Structural Engineering and Mechanics, *Vol. 57, No. 5 (2016) 897-920*
DOI: http://dx.doi.org/10.12989/sem.2016.57.5.897

Absolute effective elastic constants of composite materials

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(Received June 9, 2015, Revised January 12, 2016, Accepted January 26, 2016)

Abstract. The objective is to determine the mechanical properties of the composites formed in two types, theoretically. The first composite includes micro-particles in a matrix while the second involves long, thin fibers. A fictitious, homogeneous, linear-elastic and isotropic single material named as effective material is considered during calculation which is based on the equality of the strain energies of the composite and effective material under the same loading conditions. The procedure is carried out with volume integrals considering a unique strain energy in a body. Particularly, the effective elastic shear modulus has been calculated exactly for small-particle composites by the same procedure in order to determine of bulk modulus thereof. Additionally, the transverse shear modulus of fiber reinforced composites has been obtained through a simple approach leading to the practical equation. The results have been compared not only with the outcomes in the literature obtained by different method but also with those of finite element analysis performed in this study.

Keywords: analytical method; composites; fiber reinforced; finite element method (FEM); static analysis

1. Introduction

Composites are vastly preferred materials in many industrial areas (Guang-hui and Xiao 2015, Kim *et al*. 2015, Simsek 2010) because they are relatively economical and easily producible. In general, the particles or fibers are embedded in a matrix material to obtain a composite. The elastic constants of the added materials are higher than those of the matrix. A certain composite displays a unique mechanical behavior, although it is obtained combining some materials which have dissimilar properties for instance (Biswas 2012, Kocak *et al*. 2013, Handlin 2013).

In this study, the mechanical properties of the composites, in the types of those mentioned above, have been determined analytically. The concentration or in another saying the ratio of the volume of the added material to that of the matrix directly related to the strength of the composites. Here, the concentration values have been assumed to be so small that the particles or fibers don't interact each other. It is assumed that all materials are homogeneous, isotropic and linear elastic and the particles or fibers are homogeneously distributed in the matrix. The problems

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http://www.techno-press.org/?journal=sem&subpage=8 ISSN: 1225-4568 (Print), 1598-6217 (Online)

considered here can be simulations of the mechanical behavior of some materials which are used in numerous applications during production whose range changes from household goods or toys to structural elements or space technology (Basaran *et al*. 2015). It can be recognized from the technical literature that the determination of effective elastic constants of composites has been still worked on. For instance, elastic properties of a certain particulate composite which are determined experimentally were approximated by some estimated modification to the formulas given by Hashin-Shtrikman (Upadhyay *et al*. 2012). In another study, the formulas derived by approximate analytical solutions were checked reducing these formulations to those for two-phase elastic composites and comparing with the bounds of Hashin and Shtrikman (Lin *et al*. 2009). Numerical approach was used to predict the elastic property of multiphase composites with random microstructure (Wang and Pan 2009). Another study about the prediction of elastic properties of composites with complex microstructure is related with the phase-field microelasticity (Ni and Chiang 2007). The effective moduli of composites including particle or fiber were studied using the strain energy change by extending the replacement method and FEM analysis (Shen and Li 2003).

It is obvious that most of the new references use the Hashin's results to validate the predictions or approximations. The bulk and shear moduli of the composites including particles were investigated by Hashin, an upper and a lower bounds for these quantities were given using the variational methods (Hashin 1962). However, at the end of calculation of the bulk modulus, two bounds were found to be the same. For the shear modulus, an approximate statement was also obtained between the upper and lower bounds. Hashin and Rosen expressed the upper and lower bounds for the shear modulus in the transverse plane of a composite including thin fibers depending on a set of equations (Hashin and Rosen 1964). Two bounds for the transverse shear modulus were given by Hashin in the case of the concentration is nearly zero (Hashin 1965).

Christensen and Lo assumed a different model which consists of a single composite sphere in an infinite medium whose effective properties are investigated (Christensen and Lo 1979). It was considered that the effective homogeneous medium has the same mechanical properties as the macroscopic properties of the sphere mentioned above. This model was used not only as the spherical model but also as the plane circular model for transversely isotropic composites and gave the effective shear moduli for both composites including spherical particles and fibers. A wide review on the determination of the effective constants was given by Hashin (1983).

In this study, the exact expression of the bulk modulus of the composites including micro-particles assumed as spheres has been analytically calculated without any bounds. The main difference of this work is that the strain energies have been calculated by volume integrals. Variation of the exact expression of the shear modulus of this type of the composites versus concentration has also been plotted. Additionally, the shear modulus in the transverse plane of the composites including thin fibers has been calculated using the same method with a simple approach. Besides, the finite element analysis has been performed by ABAQUS for comparison. Moreover, some other approximate solutions have been compared with the analytical results obtained in this study.

2. Elastic constants of the composites including microparticles

2.1 Determination of the bulk modulus

A spherical body having radius b has been considered as the representative volume element (RVE) of this type of the material which includes a spherical particle positioned at the same center having radius a. The outer part of the sphere is the matrix. An effective spherical body with radius b containing a single material is also considered. The assumption is that the effective material and RVE behave alike mechanically. For convenience, the spherical coordinates (r, θ, φ) have been used.

At first, the strain energy of the single sphere will be calculated under hydrostatic pressure stress τ_0. In this problem, only the radial component u_r^* of the displacement vector exists and the non-zero components of the strain tensor are

$$\mathbf{u}^* = u_r^* \mathbf{e}_r, \quad \varepsilon_{rr}^* = \frac{\partial u_r^*}{\partial r}, \quad \varepsilon_{\theta\theta}^* = \varepsilon_{\varphi\varphi}^* = \frac{u_r^*}{r} \tag{1}$$

where $\mathbf{e}_r$ is the unit normal vector in the r direction, ε_{ij} (i,j=r, θ, φ) denotes the components of the strain tensor, and ∂ indicates the partial derivative. The superscript * is used for the quantities which belong to the effective material in whole study.

Following Hashin (1962), neglecting body forces and solving the equations of the equilibrium, u_r^* has been found as

$$u_r^* = A^* r + \frac{B^*}{r^2} \tag{2}$$

where, A^* and B^* are the integration constants. B^* is zero since the solution must be finite at $r = 0$. The boundary condition is expressed as the surface traction vector $\mathbf{T}$ is equal to $-\tau_0 \mathbf{e}_r$ on the boundary at $r = b$. To write this equation, the stress components have to be written using the relation between stress and strain in linear elasticity which is

$$\tau_{ij}^* = \lambda^* \varepsilon_{kk}^* \delta_{ij} + 2\mu^* \varepsilon_{ij}^* \tag{3}$$

where τ_{ij} indicates the stress components, λ and μ are Lamé's constants, and δ_{ij} is Kronecker's delta. Here, the summation convention on the repeated indices is valid. So, if the Eqs. (1)-(3) are used, then the stress and strain components which are different from zero are obtained in the term of the constant A^* in the spherical coordinates as

$$\varepsilon_{rr}^* = \varepsilon_{\theta\theta}^* = \varepsilon_{\varphi\varphi}^* = A^*, \quad \tau_{rr}^* = \tau_{\theta\theta}^* = \tau_{\varphi\varphi}^* = 3K^* A^*, \quad K^* = \frac{3\lambda^* + 2\mu^*}{3} \tag{4}$$

where K^* is defined as the effective bulk modulus. The mentioned boundary condition about the surface traction $\mathbf{T}$ and stress components have been written and the constant A^* is obtained from this equation as

$$\mathbf{T} = \tau^* \cdot \mathbf{n} = -\tau_0 \mathbf{e}_r \rightarrow A^* = \frac{-\tau_0}{3\lambda^* + 2\mu^*} = \frac{-\tau_0}{3K^*} \tag{5}$$

Here, $\mathbf{n}$ is the unit outward normal vector of the spherical surface. From this, the components of stress and strain are determined for this problem. Finally, the total strain energy U^* accumulated on this body with volume V can be written using the equation below which has to be written in the spherical coordinates.

$$U^* = \frac{1}{2}\int_V \tau_{ij}^* \varepsilon_{ij}^* dV \tag{6}$$

$$dV = r^2 \sin\theta dr d\theta d\varphi \tag{7}$$

$$U^* = \frac{2}{3K^*}\pi b^3 \tau_0^2 = \frac{1}{2K^*} V \tau_0^2 \tag{8}$$

Hereafter, the strain energy of the RVE of a composite including a single particle will be calculated under hydrostatic pressure stress τ_0. Using the Eq. (2), the displacement field for the particle and matrix can be written as

$$u_r^P = -\tau_0\left(A^P r + \frac{B^P}{r^2}\right), \quad (0 \le r \le a)$$

$$u_r^M = -\tau_0\left(A^M r + \frac{B^M}{r^2}\right), \quad (a \le r \le b) \tag{9}$$

