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A unified method for stresses in FGM sphere with exponentially-varying properties

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Abstract. Using the Complementary Functions Method (CFM), a general solution for the one-dimensional steady-state thermal and mechanical stresses in a hollow thick sphere made of functionally graded material (FGM) is presented. The mechanical properties are assumed to obey the exponential variations in the radial direction, and the Poisson's ratio is assumed to be constant, with general thermal and mechanical boundary conditions on the inside and outside surfaces of the sphere. In the present paper, a semi-analytical iterative technique, one of the most efficient unified method, is employed to solve the heat conduction equation and the Navier equation. For different values of inhomogeneity constant, distributions of radial displacement, radial stress, circumferential stress, and effective stress, as a function of radial direction, are obtained. Various material models from the literature are used and corresponding temperature distributions and stress distributions are computed. Verification of the proposed method is done using benchmark solutions available in the literature for some special cases and virtually exact results are obtained.

Keywords: complementary functions method; thick sphere; functionally graded materials (FGMs); exponentially varying properties

1. Introduction

The analysis of spherical structural members (shafts, pipes, tubes, etc.) is quite important especially in engineering design. These members are widely used in different areas of engineering practice. Thick sphere and thick-walled spherical shell are common components in structural applications and device systems involving aerospace and submarine structures, civil engineering structures, machines, pipes, sensors and actuators, etc. These structures are often exposed to temperature environment and the thermal stresses are then induced. For many cases, thermal stresses will significantly depress the strength and also affect the functionality of the structures. Thus, the exact analysis of the thermal stresses is really important.

FGMs are being used as interfacial zone to improve the bonding strength of layered composites to reduce the stresses in bonded dissimilar materials and as wear resistant layers in machine and engine components. They are used in modern technologies as advanced structures. Mechanical and

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thermal stresses for FGM thick hollow spheres are the subject in the theory of elasticity and thermoelasticity. Lutz and Zimmerman (1996) solved the problem of an isotropic sphere under thermal stresses; in this problem the material properties of the sphere vary linearly with radius. Obata and Noda (1994) studied the steady thermal stresses in a hollow circular cylinder and a hollow sphere made of FGMs. The aim of this research was to understand the effect of composition on stresses and to design the optimum FGM hollow circular cylinder and hollow sphere. Eslami *et al.* (2005) presented the analytical solution for thermal and mechanical behavior in a functionally graded thick hollow sphere under one-dimensional steady-state temperature distribution with thermal and mechanical boundary conditions. They assumed the material properties are expressed as power functions of radius direction. In addition, Poultangari *et al.* (2008) studied the thermal and mechanical stresses in a FGM sphere under non-axisymmetric thermo-mechanical loads. An alternate analytical method to carry out the elastic analysis of thick-walled spherical pressure vessels subjected to internal pressure was presented by You *et al*. (2004). Güven and Baykara (2001) studied functionally graded isotropic spheres subjected to internal pressure. The objective of the study is to understand the acceptable stress distributions in a hollow sphere under internal pressure for ductile and brittle material behaviors. It is stated that in a functionally graded isotropic hollow sphere designed according to the maximum shear stress failure theory, the material usage can be improved efficiently. Uncoupled steady-state thermo-elasticity problem of an FGM hollow sphere was investigated numerically by Alavi *et al.* (2008). Atefi and Moghimi (2006) obtained a closed-form solution for the two dimensional temperature distribution in a hollow sphere subjected to periodic boundary conditions. Bagri and Eslami (2007) proposed a new unified formulation for the generalized theories of the coupled thermo-elasticity on the basis of the Lord-Shulman, Green-Lindsay and Green-Naghdi models. Using Laplace transform, the governing equations have been analytically solved in the Laplace space domain for a hollow sphere and cylinder. Jabbari *et al.* (2010) solved the classical coupled thermo-elasticity problem for hollow and solid spheres. Their approach was based on resolving the coupled equations into two groups: coupled equations with homogeneous and nonhomogeneous boundary conditions. Ding *et al.* (2002) formulated the spherically dynamic thermoelastic problem for a special non-homogeneous transversely isotropic hollow sphere by introduction of a dependent variable and separation of variables technique. The transient thermal stress problem in a hollow sphere was studied by Tanigawa and Takeuti (1982). Wang *et al.* (2003) presented numerical results to show the dynamic stress responses in the uniformly heated hollow spheres. They resolved radial displacement into two functions, one of which satisfies inhomogeneous mechanical boundary conditions while the other one fulfills homogeneous mechanical boundary conditions. Bayat *et al.* (2012), assuming a power-law based variation in material properties, presented analytical results for a FG hollow sphere under spherically symmetric steady-state thermo-mechanical loadings and compared them with numerical results obtained from finite element simulations. Nejad *et al.* (2012) performed an analytical solution for FGM thick-walled spherical shells subjected to internal and/or external pressure, assuming an exponential law variation in material properties. Dai and Rao (2011) conducted research on electromagnetothermoelastic behaviors of a hollow sphere composed of functionally graded piezoelectric material. Boroujerdy and Eslami (2013) studied thermal instability of shallow spherical shells made of FGM and surface-bonded piezoelectric actuators. They assumed that the property of the FGM varies continuously through the thickness of the shell according to a power law distribution of the volume fraction of the constituent materials. They obtained the equilibrium equations based on the first-order theory of shells and the Sanders nonlinear kinematics equation.

