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Analysis of functionally graded beam using a new first-order shear deformation theory

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Abstract. A new first-order shear deformation theory is developed for dynamic behavior of functionally graded beams. The equations governing the axial and transverse deformations of functionally graded plates are derived based on the present first-order shear deformation plate theory. The governing equations and boundary conditions of functionally graded beams have the simple forms as those of isotropic plates. The influences of the volume fraction index and thickness-to-length ratio on the fundamental frequencies are discussed.The accuracy of the present solutions is verified by comparing the obtained results with the existing solutions.

Keywords: functionally graded beam; first shear deformation theory; Hamilton’s principle; vibration

1. Introduction

In material sciences, a functionally graded material (FGM) is a type of material whose composition is designed to change continuously within the solid. The concept is to make a composite material by varying the microstructure from one material to another material with a specific gradient. This enables the material to have good specifications of both materials. If it is for thermal or corrosive resistance or malleability and toughness, both strengths of the material may be used to avoid corrosion, fatigue, fracture and stress corrosion cracking. The transition between the two materials can usually be approximated by means of a power series. The aircraft and aerospace industry and the computer circuit industry are very interested in the possibility of materials that can withstand very high thermal gradients. This is normally achieved by using a ceramic layer connected with a metallic layer. The concept of FGM was first considered in Japan in 1984 during a space plane project. The FGM materials can be designed for specific applications. For example, thermal barrier coatings for turbine blades (electricity production), armor protection for military applications, fusion energy devices, space/aerospace industries, automotive applications, etc.

Dynamic analyses of FGM structures have attracted increasing research effort in the last decade

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because of the wide application areas of FGMs. For instance, Sankar *et al.* (2001) gave an elasticity solution based on the Euler-Bernoulli beam theory for functionally graded beam subjected to static transverse loads by assuming that Young's modulus of the beam vary exponentially through the thickness. Aydogdu and Taskin (2007) investigated the free vibration behavior of a simply supported FG beam by using Euler-Bernoulli beam theory, parabolic shear deformation theory and exponential shear deformation theory. Zhong and Yu (2007) presented an analytical solution of a cantilever FG beam with arbitrary graded variations of material property distribution based on two-dimensional elasticity theory. Taj *et al.* (2013) conducted static analysis of FG plates using higher order shear deformation theory. Recently, Tounsi and his co-workers (Hadji *et al.* 2011, Houari *et al.* 2011, El Meiche *et al.* 2011, Bourada *et al*. 2012, Bachir Bouiadjra *et al*. 2012, Fekrar *et al*. 2012, Klouche Djedid *et al*. 2014, Ait Yahia *et al*. 2015) developed new shear deformation plates theories involving only four unknown functions. Bourada *et al.* (2015) study a new simple shear and normal deformations theory for functionally graded beams. Belabed *et al.* (2014) presented an efficient and simple higher order shear and normal deformation theory for functionally graded material (FGM) plates. Hebali *et al.* (2014) studied the static and free vibration analysis of functionally graded plates using a new quasi-3D hyperbolic shear deformation theory. Mahi *et al.* (2015) studied the bending and free vibration analysis of isotropic, functionally graded, sandwich and laminated composite plates using a new hyperbolic shear deformation theory. Tounsi *et al.* (2013) use a refined trigonometric shear deformation theory for thermoelastic bending of functionally graded sandwich plates. Ait Amar Meziane *et al.* (2014) proposed an efficient and simple refined theory for buckling and free vibration of exponentially graded sandwich plates under various boundary conditions. Zidi *et al.* (2014) study hygro-thermo-mechanical loading for the Bending of FGM plates using a four variable refined plate theory. Bouderba *et al.* (2013) studied the thermomechanical bending response of FGM thick plates resting on Winkler-Pasternak elastic foundations. Hamidi *et al.* (2015) investigated a sinusoidal plate theory with 5-unknowns and stretching effect for thermomechanical bending of functionally graded sandwich plates. Bousahla *et al.* 2014 used a novel higher order shear and normal deformation theory based on neutral surface position for bending analysis of advanced composite plates. Bennoun *et al.* (2016) analyzed the vibration of functionally graded sandwich plates using a novel five variable refined plate theory. Ait atmane *et al.* (2016) studied the effect of thickness stretching and porosity on mechanical response of a functionally graded beams resting on elastic foundations

In the present study, free vibration of simply supported FG beams was investigated by using new first shear deformation beam theory. This theory enforces traction free Boundary conditions at beams surfaces using shear correction factors. Then, the equations governing the axial and transverse deformations of functionally graded plates are derived based on the present first-order shear deformation plate theory. Analytical solutions for free vibration are obtained. Numerical examples are presented to verify the accuracy of the present theory.