The superscripts *P* and *M* denote the quantities which belong to particle and matrix, respectively in this article. The coefficient τ_0 has been used in these expressions for the convenience of the results. At $r = 0$, the displacement must be finite so that B^P must be zero. The components of the strain and stress can be written separately using Eq. (9) for the particle and matrix. There are three unknown constants A^P, $A^M$and B^M so three boundary conditions have to be written to determine them which are

$$\tau_{rr}^M = -\tau_0 \quad \text{for} \quad r = b,$$

$$u_{rr}^M = u_{rr}^P, \quad \tau_{rr}^M = \tau_{rr}^P \quad \text{for} \quad r = a \tag{10}$$

If these conditions are written using the obtained stress and displacement components, three equations occur. Defining the concentration as a^3/b^3 which is indicated by *c* and the constant *D* as B^M/a^3, the solution of this system is

$$D = \frac{K^M - K^P}{4\mu^M\left[K^M + \left(K^P - K^M\right)c\right] + 3K^M K^P} \tag{11}$$

$$A^P = \frac{4\mu^M + 3K^M}{3\left\{4\mu^M\left[K^M + \left(K^P - K^M\right)c\right] + 3K^M K^P\right\}} \tag{12}$$

$$A^M = \frac{4\mu^M + 3K^P}{3\left\{4\mu^M\left[K^M + \left(K^P - K^M\right)c\right] + 3K^M K^P\right\}} \tag{13}$$

Now, the strain energy accumulated on the whole body can be written in terms of the bulk and shear moduli and concentration. To do this, Eq. (6) is used for the two parts of RVE and the summation of the results of the total energy. Here, the integrals are evaluated over φ and θ from

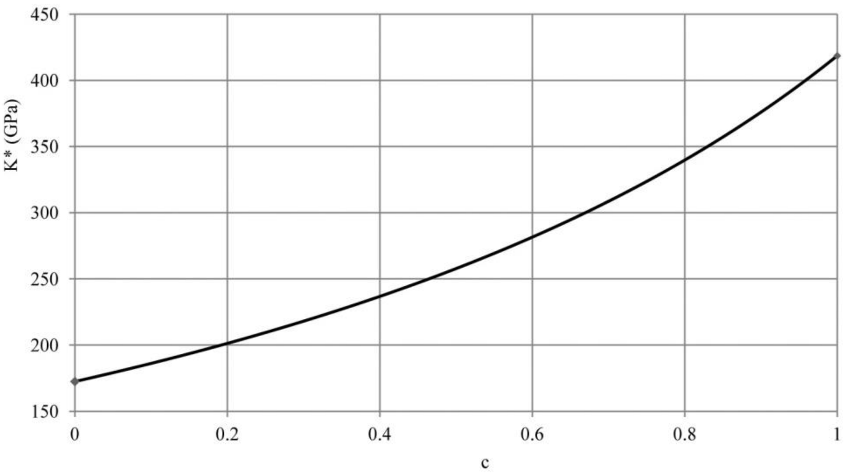

Fig. 1 Variation of K^* of particle reinforced composite versus c for the first set of materials

zero to 2π and from zero to π, respectively for both of the parts. The radius is changed from zero to *a* for the particle and from *a* to *b* for the matrix. Using the stress and strain components and Eqs. (11)-(13), the total strain energy is calculated as

$$U^{RVE} = \frac{9}{2}\left[VcK^P\left(A^P\right)^2 + VK^M\left(A^M\right)^2(1-c)\right] + 6V\mu^M D^2 c(1-c) \tag{14}$$

where V is the total volume of the sphere having radius b. If the strain energies of the effective material and the RVE given in the Eq. (8) and Eq. (14) are equated and resulting expression is rearranged for K^* then the result will be

$$K^* = \frac{1}{9\left[cK^P\left(A^P\right)^2 + (1-c)K^M\left(A^M\right)^2\right] + 12c(1-c)\mu^M D^2} \tag{15}$$

As the first set of material constants, $K^P = 418.610$ GPa, $K^M = 172.368$ GPa, $\mu^P = 288.223$ GPa and $\mu^M = 79.555$ GPa are selected for comparison with Hashin (1962) and variation of K^* versus concentration for these materials is given in Fig. 1. Here, if one draws the same variation using the expressions (35) or (36) in the article of Hashin (1962), then the same graph will be obtained exactly because Hashin's bounds coincide and express the exact solution. In this study, the calculation has been acquired without defining any bounds and the variation mentioned above shows the agreement.

2.2 Determination of the shear modulus

An effective single spherical body having radius *b* is firstly considered under the state of simple shear for the plane stress. Rectangular and spherical coordinates are used in the solution. The surface traction defined by the stress tensor $\boldsymbol{\tau}$ and the unit outward normal vector $\mathbf{n}$ is expressed as

$$\mathbf{T} = \boldsymbol{\tau} \cdot \mathbf{n} = \begin{bmatrix} 0 & \tau_0 & 0 \\ \tau_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix} = \tau_0 \begin{bmatrix} \sin\theta\sin\varphi \\ \sin\theta\cos\varphi \\ 0 \end{bmatrix} \tag{16}$$

Spherical harmonic functions have been used to obtain the solution so that the displacement vector $\mathbf{u}^*$ has been assumed that it is derived from the gradient of a function ω. This ω function is named as the spherical harmonic function and it has been stated for this problem as

$$\omega = x_1 x_2 r^n \tag{17}$$

where x_i, (i=1,2,3) indicates the Cartesian coordinates and r is the magnitude of the position vector.

There are three types of solutions related to this function which are

$$\mathbf{u}^* = r^2 \nabla \omega + \alpha^* \mathbf{r} \omega$$
$$\mathbf{u}^* = \nabla \omega$$
$$\mathbf{u}^* = r \times \nabla \omega \tag{18}$$

where ∇ is del operator, α is a constant, $\mathbf{r}$ is the position vector, and $\times$ indicates the vector product.

If the displacement vectors obtained substituting Eq. (17) into each solution given in Eq. (18) are used in the equilibrium equation (Achenbach 1973), then two roots are founded for n which are zero and -5. So, the field of displacements which provides the boundary condition given in Eq. (16) has been written as the combination of the mentioned types of the solutions for two roots as

$$\mathbf{u}^* = \mathbf{u}^{1*} + \mathbf{u}^{2*} + \mathbf{u}^{3*} + \mathbf{u}^{4*} + \mathbf{u}^{5*} + \mathbf{u}^{6*} \tag{19}$$

where these $\mathbf{u}^{i*}$, (i=1,2,3,..,6) have been written in the closed form as

$$\mathbf{u}^{1*} = D_1 \left[r^2 \nabla \left(x_1 x_2 \right) + \alpha_1^* \mathbf{r} x_1 x_2 \right] \tag{20}$$

$$\mathbf{u}^{2*} = D_2 \left[r^2 \nabla \left(\frac{x_1 x_2}{r^5} \right) + \alpha_2^* \mathbf{r} \frac{x_1 x_2}{r^5} \right] \tag{21}$$

$$\mathbf{u}^{3*} = D_3 \nabla \left(x_1 x_2 \right) \tag{22}$$

$$\mathbf{u}^{4*} = D_4 \nabla \left(\frac{x_1 x_2}{r^5} \right) \tag{23}$$

$$\mathbf{u}^{5*} = D_5 \left[\mathbf{r} \times \nabla \left(x_1 x_2 \right) \right] \tag{24}$$

$$\mathbf{u}^{6*} = D_6 \left[\mathbf{r} \times \nabla \left(\frac{x_1 x_2}{r^5} \right) \right] \tag{25}$$

Here, D_i, (i=1,2,3,...,6) are the integration constants. To obtain α_1^* and α_2^*, Eqs. (20) and (21) have been separately substituted into the equilibrium equations in terms of displacement according to the Achenbach (1973) so that these constants have been obtained in terms of the effective Lamé's constants as

$$\alpha_1^* = -\frac{4\lambda^* + 14\mu^*}{5\lambda^* + 7\mu^*}, \quad \alpha_2^* = \frac{3\lambda^* + 8\mu^*}{\mu^*} \tag{26}$$

The solution of the displacement vector written in Eq. (19) is valid everywhere in the sphere so that this must be finite at r=0. So

$$D_2 = D_4 = D_6 = 0 \tag{27}$$

The strain tensor is also a combination of the strain tensors obtained from the different types of the displacement solutions. This can be stated as

$$\boldsymbol{\varepsilon}^* = \boldsymbol{\varepsilon}^{1*} + \boldsymbol{\varepsilon}^{2*} + \boldsymbol{\varepsilon}^{3*} + \boldsymbol{\varepsilon}^{4*} + \boldsymbol{\varepsilon}^{5*} + \boldsymbol{\varepsilon}^{6*} \tag{28}$$

and the components of these symmetric matrices have been calculated as follows

$$\varepsilon_{11}^{1*} = \varepsilon_{22}^{1*} = D_1 \frac{2x_1x_2\left(\lambda^* - 7\mu^*\right)}{5\lambda^* + 7\mu^*}, \quad \varepsilon_{33}^{1*} = -D_1 \frac{2x_1x_2\left(2\lambda^* + 7\mu^*\right)}{5\lambda^* + 7\mu^*} \tag{29}$$

$$\varepsilon_{12}^{1*} = D_1 \frac{\left(x_1^2 + x_2^2\right)\left(8\lambda^* + 7\mu^*\right) + x_3^2\left(5\lambda^* + 7\mu^*\right)}{5\lambda^* + 7\mu^*} \tag{30}$$