The conventional approach of modeling FGM structural elements includes shell theories or dividing the material into homogeneous sub elements of different properties emulating the graded behavior. Series expansion methods and finite element analysis are first solution methods used in the literature. Mechanical and thermal stresses for FGMs in axisymmetric cylindrical coordinates are of interesting subjects in the theory of elasticity and thermoelasticity. The classical method of analysis is to combine the equilibrium equations with the stress-strain and strain-displacement relations to arrive at the governing equations in term of the displacement components, called the Navier equations. The material is assumed to be functionally graded in the radial direction with the grading function, in the most general sense, being an arbitrary continuous function of the radial coordinate. Forcing functions applied on the inner or outer boundaries are internal pressures. These assumptions yield governing differential equations with variable coefficients. Under these conditions analytical solutions cannot be obtained except for certain simple grading functions and pressures. The present paper uses a novel and efficient method CFM is employed in the analysis of axisymmetric elastic responses of functionally graded sphere subject to internal pressures applied on the inner boundary. It is also used for FGM cylinders with simple power-law properties which is easily analyzed by a simple direct method (Eslami *et al.* 2005). Governing differential equations thus obtained in spatial coordinates, in general, have variable coefficients. They form a two-point boundary value problem. CFM allows treating such problems as a system of initial-value problems which can readily be solved by any one of the standard methods available in the literature. The fifth-order Runga-Kutta method is employed in the present study. The theoretical background for the method is available in the literature (Aktas 1972, Roberts and Shipman 1979, Agarwal 1982). The method is also successfully applied in other structural mechanics problems such as those involving curved bars (Yildirim 1997) and beams (Calim 2009, Calim and Akkurt 2011). In addition, the CFM procedure has been applied to FGM cylinders, spheres and disks under static pressure and steady-state thermal loads by Tutuncu and Temel (2009, 2013). Finally, Temel *et al.* (2014) showed that the CFM is well suited for problems in which the graded mechanical properties and applied pressures are supplied point by point in a discretized manner. The present paper uses a novel and efficient method which combines CFM is employed a thick hollow sphere of FGM under a one-dimensional steady- state temperature distribution with general types of thermal and mechanical boundary conditions. Two material models will be used: (a) simple power law with constant Poisson's ratio (Eslami *et al.* 2005) for which case analytical benchmark solutions are available, (b) exponentially-varying properties. It should be emphasized once again that the solution procedure is not confined to any particular choice of material model; it is equally suitable for arbitrary functions defining the gradient variation of material properties.

2. Solutions by the complementary functions method

The CFM transforms two-point boundary-value problems to system of initial-value problems. It reduces to a particularly simple solution scheme when applied to the present class of problems. For an annular sphere of inner radius r_i and outer radius r_o. As it will be shown in the proceeding sections, under axisymmetric conditions, the governing differential equation of the dependent variable $u(r)$ in its most general form is

$$u'' + P(r)u' + Q(r)u = R(r) \tag{1}$$

subject to boundary conditions on the inner ($r = r_i$) and outer ($r = r_o$) surfaces. Here $()'$ denotes the derivative with respect to *r*. General closed-form solution of the above equation cannot be obtained. The complete solution of Eq. (1) is

$$u = b_j u_j + u_p \quad , \qquad j = 1{,}2 \tag{2}$$

where u_j and u_p are, respectively, homogenous and particular solutions. The coefficients b_j are determined via the boundary conditions. CFM begins by assuming $u_i = Y_1^{(i)}$ and $u_i^{'} = Y_2^{(i)}$, which means

$$(Y_1^{(i)})' = Y_2^{(i)} \tag{3a}$$

Here, the index i=1,2 refers to homogeneous solutions and i=p will mean the particular solution. To determine the homogeneous solutions, the right-hand side of Eq. (1) is set equal to zero and the following is obtained