2. Theoretical formulations

Consider a functionally graded beam with length L and rectangular cross section $b \times h$, with b being the width and h being the height as shown in Fig. 1. The beam is made of isotropic material with material properties varying smoothly in the thickness direction.

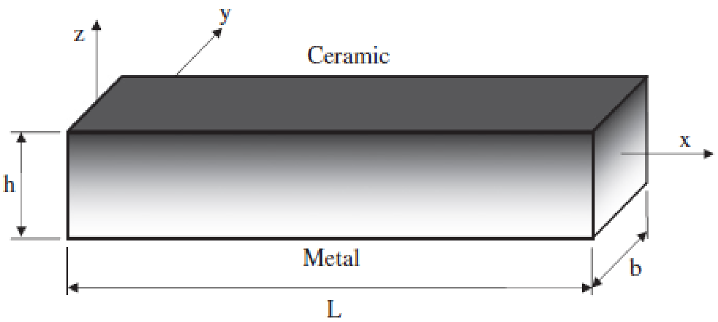

Fig. 1 Geometry and coordinate of a FG beam

2.1 Material properties

The properties of FGM vary continuously due to the gradually changing volume fraction of the constituent materials (ceramic and metal), usually in the thickness direction only. The power-law function is commonly used to describe these variations of materials properties. The expression given below represents the profile for the volume fraction.

$$V_C = \left(\frac{z}{h} + \frac{1}{2}\right)^k \tag{1a}$$

k is a parameter that dictates material variation profile through the thickness. The value of k equal to zero represents a fully ceramic beam, whereas infinite k indicates a fully metallic beam, and for different values of k one can obtain different volume fractions of metal.

The material properties of FG beams are assumed to vary continuously through the depth of the beam by the rule of mixture (Marur. 1999) as

$$P(z) = (P_t - P_b)\mathrm{V_C} + P_b \tag{1b}$$

where P denotes a generic material property like modulus, P_t and P_b denotes the property of the top and bottom faces of the beam respectively, Here, it is assumed that modules E, $G$and v vary according to the Eq. (1b). However, for simplicity, Poisson’s ratio of beam is assumed to be constant in this study for that the effect of Poisson’s ratio v on deformation is much less than that of Young’s modulus (Delale and Erdogan. 1983, Benachour *et al*. 2011).

2.1 Basic assumptions

The assumptions of the present theory are as follows:
- The origin of the Cartesian coordinate system is taken at the median surface of the FG beam.
- The displacements are small in comparison with the height of the beam and, therefore, strains involved are infinitesimal.
- The transverse normal stress σ_z is negligible in comparison with in-plane stresses σ_x.
- This theory assumes constant transverse shear stress and it needs a shear correction factor to satisfy the plate boundary conditions on the lower and upper surface.

2.2 Kinematics and constitutive equations

Based on the assumptions made in the preceding section, the displacement field can be obtained as

$$\begin{aligned} u(x,z,t) &= u_0(x,t) - z\frac{\partial \phi}{\partial x} \\ w(x,z,t) &= w(x,t) \end{aligned} \tag{2}$$

where u, w are displacements in the x, z directions, u_0 is the neutral surface displacements. ϕ is function of coordinates x and time t.

The strains associated with the displacements in Eq. (4) are

$$\varepsilon_x = \varepsilon_x^0 + z\,k_x \tag{3a}$$

$$\gamma_{xz} = \gamma_{xz}^s \tag{3b}$$

where

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial x}, \qquad k_x = -\frac{\partial^2 \phi}{\partial x^2} \tag{4a}$$

$$\gamma_{xz}^s = \frac{\partial w}{\partial x} - \frac{\partial \phi}{\partial x} \tag{4b}$$

By assuming that the material of FG beam obeys Hooke's law, the stresses in the beam become

$$\sigma_x = Q_{11}(z)\,\varepsilon_x \ \text{ and } \ \tau_{xz} = k_S Q_{55}(z)\,\gamma_{xz} \tag{5a}$$

k_S is a shear correction factor which is analogous to shear correction factor proposed by Mindlin (1951). Using the material properties defined in Eq. (1b), stiffness coefficients, Q_{ij} can be expressed as

$$Q_{11}(z) = \frac{E(z)}{1-\nu^2} \qquad \text{and} \qquad Q_{55}(z) = \frac{E(z)}{2(1+\nu)} \tag{5b}$$