$$\varepsilon_{13}^{1*} = D_1 \frac{3\lambda^* x_2 x_3}{5\lambda^* + 7\mu^*}, \quad \varepsilon_{23}^{1*} = D_1 \frac{3\lambda^* x_1 x_3}{5\lambda^* + 7\mu^*} \tag{31}$$

$$\varepsilon_{11}^{2*} = D_2 \frac{3x_1x_2\left[\left(x_2^2 + x_3^2\right)\left(2\lambda^* + \mu^*\right) - x_1^2\left(3\lambda^* + 4\mu^*\right)\right]}{\mu^* r^7} \tag{32}$$

$$\varepsilon_{22}^{2*} = D_2 \frac{3x_1x_2\left[\left(x_1^2 + x_3^2\right)\left(2\lambda^* + \mu^*\right) - x_2^2\left(3\lambda^* + 4\mu^*\right)\right]}{\mu^* r^7} \tag{33}$$

$$\varepsilon_{33}^{2*} = D_2 \frac{3x_1x_2\left(\lambda^* + \mu^*\right)\left(x_1^2 + x_2^2 - 4x_3^2\right)}{\mu^* r^7} \tag{34}$$

$$\begin{aligned}\varepsilon_{12}^{2*} = \frac{D_2}{2\mu^* r^7}\{&x_1^4\left(3\lambda^* + 2\mu^*\right) + x_1^2\left[x_3^2\left(3\lambda^* + 4\mu^*\right) - 2x_2^2\left(12\lambda^* + 13\mu^*\right)\right] \\ &+ x_2^4\left(3\lambda^* + 2\mu^*\right) + x_2^2x_3^2\left(3\lambda^* + 4\mu^*\right) + 2\mu^* x_3^4\}\end{aligned} \tag{35}$$

$$\varepsilon_{13}^{2*} = \frac{D_2}{2\mu^* r^7}\left\{3x_2x_3\left[\lambda^*\left(x_2^2 + x_3^2\right) - x_1^2\left(9\lambda^* + 10\mu^*\right)\right]\right\} \tag{36}$$

$$\varepsilon_{23}^{2*} = \frac{D_2}{2\mu^* r^7}\left\{3x_1x_3\left[\lambda^* x_1^2 - x_2^2\left(9\lambda^* + 10\mu^*\right) + \lambda^* x_3^2\right]\right\} \tag{37}$$

$$\varepsilon_{12}^{3*} = D_3, \quad \varepsilon_{11}^{3*} = \varepsilon_{22}^{3*} = \varepsilon_{33}^{3*} = \varepsilon_{13}^{3*} = \varepsilon_{23}^{3*} = 0 \tag{38}$$

$$\varepsilon_{11}^{4*} = D_4 \frac{5x_1x_2\left[4x_1^2 - 3\left(x_2^2 + x_3^2\right)\right]}{r^9} \tag{39}$$

$$\varepsilon_{22}^{4*} = -D_4 \frac{5x_1x_2\left(3x_1^2 - 4x_2^2 + 3x_3^2\right)}{r^9} \tag{40}$$

$$\varepsilon_{33}^{4*} = -D_4 \frac{5x_1x_2\left(x_1^2 + x_2^2 - 6x_3^2\right)}{r^9} \tag{41}$$

$$\varepsilon_{12}^{4*} = -D_4 \frac{4x_1^4 + 3x_1^2\left(x_3^2 - 9x_2^2\right) + 4x_2^4 + 3x_2^2x_3^2 - x_3^4}{r^9} \tag{42}$$

$$\varepsilon_{13}^{4*} = D_4 \frac{5x_2x_3\left(6x_1^2 - x_2^2 - x_3^2\right)}{r^9} \tag{43}$$

$$\varepsilon_{23}^{4*} = -D_4 \frac{5x_1x_3\left(x_1^2 - 6x_2^2 + x_3^2\right)}{r^9} \tag{44}$$

$$\varepsilon_{11}^{5*} = -\varepsilon_{22}^{5*} = -D_5x_3, \quad \varepsilon_{33}^{5*} = \varepsilon_{12}^{5*} = 0 \tag{45}$$

$$\varepsilon_{13}^{5*} = D_5 \frac{x_1}{2}, \quad \varepsilon_{23}^{5*} = -D_5 \frac{x_2}{2} \tag{46}$$

$$\varepsilon_{11}^{6*} = D_6 \frac{x_3\left(4x_1^2 - x_2^2 - x_3^2\right)}{r^7}, \quad \varepsilon_{22}^{6*} = D_6 \frac{x_3\left(x_1^2 - 4x_2^2 + x_3^2\right)}{r^7} \tag{47}$$

$$\varepsilon_{33}^{6*} = D_6 \frac{5x_3\left(x_2^2 - x_1^2\right)}{r^7}, \quad \varepsilon_{12}^{6*} = 0 \tag{48}$$

$$\varepsilon_{13}^{6*} = D_6 \frac{x_1\left[3\left(x_2^2 + x_3^2\right) - 2x_1^2\right]}{r^7}, \quad \varepsilon_{23}^{6*} = -D_6 \frac{x_2\left(3x_1^2 - 2x_2^2 + 3x_3^2\right)}{r^7} \tag{49}$$

The stress-strain relation given in Eq. (3) has been used to obtain the associated stress components. These expressions of the stress components have not been written here but they can be obtained simply by constitutive equations Eq. (3) in Cartesian coordinates. Here, the stress tensor has been similarly written using Eq. (27) in a combination of the stress tensors $\boldsymbol{\tau}^{1*}$, $\boldsymbol{\tau}^{3*}$, and $\boldsymbol{\tau}^{5*}$ obtained from the strain tensors $\boldsymbol{\varepsilon}^{1*}$, $\boldsymbol{\varepsilon}^{3*}$, and $\boldsymbol{\varepsilon}^{5*}$ because the coefficients of $\boldsymbol{\tau}^{2*}$, $\boldsymbol{\tau}^{4*}$, and $\boldsymbol{\tau}^{6*}$ were obtained as zero.

$$\boldsymbol{\tau}^{*} = \boldsymbol{\tau}^{1*} + \boldsymbol{\tau}^{3*} + \boldsymbol{\tau}^{5*} \tag{50}$$

Strain energy can be calculated using the integral expression of the strain energy which is stated as

$$U^* = \frac{1}{2}\int_{r=0}^{b}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\left[\left(\tau_{ij}^{1*} + \tau_{ij}^{3*} + \tau_{ij}^{5*}\right)\left(\varepsilon_{ij}^{1*} + \varepsilon_{ij}^{3*} + \varepsilon_{ij}^{5*}\right)\right]dV \tag{51}$$

Before evaluating this integral, the Cartesian coordinates in the stress and strain expressions have been transformed to the spherical using following relations

$$x_1 = r\sin\theta\cos\varphi, \quad x_2 = r\sin\theta\sin\varphi, \quad x_3 = r\cos\theta \tag{52}$$

Using these in the Eqs. (29)-(49) and expressions of stress components, energy has been written in terms of the unknown constants evaluating the integral in Eq. (51).

$$U^* = \frac{4\pi\mu^*}{5\left(5\lambda^* + 7\mu^*\right)^2}[14b^5 D_1 D_3\left(\lambda^* + \mu^*\right)\left(5\lambda^* + 7\mu^*\right)$$
$$+\frac{5}{3}b^3\left(5\lambda^* + 7\mu^*\right)^2\left(D_3^2 + \frac{D_5^2 b^2}{2}\right)$$
$$+\frac{1}{7}b^7 D_1^2\left(263\left(\lambda^*\right)^2 + 784\lambda^*\mu^* + 441\left(\mu^*\right)^2\right)] \tag{53}$$

The constants D_1, D_3, and D_5 have been calculated from the boundary condition that the stress vector at r=b which defines the surface traction on the outer boundary is equal to the surface traction obtained in Eq. (16). This equality is expressed as

$$\mathbf{T} = \left\{D_1\boldsymbol{\tau}^{1*}\left(b,\theta,\varphi\right) + D_3\boldsymbol{\tau}^{3*}\left(b,\theta,\varphi\right) + D_5\boldsymbol{\tau}^{5*}\left(b,\theta,\varphi\right)\right\}\cdot\mathbf{n} = \tau_0\begin{bmatrix}\sin\theta\sin\varphi\\ \sin\theta\cos\varphi\\ 0\end{bmatrix} \tag{54}$$

Although there are three equations in this expression, if the multipliers of the linearly independent functions of θ and φ are equated to zero in all equations, nine equalities are obtained. However, these are recurrences of three linearly independent equations. So, the solution of the set of these three equations gives the constants as

$$D_1 = 0, \quad D_3 = \frac{\tau_0}{2\mu}, \quad D_5 = 0 \tag{55}$$

Finally, the strain energy of the sphere having radius b under the simple shear has been obtained from Eqs. (53) and (55) as

$$U^* = \frac{\tau_0^2}{3\mu^*}\pi b^3 \tag{56}$$

Henceforth, the strain energy accumulated on a sphere including single spherical particle positioned on the same center will be calculated under the state of the simple shear. The radii of these spheres are b and a, respectively. The surface traction is the same with the previous problem. The spherical harmonic solutions have been used again. So, the displacement vectors for the matrix and particle have been written as

$$\mathbf{u}^M = \mathbf{u}^{1M} + \mathbf{u}^{2M} + \mathbf{u}^{3M} + \mathbf{u}^{4M} + \mathbf{u}^{5M} + \mathbf{u}^{6M}$$
$$\mathbf{u}^P = \mathbf{u}^{1P} + \mathbf{u}^{2P} + \mathbf{u}^{3P} + \mathbf{u}^{4P} + \mathbf{u}^{5P} + \mathbf{u}^{6P} \tag{57}$$

