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Influence of a soft FGM interlayer on contact stresses under a beam on an elastic foundation

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Abstract. Contact interaction of a beam (flexible element) with an elastic half-plane is considered, when a soft inhomogeneous (functionally graded) interlayer is present between them. The beam is bent under the action of a distributed load applied to the surface and a reaction of the elastic interlayer and the half-space. Solution of the contact problem is obtained for different values of thickness and parameters of inhomogeneity of the layer. The interlayer is assumed to be significantly softer than the underlying half-plane; case of 100 times difference in Young's moduli is considered as an example. The influence of the interlayer thickness and gradient of elastic properties on the distribution of the contact stresses under the beam is studied.

Keywords: bending of a beam; analytic solution; dual integral equation; functionally graded layer; soft layer; elastic half-plane

1. Introduction

Contact problems occupy key position in solid mechanics. Solution for classic contact problems on interaction of a punch with a half-space are well-known (Galin 1961, Sneddon, 1951). A lot of studies in this area were connected with contact problems where half-space is homogeneous, and the punch is rigid and axisymmetric. As derivation from this model, the static beam-half-space interaction was first examined by Biot (1937) and later by Rvachev (1958), Vesic (1961) and others, see Selvadurai (1979, 1984) and Wang *et al.* (2005) for details. Several methods were developed for solution of contact problems involving interaction of flexible elements with elastic foundations, including asymptotic methods (Aleksandrov and Salamatova 2009,

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Aleksandrov and Solodovnik 1974), collocation and orthogonal polynomials methods (Aleksandrov and Salamatova 2009, Bosakov 2008), numerical methods (Kim 2009, Tullini *et al.* 2013 and others).

The sudden change in material properties across the interfaces among different materials can result in large interlaminar stresses. To overcome this issue, functionally graded materials (FGMs) are composed of two or more phases with different material properties and continuously varying composition distribution, taking advantage of the desired material properties of each constituent material without interface problems. Nowadays, FGMs are widely used as core layers and face layers of structures. One of novel applications is to use functionally graded (FG) surface layers to suppress thermoelastic instability in sliding contact (Mao *et al.* 2014, 2015). Other objects of common interest, closely related to FGMs, are composite structures, such as sandwich plates (Altenbach *et al.* 2015), which are commonly treated with appropriate plate theories.

There are a number of recent studies in the literature on investigation of mechanical characteristics of FG structures with various solution methods. Usually, the essential way is to use integral transformation technique to reduce a problem to an integral equation. Such an equation then can be solved numerically, for example, using collocation technique. This approach was used in a number of recent researches. Ke *et al.* (2008) considered frictionless contact with a functionally graded piezoelectric layered half-plane. Guler *et al.* (2012) considered shear of a thin film bonded to a coating, which has exponential variation of the shear modulus by depth and is perfectly bonded to a homogeneous substrate. The axisymmetric partial slip contact of a rigid punch and an arbitrarily graded coating was analyzed by Liu *et al.* (2012).

Another approach is to approximate kernel transform of the integral equation by analytic expression of a special kind, which allows a closed-form approximate analytic solution to be obtained. The resolvent kernel method was adopted for this purpose by Tokovyy and Ma (2015). The bilateral asymptotic method for solution of dual integral equations was developed by Aizikovich *et al.* (1984, 2009), Aizikovich and Vasiliev (2013). The latter method was applied for a wide range of elasticity (Vasiliev *et al.* 2014), electroelasticity (Vasiliev *et al.* 2016), thermoelasticity problems (Krenev *et al.* 2015).

Nowadays, advanced mathematical modeling of flexible structural elements, namely beams and plates, remains one of research priorities in the field of structural engineering. Flexible elements with soft thin layers can be found in a number of modern devices, for example, solar cells (Naumenko and Eremeyev 2014). To lower contact stresses concentration in local areas of contact and extend durability of devices, soft protective coatings are commonly used. Usually, the ratio of elastic moduli for the coating and the substrate varies in the range 10 to 100.

This paper deals with a plane contact problem on a beam bending on an elastic soft functionally graded strip bonded to an elastic homogeneous half-plane. Softness here means significant difference in elastic modulus at interface between the layer and the substrate. To obtain solution of the considered beam bending problem, an approach is developed based on the aforementioned bilateral asymptotic method for solution of dual integral equations (Aizikovich *et al.* 2009). It allows approximated analytical solution of the problem to be obtained in a unified analytic form, effective in a whole range of physical and geometric parameters of the problem and applicable both for flexible and stiff beams, thick and thin coatings.