2.3 Equations of motion

Hamilton's principle is used herein to derive the equations of motion. The principle can be stated in analytical form as Reddy (2002)

$$\delta\int_{t_1}^{t_2}(U-K)\,dt = 0 \tag{6}$$

where t is the time; t_1 and t_2 are the initial and end time, respectively; δU is the virtual variation of the strain energy; and δK is the virtual variation of the kinetic energy. The variation of the strain energy of the beam can be stated as

$$\delta U = \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_x \delta\, \varepsilon_x + \tau_{xz} \delta\, \gamma_{xz}) dz_{ns} dx$$
$$= \int_0^L \left(N_x \frac{d\delta\, u_0}{dx} - M_x \frac{d^2 \delta\, w_b}{dx^2} + Q_{xz} \frac{d\delta\,(w-\phi)}{dx} \right) dx \tag{7}$$

Where N, M and Q are the stress resultants defined as

$$(N_x, M_x) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (1, z)\, \sigma_x dz_{ns} \quad \text{and} \quad Q_{xz} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz} dz \tag{8}$$

The variation of the kinetic energy can be expressed as

$$\delta K = \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(z_{ns}) [\dot{u} \delta\, \dot{u} + \dot{w} \delta\, \dot{w}] dz_{ns} dx$$
$$= \int_0^L \left\{ I_0 [\dot{u}_0 \delta \dot{u}_0 + (\dot{w})(\delta\, \dot{w})] - I_1 \left(\dot{u}_0 \frac{d\delta\dot{\phi}}{dx} + \frac{d\dot{\phi}}{dx} \delta\, \dot{u}_0 \right) \right. \tag{9}$$
$$\left. + I_2 \left(\frac{d\,\dot{\phi}}{dx} \frac{d\,\delta\dot{\phi}}{dx} \right) \right\} dx$$

Where dot-superscript convention indicates the differentiation with respect to the time variable t; $\rho(z)$ is the mass density; and (I_0, I_1, I_2) are the mass inertias defined as

$$(I_0, I_1, I_2) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (1, z, z^2) \rho(z) dz \tag{10}$$

Substituting the expressions for δU and δK from Eqs. (7) and (9) into Eq.(6) and integrating by parts versus both space and time variables, and collecting the coefficients of δu_0, $\delta\phi$, and δw, the following equations of motion of the functionally graded beam are obtained

$$\delta\, u_0 : \ \frac{\partial N_x}{\partial x} = I_0 \ddot{u}_0 \tag{10a}$$

$$\delta\, \phi : \ \frac{d^2 M_x}{dx^2} - \frac{\partial Q_{xz}}{\partial x} = I_1 \frac{\partial \ddot{u}_0}{\partial x} - I_2 \frac{\partial \ddot{\phi}}{\partial x^2} \tag{10b}$$

$$\delta\, w : \ \frac{\partial Q_{xz}}{\partial x} = I_0\, \ddot{w} \tag{10c}$$

Eq. (10) can be expressed in terms of displacements (u_0, ϕ, w) by using Eqs. (3), (4), (5) and (8) as follows

$$A_{11} d_{11} u_0 = I_0 \ddot{u}_0 \tag{11a}$$

$$-D_{11} d_{1111} \phi - A_{55}^S d_{11}(w - \phi) = I_1 \frac{\partial \ddot{u}_0}{\partial x} - I_2 \frac{\partial \ddot{\phi}}{\partial x^2} \tag{11b}$$

$$A_{55}^S d_{11}(w - \phi) = I_0 \ddot{w} \tag{11c}$$

where A_{11}, D_{11}, etc., are the beam stiffness, defined by

$$(A_{11}, D_{11}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{11}(1, z^2) dz \tag{12a}$$

and

$$A_{55}^s = k_S \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E(z)}{2(1+\nu)} dz \tag{12b}$$

3. Analytical solution

The equations of motion admit the Navier solutions for simply supported beams. The variables u_0, ϕ, w can be written by assuming the following variations

$$\begin{Bmatrix} u_0 \\ \phi \\ w \end{Bmatrix} = \sum_{m=1}^{\infty} \begin{Bmatrix} U_m \cos(\lambda x) e^{i \omega t} \\ \psi_m \sin(\lambda x) e^{i \omega t} \\ W_m \sin(\lambda x) e^{i \omega t} \end{Bmatrix} \tag{13}$$

where U_m, ψ_m, and W_m are arbitrary parameters to be determined, ω is the eigenfrequency associated with m th eigenmode, and $\lambda = m\pi/L$.