The expressions of $\mathbf{u}^{iM}$ and $\mathbf{u}^{iP}$ are the same with the expressions given in Eqs. (20)-(25)

except the unknown constants. These constants are changed with B_i and A_i (*i*=1,2,....,6) for the matrix and the particle, respectively. So, there are twelve unknown constants here. The expressions of α_1 and α_2 are the same with those in Eq. (26) for the matrix and particle. Because the displacement vector in the particle must be finite at *r*=0, three of the constants have been obtained as

$$A_2 = A_4 = A_6 = 0 \tag{58}$$

The number of the constants decreases to nine. Indicating the strain tensor as $\boldsymbol{\varepsilon}^{iP}$ (*i*=1,2,...,6) for the particle, the components of them can be calculated using the strain-displacement relations. The expressions are the same with those in Eqs. (29)-(49) except that the integration and Lamé's constants which have been changed with A_i, λ^P and μ^P, respectively. $\boldsymbol{\varepsilon}^{iM}$ can be obtained by the same previous procedure with the constants B_i, λ^M and μ^M. The stress-strain relations are used to obtain the stress components of the tensors $\boldsymbol{\tau}^{iM}$ and $\boldsymbol{\tau}^{iP}$.

To write the boundary conditions, the components in Cartesian coordinates are restated in the spherical coordinates using the transformation relations given in Eq. (52). For *r*=*a*, the displacement vectors are equal for the particle and matrix. Besides, surface traction vectors of the matrix and particle are the same in the magnitude with opposite sign for *r*=*a* and the surface traction vector in the matrix is equal to $\mathbf{T}$ given in Eq. (54) at *r*=*b*. These expressions are given as below.

$$\begin{aligned} \mathbf{u}^P(r=a) &= \mathbf{u}^M(r=a) \\ \mathbf{T}^P(r=a) &= -\mathbf{T}^M(r=a) \\ \mathbf{T}^M(r=b) &= \mathbf{T} \end{aligned} \tag{59}$$

where

$$\begin{aligned} \mathbf{T}^P(r=a) &= \boldsymbol{\tau}^P(r=a)\cdot\mathbf{n}_1 \\ \mathbf{T}^M(r=a) &= \boldsymbol{\tau}^M(r=a)\cdot\mathbf{n}_2 \\ \mathbf{T}^M(r=b) &= \boldsymbol{\tau}^M(r=b)\cdot\mathbf{n}_1 \end{aligned} \tag{60}$$

It must be emphasized that the surface normals are given as $\mathbf{n}_1 = \mathbf{e}_r$ and $\mathbf{n}_2 = -\mathbf{e}_r$ at *r*=*a* spherical boundary. Though there are nine equations, if the multipliers of the linearly independent functions of θ and φ are equated in each equality, then twenty-seven equations are obtained. However, these are recurrences of nine linearly independent equations which are simplified as

$$B_5 a^2 + \frac{B_6}{a^3} = A_5 a^2$$

$$\frac{2A_1 a^3(2\lambda^P + 7\mu^P)}{5\lambda^P + 7\mu^P} = \frac{2B_1 a^3(2\lambda^M + 7\mu^M)}{5\lambda^M + 7\mu^M} - \frac{3B_2(\lambda^M + \mu^M)}{a^2\mu^M} + \frac{5B_4}{a^4}$$

$$A_1 a^7 + A_3 a^5 = B_1 a^7 + B_2 a^2 + B_3 a^5 + B_4$$

$$\frac{\mu^M\left(B_5 a^5 - 4B_6\right)}{a^5} = A_5 \mu^P$$

$$\frac{2A_1a^2\mu^P\left(19\lambda^P+14\mu^P\right)}{5\lambda^P+7\mu^P}=\frac{2B_1a^2\mu^M\left(19\lambda^M+14\mu^M\right)}{5\lambda^M+7\mu^M}+\frac{24B_2\left(\lambda^M+\mu^M\right)}{a^3}-\frac{40B_4\mu^M}{a^5}$$

$$2A_3\mu^P+\frac{2A_1a^2\mu^P\left(8\lambda^P+7\mu^P\right)}{5\lambda^P+7\mu^P}=\frac{2B_1a^2\mu^M\left(8\lambda^M+7\mu^M\right)}{5\lambda^M+7\mu^M}-\frac{B_2\left(3\lambda^M+2\mu^M\right)}{a^3}$$

$$+\frac{2\mu^M\left(B_3a^5-4B_4\right)}{a^5}$$

$$4B_6-B_5b^5=0$$

$$2B_1b^7\mu^M\left(8\lambda^M+7\mu^M\right)+\left(5\lambda^M+7\mu^M\right)[B_2b^2\left(3\lambda^M+2\mu^M\right)+2B_3b^5\mu^M$$

$$-8B_4\mu^M]=\left(5\lambda^M+7\mu^M\right)b^5\tau_0$$

$$B_1b^7\mu^M\left(19\lambda^M+14\mu^M\right)+4\left(5\lambda^M+7\mu^M\right)\left[3B_2b^2\left(\lambda^M+\mu^M\right)-5B_4\mu^M\right]=0 \tag{61}$$

From the solution of the first, fourth and seventh of Eq. (61) the results have been obtained as

$$A_5=B_5=B_6=0 \tag{62}$$

The number of the remaining constants is six. These constants have not been determined from the remaining six equations because the solution of the set of them is analytically difficult but one can solve them for a given value of concentration which is equal to a^3/b^3 and certain values of λ^M, λ^P, μ^M and μ^P. Instead of that, the total strain energy accumulated on the considered body has been written in terms of these constants using the integral expression of the energy written as

$$U^{RVE}=\frac{1}{2}\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}[(\tau_{ij}^{1P}+\tau_{ij}^{3P})(\varepsilon_{ij}^{1P}+\varepsilon_{ij}^{3P})]r^2\sin\theta drd\theta d\varphi$$

$$+\frac{1}{2}\int_{r=a}^{b}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}[(\tau_{ij}^{1M}+\tau_{ij}^{2M}+\tau_{ij}^{3M}+\tau_{ij}^{4M})(\varepsilon_{ij}^{1M}+\varepsilon_{ij}^{2M}+\varepsilon_{ij}^{3M}+\varepsilon_{ij}^{4M})]dV \tag{63}$$

Here, the terms different from zero are considered only. The first integral in this expression is the strain energy of the particle whilst the other is that of the matrix. This has been calculated as

$$U^{RVE}=\frac{4\pi\mu^P}{5\left(5\lambda^P+7\mu^P\right)^2}[14a^5A_1A_3\left(\lambda^P+\mu^P\right)\left(5\lambda^P+7\mu^P\right)$$

$$+\frac{5}{3}a^3A_3^2\left(5\lambda^P+7\mu^P\right)^2$$

$$+\frac{1}{7}a^7A_1^2\left(263\left(\lambda^P\right)^2+784\lambda^P\mu^P+441\left(\mu^P\right)^2\right)]$$

$$+\frac{2\pi}{315a^7b^7\mu^M\left(5\lambda^M+7\mu^M\right)^2}\{-1764a^{12}b^7B_1B_3\left(\mu^M\right)^2\left(\lambda^M+\mu^M\right)\left(5\lambda^M+7\mu^M\right)$$

$$-210a^{10}b^7B_3^2\left(\mu^M\right)^2\left(5\lambda^M+7\mu^M\right)^2+1760b^7B_4^2\left(\mu^M\right)^2\left(5\lambda^M+7\mu^M\right)^2$$
$$-2112a^2b^7B_2B_4\mu^M\left(\lambda^M+\mu^M\right)\left(5\lambda^M+7\mu^M\right)^2$$
$$-18a^9b^7B_1B_2\mu^M\left(5\lambda^M+7\mu^M\right)\left(-27\left(\lambda^M\right)^2+204\lambda^M\mu^M+196\left(\mu^M\right)^2\right)$$
$$+3a^4b^7B_2^2\left(5\lambda^M+7\mu^M\right)^2\left(263\left(\lambda^M\right)^2+664\lambda^M\mu^M+436\left(\mu^M\right)^2\right)$$
$$-18a^{14}b^7B_1^2\left(\mu^M\right)^2\left(263\left(\lambda^M\right)^2+784\lambda^M\mu^M+441\left(\mu^M\right)^2\right)$$
$$+a^7[1764b^{12}B_1B_3\left(\mu^M\right)^2\left(\lambda^M+\mu^M\right)\left(5\lambda^M+7\mu^M\right)$$
$$+210b^{10}B_3^2\left(\mu^M\right)^2\left(5\lambda^M+7\mu^M\right)^2-1760B_4^2\left(\mu^M\right)^2\left(5\lambda^M+7\mu^M\right)^2$$
$$+2112b^2B_2B_4\mu^M\left(\lambda^M+\mu^M\right)\left(5\lambda^M+7\mu^M\right)^2$$
$$+18b^9B_1B_2\mu^M\left(5\lambda^M+7\mu^M\right)\left(-27\left(\lambda^M\right)^2+204\lambda^M\mu^M+196\left(\mu^M\right)^2\right)$$
$$-3b^4B_2^2\left(5\lambda^M+7\mu^M\right)^2\left(263\left(\lambda^M\right)^2+664\lambda^M\mu^M+436\left(\mu^M\right)^2\right)$$
$$+18b^{14}B_1^2\left(\mu^M\right)^2\left(263\left(\lambda^M\right)^2+784\lambda^M\mu^M+441\left(\mu^M\right)^2\right)]\} \tag{64}$$