In previous researches of contact problems for soft layers, the foundation is often assumed to be undeformable (see Vorovich and Ustinov 1959). This assumption significantly simplifies solution of the problem. In this work, the foundation is considered to be elastic, which allows us to obtain more realistic model of deformation of bodies with soft coatings. The case of 100-fold difference

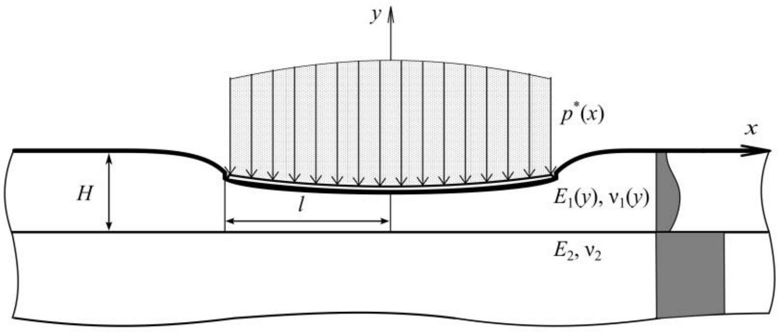

Fig. 1 Geometry of the problem

in elastic moduli of the coating and the substrate on their interface is considered.

The bilateral asymptotic method was used before to construct approximate analytical solution for problem on axisymmetric bending of a circular plate on an inhomogeneous foundation (Aizikovich *et al.* 2011). Axisymmetric contact problem on a rigid punch indentation into a soft functionally graded layer was considered before by Volkov *et al.* (2013), Vasiliev *et al.* (2014).

2. Mathematical formulation of the problem

Consider a beam of length $2l$ and constant thickness h resting on the boundary of an inhomogeneous elastic foundation, consisting of inhomogeneous layer (coating) with thickness H and homogeneous half-plane (substrate). Cartesian coordinate system $[x,y]$ is connected with the foundation boundary, and the y axis passes through the center of the beam. The beam is bent under the action of a distributed load $p^*(x)$ and the response from the layer. The applied load is symmetrical about the y axis (Fig. 1).

Young's modulus E and Poisson's ratio ν of the foundation vary with depth according to the following law

$$\mathrm{E}(y), \nu(y) = \begin{cases} E_1(y), \nu_1(y), & -H \le y \le 0 \\ E_2, \nu_2, & -\infty < y < -H \end{cases}, \quad E_2, \nu_2 = \mathrm{const} \tag{1}$$

where E_1(y), v_1(y) are arbitrary continuously differentiable functions. Hereafter, indexes $_1$ and $_2$ correspond to the layer and to the substrate, respectively.

The layer and the substrate are assumed to be glued without sliding

$$y = -H: \ \tau_{xy}^1 = \tau_{xy}^2, \ \sigma_y^1 = \sigma_y^2, v^1 = v^2, \ u^1 = u^2 \tag{2}$$

Outside of the beam, the surface is traction-free

$$y = 0: \ \tau_{xy}^1 = 0, \ \begin{cases} \sigma_y^1 = 0, & x > l \\ v^1 = -w^*(x), & x \le l \end{cases} \tag{3}$$

In Eqs. (2) and (3) τ_{xy}, σ_y are components of the stress tensor, and v,u are components of the displacement tensor.

The stresses and the displacements vanish at $x\to\infty$ and $y\to\infty$.

The quantities of primary interest are the contact stresses under the beam $q^*(x)=\sigma_y\big|_{y=0}$, and the deflections of the beam $w^*(x)$.

The following condition is fulfilled

$$w^{*\prime\prime}(\pm l)=w^{*\prime\prime\prime}(\pm l)=0 \qquad (4)$$

which corresponds to free edges of the beam.

3. Solution of the problem

3.1 Dual integral equation

To reduce the problem to the solution of the dual integral equation we use the classical approach based on the Fourier integral transformations technique

$$v^1(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}V_1(\gamma,y)e^{-\mathrm{i}\gamma x}d\gamma\,, \qquad (5)$$

$$Q^*(\gamma)=\int_{-l}^{l}q^*(\rho)e^{\mathrm{i}\gamma\rho}d\rho\,,\quad q^*(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}Q^*(\gamma)e^{-\mathrm{i}\gamma x}d\gamma \qquad (6)$$

Then Eq. (3) leads to the following

$$\int_{-\infty}^{\infty}V_1(\gamma,0)e^{-\mathrm{i}\gamma x}d\gamma=-2\pi w^*(x),\ |x|\le l \qquad (7)$$