$$(Y_2^{(i)})' = -P(r)Y_2^{(i)} - Q(r)Y_1^{(i)} \tag{3b}$$

The system of Eqs. (3a) and (3b) can be solved numerically for each homogeneous solution. Kronecker delta initial conditions given below will be used to assure the linear independence of the solutions (Roberts and Shipman 1979)

$$Y_1^{(i)} = \delta_{ji} \quad , \quad j,i = 1,2 \tag{4}$$

To obtain the particular solution, Eq. (3b) is modified as

$$(Y_2^{(p)})' = -P(r)Y_2^{(p)} - Q(r)Y_1^{(p)} + R(r) \tag{5}$$

A particular solution needs only to satisfy the differential equation and homogeneous initial conditions

$$Y_j^{(p)} = 0 \quad , \quad j = 1{,}2 \tag{6}$$

are to be imposed. Eqs. (3a), (5), (6) constitute the system of equations for the particular solution along with the initial conditions. The fifth-order Runge-Kutta method (RK5) will be used for all cases considered. Note that by this procedure not only the solution $u(r)$ itself but also its first derivative are readily calculated. Applying the boundary conditions prescribed for the particular problem on hand results in the following system of algebraic equations for the coefficients b_1 and b_2

$$\begin{bmatrix} A_{11} A_{12} \\ A_{21}\, A_{22} \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} = \begin{Bmatrix} RHS1 \\ RHS2 \end{Bmatrix} \tag{7}$$

Here, A_{ij} includes the values of the homogeneous solutions at the boundary points. *RHS*1 and *RHS*2 contain values of the particular solutions. If the sphere is subjected to internal and external pressures, they will also be included in the right hand-side terms. On the other hand, implementing CFM in the heat conduction problem will yield *RHS*1 and *RHS*2 as the prescribed temperatures along the boundaries. These points will be illustrated in the following sections.

3. Heat conduction in radial direction

The heat conduction equation in the steady-state condition for the one-dimensional problem in polar coordinates and thermal boundary conditions for a FGM hollow sphere are given, respectively, as

$$\frac{1}{r^2}\left(r^2 k(r) T'(r)\right)' = 0 \quad , \quad r_i \le r \le r_o$$
$$C_{11}T(r_i) + C_{12}T'(r_i) = f_1 \ ,$$
$$C_{11}T(r_o) + C_{12}T'(r_o) = f_2 \tag{8}$$

where k=$k(r)$ is the thermal conduction coefficient, r_i and r_o are the inner and outer radii of the hollow sphere C_{ij} are the constant thermal parameters related to the conduction and convection coefficients. The constants f_1 and f_2 are known constants on inside and outside radii.

It is assumed that the nonhomogeneous thermal conduction coefficient $k(r)$ is exponential function of r as

$$k(r) = k_0 e^{\beta r} \tag{9}$$

where k_0 and β are the material parameters. Using Eq.(9), the heat conduction equation becomes

$$\frac{1}{r^2}(r^2 e^{\beta r} T'(r))' = 0 \tag{10}$$

Steady-state axisymmetric heat conduction without heat generation will be considered. The heat balance equation in the radial direction for a non-uniform disk yields

$$T'' + B(r)T' = 0 \tag{11}$$

where $B(r) = \left(\frac{2}{r} + \beta\right)$ with β being the thermal conduction coefficient and it is varying as a function of the radial coordinate r. The boundary conductions are temperatures prescribed on the inner and outer surfaces as

$$T(r_i) = T_i \ and \ T(r_o) = T_o \tag{12}$$

The complete solution is the homogeneous solution

$$T = b_j T_j \quad , \quad j=1,2 \tag{13a}$$

with

$$T' = b_j T_j' \quad , \quad j=1,2 \tag{13b}$$

Following the steps outlined in previous section, the temperature change distribution is obtained at the collocation points. The constants b_j can now be found by imposing the boundary conditions. This process results in the system given by Eq. (7) were

$A_{11} = T_1(r_i)$	$A_{12} = T_2(r_i)$,
$A_{21} = T_1(r_o)$	$A_{22} = T_2(r_o)$,
$RHS1 = Ti$	$RHS2 = To$

4. Governing equation

Consider a thick walled sphere of inside radius r_i and outside radius r_o made of FGM