This total energy has been equated to the strain energy of the effective body given in Eq. (56). The result expression of this equation depends on the unknown constants. The variation of μ^* versus concentration has been obtained solving the remaining equations in Eqs. (61) for every 1% increment in the concentration and is given in Fig. 2 for the first set of materials whose constants' values are given in the previous section. There are also Hashin's bounds which have been drawn by the data obtained from Fig. 5 in the Hashin's article (1962).

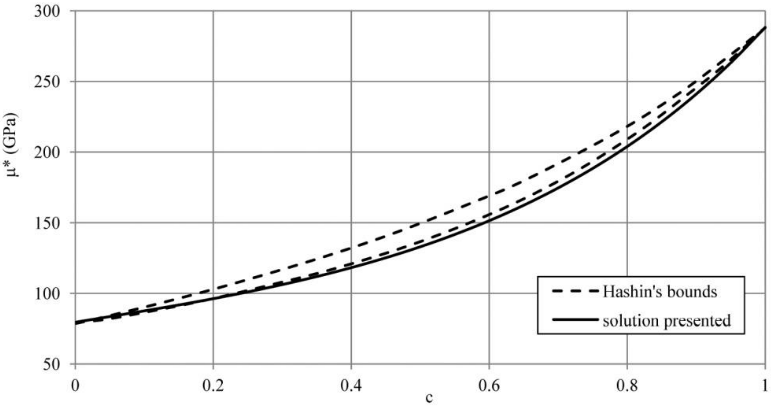

Fig. 2 Variation of μ^* of particle reinforced composite versus c compared to the Hashin's bounds (1962) for the set of first materials

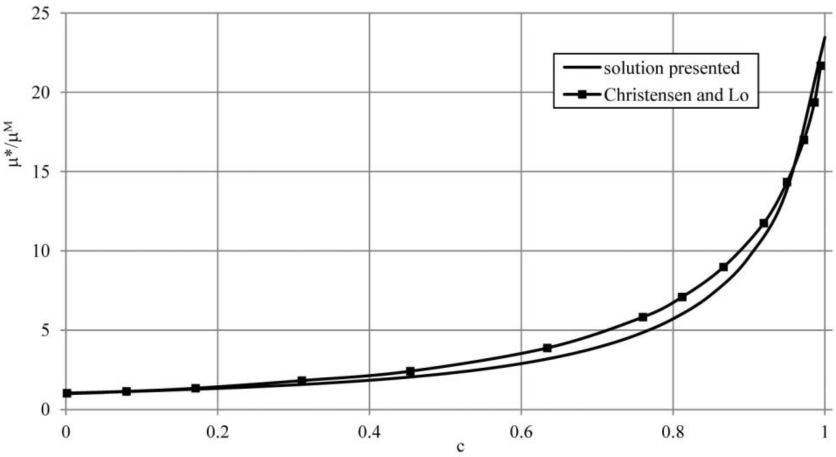

Fig. 3 Variation of μ^*/μ^M of particle reinforced composite versus c compared to Christensen and Lo's curve (1967) for their materials

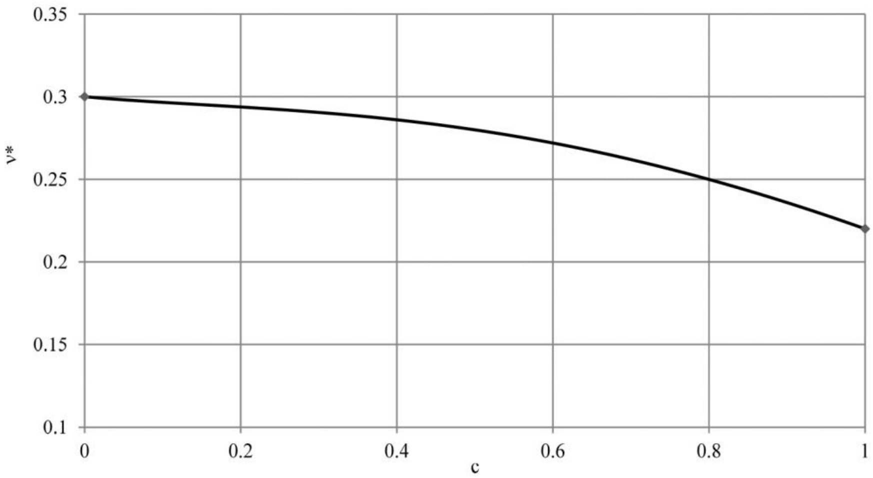

Fig. 4 Variation of effective Poisson's ratio of particle reinforced composite versus c for the first set of materials

Another graph for comparison with variation of the ratio of the effective shear modulus to the related modulus of the matrix versus concentration from the Ref. (Christensen and Lo 1979) has been given in Fig. 3. This variation has been drawn by the data from Fig. 2 in the article by Christensen and Lo (1979) and the properties of the material have been selected the same as those in that article.

As a result of determining of the effective moduli, the Poisson's ratio has been calculated determining f as K^*/μ^* as

$$\upsilon^* = \frac{3f-2}{6f+2} \tag{65}$$

So the variation of the effective Poisson's ratio versus concentration of the composites including particles has been drawn in Fig. 4.

3. Shear modulus of the fiber reinforced composite

A single cylindrical body with height h and radius b has been considered as the effective body of a composite including long fibers under the state of simple torsion. There is a uniformly distributed torsion moment m which is applied to this cylinder at the top and bottom faces in respectively x_3 and $-x_3$ direction which is the axis of the body. $M_0 = m\pi b^2$ is the total torsion moment acting on both surfaces. For convenience, the cylindrical coordinates (r, φ, z) will be used. The components of the symmetric stress tensor in the cylindrical coordinates are zero except $\tau_{r\varphi}$ and $\tau_{\varphi z}$. Here, all unknowns of the problem are independent of φ and displacement vector has only u_φ component.

A differential element in the cylindrical reference system having the height dz, and radius r is considered for a constant z coordinate. $M_z + dM_z$ and M_z represent the resulting torsion moments at the top and bottom surfaces of this differential element, respectively. Let $\tau_{z\varphi}$ denotes the shear stress acting on the bottom surface. The resultant of these stresses is M_z (Fig. 5). If this relation is written

$$M_z(r,z) = 2\pi \int_0^r \tau_{z\varphi}(\xi, z)\,\xi^2 d\xi \tag{66}$$

is obtained (Bulut *et al*. 2013). After some arrangements, the r derivative of this equation gives

$$\tau_{z\varphi}(r,z) = \frac{1}{2\pi r^2}\frac{\partial M_z}{\partial r} \tag{67}$$

Additionally, the moment equilibrium along the z axis gives

$$M_z + dM_z - M_z + \tau_{r\varphi} 2\pi r^2 dz = 0 \rightarrow \tau_{r\varphi} = -\frac{1}{2\pi r^2}\frac{\partial M_z}{\partial z} \tag{68}$$

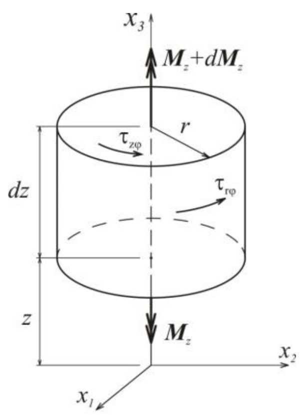

Fig. 5 A differential element in the cylindrical reference system having the height *dz*, and radius *r*

These two shear stress components expressed in Eqs. (67) and (68) can also be written in terms of the components of the displacement. To do this, the strain components have been firstly expressed in terms of displacement components. Because only u_φ exists, the strains different from zero are written as

$$\varepsilon_{r\varphi} = \frac{1}{2}\left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right) = \frac{1}{2}\gamma_{r\varphi} \tag{69}$$

$$\varepsilon_{z\varphi} = \frac{1}{2}\left(\frac{\partial u_\varphi}{\partial z}\right) = \frac{1}{2}\gamma_{z\varphi} \tag{70}$$

where $\gamma_{r\varphi}$ and $\gamma_{z\varphi}$ denotes the shear strains.