Substituting Eq. (13) into Eq. (11a)-(11c), the closed form solutions can be obtained from

$$([C] - \omega^2 [M])\{\Delta\} = 0 \tag{14}$$

where $\{\Delta\} = \{U_m, \psi_m, W_m\}^t$ and $[C]$ and $[M]$ are the symmetric matrixes given by

$$[C] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}, \quad [M] = \begin{bmatrix} m_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \tag{15}$$

where

$$a_{11} = -A_{11}\lambda^2, \quad a_{12} = 0,$$

$$a_{13} = 0\,,\;\; a_{22} = -\left(D_{11}\lambda^4 + A_{55}^S\lambda^2\right) \tag{16a}$$

$$a_{23} = A_{55}^S\lambda^2\,,\;\; a_{33} = -A_{55}^S\lambda^2$$

$$m_{11} = -I_0\,,\;\; m_{12} = I_1\lambda$$

$$m_{13} = 0\,,\;\; m_{22} = -I_2\lambda^2$$

$$m_{23} = 0\,,\;\; m_{33} = -I_0 \tag{16b}$$

4. Numerical results and discussions

In this section, various numerical examples are presented and discussed to verify the accuracy of present theories in predicting the free vibration response of simply supported FG beams. The FG beam is taken to be made of aluminum and alumina with the following material properties:

Ceramic (P_C: Alumina, Al_2O_3): E_c=380 GPa; ρ_c=3800 kg/m^3; v=0.3;

Metal (P_M: Aluminium, Al): E_m=70 GPa; ρ_c=2707 kg/m^3; v=0.3;

And their properties change through the thickness of the beam according to power-law. The bottom surfaces of the FG beams are aluminium rich, whereas the top surfaces of the FG beams are alumina rich.

For convenience, the following dimensionless form is used:

$$\bar{\omega} = \frac{\omega L^2}{h}\sqrt{\frac{\rho_c}{E_c}}$$

Table 1 Non-dimensional natural frequencies of simply supported homogenous beam versus thickness-to-length ratio (k=0) $\bar{\omega} = \frac{\omega L^2}{h}\sqrt{\frac{\rho_c}{E_c}}$

h/L	Euler-Bernoulli (Reddy1999)	FSDT (Koochaki 2011)	Present
0.01	2.985526	2.986137	2.9861309
0.0125	2.985232	2.985827	2.9858301
0.0142	2.984340	2.985556	2.9855685
0.0166	2.984865	2.985155	2.9851691
0.02	2.983701	2.984505	2.9845053
0.025	2.982588	2.983285	2.9832857
0.033	2.979668	2.980657	2.9806569
0.04	2.976570	2.978020	2.9780219
0.05	2.971688	2.973193	2.9731933
0.066	2.961235	2.962858	2.9628589
0.1	2.931568	2.934044	2.9340444

The effects of thickness-to-length ratio *h/L* and the volume fraction index *p* on the natural frequency of simply supported FG beam is investigated, and the non-dimensional natural frequencies obtained using the new first shear deformation theory (NFSDBT) for homogenous beam (*k*=0) are compared with Euler-Bernoulli beam theory results (Reddy 1999) and the first order shear deformation (Koochaki 2011) in Table 1.

As can be seen the results of the new first shear deformation beam theory is in good agreement with the Euler-Bernoulli beam and the first order shear deformation theory results. Also, the frequencies predicted by the two shear deformation theories are very close to each other. The values of the non-dimensional natural frequency of FG beams for various values of *k* based on the new first shear deformation theory are shown in Fig. 2.

The natural frequencies decrease with increasing the thickness-to-length ratio *h/L* and volume fraction index *k*.