Let us introduce the following notations

$$V_1^*(\gamma,y)=-\Theta\frac{\gamma V_1(\gamma,y)}{Q^*(\gamma)}\,,\quad \Theta=\frac{E(0)}{2(1-\nu^2(0))} \qquad (8)$$

Using Eq. (8) we rewrite equation Eq. (7) in the form

$$\int_{-\infty}^{\infty}\frac{1}{|\gamma|}V_1^*(\gamma,0)Q^*(\gamma)e^{-\mathrm{i}\gamma x}d\gamma=2\pi\Theta w^*(x),\ |x|\le l \qquad (9)$$

Substituting Eq. (6) into Eq. (9) we get

$$\int_{-l}^{l}q^*(\rho)e^{\mathrm{i}\gamma\rho}d\rho\int_{-\infty}^{\infty}\frac{1}{|\gamma|}V_1^*(\gamma,0)e^{-\mathrm{i}\gamma x}d\gamma=2\pi\Theta w^*(x),\quad |x|\le l \qquad (10)$$

Let us introduce the dimensionless variables and functions:

$$\gamma H=u,\lambda=H/l\,,x'=x/l\,,\rho'=\rho/l\,,\mathrm{V}_1^*(u/H,0)=L(u),q^*(x)=q(x')Dl^{-3},$$
$$y'=y/H,w^*(x)=w(x')l\,,\ p^*(x)=p(x')Dl^{-3},\alpha=u\lambda^{-1},\ s=\Theta l^3/D$$

where D is a parameter characterizing stiffness of the beam, parameter s is dimensionless bending stiffness of the plate. Function $L(u)$ is the kernel transform of the integral equation.

Using the above dimensionless parameters, we rewrite Eq. (10) and obtain the Fredholm integral equation of the first kind over the function $q(\rho)$

$$\int_{-1}^{1} q(\rho)d\rho \int_{-\infty}^{\infty} \frac{1}{|\alpha|} L(\lambda\alpha) e^{\mathrm{i}\alpha(\rho-x)} d\alpha = 2\pi s w(x), \ |x| \le 1 \tag{11}$$

Eq. (11) is equivalent to the following dual integral equation:

$$\begin{cases} \displaystyle\int_{-\infty}^{\infty} \frac{Q(\alpha)}{|\alpha|} L(\alpha\lambda) e^{-\mathrm{i}x\alpha} d\alpha = 2\pi s w(x), & |x| \le 1 \\ \displaystyle\int_{-\infty}^{\infty} Q(\alpha) e^{-\mathrm{i}x\alpha} d\alpha = 0, & |x| > 1 \end{cases} \tag{12}$$

According to the Kirchhoff's plate model the deflection of the plate has to satisfy the differential equation of a bending of the beam

$$w^{\mathrm{IV}}(x) = p(x) - q(x), \ x \in [-1,1] \tag{13}$$

where $w^{\mathrm{IV}}(x)$ is the fourth derivative of $w(x)$.

3.2 Solution of the problem

Let us represent deflection function in a form of series with respect to eigenfunctions of oscillations of a beam with free edges

$$w(x) = \sum_{n=0}^{\infty} w_n \varphi_n(x), \ \ \varphi_0 = 1, \ \ \varphi_n(x) = \frac{\cos r_n x}{\cos r_n} + \frac{\operatorname{ch} r_n x}{\operatorname{ch} r_n}, \quad n = 1,2,\ldots, \quad |x| \le 1 \tag{14}$$

where r_n are zeroes of the characteristic equation

$$\operatorname{tg} r_n = -\operatorname{th} r_n, \ \ n = 1,2,\ldots; \ r_n \approx (n - 0.25)\pi, n \ge 1 .$$

Due to the linearity of the problem, the contact stresses $q(x)$ and its Fourier transform $Q(\alpha)$ can be represented as

$$q(x) = \sum_{n=0}^{\infty} w_n q_n(x), \ Q(\alpha) = \sum_{n=0}^{\infty} w_n Q_n(\alpha) \ \ |x| \le 1, \tag{15}$$

Substituting Eqs. (14) and (15) into Eq. (12) we get the dual integral equation over the function $Q_n(\alpha)$

$$\begin{cases} \displaystyle\int_{-\infty}^{\infty} \frac{Q_n(\alpha)}{|\alpha|} L(\lambda\alpha) \cos(x\alpha) d\alpha = 2\pi s \varphi_n(x), & |x| \le 1 \\ \displaystyle\int_{-\infty}^{\infty} Q_n(\alpha) \cos(x\alpha) d\alpha = 0, & |x| > 1 \end{cases} \tag{16}$$

