\varphi<\varphi_{cr}$. Therefore, the ratio $V/|U|$ is bounded and no solution satisfying the regime of sticking at $\rho=1$ exists if this ratio is large enough. In order to determine the maximum possible value of the ratio $V/|U|$ at which the solution at sticking exists, it is necessary to find a maximum of the right hand side of Eq. (34) with the subsidiary condition (35).

Differentiating the right hand side of Eq. (34) with respect to φ_a gives

$$\begin{aligned}\frac{d(V/U)}{bd\varphi_a} = {} & \frac{2(\cos 2\varphi_b - 2m)\tan 2\varphi_b}{\sin 2\varphi_a}\left(\frac{\cot\varphi_b}{\cot\varphi_a}\right)^{2m}\frac{d\varphi_b}{d\varphi_a} - \frac{2(\cos 2\varphi_a - 2m)}{\cos 2\varphi_a} - \\ & \frac{4(\cos 2\varphi_a - 2m)}{\cot^{2m}\varphi_a \sin^2 2\varphi_a}\int_{\varphi_a}^{\varphi_b}(\cos 2\varphi - 2m)\cot^{2m}\varphi\tan 2\varphi d\varphi.\end{aligned} \tag{41}$$

Differentiating Eq. (35) with respect to φ_a results in

$$b^2(\cos 2\varphi_b - 2m)\cot^{2m}\varphi_b\frac{d\varphi_b}{d\varphi_a} = (\cos 2\varphi_a - 2m)\cot^{2m}\varphi_a. \tag{42}$$

It follows from Eqs. (36), (41) and (42) that the ratio V/U considered as a function of φ_a is stationary at $\varphi_a=\varphi_{cr}$. The derivative $d^2\varphi_b/d\varphi_a^2$ at the point $\varphi_a=\varphi_{cr}$ is found from Eq. (42) as

$$\frac{d^2\varphi_b}{d\varphi_a^2} = -\frac{2\sin 2\varphi_{cr}\cot^{2m}\varphi_{cr}}{b^2(\cos 2\varphi_{bc} - 2m)\cot^{2m}\varphi_{bc}} \tag{43}$$

where φ_{bc} is the value of φ_b at $\varphi_a=\varphi_{cr}$. This value depends on b and should be found from the solution of Eq. (35) in which φ_a should be replaced with φ_{cr}. Differentiating the right hand side of Eq. (41) with respect to φ_a, putting $\varphi_a=\varphi_{cr}$ and using Eq. (43) give

$$Q = \frac{d^2(V/U)}{bd\varphi_a^2} = -\frac{4}{b^2}\tan 2\varphi_{bc} + \frac{2\sin 2\varphi_{cr}}{m} + 8\int_{\varphi_{cr}}^{\varphi_{bc}}\frac{(\cos 2\varphi - 2m)\tan 2\varphi}{\sin 2\varphi_{cr}}\left(\frac{\cot\varphi}{\cot\varphi_{cr}}\right)^{2m}d\varphi \tag{44}$$

at $\varphi_a=\varphi_{cr}$. If $Q>0$ then the function V/U attains a minimum at $\varphi_a=\varphi_{cr}$. Therefore, the function $V/|U|$ attains a maximum at this point. The maximum possible value of $V/(|U|b)$ is denoted by s. The dependence of s on b for several values of c_1 is depicted in Fig. 1. The broken line corresponds to $c_1=0.022$ (Spitzig and Richmond 1979).