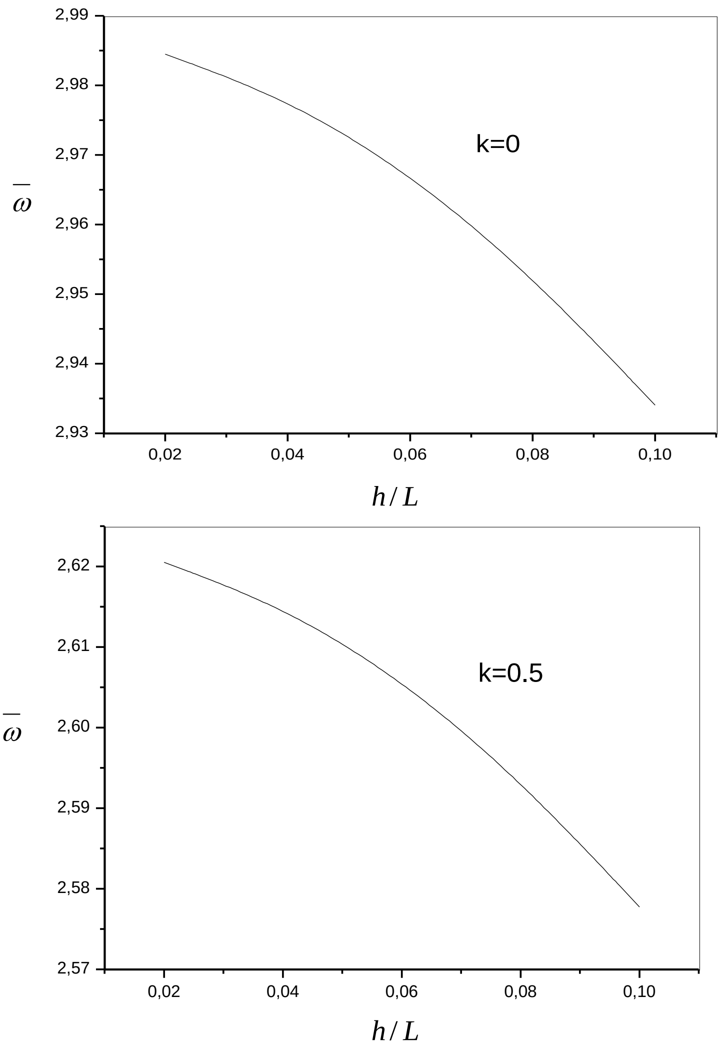

Fig. 2 The values of the non-dimensional natural frequency $\bar{\varpi}$ of FG beams for various values of *k*