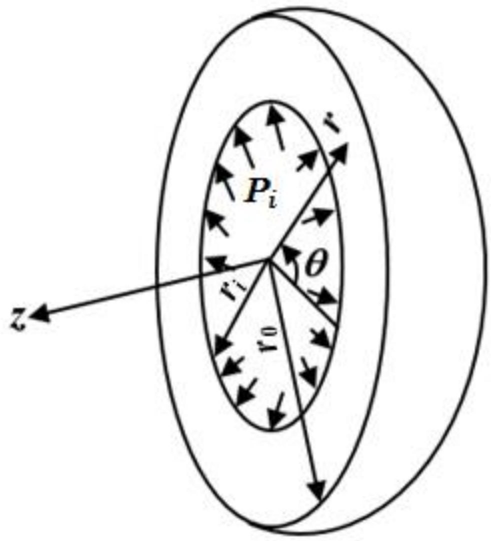

Fig. 1 Schematic diagram of a thick-walled sphere

(see Fig. 1). The material is graded through the r-direction. Let u be displacement component in the radial direction. Then the strain-displacement relations are

$$\varepsilon_{rr} = \frac{du}{dr} \qquad \varepsilon_{\theta\theta} = \frac{u}{r} \tag{14}$$

The stress-strain relations are

$$\begin{aligned} \sigma_{rr} &= (\lambda + 2\mu)\varepsilon_{rr} + 2\lambda\varepsilon_{\theta\theta} - (3\lambda + 2\mu)\alpha T(r) \\ \sigma_{\theta\theta} &= 2(\lambda + \mu)\varepsilon_{\theta\theta} + \lambda\varepsilon_{rr} - (3\lambda + 2\mu)\alpha T(r) \end{aligned} \tag{15}$$

where σ_{ij} and ε_{ij} $(i,j=r,\theta)$ are stress and strain tensors, $T(r)$ is the temperature distribution determined from the heat conduction equation, α is the coefficient of thermal expansion, and λ and μ are Lame coefficients related to the modulus of elasticity E and Poisson's ratio v as

$$\lambda = \frac{v\,E(r)}{(1+v)(1-2v)} \quad , \quad \mu = \frac{E(r)}{2(1+v)} \tag{16}$$

The equilibrium equation in the radial direction, disregarding the body force and inertia term, is

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{2\,(\sigma_{rr} - \sigma_{\theta\theta})}{r} = 0 \tag{17}$$

To obtain the equilibrium equation in terms of the displacement component for the FGM cylinder, the functional relationship of the material properties must be known. To ascertain the effect of the inhomogeneity, the properties are considered to vary exponentially across the thickness

$$E(r) = E_0 e^{\beta r}\,,\ \alpha(r) = \alpha_0 e^{\beta r} \tag{18}$$

where E_0 and α_0 are material constants and β is the inhomogeneity parameter. We may further assume that Poisson's ratio is constant.

Table 1 Comparison of CFM with Eslami *et al.* (2005) results for homogenous sphere (m=0 and β=0)

r/r_i	$T/T(r_i)$		u/r_i		σ_{rr}/P_i		$\sigma_{\theta\theta}/P_i$	
	CFM	Eslami	CFM	Eslami	CFM	Eslami	CFM	Eslami
1	1	1	0.000528	0.000528	-1	-1	2.522017	2.522017
1.01	0.940594	0.940594	0.000522	0.000522	-0.93091	-0.93091	2.491538	2.491538
1.02	0.882353	0.882353	0.000516	0.000516	-0.86441	-0.86441	2.462292	2.462292
1.03	0.825243	0.825243	0.000511	0.000511	-0.8004	-0.8004	2.434198	2.434198
1.04	0.769231	0.769231	0.000505	0.000505	-0.73876	-0.73876	2.407214	2.407214
1.05	0.714286	0.714286	0.0005	0.0005	-0.67937	-0.67937	2.381289	2.381289
1.06	0.660377	0.660377	0.000495	0.000495	-0.62213	-0.62213	2.356368	2.356368
1.07	0.607477	0.607477	0.00049	0.00049	-0.56694	-0.56694	2.332403	2.332403
1.08	0.555556	0.555556	0.000485	0.000485	-0.51371	-0.51371	2.309347	2.309347
1.09	0.504587	0.504587	0.000481	0.000481	-0.46236	-0.46236	2.287161	2.287161
1.1	0.454545	0.454545	0.000476	0.000476	-0.41278	-0.41278	2.265813	2.265813
1.11	0.405405	0.405405	0.000472	0.000472	-0.36493	-0.36493	2.245248	2.245248
1.12	0.357143	0.357143	0.000468	0.000468	-0.3187	-0.3187	2.225446	2.225446
1.13	0.309735	0.309735	0.000464	0.000464	-0.27404	-0.27404	2.206369	2.206369
1.14	0.263158	0.263158	0.00046	0.00046	-0.23088	-0.23088	2.187978	2.187978
1.15	0.217391	0.217391	0.000456	0.000456	-0.18915	-0.18915	2.170254	2.170254
1.16	0.172414	0.172414	0.000452	0.000452	-0.14879	-0.14879	2.153161	2.153161
1.17	0.128205	0.128205	0.000449	0.000449	-0.10975	-0.10975	2.136668	2.136668
1.18	0.084746	0.084746	0.000446	0.000446	-0.07198	-0.07198	2.120758	2.120758
1.19	0.042017	0.042017	0.000442	0.000442	-0.03541	-0.03541	2.105405	2.105405
1.2	0	0	0.000439	0.000439	3.7E-06	3.7E-06	2.090585	2.090585