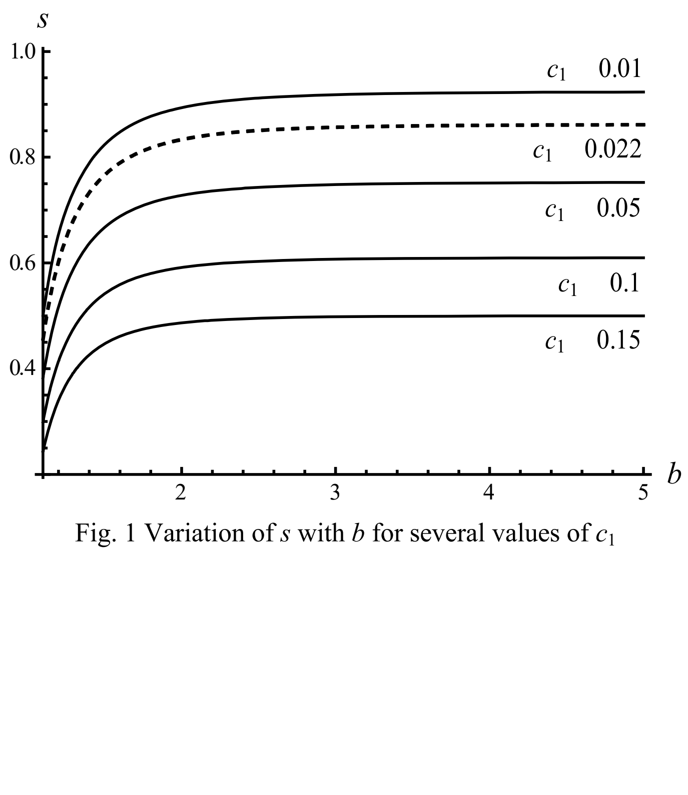

Fig. 1 Variation of s with b for several values of c_1

The integrand in Eq. (34) is negative in the range $\varphi_{cr}<\varphi<\pi/4$. This condition and Eq. (39) ensure that U is negative. However, this case is not important for applications since the range in which b can vary is very small. For example, if $c_1=0.022$ (Spitzig and Richmond 1979) it is possible to find from Eq. (35) that $1<b<1.00036$.

6. Solution for the yield criterion (3).

It is convenient to put $\sigma_0=(C+T)/2$. Then, using Eq. (25) it is possible to transform Eq. (23) to

$$\frac{2I_2}{\sqrt{3}(C+T)}\sin 2\varphi = \frac{r_0^2}{a_0^2\rho^2}. \tag{45}$$

Eliminating I_1 in Eq. (14) by means of Eq. (3) yields

$$\sigma_{rr} = \frac{CT}{3(C-T)} - \frac{I_2^2}{3(C-T)} + \frac{I_2}{\sqrt{3}}\cos 2\varphi, \quad \sigma_{\theta\theta} = \frac{CT}{3(C-T)} - \frac{I_2^2}{3(C-T)} - \frac{I_2}{\sqrt{3}}\cos 2\varphi . \tag{46}$$

Eliminating I_2 in these equations by means of Eq. (45) and substituting the resulting expressions for the normal stresses into Eq. (9)[1] yield

$$\frac{d\rho}{d\varphi} = \frac{\rho\left(\rho^2 - 2\alpha\cot 2\varphi\right)}{2\alpha}, \quad \alpha = \frac{(C+T)}{2(C-T)}\frac{r_0^2}{a_0^2}. \tag{47}$$

Note that $C>T$ (Raghava *et al.* 1973) and, therefore, $\alpha>0$. The solution to this equation satisfying the condition $\rho=1$ for $\varphi<\varphi_a$ is

$$\rho^2 = g(\varphi) = \frac{2\alpha}{\sin 2\varphi}\left[\frac{2\alpha}{\sin 2\varphi_a} - \ln\left(\frac{\tan\varphi}{\tan\varphi_a}\right)\right]^{-1}. \tag{48}$$

Then

$$b^2 = \frac{2\alpha}{\sin 2\varphi_b}\left[\frac{2\alpha}{\sin 2\varphi_a} - \ln\left(\frac{\tan\varphi_b}{\tan\varphi_a}\right)\right]^{-1}. \tag{49}$$

This equation connects φ_a and φ_b. Using Eqs. (25), (47) and (48) it is possible to rewrite Eq. (24) as

$$\frac{dw}{d\varphi} = \frac{2\alpha - g(\varphi)\tan 2\varphi}{\alpha g(\varphi)}. \tag{50}$$

Integrating and using the boundary condition (5) give

$$w = \frac{1}{\alpha}\int_{\varphi_a}^{\varphi}\left[\frac{2\alpha}{g(\chi)} - \tan 2\chi\right]d\chi. \tag{51}$$