The kernel transform $L(\alpha)$ depends on the properties of the nonhomogeneous materials. In (Aizikovich and Aleksandrov 1982) it was shown that $L(\alpha)$ possesses the following properties

$$\begin{aligned} L(\alpha) &= A + B|\alpha| + C\alpha^2 + O(|\alpha|^3),\ \alpha \to 0 \\ L(\alpha) &= 1 + D|\alpha|^{-1} + E\alpha^{-2} + O(|\alpha|^{-3}),\ \alpha \to \infty \end{aligned} \tag{17}$$

where A, B, C and D are certain constants which values depend on the material properties. It was shown (Aizikovich and Vasiliev 2013, Vasiliev *et al.* 2014) that the kernel transform $L(\alpha)$ can be approximated with high accuracy by the expression

$$\begin{aligned} &L(\alpha\lambda) \approx L_N(\alpha\lambda) = \frac{P_1(\alpha^2\lambda^2)}{P_2(\alpha^2\lambda^2)} \\ &P_1(\alpha^2\lambda^2) = \prod_{i=1}^{N}(\alpha^2\lambda^2 + a_i^2), \quad P_2(\alpha^2\lambda^2) = \prod_{i=1}^{N}(\alpha^2\lambda^2 + b_i^2) \end{aligned} \tag{18}$$

where a_i and b_i are certain constants. Detailed description of the process of determining coefficients a_i, b_i (i=1...N) is provided in (Aizikovich and Vasiliev 2013).

It was shown in (Aizikovich 2009) that the solution of approximate dual integral equation resulting from replacing transform $L(\alpha)$ in Eq. (16) by its approximation $L_N(\alpha)$ is asymptotically exact for both thin and thick coatings, i.e., for $\lambda\to 0$ and $\lambda\to\infty$.

Let us introduce the functions

$$d_n(x) = \frac{1}{2\pi s}\int_{-\infty}^{\infty}\frac{Q_n(\alpha)}{|\alpha|}\cos(x\alpha)d\alpha \tag{19}$$

then using operational analysis (Aleksandrov 1973) and Eqs. (18), (19), we represent first equation in Eq. (16) in operator form

$$P_1(-\lambda^2 D)d_n(x) = P_2(-\lambda^2 D)\varphi_n(x), \quad D = \frac{d^2}{dx^2}, \quad x \in [-1,1] \tag{20}$$

The solution of the differential Eq. (20) for d_n, taking into account parity of $p(x)$, $q(x)$ and $\varphi_n(x)$, has the form

$$\begin{aligned} d_0(x) &= \sum_{j=1}^{N}\frac{C_j^n}{a_j\lambda^{-1}}\operatorname{ch}(a_j\lambda^{-1}x) + L_N^{-1}(0) \\ d_n(x) &= \sum_{j=1}^{N}\frac{C_j^n}{a_j\lambda^{-1}}\operatorname{ch}(a_j\lambda^{-1}x) + L_N^{-1}(\lambda r_n)\frac{\cos(r_n x)}{\cos(r_n)} + L_N^{-1}(\mathrm{i}\lambda r_n)\frac{\operatorname{ch}(r_n x)}{\operatorname{ch}(r_n)} \end{aligned} \tag{21}$$

where i is an imaginary unit, and coefficients C_j^n are arbitrary constants.

By differentiating Eq. (19) by x, Eq. (16) can be rewritten as follows

$$\begin{cases} \displaystyle\int_0^{\infty} Q_n(\alpha)\sin(x\alpha)d\alpha = -\pi s d_n'(x) & 0 \le x \le 1 \\ \displaystyle\int_0^{\infty} Q_n(\alpha)\cos(x\alpha)d\alpha = 0, & x > 1 \end{cases} \tag{22}$$

Here we change integration limits from $(-\infty, \infty)$ to $(0, \infty)$ due to the parity of integrands.