The associated stress components can be written using Hooke's Law as

$$\tau_{z\varphi} = \mu \frac{\partial u_\varphi}{\partial z} \tag{71}$$

$$\tau_{r\varphi} = \mu\left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right) \tag{72}$$

If, the expressions of $\tau_{z\varphi}$ and $\tau_{r\varphi}$ obtained respectively in Eqs. (67), (71), (68), and (72) are equated to each other, the equalities

$$\frac{1}{2\pi r^2}\frac{\partial M_z}{\partial r} = \mu \frac{\partial u_\varphi}{\partial z} \tag{73}$$

$$-\frac{1}{2\pi r^2}\frac{\partial M_z}{\partial z} = \mu\left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right) \tag{74}$$

are obtained. Calculating the *r* derivative of Eq. (73) and the *z* derivative of Eq. (74), the following differential equation is obtained for M_z eliminating the terms of u_φ.

$$\frac{\partial^2 M_z}{\partial r^2} - \frac{3}{r}\frac{\partial M_z}{\partial r} + \frac{\partial^2 M_z}{\partial z^2} = 0 \tag{75}$$

The solution of this equation for this problem is

$$M_z = Cr^4 + B \tag{76}$$

where *C* and *B* are integration constants. M_z should vanish for *r*=0. Then, *B* becomes zero. M_z must be equal to M_0 for *r*=*b*. Writing this, the other constant is obtained as

$$M_z(r = b) = M_0 = m\pi b^2 \rightarrow C = \frac{m\pi}{b^2} \tag{77}$$

So, the shear stresses in Eqs. (67) and (68) are obtained as

$$\tau_{z\varphi} = \frac{2mr}{b^2}, \qquad \tau_{r\varphi} = 0 \tag{78}$$

The shear strain different from zero is also obtained using Hooke's Law as

$$\gamma_{z\varphi} = \frac{2mr}{\mu b^2} \tag{79}$$

The total strain energy accumulated on the effective body under simple torsion can be calculated using the results given in Eqs. (78) and (79) as

$$U^* = \frac{1}{2}\int_{\varphi=0}^{2\pi}\int_{r=0}^{b}\int_{z=-h/2}^{h/2} \tau^*_{z\varphi}\gamma^*_{z\varphi} r dz dr d\varphi = \frac{\pi h m^2}{\mu^*} \tag{80}$$

whose quantities are effective.

To calculate the strain energy of a cylindrical body including a coaxial cylindrical fiber under the same state of stress, the total torsion moment is divided into two parts, M^M and M^F where M and F denote matrix and fiber, respectively. The solution for the single cylindrical body given in Eq. (76) can be used. For the fiber, the moment and the non-zero components of the shear stress and strain are written as

$$M^F(r,z) = C^F r^4, \qquad \tau^F_{z\varphi}(r,z) = \mu^F \gamma^F_{z\varphi} = \frac{2C^F r}{\pi} \tag{81}$$

Here, r may change from zero to a. For the matrix which has the geometry of a hollow cylinder, the moment and the component of stress and strain are

$$M^M(r,z) = C^M r^4 + D^M, \qquad \tau^M_{z\varphi}(r,z) = \mu^M \gamma^M_{z\varphi} = \frac{1}{2\pi r^2}\frac{\partial M^M}{\partial r} = \frac{2C^M r}{\pi} \tag{82}$$

The total torsion moment on the surface due to these shear stresses is calculated as

$$\int_0^a 2\pi\tau^F_{z\varphi} r^2 dr + \int_a^b 2\pi\tau^M_{z\varphi} r^2 dr = M_0 \rightarrow C^F a^4 + C^M\left(b^4 - a^4\right) = M_0 \tag{83}$$

Additionally, if the stress expressions of (81) and (82) are substituted into Eq. (71), then

$$\tau^F_{z\varphi}(r,z) = \frac{2C^F r}{\pi} = \mu^F \frac{\partial u^F_\varphi}{\partial z} \tag{84}$$

$$\tau^M_{z\varphi}(r,z) = \frac{2C^M r}{\pi} = \mu^M \frac{\partial u^M_\varphi}{\partial z} \tag{85}$$

are obtained. When these equations are integrated over z

$$u^F_\varphi = \frac{2C^F r}{\pi\mu^F} z + f_1(r) \tag{86}$$

$$u^M_\varphi = \frac{2C^M r}{\pi\mu^M} z + f_2(r) \tag{87}$$

are obtained. Here, $f_1(r)$ and $f_2(r)$ are functions which depend only on r. The components u^F_φ and u^M_φ are zero at z=0. This gives

$$f_1(r) = f_2(r) = 0 \tag{88}$$

At $r=a$, the displacement vectors of the matrix and the fiber must be equal due to continuity. It results as

$$\frac{C^F}{C^M} = \frac{\mu^F}{\mu^M} \rightarrow C^F = C^M \frac{\mu^F}{\mu^M} \tag{89}$$

The solution of the set of Eq. (83) and Eq. (89) gives

$$C^F = \frac{M_0 \mu^F}{\mu^F a^4 + \mu^M \left(b^4 - a^4\right)} \tag{90}$$

$$C^M = \frac{M_0 \mu^M}{\mu^F a^4 + \mu^M \left(b^4 - a^4\right)} \tag{91}$$

The components of the stress and strain tensor can be calculated and the total strain energy is calculated as

$$U^{RVE} = \frac{1}{2} 2\pi h \left[\int_0^a \tau_{z\varphi}^F \gamma_{z\varphi}^F dr + \int_a^b \tau_{z\varphi}^M \gamma_{z\varphi}^M dr \right] = \frac{b^4 m^2 \pi h}{a^4 \left(\mu^F - \mu^M\right) + b^4 \mu^M} \tag{92}$$

If this energy expression is equated to the effective energy given in Eq. (80), and if the resulting equality is rearranged, then the expression of the effective shear modulus are obtained as

$$\frac{1}{\mu^*} = \frac{b^4}{a^4 \mu^F + \left(b^4 - a^4\right)\mu^M} \tag{93}$$

For this problem, concentration c can be stated as a^2/b^2. If a in the above equation is changed with cb^2, then the effective shear modulus exists as

$$\mu^* = \mu^F c^2 + \mu^M \left(1 - c^2\right) \tag{94}$$

The variation of the ratio μ^*/μ^M versus concentration for the fiber reinforced composite has been drawn in Fig. 6. In this figure, Hashin's bounds have been obtained using the Eqs. (4.27) and (4.28) in the article of (Hashin 1965) and Christensen and Lo's variation has been drawn by the data from Fig. 4 in the article (Christensen and Lo 1979). The material constants of the matrix and fibers have been obtained from the latter reference.

4. Finite element analysis

Here, two different finite element models are used. The first one is a 16 cm×16 cm cubic matrix including uniformly and symmetrically distributed spherical particles with 1 cm radius each (Fig. 7(a)). The second one is cylindrical body with 2 cm diameter and 7 cm length and it involves cylindrical fibers with radius 0.1 cm. The symmetry axes are the same for both the matrix and fibers (Fig. 7(b)). These models can be considered as RVEs. A similar analysis acquired by

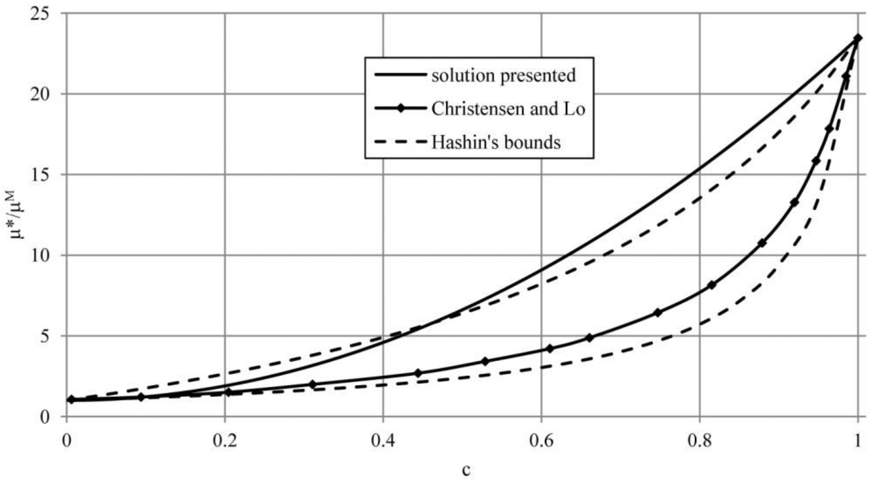

Fig. 6 Variation of μ^*/μ^M of fiber reinforced composite versus c compared to the Hashin's bounds (1965) and the curve of Christensen and Lo (1979) for their materials