Using relations (14)-(18), Navier equation in term of the displacement is

$$u'' + P(r)u' + Q(r)u = R(r) \tag{19}$$

$$P(r) = (\beta r + 2)\frac{1}{r} \quad , \qquad Q(r) = \left(\frac{2(\nu(1+\beta r) - 1}{1-\nu}\right)\frac{1}{r^2} \quad ,$$
$$R(r) = \frac{\alpha_0(1+\nu)}{1-\nu}(2e^{\beta r}\beta T + e^{\beta r}T') \tag{20}$$

Following the steps outlined in solutions by the CFM section, the complete displacement is obtained at the collocation points as

$$u = b_1 u_1 + b_2 u_2 + u_p, \tag{21}$$

with

$$u' = b_1 u_1{}' + b_2 u_2{}' + u_p{}' \tag{22}$$

The coefficients b_1, b_2 will be determined using the stress free conditions on inner ($\sigma_{rr}(r_i) = -P_i$)

Table 2 Comparison of FGM results with Eslami *et al.* (2005) results for FGM spheres with constant Poisson's ratio and elastic modulus obeying a simple power law (m=-2 and β=-2)

r/r_i	$T/T(r_i)$		u/r_i		σ_{rr}/P_i		$\sigma_{\theta\theta}/P_i$	
	CFM	Eslami	CFM	Eslami	CFM	Eslami	CFM	Eslami
1	1	1	0.000625	0.000625	-1	-1	3.07393	3.07393
1.01	0.95	0.95	0.000618	0.000618	-0.92076	-0.92076	2.970641	2.970641
1.02	0.9	0.9	0.000611	0.000611	-0.8458	-0.8458	2.872551	2.872551
1.03	0.85	0.85	0.000605	0.000605	-0.77485	-0.77485	2.779339	2.779339
1.04	0.8	0.8	0.000599	0.000599	-0.70768	-0.70768	2.690724	2.690724
1.05	0.75	0.75	0.000593	0.000593	-0.64407	-0.64407	2.606408	2.606408
1.06	0.7	0.7	0.000587	0.000587	-0.58379	-0.58379	2.526168	2.526168
1.07	0.65	0.65	0.000581	0.000581	-0.52666	-0.52666	2.449749	2.449749
1.08	0.6	0.6	0.000575	0.000575	-0.47246	-0.47246	2.376949	2.376949
1.09	0.55	0.55	0.00057	0.00057	-0.42106	-0.42106	2.307546	2.307546
1.1	0.5	0.5	0.000565	0.000565	-0.37228	-0.37228	2.241353	2.241353
1.11	0.45	0.45	0.00056	0.00056	-0.32598	-0.32598	2.178194	2.178194
1.12	0.4	0.4	0.000555	0.000555	-0.282	-0.282	2.117902	2.117902
1.13	0.35	0.35	0.00055	0.00055	-0.24022	-0.24022	2.060317	2.060317
1.14	0.3	0.3	0.000545	0.000545	-0.20052	-0.20052	2.005291	2.005291
1.15	0.25	0.25	0.000541	0.000541	-0.16278	-0.16278	1.952685	1.952685
1.16	0.2	0.2	0.000537	0.000537	-0.12691	-0.12691	1.902372	1.902372
1.17	0.15	0.15	0.000532	0.000532	-0.09278	-0.09278	1.85423	1.85423
1.18	0.1	0.1	0.000528	0.000528	-0.06031	-0.06031	1.80815	1.80815
1.19	0.05	0.05	0.000524	0.000524	-0.02941	-0.02941	1.764018	1.764018
1.2	0	0	0.000521	0.000521	1.07E-06	1.07E-06	1.721733	1.721733

and outer ($\sigma_{rr}(r_o) = -P_o$) boundaries. This step is particularly simple since during the solution process the first derivative of the radial displacement has already been calculated.