Then, the boundary condition (7) leads to

$$\frac{V}{Ub} = \frac{1}{\alpha}\int_{\varphi_a}^{\varphi_b}\left[\frac{2\alpha}{g(\varphi)} - \tan 2\varphi\right]d\varphi. \tag{52}$$

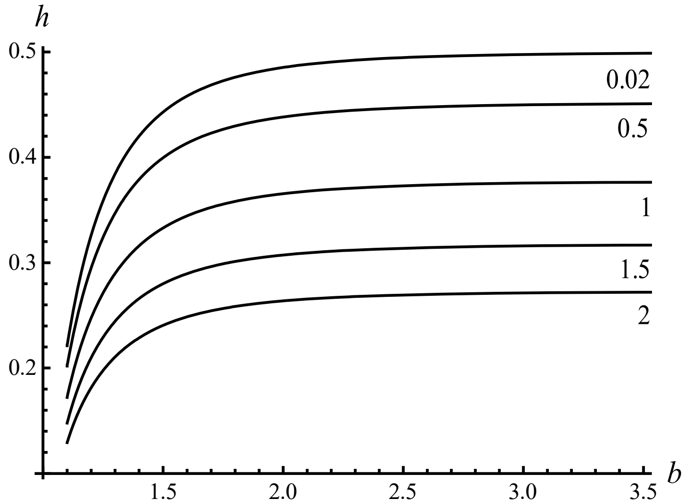

Fig. 2 Variation of h with b for several values of α

Consider the expansion of the cylinder. In this case Eq. (16) is valid. Therefore, $\tan 2\varphi<0$ and the integrand involved in Eq. (52) is positive. Moreover, it is seen from Eq. (47) that $d\rho/d\varphi>0$ in the range $\pi/4\le\varphi\le\pi/2$. Therefore, $\varphi_a<\varphi_b$, the integral in Eq. (52) is positive and the right hand side of this equation approaches infinity as $\varphi_a\to\pi/4$. Thus it is always possible to find the solution satisfying the regime of sticking at the friction surface $\rho=1$.

Consider the contraction of the cylinder. In this case Eq. (17) is valid and, therefore, $\tan 2\varphi>0$. It is seen from Eq. (47) that $d\rho/d\varphi=0$ at $\rho^2=\rho_{cr}^2=2\alpha\cot 2\varphi_{cr}$. Differentiating Eq. (47) with respect to φ and putting $\rho=\rho_{cr}$ give $d^2\rho/d\varphi^2>0$ at $\varphi=\varphi_{cr}$. Therefore, the function $\rho(\varphi)$ attains a minimum at $\varphi<\varphi_{cr}$. Since $\rho\ge 1$, the condition $\varphi=\varphi_{cr}$ can be satisfied if and only if $\varphi_{cr}=\varphi_a$. Thus Eqs. (39) and (40) are valid. In the case of Eq. (40) the integral involved in Eq. (52) is convergent as $\varphi\to\varphi_{cr}$ independently of the value of φ_b. Therefore, the ratio $V/|U|$ is bounded at $\varphi_a=\varphi_{cr}$ and no solution satisfying the regime of sticking at $\rho=1$ exists if

$$\frac{V}{|U|b\alpha^2}>h \tag{53}$$

The variation of h with b is shown in Fig. 2 for several values of α.

In the case of Eq. (39) the integral involved in Eq. (52) is divergent if $\varphi_b=\pi/4$. Substituting this value of φ_b and $\varphi_a=\varphi_{cr}$ into Eq. (49) gives

$$b_m=\sqrt{\frac{2\alpha\sin 2\varphi_{cr}}{2\alpha+\sin 2\varphi_{cr}\ln\left(\tan\varphi_{cr}\right)}}. \tag{54}$$