Applying operator $\int_0^t \frac{xdx}{\sqrt{t^2-x^2}}$ to the first of Eq. (22) and $\int_t^\infty \frac{xdx}{\sqrt{x^2-t^2}}$ to the second one, we obtain

$$\begin{cases} \int_0^\infty Q_n(\alpha)J_1(t\alpha)d\alpha = -\frac{2s}{t}\int_0^t \frac{xd_n'(x)dx}{\sqrt{t^2-x^2}}, \quad 0 \le t \le 1 \\ \int_0^\infty Q_n(\alpha)J_1(t\alpha)d\alpha = 0, \quad t > 1 \end{cases} \tag{23}$$

Introducing change of variables $Q_n^*(\alpha)=\alpha^{-1}Q_n(\alpha)$ in Eq. (23) and inverting the Hankel transform, we obtain

$$Q_n(\alpha) = -2s\alpha\int_0^1 J_1(\alpha t)dt\int_0^t \frac{xd_n'(x)dx}{\sqrt{t^2-x^2}} \tag{24}$$

The homogeneous dual integral Eq. (22) in case of n=0 also has a solution $Q^*(\alpha)=FJ_0(\alpha)$. The unknown constant F is determined from the condition

$$P = \int_{-1}^{1} p(x)dx = \int_{-1}^{1} q(x)dx = w_0 F \tag{25}$$

where P is the applied force, defined by the first equality in (25). Inverting the Fourier transform in Eq. (24) and calculating obtained integrals, taking form of $Q^*(\alpha)$ into account we obtain

$$\begin{aligned} q_0(x) &= F\left[\frac{1}{\pi\sqrt{1-x^2}} + \sum_{m=1}^{N} C_m^0 \Psi(a_m\lambda^{-1}, x)\right], \\ q_n(x) &= s\left[\mathrm{i}\frac{r_n\Psi(\mathrm{i}r_n, x)}{\cos(r_n)L_N(\lambda r_n)} + \frac{r_n\Psi(r_n, x)}{\mathrm{ch}(r_n)L_N(\mathrm{i}\lambda r_n)} + \sum_{m=1}^{N} C_m^n \Psi(a_m\lambda^{-1}, x)\right], \; n = 1,2,\ldots \end{aligned} \tag{26}$$

where:

$$\Psi(A,x) = \frac{I_1(A)}{\sqrt{1-x^2}} - A\int_x^1 \left[I_0(A)\alpha\,\mathrm{ch}\,A(\alpha-x) - I_1(A)\,\mathrm{sh}\,A(\alpha-x)\right]\frac{d\alpha}{\sqrt{1-\alpha^2}}$$

$I_0(x)$ and $I_1(x)$ are the modified Bessel functions of the first kind.

Accounting for Eqs. (25), (26), we finally represent the contact stresses from Eq. (15) in form

$$q(x) = P\left[\frac{1}{\pi\sqrt{1-x^2}} + \sum_{m=1}^{N} C_m^0 \Psi(a_m\lambda^{-1}, x)\right] + \sum_{n=1}^{\infty} w_n q_n(x) \tag{27}$$

When w_n=0, Eq. (27) transforms to the formula for determining distribution of the contact stresses under a rigid punch.

To determine constants C_i^n, we substitute Eqs. (26) into Eq. (16). The set of constants

$C_i^n\,(n=0,1,2,...;\ i=1..N)$ is determined from the system of linear algebraic equations below.

$$\sum_{m=1}^{N} C_m^0\,\mathrm{Z}(a_m\lambda^{-1},b_k\lambda^{-1}) = \mathrm{K}_0(b_k\lambda^{-1})\lambda b_k^{-1}\pi^{-1},\quad k=1,2,\ldots,N$$
$$\sum_{m=1}^{N} C_m^n\,\mathrm{Z}(a_m\lambda^{-1},b_k\lambda^{-1}) = -\frac{r_n Z(r_n,b_k\lambda^{-1})}{\mathrm{ch}(r_n)L_N(\mathrm{i}\lambda r_n)} - \frac{\mathrm{i}r_n Z(\mathrm{i}r_n,b_k\lambda^{-1})}{\cos(r_n)L_N(\lambda r_n)}, \tag{28}$$
$$k=1,2,\ldots,N;\quad n=1,2,\ldots$$

where

$\mathrm{Z}(a,b)=[aI_0(a)\,\mathrm{K}_1(b)+I_1(a)b\,\mathrm{K}_0(b)]\left(a^2-b^2\right)^{-1}$,

K_0, K_1 are the Macdonald's functions or the modified Bessel's functions of second kind.