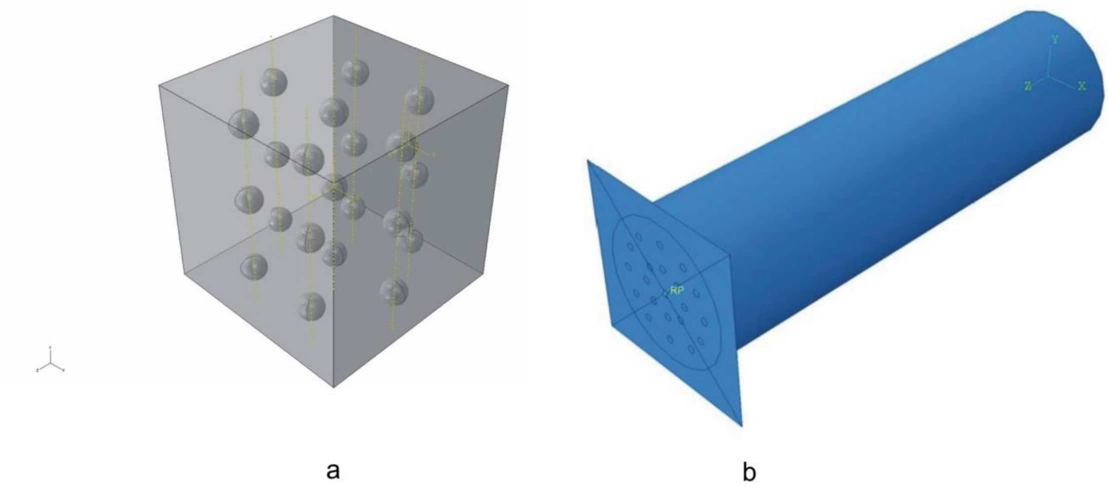

Fig. 7(a), (b) The FEA models of the two types of composites

Seguardo and Llorca (2002) defining a cubic unit element which includes spherical particles distributed into that according to an algorithm. This body has been considered as a linear, elastic and isotropic body. Three-dimensional models for three different inclusions; voids, rigid particles, and glass particles, were simulated by ANSYS in that study and variations of effective constants versus concentration were given graphically.

In this study, for the first model, one side face of the cube in the *xz*-plane and for the second model one circular face in the *xy*-plane are fixed. Both models are loaded simple tension in the direction being perpendicular to the fixed surfaces. The normal stresses, of which resultants are the tension forces, have been considered uniformly distributed at the top surfaces which are the opposite surfaces of the fixed ones. The concentrations are 2.5, 3.7, 6.1, 8.3, and 19.6 for the first model and 1.00, 1.25, 2.25, 4.25, and 5.25 for the second one. Two types of material have been selected which are linear-elastic, homogeneous and isotropic. For the first FEA model, Hashin's material has been used (Hashin 1962), while for the second model, a different materials set has been used which are polyester as the matrix and Kevlar fiber. The elastic constants for the model of the fiber reinforced composite are

$$E^M = 1.820 \text{ GPa}, \ \nu^M = 0.27$$

$$E^P = 100.000 \text{ GPa}, \ \nu^P = 0.33 \tag{95}$$

To solve these FEA models, ABAQUS has been used. The average number of the mesh elements and type of it are 47310 and C3D4 (4-node tetrahedron) for the first model while those for the second model are 82524 and C3D8R (8-node linear brick). The stresses and strains have been obtained at the nodes in the mid portion of the models. Using them, the associated effective moduli have been calculated by elasticity formulas. The results have been discussed in the next section.

5. Conclusions

Some mechanical properties of two types of composites have been examined. The aim is to consider the composite as a unique effective material and to calculate the material constants of it. These effective constants heavily depend on those of the particles and matrix.

The first composite consists of a matrix and spherical particles embedded into that. To determine the bulk and shear moduli of this composite, a RVE, which is a sphere of matrix involving a spherical particle with the same center, has been considered. Two types of loading on this sphere have been examined that the first is hydrostatic pressure while second is simple shear. After solving these two problems analytically, the strain energies accumulated in this RVE have been calculated performing volume integrals for each loading. Same problems were solved by Hashin (1962), but some quantities were calculated by performing two different types of surface integrals. After performing these integrals, total strain energy equated to the strain energy which belong to the single sphere of the unique material representing composite for each loading. Then this equality gives the effective bulk modulus K^* for the first loading and effective shear modulus μ^* for the second one in terms of μ^P, μ^M, K^P and K^M.

In the case of hydrostatic pressure problem which is used for the calculation of K^*, Hashin's two results and the result obtained here are the same (Hashin 1962). This means Hashin's strain energy are the same with it calculated here. However, for calculation of shear modulus which performed by solving simple shear problem, two values for each concentration value were given by Hashin. It is expected that the result which is found here must be between these two values for the same concentration. But, the solution presented is nearly under the lower boundary of Hashin. It is thought that this difference, which is very small, comes out from rounding errors in Hashin's paper. This fact can be seen if one calculates the shear and bulk moduli in terms of those given by Hashin. It is thought that the differences given in Fig. 2 arises due to these rounding errors.

Solutions in the literature were based on some energy definitions. In this study, it has been found on the uniqueness of the energy for a unique problem and it has been calculated performing volume integrals. The latter approach is supported by the result of Hashin (1962) which belongs to the effective bulk modulus through the coincidence of two boundaries.

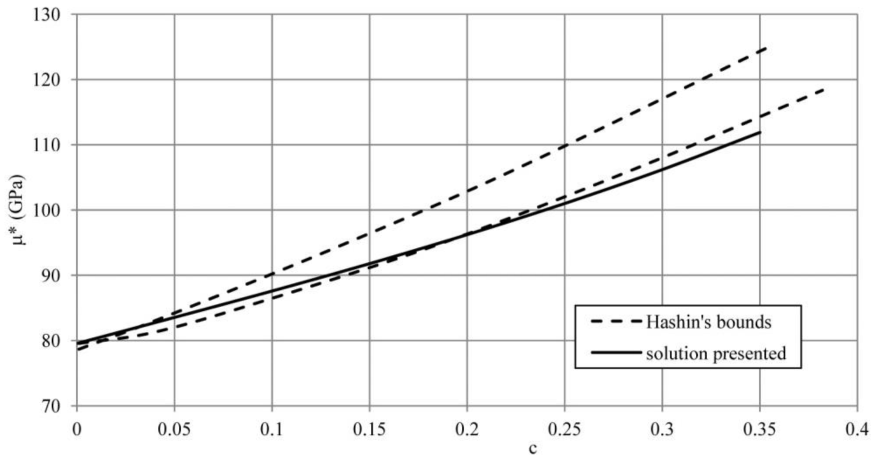

Fig. 8 Variation of μ^* of particle reinforced composite versus c compared to the Hashin's bounds (1962) for the set of first materials in smaller scale than that in Fig. 2

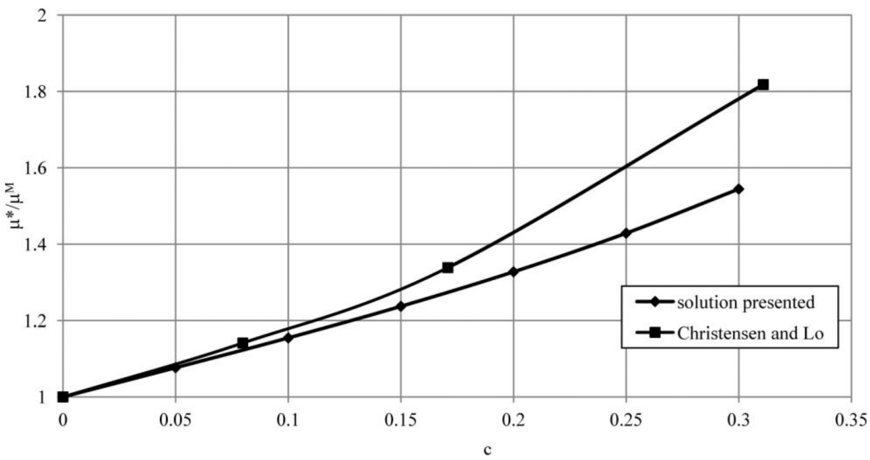

Fig. 9 Variation of μ^*/μ^M of particle reinforced composite versus c compared to Christensen and Lo's curve (1979) for their materials for smaller scale than that in Fig. 4

The solution for the effective shear modulus of the composites including micro-particles given in the article by Christensen and Lo (1979) were expressed by displacement with three integration constants D_1, D_3, and D_4 for the equivalent infinite homogeneous medium. The coordinate system is not specified clearly in the paper. Moreover, there are some differences in the number of equations and unknown constants. The curves of variations given in Fig. 3 are very close to each other for low concentrations.