The dependence of b_m on α is depicted in Fig. 3. The physical sense of b_m is as follows. If $b<b_m$ then Eq. (49) can be solved for φ_b at $\varphi_a=\varphi_{cr}$ and $\varphi_b<\pi/4$. In this case the ratio $V/|U|$ is bounded at $\varphi_a=\varphi_{cr}$ and no solution satisfying the regime of sticking at $\rho=1$ exists if

$$\frac{V}{|U|b\alpha^2}>k \tag{55}$$

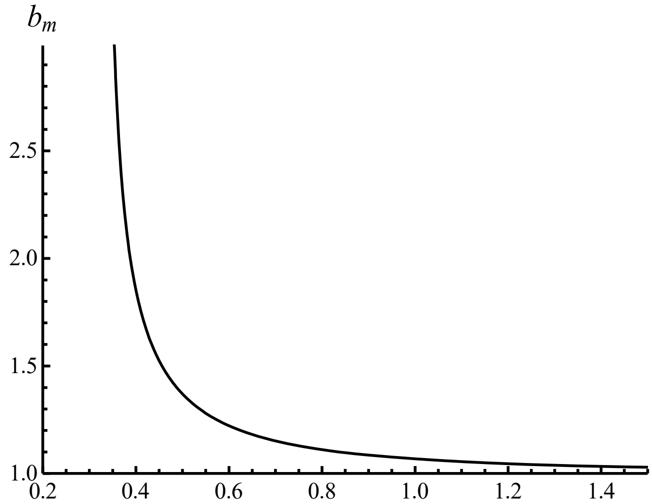

Fig. 3 Variation of $b_m$with α

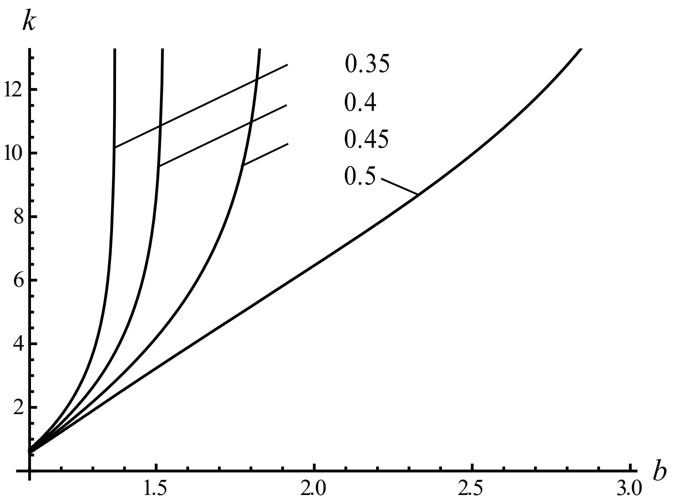

Fig. 4 Variation of k with b for several values of α

The variation of k with b is shown in Fig. 4 for several values of α. If $b=b_m$ then $V/|U|\to\infty$ as $\varphi_a\to\varphi_{cr}$. Therefore, the regime of sticking always occurs at $\rho=1$. Finally, no solution exists if $b>b_m$.

7. Conclusions

Using the boundary value problem formulated in Alexandrov and Harris (2006) the qualitative behaviour of solutions in the vicinity of the friction interface for the yield criteria proposed in Raghava *et al*. (1976), Spitzig and Richmond (1979) has been studied. From this work, the following conclusions can be drawn.

1. In the case of expansion of the cylinder, the solution satisfying the regime of sticking at the

inner radius exists independently of the other boundary conditions for both yield criteria considered in the present paper.

2. In the case of contraction of the cylinder obeying the yield criterion (2), the solution satisfying the regime of sticking does not exist if the ratio of the circumferential velocity at the outer radius to the radial velocity of the inner radius is large enough. The dependence of $s=V/(|U|b)$ on b is illustrated in Fig. 1. It is worthy of note that s depends on the material parameter c_1 but is independent of p involved in the stress boundary condition (8).
3. In the case of contraction of the cylinder obeying the yield criterion (3), the solution satisfying the regime of sticking does not exist if the ratio of the circumferential velocity at the outer radius to the radial velocity of the inner radius is large enough. The dependences of $h=V/(|U|b\alpha^2)$ and $k=V/(|U|b\alpha^2)$ on b are illustrated in Figs. 2 and 4, respectively. The different symbols, h and k, are used to distinguish two cases corresponding to Eqs. (39) and (40). In contrast to the yield criterion (2), both h and k are affected by p because the definition for α involves r_0 and this constant of integration should be found by means of the boundary condition (8).

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