The contact stresses $q(x)$ and the pressure applied to the beam $p(x)$ can be represented as following series

$$q(x)=\sum_{n=0}^{\infty} y_n\varphi_n(x),\quad y_n=\frac{1}{2}\int_{-1}^{1} q(x)\varphi_n(x)dx \tag{29}$$

$$p(x)=\sum_{n=0}^{\infty} p_n\varphi_n(x),\quad p_n=\frac{1}{2}\int_{-1}^{1} p(x)\varphi_n(x)dx \tag{30}$$

Substituting Eqs. (14), (15), (29) and (30) into Eq. (13) and two times differentiating it, we get the infinite system of linear algebraic equations over w_n

$$r_n^4 w_n = p_n - \frac{P}{2}E_0^n - s\frac{1}{2}\sum_{j=1}^{\infty} w_j \mathrm{E}_j^n,\quad n=1,2,\ldots \tag{31}$$

where

$$\mathrm{E}_j^n = f_j^n + \sum_{m=1}^{N} C_m^j D(r_n, a_m\lambda^{-1})$$

$$f_j^n = r_j\left(\mathrm{i}\frac{D(r_n,\mathrm{i}r_j)}{\cos(r_j)L_N(\lambda r_j)} + \frac{D(r_n,r_j)}{\mathrm{ch}(r_j)L_N(\mathrm{i}\lambda r_j)})\right),$$

$$D(a,b)=\pi\big(\mathrm{R}(a,b)+\mathrm{R}(\mathrm{i}a,b)\big),$$

$$R(a,b)=a\,\mathrm{ch}^{-1}(a)[bI_0(b)I_1(a)-I_1(b)aI_0(a)]\left(b^2-a^2\right)^{-1},$$

In particular,

$$E_0^n = \left(\cos^{-1}(r_j)\,\mathrm{I}_0(\mathrm{i}r_j)+\mathrm{ch}^{-1}(r_j)\,\mathrm{I}_0(r_j)+\sum_{m=1}^{N} C_m^0 D(r_n,a_m\lambda^{-1})\right)$$

Applying the reduction method to the infinite system (31), we get the finite system for determination of w_n (n=1,...,K)

$$w_n r_n^4 + s\frac{1}{2}\sum_{j=1}^{K} w_j \mathrm{E}_j^n = p_n - \frac{P}{2} E_0^n, \quad n = 1, 2, \ldots, K \tag{32}$$

After determining parameters w_n (n=1,...,K) for a fixed value K and substituting them in Eq. (27) , we finally get the contact stresses q(x). But the expression for the beam deflection $w(x)$ (14) still has one unknown constant w_0. To determine it, we need to put an additional condition on $w(x)$, for example, $w(0)=\delta$, where δ is a known constant.

4. Some numerical results for a soft foundation

To illustrate application of the obtained solution, let us consider three typical inhomogeneity laws for variation of elastic properties in the interlayer:

coating 1: $E_1(y) = 1$;

coating 2: $E_1(y) = 1/3{,}5 - 2{,}5y/(3{,}5H)$;

coating 3: $E_1(y) = 3{,}5 + 2{,}5\,y/H$.

Variation of Young's moduli of these coatings by depth is illustrated on Fig. 2.

We assume that the coating is much softer than the substrate ($E_2 >> E_1(-H)$) and use parameter $\beta = E_2/E_1(-H)$ to characterize its softness. Let us consider the case of β=100. Poisson's ratio of the considered coatings is assumed to be the same as one of the substrate: $\nu_1(y)=\nu_2=0.3$. The beam is subjected to the uniform load $p(x)$=1.