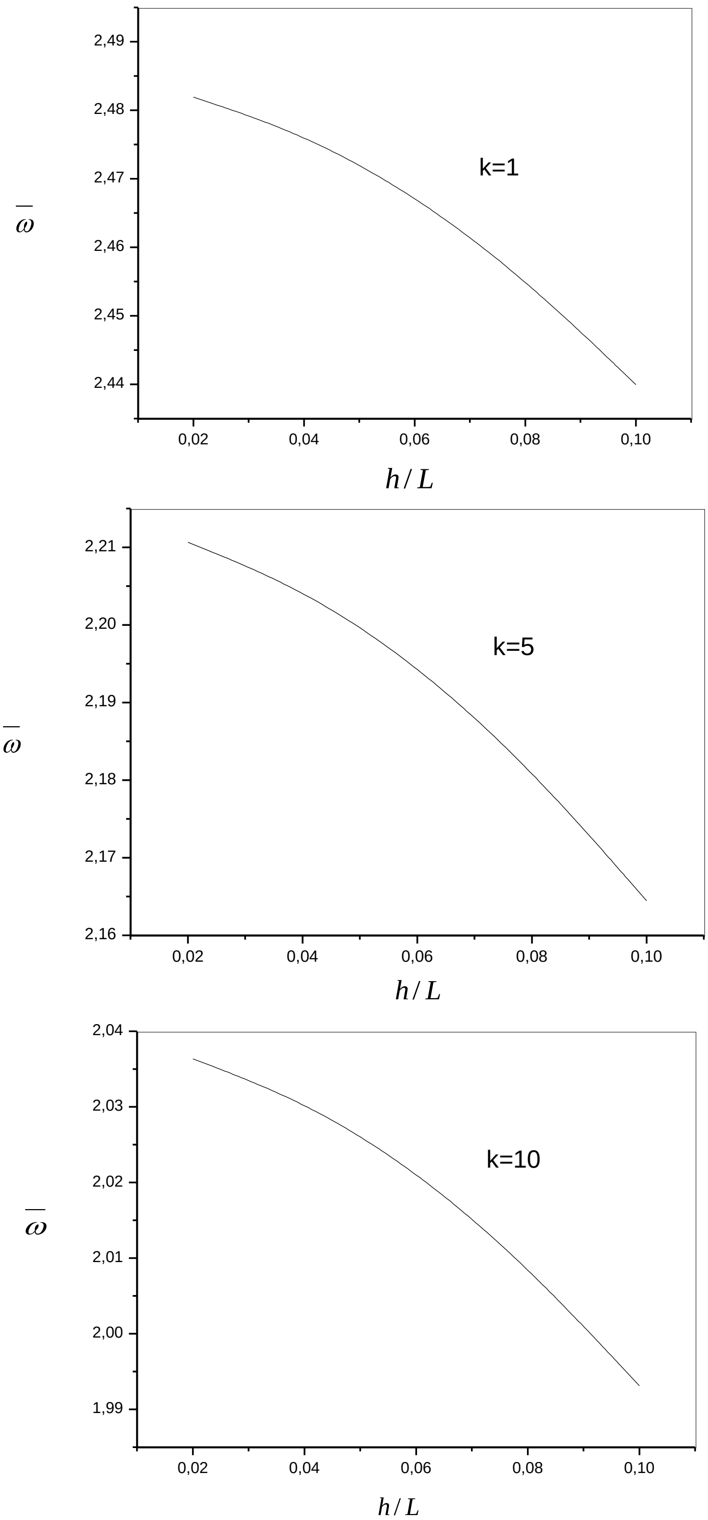

Fig. 2 Continued

5. Conclusions

A New first shear deformation beam theory (NFSDT) is developed for dynamic behavior of FG beam. Based on the present theory, the equations of motion are derived from Hamilton's principle. The effects of volume fraction ratio and thickness-to-length ratio on fundamental frequencies are investigated. The accuracy of the present theory is verified by comparing the obtained results with those reported in the literature. Finally, it can be concluded that the NFSDT is not only accurate but also simple in predicting the dynamic behavior of FG beam.

References

Ait Amar Meziane, M., Abdelaziz, H.H. and Tounsi, A. (2014), "An efficient and simple refined theory for buckling and free vibration of exponentially graded sandwich plates under various boundary conditions", *J. Sandw. Struct. Mater.*, **16**(3), 293-318.

Ait Atmane, H., Tounsi, A. and Bernard, F. (2016), "Effect of thickness stretching and porosity on mechanical response of a functionally graded beams resting on elastic foundations", *Int. J. Mech. Mater. Des*., 1-14.

Ait Yahia, S., Ait Atmane, H., Houari, M.S.A. and Tounsi, A. (2015), "Wave propagation in functionally graded plates with porosities using various higher-order shear deformation plate theories", *Struct. Eng. Mech.*, **53**(6), 1143-1165.

Aydogdu, M. and Taskin, V. (2007), "Free vibration analysis of functionally graded beams with simply supported edges", *Mater. Des.*, **28**(5), 1651-1656.

Bachir Bouiadjra, M., Houari, M.S.A. and Tounsi, A., (2012), "Thermal buckling of functionally graded plates according to a four-variable refined plate theory", *J. Therm. Stress.*, **35**, 677-694.

Belabed, Z., Houari, M.S.A., Tounsi, A., Mahmoud, S.R. and Anwar Bég, O. (2014), "An efficient and simple higher order shear and normal deformation theory for functionally graded material (FGM) plates", *Compos. Part B*, **60**, 274-283.

Benachour, A., Tahar, H.D., Atmane, H.A., Tounsi, A. and Ahmed, M.S. (2011), "A four variable refined plate theory for free vibrations of functionally graded plates with arbitrary gradient", *Compos. Part B: Eng.*, **42**(6), 1386-1394.

Bennoun, M., Houari, M.S.A. and Tounsi, A. (2016), "A novel five variable refined plate theory for vibration analysis of functionally graded sandwich plates", *Mech. Adv. Mater. Struct.*, **23**(4), 423-431.

Bouderba, B., Houari, M.S.A. and Tounsi, A. (2013) "Thermomechanical bending response of FGM thick plates resting on Winkler-Pasternak elastic foundations", *Steel Compos. Struct.*, **14**(1), 85-104.

Bourada, M., Tounsi, A., Houari, M.S.A. and Adda Bedia, E.A., (2012), "A new four-variable refined plate theory for thermal buckling analysis of functionally graded sandwich plates", *J. Sandw. Struct. Mater.*, **14**, 5-33.