5. Results and discussions

Consider a thick hollow sphere of inner radius r_i=1 m and outer radius r_o=1.2 m. Poisson's ratio is taken to be 0.3, and the modulus of elasticity and the thermal coefficient of expansion at the inner radius are E_o=200 GPa and α_o=1.2×10^{-6}/^{o}C, respectively. The properties are considered to vary exponentially across the thickness. The boundary conditions for temperature are taken as $T(r_i)$=10 ^{o}C and $T(r_o)$=0 ^{o}C. The hollow sphere has pressure on its inner surface so the boundary conditions for stresses are assumed as $\sigma_{rr}(r_i) = -50$ MPa and $\sigma_{rr}(r_o) = 0$ MPa.

The accuracy of the present method is first compared to analytical results presented for a simple power-law material model and a step-type inside pressure (Eslami *et al.* 2005). The comparing will be illustrated in the Tables 1-3. It can be observed from the Tables, the results are in good agreement with the same results from Eslami *et al.* (2005). The numerical results have

Table 3 Comparison of FGM results with Eslami *et al.* (2005) results for FGM spheres with constant Poisson's ratio and elastic modulus obeying a simple power law ($m=2$ and $\beta=2$)

r/r_i	$T/T(r_i)$		u/r_i		σ_{rr}/P_i		$\sigma_{\theta\theta}/P_i$	
	CFM	Eslami	CFM	Eslami	CFM	Eslami	CFM	Eslami
1	1	1	0.000442	0.000442	-1	-1	2.030027	2.030027
1.01	0.930192	0.930192	0.000437	0.000437	-0.94007	-0.94007	2.052493	2.052493
1.02	0.863095	0.863095	0.000432	0.000432	-0.88147	-0.88147	2.075142	2.075142
1.03	0.798578	0.798578	0.000427	0.000427	-0.82411	-0.82411	2.097975	2.097975
1.04	0.736519	0.736519	0.000422	0.000422	-0.76797	-0.76797	2.12097	2.12097
1.05	0.676801	0.676801	0.000418	0.000418	-0.71298	-0.71298	2.144144	2.144144
1.06	0.619316	0.619316	0.000413	0.000413	-0.65911	-0.65911	2.167485	2.167485
1.07	0.56396	0.56396	0.000409	0.000409	-0.6063	-0.6063	2.191004	2.191004
1.08	0.510635	0.510635	0.000405	0.000405	-0.55452	-0.55452	2.21469	2.21469
1.09	0.459249	0.459249	0.000401	0.000401	-0.50373	-0.50373	2.238545	2.238545
1.1	0.409714	0.409714	0.000397	0.000397	-0.45388	-0.45388	2.262578	2.262578
1.11	0.361949	0.361949	0.000394	0.000394	-0.40493	-0.40493	2.286778	2.286778
1.12	0.315874	0.315874	0.00039	0.00039	-0.35687	-0.35687	2.31114	2.31114
1.13	0.271416	0.271416	0.000387	0.000387	-0.30964	-0.30964	2.335689	2.335689
1.14	0.228504	0.228504	0.000384	0.000384	-0.26322	-0.26322	2.360397	2.360397
1.15	0.187071	0.187071	0.000381	0.000381	-0.21758	-0.21758	2.385274	2.385274
1.16	0.147056	0.147056	0.000378	0.000378	-0.17267	-0.17267	2.41033	2.41033
1.17	0.108396	0.108396	0.000375	0.000375	-0.12849	-0.12849	2.435548	2.435548
1.18	0.071036	0.071036	0.000372	0.000372	-0.08501	-0.08501	2.460937	2.460937
1.19	0.034921	0.034921	0.000369	0.000369	-0.04219	-0.04219	2.486489	2.486489
1.2	0	0	0.000366	0.000366	3.49E-06	3.49E-06	2.512222	2.512222