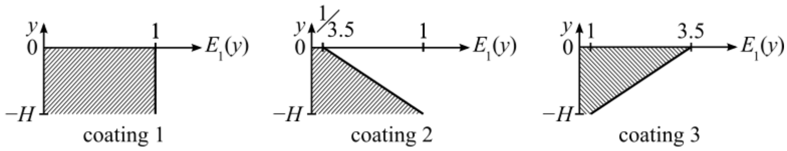

Fig. 2 Variation of Young's modulus by depth of the coatings 1-3

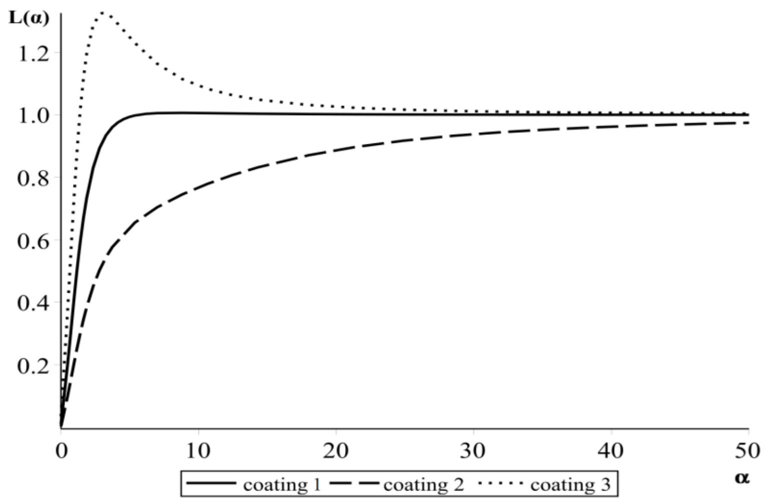

Fig. 3 Kernel transforms of the integral equation of the problem for coatings 1-3

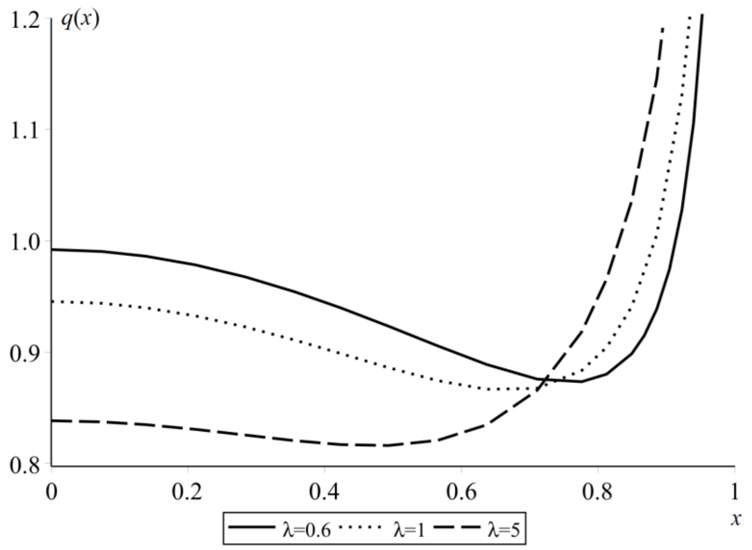

Fig. 4 Graphs of pressure distribution $q(x)$ versus x for coating 1.The graphs are presented for soft coatings with β=100, beam of intermediate flexibility s=5 and some coating thicknesses λ

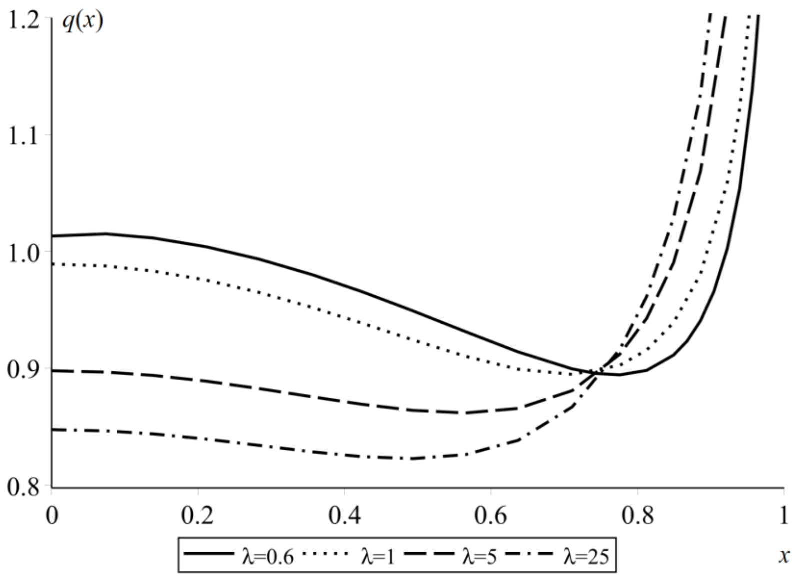

Fig. 5 Graphs of pressure distribution $q(x)$ versus x for coating 2.The graphs are presented for soft coatings with β=100, beam of intermediate flexibility s=5 and some coating thicknesses λ

Coating 1 is homogeneous in depth. Coatings 2 and 3 have Young's moduli linearly changing with depth. Kernel transforms for the considered coatings are plotted on Fig. 3. Method for numerical evaluation of the kernel transform is described in (Aizikovich and Alexandrov 1984).