For low concentrations, whole results given by various authors are almost the same in big scale. In fact, the curve of variation of the moduli need not be given in the range of concentration values from zero to 1 because after a value of it, inclusions interact each other. Additionally, the aim of the production of composites becomes meaningless if the material having high elastic properties relative to the other one has bigger concentration than the matrix. Nevertheless, the variations have

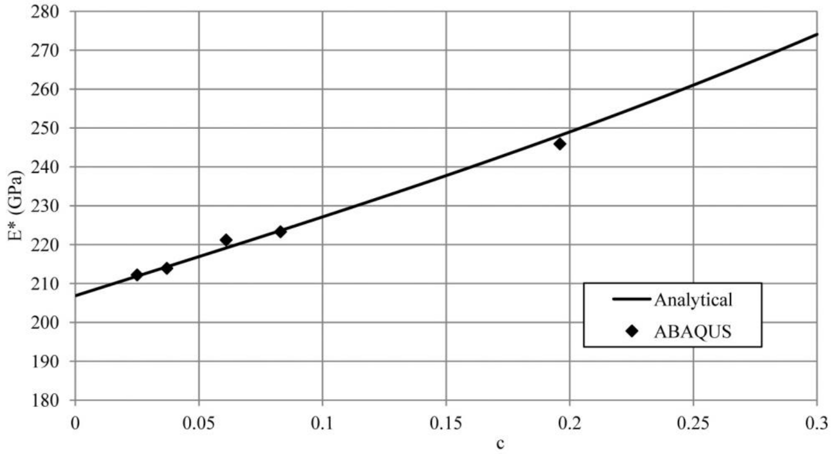

Fig. 10 The comparison of variation of the modulus of elasticity of particle reinforced composite versus *c* with those obtained from ABAQUS analysis

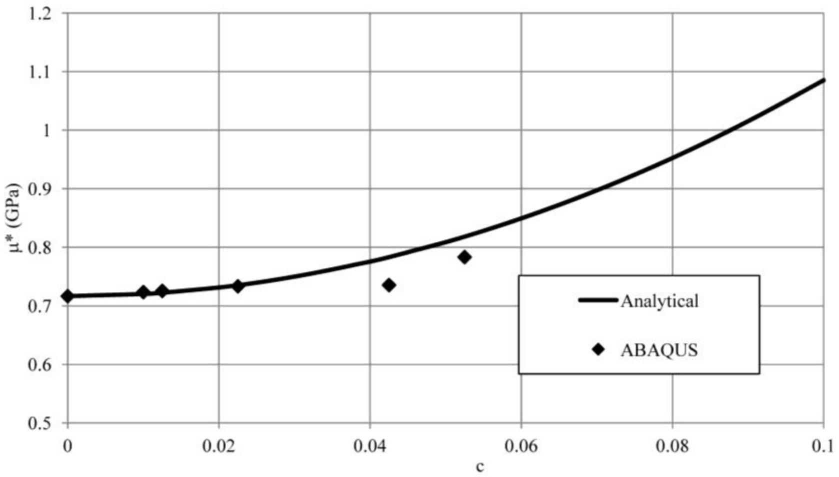

Fig. 11 The comparison of variation of the shear modulus of fiber reinforced composite versus *c* with those obtained from ABAQUS analysis

been given in full range in the previous figures due to keeping the tradition of the previous articles. Through this fact, the variation of μ^* has been given again in Fig. 8 with smaller scale than that used in Fig. 2. The differences around the small concentrations can be seen easily. Fig. 3 has been also given in Fig. 9 in a smaller scale.

The same procedure has been conducted for the second types of composites. Here, RVE has cylindrical geometry and includes a cylindrical fiber whose main axis coinciding with that of the element. The effective material is also a cylinder having the same radius as that of the RVE. In fact, RVE is not an isotropic body. However, there is a transversely isotropy which is in the plane of cross-section. Due to the loading which is simple torsion, the stress distribution does not depend on the coordinates θ and z in the cylindrical coordinates. So the relation between the stress and strain can be expressed as in terms of only shear modulus in this plane having isotropy. This

Table 1 Results from the analytical solution and FEA for the particulate composite

c (%)	E^*_{FE} (GPa)	E^*_A (GPa)
2.5	212.20	211.87
3.7	213.90	214.28
6.1	221.20	219.13
8.3	223.30	223.62
19.6	245.90	248.07

Table 2 Results from the analytical solution and FEA for the fiber reinforced composite

c (%)	E^*_{FE} (GPa)	υ^*_{FE}	μ^*_{FE} (GPa)	μ^*_A (GPa)
0	1.820	0.270	0.717	0.717
1	1.833	0.267	0.724	0.720
1.25	1.836	0.265	0.726	0.722
2.25	1.848	0.260	0.733	0.735
4.25	1.851	0.258	0.736	0.783
5.25	1.904	0.215	0.783	0.818

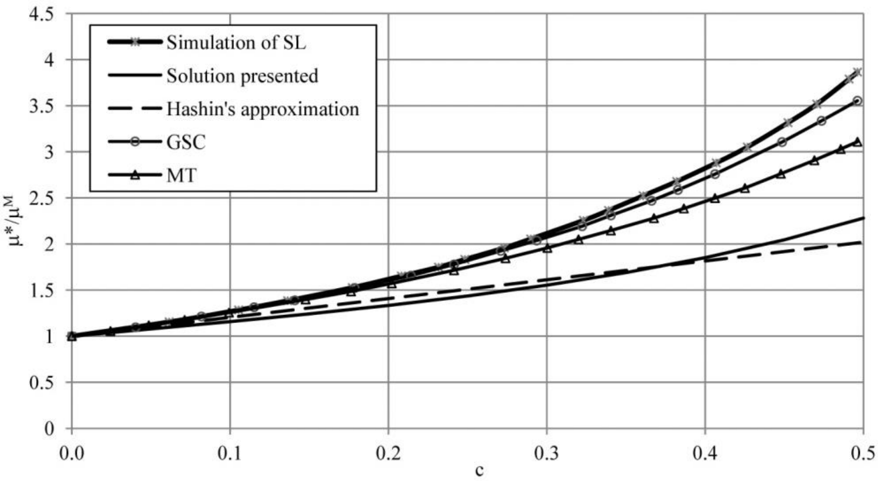

Fig. 12 The comparison of the μ^*/μ^M resulted from different analyses for epoxy resin/glass spheres (B^M=4.167 GPa, B^P=38.890 GPa, μ^M=1.087 GPa, μ^P=29.167 GPa)

approach has been given by a simple expression for μ^* in terms of μ^M and μ^F and this result can be easily used for low concentrations (Fig. 6).

The results from the FEM analysis have been given in Fig. 10 for the composites having particles and in Fig. 11 for those including fibers. In Fig. 10, the effective modulus of elasticity has been given calculating from the constants determined and the materials have been selected as the first set of materials given in the Sect. 2.1. In the next figure, variations of shear modulus obtained from the analytical solution and FEM analysis have been compared. The data provide good agreement for low concentrations. This situation can be seen in Table 1 for the particulate

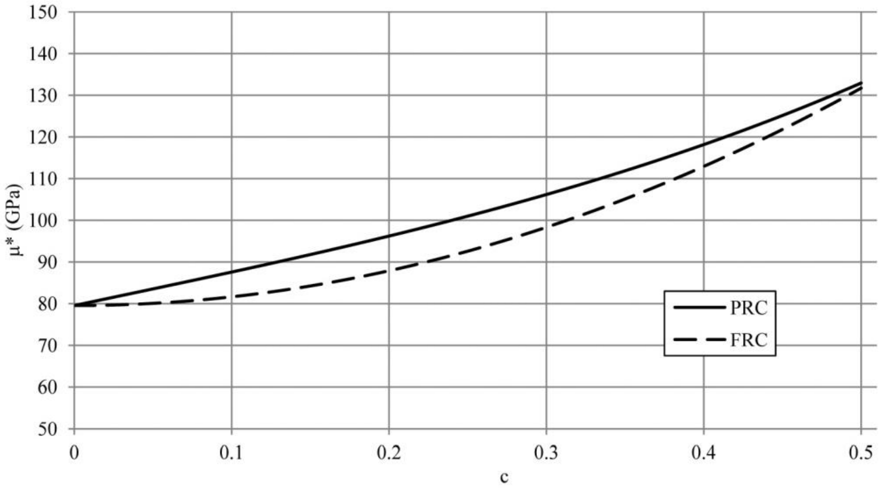

Fig. 13 The comparison of the shear moduli of the particle reinforced and fiber reinforced composites for the first set of materials

composites and in Table 2 for the fiber reinforced composite. *FE* and *A* indicate the results from finite element analysis and analytical solutions in these tables, respectively.

In Fig. 12, using epoxy matrix and glass particles which are the same in Fig. 6(b) in the article of Seguardo and Llorca (2002), variation of μ^*/μ^M versus concentration has been given. There are curves from three-dimensional simulation of Seguardo Llorca (Simulation of SL), Christensen and Lo's Generalized Self consistent solution (GSC) (Christensen and Lo 1979), and Mori-Tanaka's method (MT) which were applied to the composites by Benveniste (1987). These curves have been graphed using data obtained from the mentioned figure in the article of Seguardo and Llorca (2002), although the vertical axis was given to be *G* mistakenly instead of G/G^M in that work. Hashin's approximation has been drawn in there using Eq. (54) in the article of Hashin (1962). Present solution is close to Hashin's approximation and this approximation was recommended for a curve lying between the two bounds of the effective shear modulus. Taking into account the factor of safety in design, the lower logical values are more convenient so it can be thought that the solution presented here is more effective.

Lastly, variation of the effective shear modulus of the composites including particles and fibers have been compared in Fig. 13 for the first set of materials. In fact, given results in here are valid for the values of the lower concentrations and the curves intersect at the concentration value of 0.53. So, a statement can be said that the composites including particles have high shear modulus than that of the composite having fibers for low concentrations.

Acknowledgements

The authors would like to thank the Management of Scientific Research Projects of Istanbul Tech Univ and the Scientific and Technological Research Council of Turkey for the supports.

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