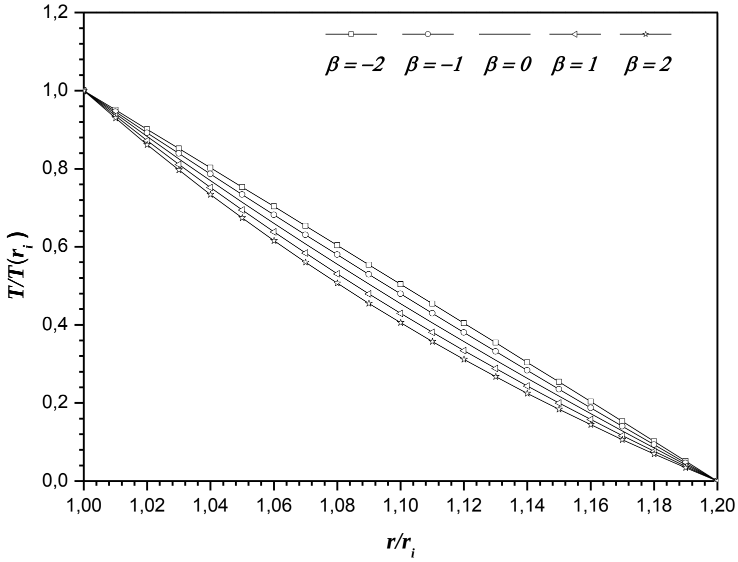

Fig. 2 Radial distribution of temperature for sphere

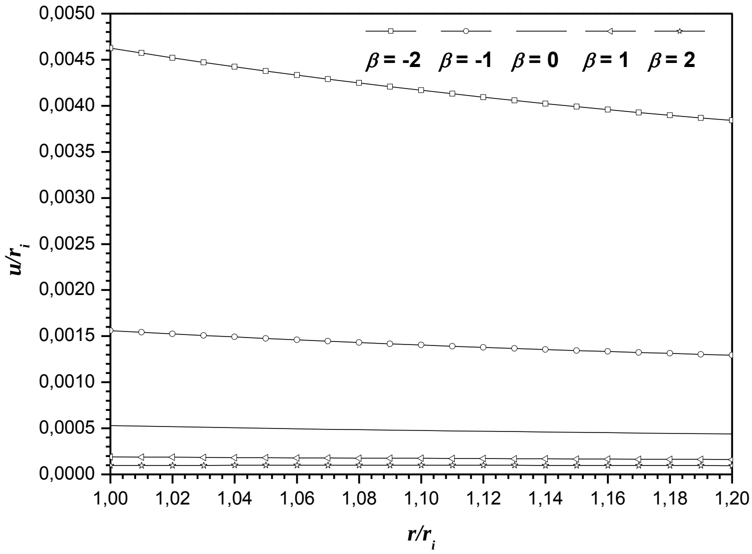

Fig. 3 Radial distribution of radial displacement for sphere

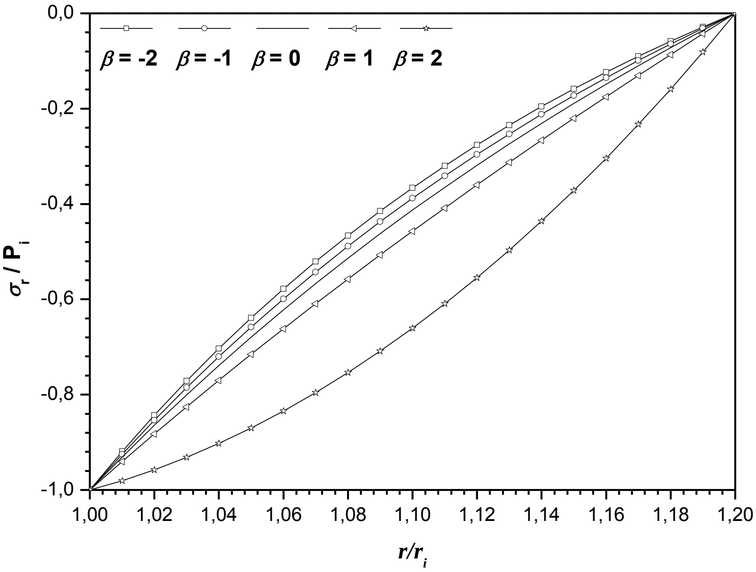

Fig. 4 Radial distribution of radial stress for sphere

been obtained to six-digit accuracy by picking only 11 collocation points.

Fig. 2 shows the variations of the temperature along the radial direction for different values inhomogeneity parameter (β). The figure shows that as the inhomogeneity parameter β increases, the temperature decreased. Fig. 3 shows the plot of the radial displacement along the radius. The magnitude of the radial displacement is decreased as the inhomogeneity parameter β is increased. The radial and circumferential stresses are plotted along the radial direction and shown in Figs. 4 and 5. The magnitude of the radial stress is decreased as β is increased. It is seen that for $\beta< 1$ the