Figs. 4-6 present the distribution of the contact stresses for coatings 1-3, respectively. Coordinate x is varied in range [0..0.99]. All presented curves are plotted for case of flexibility

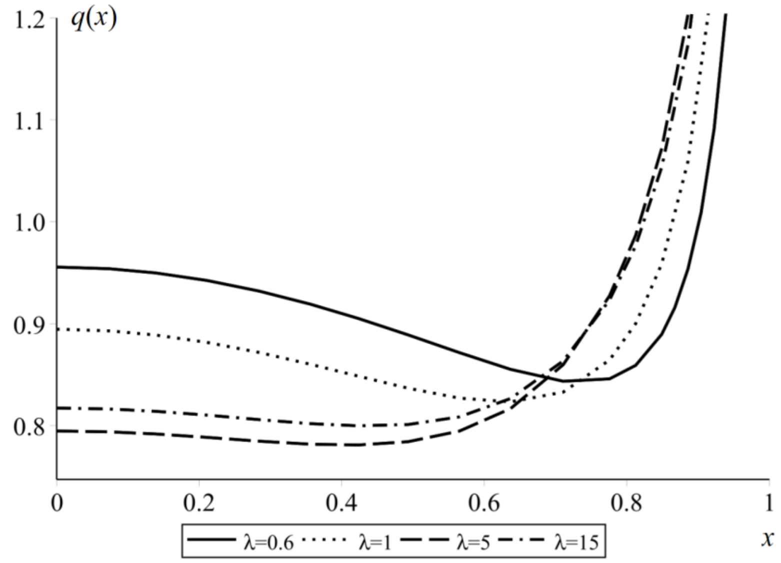

Fig. 6 Graphs of pressure distribution $q(x)$ versus x for coating 3. The graphs are presented for soft coatings with β=100, beam of intermediate flexibility s=5 and some coating thicknesses λ

parameter s=5 and some fixed values of λ. For all of the considered cases, one can note decrease of the stresses magnitude under the beam with increase of distance from its center up to point of inflection, and then stresses increase while getting closer to the edge of the beam.

As it can be seen from Figs. 4-5, the contact stresses under the center of the beam are greater for thinner coatings (corresponding to small values of λ) than for thicker ones. On the contrary, coating stresses near the edges of the beam are greater for thicker coatings. Because of the edge of the beam cuts into the half-plane, the contact stresses tend to infinity as $x \to 1$.

For all considered values of coating thickness and kinds of elastic properties variation, the contact stresses express nonmonotone variation along the x axis. Minimum values are achieved in a point $x_0 \in [0.5,0.85]$. Value of x_0 is greater for thin coatings than for thick ones. As thickness of the coating increases, x_0 moves to the left along the x axis, and the magnitude of stresses grows, i.e. the contact stresses tend to monotone distribution. Such behavior for soft thin coatings was mentioned before for the problem of a rigid punch indentation (Kudish *et al.* 2016).

The described effects are caused by the fact that considered coatings are significantly softer than the substrate, with 100-fold difference in elastic moduli. So, thinner the coating, more considerable is the contribution of the substrate to the elastic response. In a limiting case of $\lambda \to 0$, the coating almost disappears, and the elastic response is caused exclusively by the substrate. Conversely, in a case of a thick coating, most of the elastic deformation is damped by the soft material of the coating.

The behavior of the coating 3 (Fig. 6) is somewhat different. For average values of λ (λ=5), the contact stresses under the center of the beam are lower than for large λ (see case of λ=15 for comparison). It is implied by the behavior of Young's modulus, which has minimum on the coating-substrate interface, so Young's modulus of the coating-substrate system has nonmonotone variation along the y axis. This is the reason for the contact stresses in a point also to express nonmonotone behavior. It can be explained from a physical point of view. For very large values of

λ, the coating-substrate system behaves much like a homogeneous half-plane with Young's modulus $E=E_1(0)=3.5$, for very small λ-as a half-plane with $E=E_2=100$. But for average values of λ, most of the elastic response is produced by the material near the lower boundary of the coating, where values of Young's modulus are minimal.

5. Conclusions

Analytical expressions for the contact stresses appearing under the beam and the deflection function were constructed using the bilateral asymptotic method. The method allows one to analyze interaction with an elastic layer resting on a much stiffer substrate. Using the high-accuracy approximations for the kernel transform (see Aizikovich and Vasiliev 2013), it is possible to obtain a solution of the problem which is applicable for all possible values of λ and any stiffness of the beam. The same method was successfully applied to a wide class of contact problems for materials with functionally-graded coatings (Volkov *et al*, 2013, Vasiliev *et al.* 2014, 2015, Kudish *et al.* 2016).

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