Bourada, M., Kaci, A., Houari, M.S.A. and Tounsi, A. (2015), "A new simple shear and normal deformations theory for functionally graded beams", *Steel Compos. Struct.*, **18**(2), 409-423.

Bousahla, A.A., Houari, M.S.A., Tounsi, A. and Adda Bedia, E.A., (2014), "A novel higher order shear and normal deformation theory based on neutral surface position for bending analysis of advanced composite plates", *Int. J. Comput. Meth.*, **11**(6), 1350082.

Delale, F. and Erdogan, F. (1983), "The crack problem for a nonhomogeneous plane", *J. Appl. Mech*., **50**(3), 609-614.

El Meiche, N., Tounsi, A., Ziane, N., Mechab, I. and Adda Bedia, E.A., (2011), "A new hyperbolic shear deformation theory for buckling and vibration of functionally graded sandwich plate", *Int. J. Mech. Sci.*, **53**, 237-247.

Hadji, L., Atmane, H.A., Tounsi, A., Mechab, I. and Adda Bedia, E.A., (2011), "Free vibration of functionally graded sandwich plates using four variable refined plate theory", *Appl. Math. Mech.*, **32**, 925-942.

Hamidi, A., Houari, M.S.A., Mahmoud, S.R. and Tounsi, A. (2015), "A sinusoidal plate theory with 5-unknowns and stretching effect for thermomechanical bending of functionally graded sandwich plates", *Steel Compos. Struct.*, **18**(1), 235-253.

Hebali, H., Tounsi, A., Houari, M.S.A., Bessaim, A. and Adda Bedia, E.A. (2014), "A new quasi-3D hyperbolic shear deformation theory for the static and free vibration analysis of functionally graded plates", *ASCE J. Eng. Mech.*, **140**, 374-383.

Houari, M.S.A., Benyoucef, S., Mechab, I., Tounsi, A. and Adda Bedia, E.A. (2011), "Two variable refined plate theory for thermoelastic bending analysis of functionally graded sandwich plates", *J. Therm. Stress.*, **34**, 315-334.

Klouche Djedid, I., Benachour, A., Houari, M.S.A., Tounsi, A. and Ameur, M. (2014), "A *n*-order four variable refined theory for bending and free vibration of functionally graded plates", *Steel Compos. Struct.*, **17**(1), 21-46.

Koochaki, G.R. (2011), "Free vibration analysis of functionally graded beams", *World Acad. Sci. Eng. Tech.*, **74**, 514-517.

Mahi, A., Adda Bedia, E.A. and Tounsi, A. (2015), "A new hyperbolic shear deformation theory for bending and free vibration analysis of isotropic, functionally graded, sandwich and laminated composite plates", *Appl. Math. Model.*, **39**, 2489-2508.

Marur, P.R. (1999), "Fracture behaviour of functionally graded materials", Ph.D. Dissertation, Auburn University, Auburn, AL, USA.

Mindlin, R.D. (1951), "Influence of rotary inertia and shear on flexural motions of isotropic elastic plates", *ASME J. Appl. Mech.*, **18**, 31-38.

Reddy, J. N. (1999), *Theory and Anaslysis of Elastic Plates*, Taylor & Francis Publication, Philadelphia.

Sankar, B.V. (2001), "An elasticity solution for functionally graded beams", *Compos. Sci. Tech.*, **61**(5), 689-696.

Reddy, J.N. (2002), *Energy Principles and Variational Methods in Applied Mechanics*, John Wiley & Sons Inc.

Taj, M.N.A., Chakrabarti, A. and Sheikh, A., (2013), "Analysis of functionally graded plates using higher order shear deformation theory", *Appl. Math. Model.*, **30**, 1-11.

Tounsi, A., Houari, M.S.A., Benyoucef, S. and Adda Bedia, E.A. (2013), A refined trigonometric shear deformation theory for thermoelastic bending of functionally graded sandwich plates", *Aerosp. Sci. Tech.*, **24**, 209-220.

Zidi, M., Tounsi, A., Houari, M.S.A., Adda Bedia, E.A. and Anwar Bég, O. (2014), "Bending analysis of FGM plates under hygro-thermo-mechanical loading using a four variable refined plate theory", *Aerosp. Sci. Tech.*, **34**, 24-34.

Zhong, Z. and Yu, T. (2007), "Analytical solution of a cantilever functionally graded beam", *Compos. Sci. Tech.*, **67**(3-4), 481-488.

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