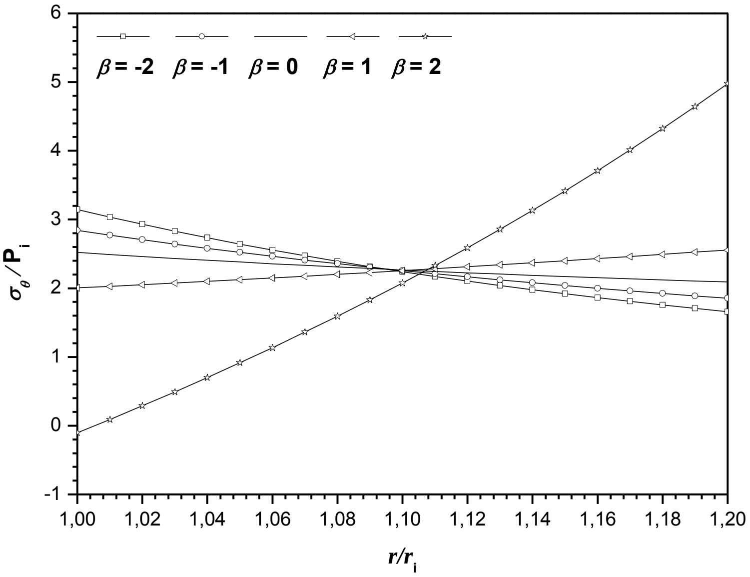

Fig. 5 Radial distribution of hoop stress for sphere

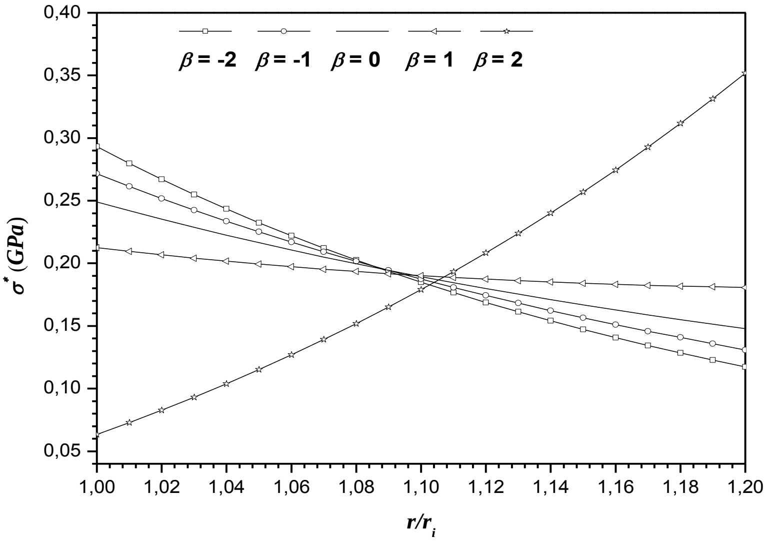

Fig. 6 Effective stress distribution for r_o/r_i=1.2 for sphere

circumferential stress decreases along the radial direction. For β>1, the circumferential stress increases as the radius increases, since the modulus of elasticity is an increasing function of the radius, see Eq. (18). Physically, this means that the outer layers of the sphere are biased to maintain the stress due to their higher stiffness. There is a limiting value for β, where the circumferential stress remains almost constant along the radius. The curve associated with β =1 shows that the variation of circumferential stress along the radial direction is minor, and is almost uniform across the radius. To investigate the pattern of the stress distribution along the sphere radius, the effective stress $\sigma^* = \sqrt{2}|\sigma_r - \sigma_\theta|$ is plotted along the radial direction for different values of r_o /r_i and the inhomogeneity parameter β. Figure 6 is plotted for r_o /r_i=1.2. It is interesting to note from Fig. 6 that for β =1 the effective stress is almost uniform along the radius of sphere.

It should be pointed out once again that the purpose of the present work has been the introduction of CFM to the solution procedure of the class of problems on hand. Converting the two-point boundary value problem to a system of initial-value problem gave way to the implementation of well-established numerical schemes. Runge-Kutta method of fifth-order (RK5) has been used to solve the system of equations. The procedure is simple, efficiently implemented and accurate to the extent that exact numerical results have been obtained to six-digit accuracy by picking only 11 collocation points in RK5.

6. Conclusions

This paper presents a numerical solution for the calculation of the axisymmetric thermal and mechanical stresses in a thick hollow sphere made of FGM. The material properties through the graded direction are assumed to be nonlinear with a power law distribution and exponentially-varying properties. The mechanical and thermal stresses are obtained through the CFM of solution of the Navier equation. The comparisons of temperature distributions and stress distributions are presented in the form of tables. The numerical results for all cases are shown to exactly match this reported in Eslami *et al.* (2005). At the result, we can say that:

• With the unified approach presented in the present study, one would not have to compromise on the functional continuity of the material properties. Analysis of any material model in the form of an arbitrary function subject to an internal pressure has been analyzed efficiently and accurately by employing CFM.
• The unified method used is simple, accurate and efficiently implemented.
• The method employed in this study allows solutions of continuous functions.
• The CFM of solving the differential equation provides a complete solution, yielding both thermal stresses and temperature distributions.

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